WARWICK

ECONOMICS

CRETA

Centre for Research in Economic Theory and its Applications

Discussion Paper Series

Utility Representation in Abstract Wiener Space

G. Charles-Cadogan

April 2021 No: 70

CRETA
Centre for Research in Economic Theory and its Applications
Department of Economics
University of Warwick, Coventry,
CV4 7AL, United Kingdom

warwick.ac.uk/fac/soc/economics/research/centres/creta/papers

Utility Representation in Abstract Wiener Space*

G. Charles-Cadogan †
Preliminary draft

April 21, 2021

Abstract

We extend Machina's (1982) preference functional to abstract Wiener space. This has the advantage of extending utility functions to: infinite dimensional spaces; providing estimates for Machina's (1982) nonlinear utility functional; and establishing a nexus between microfoundations of local utility, subjective probability, prospect theory, and elements of quantum decision theory without complex valued Hilbert spaces. For example, the class of Markowitz nonconvex utility functions (for which prospect theory's value function is a special case) are vector valued functions in abstract Wiener space. Instead of preferences over probability distributions, the problem is transformed into one of preferences over states. Under Arzela-Ascoli Theorem, Wiener measure is the limit and unique conjugate prior in Wiener space. By a change of measure local subjective (posterior) probability is a Wiener integral. So, binary choice is stochastic. This poses a challenge for the transitivity axiom because intransitive preferences will occur in that space almost surely. Savage's (1972) SEU fails in the space because probability is state dependent. Decision weights for Rank Dependent Expected Utility are additive only if their sum grows like the law of iterated logarithm $O_p\left(\left(2\sigma_n^2 \log\log\sigma_n^2\right)^{-1/2}\right)$ where σ_n^2 is the sample variance of decision weights.

Keywords: decision theory, local utility, nonlinear subjective probability, abstract Wiener spaces

JEL Classification: C02, D81

1 Introduction

Machina (1982) introduced a utility functional and exploited its Frechet derivative to establish a local utility representation in a *de facto* Banach space. In this paper we show how Machina's (1982) seminal model of local expected utility extends to abstract Wiener space. In that setup, local utility is an abstract Fourrier coefficient, and local probability is a stochastic integral so it is highly nonlinear. Furthermore, Born rule type results for nonlinear probabilities are obtained without appeal to complex valued Hilbert spaces (cf. Charles-Cadogan, 2018).[1] We also present a

*I have no competing interest or conflicts of interest to declare. I thank Hui-Hsiung Kuo and Peter Hammond for their encouragement on this topic, and Leonard Gross for bringing my attention to application of abstract Wiener space in mathematical finance. Any errors which may remain are my own.

†Division of Finance, School of Business, University of Leicester, Leicester, LE1 7RH, United Kingdom; and Institute for Innovation and Technology Management (IITM), Ted Rogers School of Management, Ryerson University, 575 Bay, Toronto, ON M5G 2C5; Tel: +44 (0116) 229 7385; e-mail: : gcc13@le.ac.uk

[1]There is a growing literature on quantum decision theory (QDT) which makes use of quantum logic and quantum mechanics concepts to explain decision making in the presence of risk (Aerts et al., 2018). The relationship between statistical decision theory and quantum principles is known for a while. See e.g. Holevo (1973, 2011) and references therein. QDT is often presented as a theory of decision making that resolves seemingly irrational behaviour that classic expected utility theory (EUT), and behavioural decision making theories like prospect theory (PT) and cumulative prospect theories (CPT) (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992) are unable to explain. Refer to Khrennikov (2009); Khrennikov and Haven (2009); Asano et al. (2012) and Jerome Busemeyer and his co-workers, e.g. Busemeyer and Townsend (1993); Busemeyer et al. (2006, 2011, 2014). However, much of that work takes place in complex valued Hilbert space (see e.g. Basieva et al., 2018).

new result that additive decision weights in rank dependent utility (RDU) require a normalizer that follows the law of the iterated logarithm. This result has implications for large sample economic experiments that assume additivity of decision weights without normalization.

2 The Model

In this section we introduce the basic elements in Machina (1982) choice set and show how they extend to abstract Wiener space. The main result is Theorem 2.3. Applications follow in Section 3.

2.1 Extension of Machina Choice Space to Abstract Wiener Space

Machina (1982, pg. 293) introduced a preference functional $V(F)$ on a choice set

$$D[0,M] = \{F \,|\, F:[0,M] \to [0,1], \quad F(x) = \Pr\{X \leq x\},\, x \in [0,M] \tag{2.1}$$

where the domain of V is the space of Lebesgue integrable functions $L[0,M]$ and F is a distribution function on X.. The introduction of a norm $\sup\|\cdot\|$ on $L[0,M]$ induces a Banach space $L_B[0,M] = (L[0,M], \sup\|\cdot\|)$. In particular, since $F \in L[0,M]$, we have the choice set $D[0,M] \subset L_B[0,M]$. We assume that $V(F)$ is not of bounded variation.[2]

For convenience, we normalize the range $[0,M]$ so that $D[0,M]$ is now on the unit interval $D[0,1]$. Instead of distribution functions F defined on the Lebesgue integrable space $L[0,1]$, we consider a *smaller class* of distribution functions F defined on the subspace of square integrable functions $D[0,1] \cap L^2[0,1] \subset L[0,1]$. The complex valued functions defined on $L^2([0,1],\mu)$ with respect to a measure μ, and inner product $< f,g >_H = \int f\bar{g}\mu(dx)$ is a Hilbert space (Reed and Simon, 1980, p. 40). Without loss of generality, in this paper all functions are real valued. Furthermore, Hahn-Banach extension theorem (Reed and Simon, 1980, pg. 75) allows us to extend linear functionals on $L^2[0,1]$ to $L[0,1]$. For notational convenience we let $L^2_H[0,1] = \left(L^2[0,1], <,>_H\right)$ be the Hilbert space induced by the inner product $<,>_H$ on $L^2[0,1]$, and $\widetilde{L}^2_H[0,1]$ be the extended

[2]Technically, if $V(F)$ is of bounded variation, then the Banach space is not separable (Adams, 1936, p. 421). Carmona and Tehranchi (2006, pp. 76-77) address issues arising from measure on nonseparable spaces. Besides, a separable Banach space can be embedded in a space of continuous (Bessaga and Pełczyński, 1975, p. 50, Banach-Mazur Thm) but nowhere differentiable functions (Wiener, 1923; Rodríguez-Piazza, 1995)–the subject matter of this paper.

Hilbert space over $L[0,1]$. So in principle Machina's preference functions are defined on this extended space. Let

$$\iota : \widetilde{L}^2_H[0,1] \hookrightarrow L_B[0,1] \tag{2.2}$$

be an inclusion map such that $\iota\left(\widetilde{L}^2_H[0,1]\right)$ is dense in $L_B[0,1]$. Then

$$(\iota,\, \widetilde{L}^2_H[0,1],\, L_B[0,1]) \tag{2.3}$$

is an abstract Wiener space,[3] and the projection of $L^2[0,1]$ on $\mathbb{R}$ is a Gaussian measure P_*. See e.g. (Cameron and Martin, 1944, Lemma 3., and eq (6.5), pg. 393), (Gross, 1967, Cor. 1, p. 38), and (Nualart, 2006, pp. 225-226). More formally:

Definition 2.1 (Abstract Wiener space)**.** Cameron and Martin (1944); Gross (1967).
Let Ω be a separable Banach space, $P_\star$ be a probability measure on Ω, and $\mathscr{F}$ be the σ-field of Borel measurable subsets of Ω. There exist a separable Hilbert space $\mathfrak{H}$ that is continuously and densely embedded in Ω with inclusion map $\iota : \mathfrak{H} \to \Omega$ and such that

$$\int_\Omega e^{i<x,y>} P_\star(dx) = e^{-\frac{1}{2}\|y\|^2_{\mathfrak{H}}} \tag{2.4}$$

for any $y \in \Omega^\star \subset \mathfrak{H}^\star$ where $\Omega^\star$ and $\mathfrak{H}^\star$ are dual spaces of Ω and $\mathfrak{H}$ respectively. The triple $(\Omega, \mathfrak{H}, P_\star)$ is called an abstract Wiener space. □

Theorem 2.1 (Abstract Wiener space over Banach space)**.** *(Kuo, 1975, Thm 4.4, p. 79) Let $\mathfrak{B}$ be a real separable Banach space. Then there exist a separable Hilbert space $\mathfrak{H}$ densely embedded in $\mathfrak{B}$ such that $\mathfrak{B}$-norm is measurable over $\mathfrak{H}$, i.e. $(\iota, \mathfrak{H}, \mathfrak{B})$ is an abstract Wiener space, where ι is the inclusion map from $\mathfrak{H}$ into $\mathfrak{B}$.* □

Based on the foregoing, Machina's (1982) (normalized) preference functional satisfies the following application of Theorem 2.1

Theorem 2.2 (Machina's (1982) preference functional on abstract Wiener space)**.** *Machina (1982) preference functional $V(F)$ defined on the choice set $D[0,1]$ extends to the abstract Wiener space* $(\iota,\, \widetilde{L}^2_H[0,1],\, L_B[0,1])$.

It is known that the class of Hermite polynomials $\{H_n(x)\}_{n=0}^{n=\infty}$ form an orthonormal basis

[3] See Theorem 2.1, *infra*. Without the norm $<,>_H$, $(\iota,\, L^2[0,1],\, L[0,1])$ is not an abstract Wiener space (Kuo, 1975, p. 86).

for the Hilbert space $\widetilde{L}^2_H$ (see e.g. Akheizer and Glazman, 1961, pg. 25) where

$$H_n(x) = (-1)^n\, e^{x^2} \frac{d^n\, e^{-x^2}}{dx^n} \tag{2.5}$$

Figure 1: State functions distributions

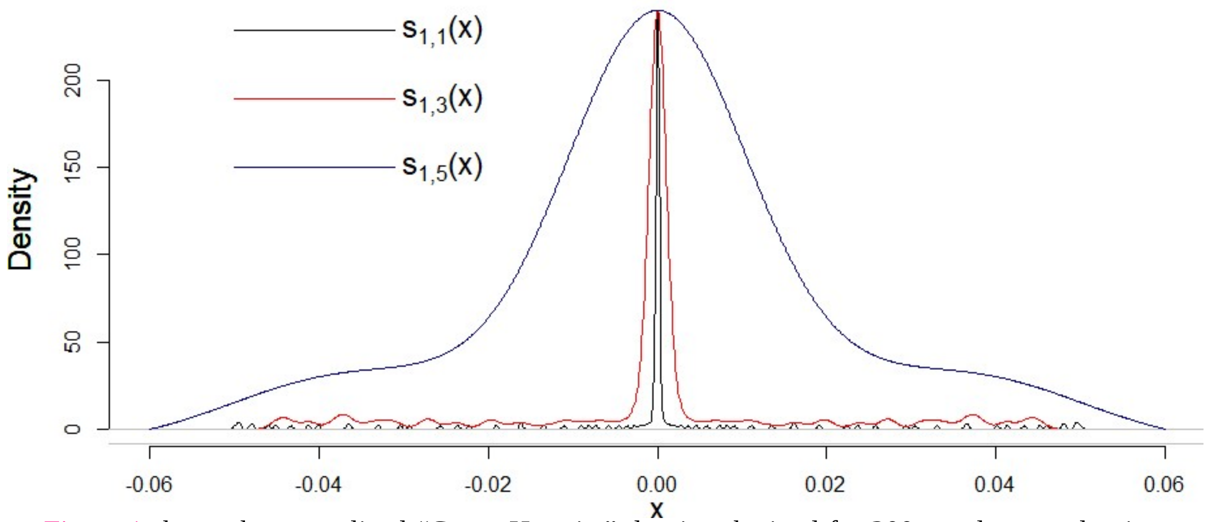

Figure 1 shows the normalized "Gauss-Hermite" density obtained for 200 evenly spaced points in the interval $[-2,2]$ with a bandwith=0.001. The plots imply a monotonic increasing rank order for risk for s_{11}, s_{13}, s_{15}, respectively. Since we are working with the positive half range $[0,2]$, the half density must be multiplied by 2 to preserve the total area under the curve being 1.

Without loss of generality, the interval $[0,M] \subset \mathbb{R}$. So the projection of $\widetilde{L}^2_H$ on $\mathbb{R}$ can be restricted to $[0,M]$ in the sequel for $x \in [0,M]$. Moreover, linear functionals on $\widetilde{L}^2_H$ can be represented by the orthogonal basis functions $H_n(x)$. See Cameron and Martin (1947). In particular, $\mathfrak{D}$ is a subspace of $\widetilde{L}^2_H$ so it inherits this basis. Thus, we write the (orthogonal) basis state functions

$$s_{1,n}(x) = A_n H_n(x) e^{-\frac{x^2}{2}} \tag{2.6}$$

which upon normalization (see e.g. Wiener (1933, p. 57), Boyd (2018, Appendix A)) gives

$$\int_{-\infty}^{\infty} |s_{1,n}(x)|^2 dx = A_n^2 \int_{-\infty}^{\infty} H_n^2(x) e^{-x^2} dx = 1, \qquad A_n = (2^n n!\,\sqrt{2\pi})^{-1} \tag{2.7}$$

Here $|s_{1,n}(x)|^2$ is a state dependent "Gauss-Hermite" probability density function on $\mathbb{R}$.[4] A plot of the densities for those functions are shown in Figure 1. The density functions are characterized by high kurtosis and symmetry around 0. A risk averse decision maker (DM) will have state preference $s_{11} \succ s_{13} \succ s_{15}$. We let S be the set of states in the sequel.

2.1.1 Vector valued utility as abstract Fourier coefficients in Hilbert space

Let $U(x)$ be a utility function defined on X taking values in $\mathbb{R}$, and $x \in X$ i.e. $U : X \to \mathbb{R}$. The abstract Fourier coefficient (Haase, 2010, p. 89) is given by

$$u_j(x,s) = \frac{< U,\ s_{1,j} >}{< s_{1,j},\ s_{1,j} >} = \frac{\int_{x \in X} U(z)\, s_{1,j}(z) dz}{\int_{x \in X} |s_{1,j}(z)|^2\, dz} \tag{2.8}$$

where $u_j(x,s)$ is a coordinate of the vector valued function (Reed and Simon, 1980, p. 40) $U(x)$ relative to the "axis" $s_{1,j}$ and $\mathbf{u}(s) = (u_1(s), u_2(s), \ldots)$ is an infinite dimensional vector. The $s'_{1,j}s$ constitutes a non-unique behavioral [orthogonal] basis, call it S, for our separable Hilbert space $\widetilde{L}^2_H$. In other words, $\mathbf{u}(s)$ is the orthogonal projection of U on S. In *quantum mechanics* objects like $s_{1,j}$ are called wave functions, and they are typically complex valued solutions to Schrödinger's equation. In particular, $s_{1,j}$ *is a state*, and a utility functional is a superposition of the set of all possible states in a given [bounded] domain. That is, the utility function traverses the state space and intersects all possible states such that it is the locus of the "cut points", and $u_j(x,s)$ is the local utility element of the infinite dimensional vector values utility function.

For the purpose of exposition we used the Gaussian projection of abstract Wiener space to qualify the Wiener measure $P_\star$, and circumvent complex valued basis functions. While our model has aspects of *quantum mechanics* it differs on the wavefunction representation. Consequently, the probability that $s_{1,n}(x)$ lies in a region $A \in \mathfrak{B}(\mathbb{R})$ (up to a normalizing constant) is given by

$$\mathrm{P}(A) = \Pr\{s_{1,n}(x) \in A\} = \int_{x \in u^{-1}_{1,n}(A)} |s_{1,n}(x)|^2 dx \tag{2.9}$$

[4] See Kreyszig (1978, p. 181) where Gram-Schmidt orthogonalization was applied to the product of a Gaussian density $e^{-x^2/2}$ and Hermite polynomial $H_n(x)$ to derive an orthogonal basis like $\{s_{1j}\}_{j=1}^{\infty}$. Even though there are other orthogonal bases, for our purposes the "Gauss-Hermite" conveniently facilitates computation of a posterior probability in the conjugate family in Section 3.

This is a nonlinear state dependent probability that resembles Born's rule (Wang et al., 2013; Charles-Cadogan, 2018) and nonlinear source dependent probability weighting popularized by prospect theory (Abdellaoui et al., 2011). For

$$U(x;\mathbf{u}) = \sum_{j=0}^{\infty} u_j s_{1,j}(x) \tag{2.10}$$

u_j is local utility, and it is the j-th coordinate of $U(x;\mathbf{u})$ such that

$$\int_{-\infty}^{\infty} |U(x;\mathbf{u})|^2 dx = \sum_{j=0}^{\infty} u_j^2 \int_{-\infty}^{\infty} |s_{1,j}(x)|^2 dx = \sum_{j=0}^{\infty} u_j^2 < \infty \tag{2.11}$$

according to Riesz-Fisher Theorem (Riesz and Sz-Nagy, 1955, p. 70). Thus,

$$\left(\sum_{j=0}^{\infty} u_j^2\right)^{-1} \int_{-\infty}^{\infty} |U(x;\mathbf{u})|^2 = 1 \tag{2.12}$$

Normalize u_j so that for $\boldsymbol{u} = (u_1, u_2, \ldots, u_n, \ldots)$, $\quad \widehat{u}_j = \frac{u_j}{\|\boldsymbol{u}\|}$ and write

$$U(x;\widehat{\mathbf{u}}) = \sum_{j=0}^{\infty} \widehat{u}_j s_{1,j}(x) \text{ where } \widehat{\boldsymbol{u}} = (\widehat{u}_0, \widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_n, \ldots) \tag{2.13}$$

where $\widehat{u}_j$ is a normalized coordinate. In that way $\int_{-\infty}^{\infty} |U(x;\widehat{\mathbf{u}})|^2 dx = 1$.[5] The foregoing analysis shows that $U(x;\mathbf{u})$ is in the class of Friedman and Savage (1948); Markowitz (1952) utility functions if we set $u_j = 0$ for even Hermite polynomials (by virtue of the behavioral basis functions $s_{1,j}$ relationship in (2.6)) because odd Hermite polynomials pass through the origin and odd powers admit the sinusoidal pattern in Markowitz's (1952) sketch.

Assumption 1. $u_2 j = 0,\ j = 1, 2, \ldots$ *for even Hermite polynomials.*

Thus, the behavioural probability that the utility path $U(x;\mathbf{u})$ is in a given set $A \in \mathfrak{B}(\mathbb{R})$ is given by

$$\mathrm{P}(A) = \Pr\{U(x;\mathbf{u}) \in A\} = \Pr\{U(x;\widehat{\mathbf{u}}) \in A\} = \int_{x \in U^{-1}(A)} |U(x;\widehat{\mathbf{u}})|^2 dx \tag{2.14}$$

But this probability is functionally equivalent to the joint probability associated with an infinite

[5] See Anscombe and Aumann (1963, p. 201) for normalization of utility functions.

dimensional cylindrical set

$$\mathrm{P}(A) = \Pr\{u_1 \in A_1, u_2 \in A_2, \ldots, u_n \in A_n, \ldots\} \tag{2.15}$$

$$= \Pr\{(u_1, u_2, \ldots, u_n, \ldots) \in A_0 \times A_1 \times \ldots A_n \times \ldots\} \tag{2.16}$$

Furthermore, by construction local utility $u_j(s)$ is state dependent for states $s \in S$ in (2.8). According to Kolmogorov's representation theorem (Gikhman and Skorokhod, 1969, pp. 107-108), there exist a probability space (Ω, F, P) and a function $g(u(s), \omega)$ such that the two objects constitute a representation of (2.16). Theorem 2.3 gives meaning to those objects for $V(F) \in \widetilde{L}^2_H[0,1]$ where we have the following

Main Result

Theorem 2.3 (Infinite dimensional state dependent Machina preference functional)**.** *Let* S *be the set of states,* $s \in \mathrm{S}$, $(V(F) \in \widetilde{L}^2_H[0,1]$ *and the projection of* $\widetilde{L}^2_H[0,1]$ *on* $[0,1]$ *be the Wiener measure* $P_\star$. *Then*

$$V(s,F) = \sum_{j=1}^{\infty} \widehat{u}_j(s) v_j(s, P_\star) \quad \widehat{u}_j(s) = \frac{u_j(s)}{\|\boldsymbol{u}(s)\|}, \ \widehat{\boldsymbol{u}}(s) = (\widehat{u}_0(s), \widehat{u}_1(s), \widehat{u}_2(s), \ldots, \widehat{u}_n(s), \ldots) \tag{2.17}$$

is a state dependent representation of $V(F)$ *on the abstract Wiener space* $\left(L[0,1], \widetilde{L}^2_H[0,1], P_\star\right)$ *where* $u_j(s)$ *is state dependent local utility,* $v_j(s, P_\star) = \int_{x \in X} s_{1,j}(x) dP_\star(x)$ *is a local state dependent probability measure drawn from the censored probability density* $2\mathcal{N}(0, \|s_{1j}\|^2)$ *for states* $s_{1,j}(x)$, *and* $E[v_j^2(s, P_\star)] = \|s_{1j}(X)\|^2$ *and normal distribution* $\mathcal{N}$, $\widehat{u}_j(s) v_j(s, P_\star)$ *is state dependent local expected utility, and* F *is a distribution function over* $\widehat{\boldsymbol{u}}(s)$. *Moreover,* $V(s,F)$ *is also a Wiener functional.* □

Remark 2.1. If $\sum_j v_j(P_\star) = 1$, then $V(F)$ coincides with Von Neuman and Morgenstern (1953) expected utility functional. Note that v_j is distributed as $2\mathcal{N}(0, \|s_{1j}\|^2)$ to censor negative probabilities that may arise from Wiener integral.

Corollary 1 (Wiener integral)**.** *Local probability* $v_j(s, P_\star) = \int_X s_{1,j}(x) dP_\star(x)$ *is a Wiener integral.*

Corollary 2 (Estimates for $V(F)$)**.** $\|V(F)\| \leq K_N^{\frac{1}{2}} \left(\sup_j \int_X s^2_{1,j}(x).dx\right)^{\frac{1}{2}}$ *for some constant* K_N. □

Proof.

$$\|V(F)\|^2 = \left\|\sum_{j=1}^{\infty} \widehat{u}_j v_j(s,P_\star)\right\|^2 \leq \left(\sum_{j=1}^{\infty} \widehat{u}_j^2\right) \sup_j \|v_j^2(P_\star)\|^2 \leq \left(\sum_{j=1}^{N} \widehat{u}_j^2\right) \sup_j \int_X s_{1,j}^2(x)|dP_\star(x)|^2 \leq \tag{2.18}$$

$$K_N\left(\sup_j \int_X s_{1,j}^2(x).dx\right) < \infty \quad \text{where } \sum_{j=1}^{\infty} \widehat{u}_j^2 = 1,\ K_N = \sum_{j=1}^{N} \widehat{u}_j^2 > 1,\ K_N \downarrow 1 \text{ and } |dP_\star(x)|^2 = dx. \tag{2.19}$$

□

3 Applications and Numerical Experiment

3.1 Subjective expected utility

3.1.1 SEU

This example is motivated by Karni (2017) who argued that state dependent utility functions are supported by Savage's (1972) postulates. In the context of Savage (1972) Subjective Expected Utility Theory (SEU) there exist a set of acts F (by abuse of notation this is different from the distribution function F), a set of states S, and a probability distribution over F such that $f \in F$ and $s \in S$, and $f(s) = x$ is constant. Acts map states into a space of outcomes (called consequences) X. So $F = X^S$. DMs have (unique) subjective probability $\pi(s)$ over F, for which they evaluate the expected utility $\sum \pi(s)u(f(s))$. We assume the usual binary relation $\succ$ to mean "strictly prefer" and $\sim$ to mean "indifferent to" and $\succeq$ to mean "weakly preferred to". So that for $x, y \in X$ we have $x \succ y \Rightarrow f(s) \succ g(s)$ where $f(s) = x$ and $g(s) = y$ for $f, g \in F$. In our set up in Theorem 2.3, $u(f(s))$ is replaced by local utility $u_j(s)$. $\pi(s)$ is replaced with $v_j(s, P_\star)$ which behaves like a posterior probability in the sense that $P_\star(x)$ is the canonical prior in abstract Wiener space, $s_{1j}(x)$ is a "Gauss-Hermite" function, and Wiener measure is predicated on the Gaussian distribution which is a canonical conjugate prior (DeGroot, 1970, §9.5). So, $v_j(s, P_\star)$ is a posterior Wiener functional in the conjugate class of Gaussian probability measures. However, in SEU theory $\pi(s)$ is independent of states (see (Baccelli, 2020) and references therein) while our $v_j(s, P_\star)$ is not by construction.

To see this, we use the kernel function technique in Raiffa and Schlaifer (1961, p. 47) and define a kernel function $\kappa(s|P_\star)$ and a function $N(s)$ such that $N(s)\kappa(s|P_\star) = s_{1j}P_\star$. By virtue of s_{1j} and $P_\star$ being members of the conjugate family of exponential functions, Raiffa and Schlaifer

(1961, Thm, eq(3.3)) posit that the posterior distribution is of the same form up to a constant of variation. Thus, $v_j(s,P_\star)$ is a posterior density with prior $P_\star$. Because $v_j(s,P_\star)$ is state dependent, it does not satisfy Savage's (1972) requirement of independence (Kreps, 1988, pp. 34-35) nor Abdellaoui and Wakker (2020) more recent generalized SEU theory. Thus, we provide an example of a decision space where Savage's (1972) SEU does not apply. If we substitute $v_j(P_\star)$ with $P_\star$, then SEU applies in the sense of Karni (2017) since $P_\star$ is independent of state dependent local utility in Theorem 2.3.

3.1.2 Cumulative Prospect Theory

One of the linchpins of Tversky and Kahneman (1992) Cumulative Prospect Theory (CPT) is the use of Quiggin's (1982) Rank Dependent Utility (RDU) transformation of nonlinear probability into linear decision weights. The additive linearized weights produced by RDU (Wakker, 2010, Chapters 7-8) fail in abstract Wiener space for the following reason. Let $(w \circ v_j)(P_\star)$ be a composite probability weighting function for some probability weighting function $w(\bullet)$. Since v_j is a Wiener integral i.e. a Wiener functional, the composite function is also a Wiener functional. RDU linearization of a Wiener process produces independent increments that are also Wiener processes (Gikhman and Skorokhod, 1969, p. 21). The RDU linearization scheme for decision weights $\pi_j(P_\star)$, for a probability distribution $v_1(P_\star), v_2(P_\star), \ldots, v_n(P_\star)$ and weighting function $w(\bullet)$ with $P_\star$ suppressed, are as follows:

$$\pi_1 = w(v_1), \qquad \pi_2 = w(v_1 + v_2) - w(v_1) \tag{3.1}$$

$$\vdots$$

$$\pi_j = w(v_1 + v_2 + \cdots + v_j) - w(v_1 + v_2 + \cdots + v_{j-1}), \ldots, \quad \pi_n = 1 - w(v_1 + v_2 + \cdots + v_{n-1}) \tag{3.2}$$

where $\pi(0) = 0$. Each π_j is itself a Wiener functional of $v(P_\star)$. Let the sum of the linearized probabilities be given by $S_n(P_\star) = \sum_{j=1}^{n} \pi_j(P_\star) = 1$. There, $P(S_n(P_\star) = 1) = 0$ almost surely and linearization fails.

Rank Dependent Utility decision weights and the law of the iterated logarithm

As shown above, without more, additivity of RDU linearized decision weights fails almost surely in Wiener space. However, if additive decision weights are transformed in the context of the law of the iterated logarithm[6] (LIL) in Banach spaces, then linearization holds. To see this, we begin with Kuo (1975, Thm 5.3) which states that $(\iota, C'[0,1,c[0,1])$, where $C[0,1]$ is the space of continuous functions on [0,1] and $C'[0,1]$ is the space of first derivative of functions defined in $C[0,1]$, is an abstract Wiener space. In particular, if we endow $C'[0,1]$ with the inner product norm, we have, for some derivative operator D and probability weighting functional $w()$, that $< Dw, dP_\star >= \int Dw\, dP_\star$ is a stochastic integral–in this case a Wiener integral. The canonical norm in $C[0,1]$ is the sup-norm $\| f \| = \sup_{0\leq t\leq 1} |f(t)|$. For $\iota : C'[0,1] \hookrightarrow C[0,1]$ we have from the Cauchy-Schwarz inequality

$$|< Dw, dP_\star > | = \left| \int DwdP_\star \right| \leq \left\| \int Dw \right\| \left\| \int dP_\star \right\| \leq \sup \left\| \int Dw \right\| \sup \left\| \int dP_\star \right\| = \sup |w| = \|w\| \tag{3.3}$$

This implies that $\|\bullet\|$ is measurable over $C'[0,1]$. The inclusion map embeds $(C'[0,1], <,>)$ in $(C[0,1], \|\bullet\|)$. Furthermore, for given v_1, v_2 and D a Frechet derivative, we have

$$\pi_2(P_\star) = \pi(v_1, v_2; P_\star) = w(v_1 + v_2; P_\star) - w(v_1; P_\star) = v_2 Dw(v_1; P_\star) + o(\|v_2\|) \tag{3.4}$$

Here, π_2 is a Banach valued random variable in $C'[0,1]$. It is also an increment of the Wiener functional $w(\bullet)$ so it is independent. Furthermore,

$$E[\pi_2(P_\star)] = E[v_2 Dw(v_1) + o(\|v_2\|)] = 0, \quad E[\pi_2^2(P_\star) = E[(v_2 Dw(v_1) + o(\|v_2\|))^2] < \infty \tag{3.5}$$

since $E[v] = 0$. By induction, the relation in (3.4) extends to all decision weights $\pi_1, \pi_2, \ldots, \pi_n$. The inclusion map embeds $\pi_j, \forall j$ in the Banach space $(C[0,1], \|\bullet\|)$ which is separable by hypothesis. The foregoing analyses in (3.4) and (3.5) support the following

Proposition 1 (RDU decisions weights and LIL)**.**
Let $\pi_1, \pi_2, \ldots$ be an independent and identically distributed set of Banach-space valued decision

[6] The interested reader is directed to Chow and Teicher (1988, §10.2) for details on construction of the LIL.

weights such that: $E[\pi] = 0$ *and* $E[\|\pi\|^2] < \infty$. *Then*

$$P\left\{\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} = 1\right\} = 1$$

where $S_n = \pi_1 + \pi_2 + \ldots + \pi_n$, $\sigma_n^2 = \sum_{j=1}^n \|\pi_j\|^2$ *and* $a_n = \sqrt{2\sigma_n^2 \log\log \sigma_n^2}$

Proof. See Kuelbs (1977, Thm 4.2(1)) □

3.2 Local expected utility

Machina (1982, pp. 294-295) proffered $V(\cdot)$ as a nonlinear functional, and posited

$$V(F^\star) - V(F) = \int U(x;F)(dF^\star(x) - dF(x)) + o(\|F^\star - F\|) \tag{3.6}$$

as a first order expansion of a Frechet derivative. If $F^\star \succeq F$, then $V(F^\star) - V(F) \geq 0$. In our case, Theorem 2.3 tells us that

$$V(F^\star) - V(F) = \sum_{j=0}^{\infty} (\widehat{u}_j^\star - \widehat{u}_j) v_j(s, P_\star) \tag{3.7}$$

is a Wiener functional where $\widehat{u}_j^\star$, $\widehat{u}_j$ and v_j depend on the state $s_{1,j}(x)$. So, if $\sum_{j=0}^{\infty} (\widehat{u}_j^\star - \widehat{u}_j) v_j(s, P_\star) > 0$, then probabilistically, our DM weakly prefers the local utilities $\widehat{u}_j^\star$, in the coordinate vector $\widehat{\mathbf{u}}^\star$ for $V(F^\star)$, to $\widehat{u}_j$ in the coordinate vector $\mathbf{u}$ for $V(F)$, i.e. $\widehat{u}_j^\star \succeq \widehat{u}_j$ implies $\Pr\{\widehat{u}_j^\star \geq \widehat{u}_j\} > 0$ almost surely. This invokes the weak stochastic transitivity axiom (WSTA) of Tversky (1969). Unlike Machina's local utility in (3.6), the aggregated local utility in (3.7) is subject to Condorcet's paradox which holds that aggregation on individual transitive choice can result in intransitive preferences for the aggregate (De Condorcet et al., 2014). So abstract Wiener space poses a challenge to decision theory and the best we can hope for on that space is a decision maker in equipoise. In the sequel we drop the s from $v_j(s, P_\star)$ for notational convenience.

Another way to see this is to let $Q_j(x)$ be a probability measure that is absolutely continuous with respect to $P_\star$. According to Radon-Nykodym Theorem (Gikhman and Skorokhod, 1969, p. 78), $dQ_j/dP_\star = s_{1j}$. So that $\int dQ_j = Q_j = v_j(P_\star) = \int s_{1j}(x) dP_\star(x)$. Thus $v_j(P_\star)$ is equivalent to a change of measure Q_j which inherits its nonlinearity from v_j. Substitution in Theorem 2.3 gives us $V(F^\star) - V(F) = \sum_{j=0}^{\infty} (\widehat{u}_j^\star - \widehat{u}_j) Q_j$ where $u_j^\star$, $\widehat{u}_j$ are local utility elements of the infinite

Figure 2: Simulated Q-meas for $2\mathscr{N}(0,\|s_{1,1}\|^2)\star\chi_{\{Q_1\geq 0\}}$

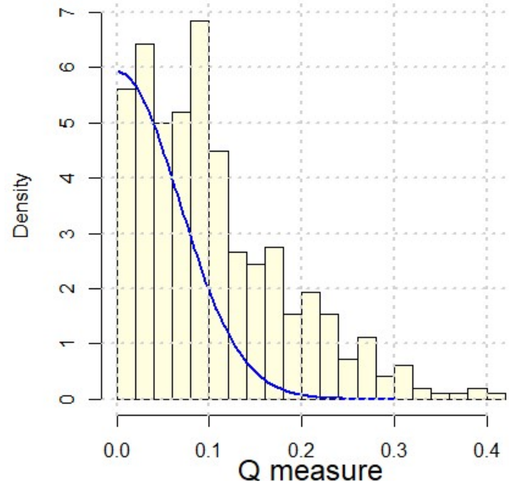

Figure 3: Nonlinear Q-measure drawn for $2\mathscr{N}(0,\|s_{1,1}\|^2)\star\chi_{\{Q_1\geq 0\}}$

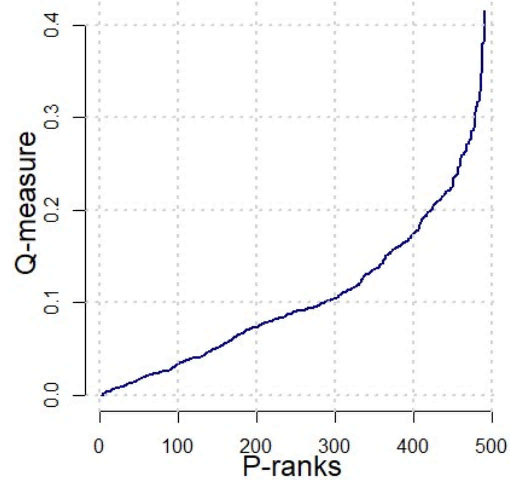

Figure 2 depicts an experiment with 1000 draws from $N(0,\|s_{1,1}\|^2)$ for $Q_1\geq 0$. So, $Q_1\sim 2*\mathscr{N}(0,\|s_{1,1}\|^2)*\chi_{\{Q_1\geq 0\}}$ where $\chi_{\{\bullet\}}$ is a characteristic function. Figure 3 portrays the nonlinear nature of the change of measure $Q=\nu_1(P_\star)$ theorized in Proposition 2.

dimensional vector valued utility functionals $V(F^\star)$ and $V(F)$, respectively. Since $Q_j\geq 0$ we have $F^\star\succeq F$ if we set $Q_j=0$ by censoring when $\widehat{u}_j^\star-\widehat{u}_j<0$ (cf. Figure 2). Since Q_j is drawn from a symmetric distribution, on average half of the $Q_j's$ would be less than 0. So, our DM is in a WSTA world of equipoise. Rank dependent utility (Quiggin, 1982, 1993), cumulative prospect theory (Tversky and Kahneman, 1992), quantum decision theory (Busemeyer et al., 2006) and Allais (1953) provide different specifications for nonlinear probabilities compared to Q_j. Thus, we proved

Proposition 2 (Nonlinear probability representation)**.** *Let $\mathscr{N}(\cdot)$ be the normal density function. Q_j is a change of probability measure with respect to Wiener measure $P_\star$, with probability density function $\phi_{Q_j}(x)=2\mathscr{N}(0,\|s_{1,j}\|^2)\star\chi_{\{Q_j\geq 0\}}$ where $\chi_{\{\bullet\}}$ is a characteristic function and $2\|s_{1,j}\|^2$ is the variance of Q_j.* □

Figures 2 and 3 depict a simple experiment with simulated distributions articulated in Proposition 2. There, we see that after around the 400 probability rank probabilities are quite nonlinear. In other words, larger probabilities jump more than lower probabilities.

Figure 4: State functions

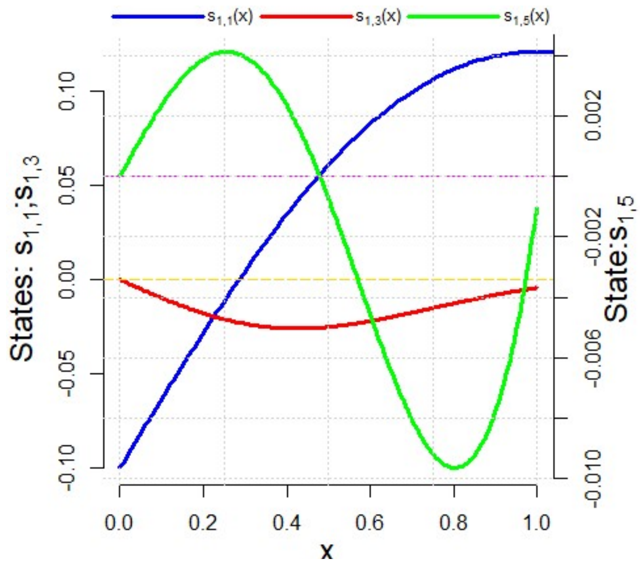

Figure 5: Approximate recovery of $U(x;\mathbf{u})$

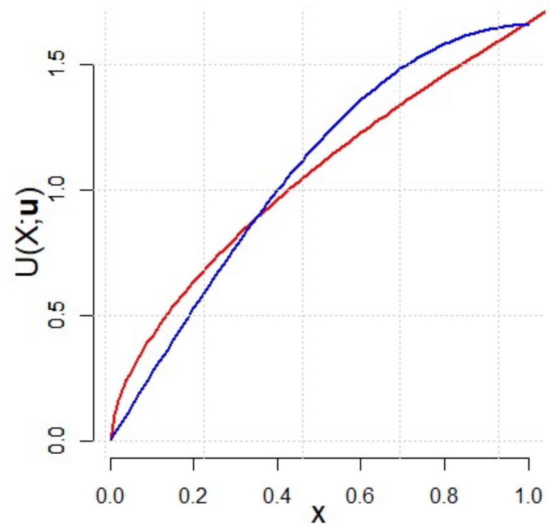

Figure 4 depicts state preference ranking $s_{1,1}(x) \succeq s_{1,5}(x) \succeq s_{1,3}(x)$ for $x \in [0,2]$. Figure 5 plots the CRRA utility function $U(x;a) = x^{1-r}/(1-r)$, $r = 0.4$, and its approximation is given by $\widetilde{U}(x;\mathbf{u}) \approx \widetilde{u}_1 s_{1,1}(x) + \widetilde{u}_3 s_{1,3}(x) + \widetilde{u}_5 s_{1,5}(x)$ where local utility $\widetilde{u}_1 \approx 8.65$, $\widetilde{u}_3 \approx 226.95$ and $\widetilde{u}_3 \approx -409579$ for $x \in [0,2]$. The local utility ranking $u_3 \succ u_1 \succ u_5$ seems to be independent of the state ranking (Karni, 2017). Let χ_a be an indicator function for a. The plotted values of $\widetilde{u}$ were constructed with the following affine transformation $\widetilde{u}(x) * \chi_{\{\widetilde{u}(x)>0\}} * \left(\max_x U(x)\chi_{\{x<1\}} / \max \widetilde{u}(x)\right)$.

Similar to Machina's (1982) DM's local utility $U(x;F)$ in (3.6), our DM in (3.7) behaves like a local expected utility maximizer with respect to the local expected utility functional $\widehat{u}_j Q_j$ on abstract Wiener space. According to Corollary 1, $v_j(x)$ is a Wiener integral so it is highly nonlinear, and $V(F)$ inherits the nonlinearity and Wiener integral feature. Furthermore, $V(F)$ is also infinite dimensional by virtue of being a Wiener integral (Shilov, 1963).

3.3 Estimation

Figure 4 depicts an experiment with the Gauss-Hermite state functions $s_{1,1}(x); s_{1,3}(x); s_{1,5}(x)$ according to (2.6) for $H_1(x) = 2x$, $H_3(x) = -12 * x + 8 * x^3$ and $H_5(x) = 12 - 48 * x^2 + 16 * x^4$ defined on $[0,1]$. We estimated the equation $\widetilde{U(x;a)} \approx \widetilde{u}_1 s_{1,1}(x) + \widetilde{u}_3 s_{1,3}(x) + \widetilde{u}_5 s_{1,5}(x)$ for CRRA utility with risk aversion coefficient calibrated at $r = 0.4$ in accord with Arrow (1971, p. 98) and the experimental literature (Holt and Laury, 2002). We used numerical integration to estimate local utility $\widetilde{u}_1 = \int_0^2 U(x) s_{1,j}(x) dx / \int_0^2 s_{1,j}^2(x) dx$, $j = 1,3,5$ by virtue of (2.8). Identifying

restrictions on U require that even values $a_{2k} = 0, k = 1, \ldots$. Also, for our purposes the Hermite polynomials $H_5(x)$ and above did not contribute much on our intervals of interest. We note in passing that under Corollary 2, the estimate
$\|V(F)\| \leq K_2^{\frac{1}{2}} \sqrt{\sup_j \int_0^2 s_{1,j}^2(x).dx} = K_2^{\frac{1}{2}} \sqrt{\sup\{\int_0^2 s_{1,1}^2(x).dx,\ \int_0^2 s_{1,3}^2(x).dx\}}$
$= 227.11\sqrt{\sup\{6.73 \times 10^{-2},\ 1.70 \times 10^{-3}} \approx 15.22$ on the interval $[0,2]$. This is a very conservative estimate since it is much higher than 2, and higher order states are not included.

4 Conclusion

Utility representation on abstract Wiener space provides us with several sharp results concerning local utility and nonlinear probabilities. In particular, local utility is an abstract Fourier coefficient, i.e. an orthogonal projection of a utility function on basis functions that represent states. It is also a coordinate element of the infinite dimensional vector valued utility function. We introduce a new representation for local nonlinear probability as a Wiener integral for which Wiener measure is a *de facto* conjugate prior. Furthermore, we find that Savage's (1972) SEU does not hold on abstract Wiener space because local utility and the local nonlinear probability are both state dependent, and SEU requires independence. We also find that decision weights under rank and sign dependent utility (RDEU) are random variables in Wiener space. So, expected (and nonexpected) utility functionals are stochastic. Decision weights are additive if they obey the law of iterated logarithm and grow like $O_p(\left(2\sigma_n^2 \log\log \sigma_n^2\right)^{-1/2})$ where σ_n^2 is the sample variance of decision weights. This result has implications for the use of RDU in economic experiments with large samples. For, if decision weights are random variables, then they may not sum to one and an appropriate normalizing transformation of the weights will be needed. To the best of our knowledge this issue has not been addressed in the literature until now.

Among other things, decision making in abstract Wiener space also poses a challenge to the transitivity axiom. For, under Borel-Cantelli lemma (or other zero-one laws of probability) it implies that intransitive preferences will eventually occur almost surely in that space. In fact, the weak transitivity axiom implies that the best we can hope for is decision maker equipoise in the abstract Wiener subspace of Machina's Banach space. It would be interesting to see what restrictions must be imposed on the topology for the transitivity axiom to hold almost surely.

Our model extends naturally to path dependent utility, i.e. utility function of Wiener process on a probability space $(\Omega, \mathscr{F}, P)$ with the Ito integral representation (Øksendal, 1997, p. 1.9) $V(W_t) = E[V] + \int \varphi(t,\omega)\, dW(t,\omega)$ where $\varphi(t,\omega) \in \widetilde{L}^2_H$. For the functional of several Wiener processes $V(W_1(t,\omega),\ldots,W_n(t,\omega))$ the utility representation falls in the Clark-Ocone-Haussman integral representation and under certain conditions $\varphi(t,\omega) = E\left[D_t V|\mathscr{F}_t\right]$ where $D_t V$ is a Malliavin derivative of V and outside the scope of this paper (cf. Nualart, 2006, p. 46); (Mataramvura, 2012).

References

Abdellaoui, M., A. Baillon, L. Placido, and P. P. Wakker (2011, Apr). The Rich Domain of Uncertainty: Source Functions and Their Experimental Implementation. *American Economic Review 101*(2), 695–723. http://doi.org/10.1257/aer.101.2.695.

Abdellaoui, M. and P. P. Wakker (2020). Savage for dummies and experts. *Journal of Economic Theory 186*, 104991. https://doi.org/10.1016/j.jet.2020.104991.

Adams, C. R. (1936). The space of functions of bounded variation and certain general spaces. *Transactions of the American Mathematical Society 40*(3), 421–438. https://doi.org/10.2307/1989632.

Aerts, D., E. Haven, and S. Sozzo (2018). A proposal to extend expected utility in a quantum probabilistic framework. *Economic Theory 65*(4), 1079–1109. https://doi.org/10.1007/s00199-017-1051-2.

Akheizer, N. and I. Glazman (1961). *Theory of Linear Operators in Hilbert Space*. Frederick Ungar Publ. Co.:New York. Dover reprint 1993.

Allais, M. (1953, Oct). Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école américaine. *Econometrica 21*(4), 503–546. http://doi.org/10.2307/1907921.

Anscombe, F. and R. Aumann (1963, Mar). A Definition of Subjective Probability. *Annals of Mathematical Statistics 34*(1), 199–205.

Arrow, K. (1971). *Essays in the Theory of Risk Bearing*. Amersterdam: North Holland.

Asano, M., I. Basieva, A. Khrennikov, M. Ohya, and Y. Tanaka (2012). Quantum-like dynamics of decision-making. *Physica A: Statistical Mechanics and its Applications 391*(5), 2083 – 2099. http://dx.doi.org/10.1016/j.physa.2011.11.042.

Baccelli, J. (2020, May). The Problem of State-Dependent Utility: A Reappraisal. *British Journal for the Philosophy of Science*. In press. https://doi.org/10.1093/bjps/axz024.

Basieva, I., P. Khrennikova, E. Pothos, M. Asano, and A. Khrennikov (2018). Quantum-like model of subjective expected utility. *Journal of Mathematical Economics*. In press. https://doi.org/10.1016/j.jmateco.2018.02.001.

Bessaga, C. and A. Pełczyński (1975). *Selected topics in infinite-dimensional topology*, Volume 58. Warsaw, Poland: Polish Scientific Publishers.

Boyd, J. P. (2018). *Dynamics of the equatorial ocean*. New York: Springer.

Busemeyer, J. R., E. M. Pothos, R. Franco, and J. S. Trueblood (2011, April). A Quantum Theoretical Explorations for Probability Judgment Errors. *Psychological Review 118*(2), 193–218.

Busemeyer, J. R. and J. T. Townsend (1993). Decision field theory: A dynamic cognitive approach to decision making in an uncertain environment. *Psychological Review 100*, 432–453.

Busemeyer, J. R., Z. Wang, A. Khrennikov, and I. Basieva (2014). Applying quantum principles to psychology. *Physica Scripta 2014*(T163), 1–10. http://stacks.iop.org/1402-4896/2014/i=T163/a=014007.

Busemeyer, J. R., Z. Wang, and J. T. Townsend (2006, June). Quantum Dynamics of Human Decision Making. *Journal of Mathematical Psychoology 50*(3), 220–241.

Cameron, R. H. and W. T. Martin (1944, April). Transformations of Weiner Integrals Under Translations. *Annals of Mathematics 45*(2), 386–396.

Cameron, R. H. and W. T. Martin (1947, April). The Orthogonal Development of Non-Linear Functionals in Series of Fourier-Hermite Functionals. *Annals of Mathematics 48*(2), 385–392.

Carmona, R. A. and M. R. Tehranchi (2006). *Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective*. Springer Finance. Springer.

Charles-Cadogan, G. (2018). Probability interference in expected utility theory. *Journal of Mathematical Economics 78*, 163–175. https://doi.org/10.1016/j.jmateco.2018.03.006.

Chow, Y. S. and H. Teicher (1988). *Probability Theory: Independence, Interchangeability, Martingales* (2nd ed.). New York, NY: Springer Science & Business Media.

De Condorcet, N. et al. (2014). *Essay on the application of analysis to the probability of decisions rendered by plurality of votes*. Cambridge, UK: Cambridge University Press. English translation of 1785 French edition.

DeGroot, M. (1970). *Optimal Statistical Decisions*. New York, NY: McGraw-Hill, Inc.

Friedman, M. and L. J. Savage (1948, Aug). The Utility Analysis of Choice Involving Risk. *Journal of Political Economy 56*(4), 279–304.

Gikhman, I. I. and A. V. Skorokhod (1969). *Introduction to The Theory of Random Processes*. Phildelphia, PA: W. B. Saunders, Co. Dover reprint 1996.

Gross, L. (1967). Abstract Wiener Spaces. In *Proc. Fifth Berkeley Symp. on Math. Statist. and Prob.*, Volume 2, Berkeley, CA, pp. 31–42. University of California Press, Inc.

Haase, M. (2010, April). Lectures in functional analysis. Mimeo. https://fa.its.tudelft.nl/ haase/files/LFA-July2010.pdf.

Holevo, A. S. (1973). Statistical decision theory for quantum systems. *Journal of Multivariate Analysis 3*(4), 337–394.

Holevo, A. S. (2011). *Probabilistic and statistical aspects of quantum theory*, Volume 1. Springer Science & Business Media.

Holt, C. A. and S. K. Laury (2002). Risk aversion and incentive effects. *American Economic Review 92*(5), 1644–1655. http://doi.org/10.1257/000282802762024700.

Kahneman, D. and A. Tversky (1979). Prospect theory: An analysis of decisions under risk. *Econometrica 47*(2), 263–291.

Karni, E. (2017). *Savage's Subjective Expected Utility Model*, pp. 1–5. London: Palgrave Macmillan UK. https://doi.org/10.1057/978-1-349-95121-5˙2467-1.

Khrennikov, A. (2009). Quantum-like model of cognitive decision making and information processing. *Biosystems 95*(3), 179 – 187. http://dx.doi.org/10.1016/j.biosystems.2008.10.004.

Khrennikov, A. Y. and E. Haven (2009). Quantum mechanics and violations of the sure-thing principle: The use of probability interference and other concepts. *Journal of Mathematical Psychology 53*(5), 378 – 388. Special Issue: Quantum Cognition http://dx.doi.org/10.1016/j.jmp.2009.01.007.

Kreps, D. (1988). *Notes on the Theory of Choice*. Westview Press.

Kreyszig, E. (1978). *Introductory Functional Analysis With Applications*. New York, NY: John Wiley & Sons, Inc.

Kuelbs, J. (1977). Kolmogorov's law of the iterated logarithm for banach space valued random variables. *Illinois Journal of Mathematics 21*(4), 784–800. https://doi.org/10.1215/ijm/1256048928.

Kuo, H. (1975). *Gaussian Measures in Banach Spaces*, Volume 463 of *Lecture Notes in Mathematics*. New York,NY: Springer-Verlag.

Machina, M. J. (1982, March). "Expected Utility" Without The Independence Axiom. *Econometrica 50*(2), 277–324.

Markowitz, H. (1952, April). The Utility of Wealth. *Journal of Political Economy 40*(2), 151–158.

Mataramvura, S. (2012). The Malliavin derivative and application to pricing and hedging a European exchange option. *Journal of Mathematical Finance 2*, 280–290. http://doi.org/10.4236/jmf.2012.24031.

Nualart, D. (2006). *The Malliavin Calculus and Related Topics* (2nd ed.). Probability and Its Applications. New York: Springer-Verlag.

Øksendal, B. (1997, May). An introduction to malliavin calculus with applications to economics. Unpublished lecture notes, Depart of Mathematics, University of Oslo.

Quiggin, J. (1982). A Theory of Anticipated Utility. *Journal of Economic Behaviour and Organization 3*(4), 323–343.

Quiggin, J. (1993). *Generalized Expected Utility Theory: The Rank Dependent Model*. Boston, MA: Kluwer Academic Press.

Raiffa, H. and R. Schlaifer (1961). *Applied Statistical Decision Theory*. Studies in Managerial Economics. Boston, MA: Harvard University Press. Division of Research, Graduate School of Business Adminitration, Harvard.

Reed, M. and B. Simon (1980). *Modern Methods of Mathematical Physics* (Revised Enlarged ed.), Volume Functional Analysis. New York: Academic Press, Inc.

Riesz, F. and B. Sz-Nagy (1955). *Functional Analysis*. New York, NY: Frederick Ungar Publishing Co. Dover Publishing Reprint 1990.

Rodríguez-Piazza, L. (1995). Every separable Banach space is isometric to a space of continuous nowhere differentiable functions. *Proceedings of the American Mathematical Society 123*(12), 3649–3654. https://doi.org/10.1090/S0002-9939-1995-1328375-8.

Savage, L. J. (1972). *The Foundations of Statistics* (2nd revised ed ed.). Mineola, NY. Dover Publications, Inc. Unabridged revised 1954 edition.

Shilov, G. E. (1963). Integration in infinite dimensional spaces and the wiener integral. *Russian Math. Surveys 18*(2), 97–118. https://doi.org/10.1070/RM1963v018n02ABEH001135.

Tversky, A. (1969, Jan.). Intransitivity of Preferences. *Psychological Review 76*(1), 31–48.

Tversky, A. and D. Kahneman (1992). Advances in Prospect Theory: Cumulative Representation of Uncertainty. *Journal of Risk and Uncertainty 5*, 297–323.

Von Neuman, J. and O. Morgenstern (1953). *Theory of Games and Economic behavior* (3rd ed.). Princeton, N.J.: Princeton University Press.

Wakker, P. P. (2010). *Prospect Theory for Risk and Ambiguity*. New York, N. Y.: Cambridge University Press.

Wang, Z., J. R. Busemeyer, H. Atmanspacher, and E. M. Pothos (2013). The Potential of Using Quantum Theory to Build Models of Cognition. *Topics in Cognitive Science 5*(4), 672–688. https://doi.org/10.1111/tops.12043.

Wiener, N. (1923). Differential-space. *Journal of Mathematics and Physics 2*(1-4), 131–174. https://doi.org/10.1002/sapm192321131.

Wiener, N. (1933). *The Fourier Integral and Certain of Its Applications*. London. Cambridge University Press.