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Prerationality as Avoiding Predictably Regrettable Consequences

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Abstract:
Following previous work on consequentialist decision theory, we consider an unrestricted domain of finite decision trees, including continuation subtrees, with: (i) decision nodes where the decision maker must act; (ii) chance nodes where a "roulette lottery" with strictly positive probabilities that are defined a priori is resolved; (iii) event nodes where a "horse lottery" is resolved. A complete family of binary conditional base relations over Anscombe–Aumann lottery consequences is defined to be "prerational" just in case there exists a behaviour rule that is defined throughout the tree domain which is explicable as avoiding, under all predictable circumstances, regrettable consequences. It is shown that a family of base relations is prerational if and only if: (i) each relation is complete and transitive; (ii) each relation satisfies the independence axiom of expected utility theory; (iii) the entire family satisfies a strict form of Anscombe and Aumann's extension of Savage's sure thing principle. Assuming that the base relations satisfy non-triviality and a generalized form of state independence that holds even when consequence domains are state dependent, prerationality combined with continuity on Marschak triangles is equivalent to representation by a class of refined subjective expected utility functions that excludes zero probabilities. `[197 words]`

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1 Introduction and Outline

1.1 Gilboa on Avoiding Regrettable Axioms

In economics rationality is usually regarded as making a decision whose consequences maximize a preference ordering. When the consequences are risky or uncertain, it is usual to invoke the "Bayesian paradigm" according to which a rational decision maximizes the subjective expected value of a von Neumann–Morgenstern utility function defined over consequences. Moreover, this utility function lies in a unique equivalence class of cardinally equivalent functions.

Prominent among the many works that depart from this Bayesian paradigm is the "case-based decision theory" developed by Gilboa and Schmeidler (1995, 2001). Later, Gilboa (2015, pp. 315–6) offers a novel idea for redefining rationality, based on applying the notion of regret to axioms:

> "Just as people may vary in valuing wealth or human life, they may vary in the way they cherish transitivity or some other decision-theoretic axiom."

Indeed, he writes (pp. 316):

> "... we seek a new definition of "rationality", and start using this term in a way that is less confusing and more conducive to practical debates. One such ... template of a definition is the following: a mode of behavior is irrational for a decision-maker, if, when the latter is exposed to the analysis of her choices, she would have liked to change her decision, or to make different choices in similar future circumstance. Note that this definition is based on a sense of regret, or embarrassment about one's decisions, and not only to observed behavior."

This approach leaves Gilboa entirely free, of course, to consider decision makers who would regret following any or even all of the axioms that underlie the Bayesian paradigm.

More recently, in his comprehensive discussion of the Allais paradox, without necessarily endorsing Gilboa's view, Mongin (2019, p. 443) suggests that Allais might have wanted to argue that:

> "... the "rational man" strives to conform his particular choices to general rules of conduct that he is prepared to answer for."

Mongin also adds this footnote on the same page:

> "The conception attributed here to Allais bears some analogy with that defended by Gilboa [and co-authors]. However, we emphasize the role of generalities in the agent's reflective process more than these authors do."[1]

1.2 Savage on Avoiding Regrettable Consequences

A similar idea concerning regret arose much earlier in the reaction that Savage (1954, p. 103) had to the preferences which he himself had reported in connection with Allais' (1953) example after realizing that these preferences violated the independence axiom of expected utility theory.[2]

> "There is, of course, an important sense in which preferences, being purely subjective, cannot be in error; but in a different, more subtle sense they can be. Let me illustrate by a simple example containing no reference to uncertainty. A man buying a car for $2,134.56 is tempted to order it with radio installed, which will bring the price to $2,228.41, feeling that the difference is trifling. But when he reflects that, if he already had the car, he certainly would not spend $93.85 for a radio for it, he realizes that he has made an error."

The predictably regrettable consequence of the error that Savage describes is, of course, that the man has effectively spent $93.85 for his car radio, instead of the lower price he would have been willing to pay for a car radio on its own.[3]

Like Gilboa, my argument will also explore the implications of viewing rationality as the avoidance of regret. Whereas Gilboa seems primarily concerned with regrettable axioms, however, the regret I shall consider concerns possible consequences of how the decision maker (DM) behaves in different decision trees. It will be shown that avoiding predictably regrettable consequences rather than regrettable axioms leads to very different conclusions.

[1] Mongin cites Gilboa (2011, p. 18), which appears to be a much less accessible version of the paper Gilboa (2010) with the same title. He also cites Gilboa and Schmeidler (2001, pp. 17–18) and Gilboa et al. (2010).

[2] See, for example, the discussion in Mongin (2019).

[3] See Dietrich et al. (2021) for a recent discussion of this response by Savage to Allais, as well as of Broome (2013).

1.3 The Key Hypothesis of Prerationality

The main assumption of the paper is that uncertainty is described by a finite set of possible states of the world and that, for each uncertain event, there exists a conditional binary base relation over consequences which, given the opportunity to decide between any pair of consequences, indicates which consequence is regrettable, if either is. The complete family of base relations for all possible events will be described as “prerational” just in case together they provide a foundation that allows behaviour in any finite decision tree to be explained as the avoidance of predictably regrettable consequences or, equivalently, as the choice of predictably desirable consequences.

It will be shown that a complete family of conditional base relations is prerational if and only if it meets several of the main precepts of the Bayesian paradigm. In particular, for each uncertain event, the corresponding conditional base relation must be complete and transitive, and also satisfy the independence axiom that Samuelson (1952) introduced to characterize expected utility. Furthermore, the complete family of conditional base relations for all possible events must satisfy Ellsberg’s (1961) extension of Savage’s (1954) sure thing principle to the framework proposed by Anscombe and Aumann (1963), where there are not only “roulette” lotteries with risky outcomes, but also “horse” lotteries whose outcome depends on an uncertain state of the world.

The latter part of the paper will investigate the logical gap between prerationality and the Bayesian rationality condition requiring that there exist subjective probabilities over uncertain events along with a cardinal equivalence class of von Neumann–Morgenstern utility functions such that preferences can be represented by subjectively expected utility. The main conclusion is that, under weak domain conditions, prerationality combined with continuity is logically equivalent to a “refined” form of Bayesian rationality that excludes zero subjective probabilities.

1.4 Outline of Paper

Following this introduction, the next Section 2 offers a brief analysis of regrettable consequences in the special case when there is only one decision to be made. There is a corresponding theory of choice based on avoiding regrettable consequences.

The next two sections set out the basic framework which allows us to state formally the prerationality condition set out in Section 1.3 and to explore its implications. Specifically, Section 3 recalls and refines somewhat the key concepts from Hammond (1988) concerning finite decision trees with decision, chance, event, and terminal nodes, as well as a mapping from each terminal node of the tree to a consequence in an event specific domain of lottery consequences. Next, in any finite decision tree with consequences, Section 4 considers decision strategies and their lottery consequences, as well as feasible sets, plans, and behaviour.

The prerationality condition of Section 1.3 is set out formally in Definition 5.5. Section 6 is then devoted to our first main result, which appears as Theorem 6.3. This adapts some key results from Hammond (1988) in order to characterize those complete families of base relations that satisfy prerationality. This main theorem has many significant implications, of which the following three are the most evident and apparently the most important:

1. First, each conditional base relation in a complete family that is prerational must be complete and transitive over the relevant space of consequence lotteries. This excludes preferences that allow consequences to be incomparable, as in Chang (1997). It also excludes the kind of intransitive preference relation that has been extensively considered in work such as Luce (1956) or Arrow and Reynaud (1986).

2. Second, each conditional base relation should satisfy the independence axiom. This implies the irrationality of preferences that, as in the extensive literature inspired by Allais (1953) and later Machina (1987, 1989), cannot be represented by an expected utility function.

3. Third, the complete family of conditional base relations that apply in different uncertain events should satisfy the extension of Savage's (1954) sure thing principle to the framework introduced by Anscombe and Aumann (1963) that combines roulette and horse lotteries.[4] This principle excludes the base relations that appear in the extensive work inspired by Ellsberg (1961), including Gilboa and Schmeidler (1995, 2001), which replace a single "subjective" probability measure over the results of a horse lottery with, for example, a range of "ambiguous" probabilities.

[4]Note that, although Ellsberg (1961) writes of the "Savage axioms", he works in this more general framework.

Section 7 is concerned with bridging the logical gap between prerationality and Bayesian rationality. Indeed, for it to be consistent with prerationality, Bayesian rationality has to be refined in order to exclude the null events considered by Savage (1954), as well as zero probabilities at any chance or event node. But then, for prerationality to be consistent with refined Bayesian rationality, the conditional base relations must meet the standard condition of continuity on triangles specified in Section 7. This continuity condition excludes the kind of lexicographic preferences considered by Fishburn (1971) and Martínez-Legaz (1998). Such preferences seem as though they should be relevant to, for example, the ethical arguments in the philosophical work on competing claims inspired by Voorhoeve (2014).

Finally, even continuous base relations which are prerational may fail to be Bayesian rational. Additional conditions that suffice for refined Bayesian rationality, including the requirement that positive subjective probabilities exist, which were the subject of Hammond (1998b, 1999), are somewhat tightened in the latter parts of Section 7. Indeed, state-dependent consequence domains can be accommodated by postulating a "generalized" state independence condition. This requires that all base relations that are conditional on a single state of the world must be restrictions to the consequence domain in that state of a single universal base relation over all roulette lotteries whose outcomes belong to one "union" consequence domain. Section 7 concludes with the second main result which, under the stated auxiliary assumptions, shows that a complete family of conditional base relations is both prerational and continuous on triangles if and only if it is refined Bayesian rationality.

The final Section 8 discusses some implications and some possible extensions of the main result.

2 Predictably Regrettable Consequences

2.1 Concepts of Regret

Predictably regrettable consequences will be those that, as in Savage's example of paying too much for a car radio that was discussed in Section 1.2, arise from what the DM realizes ex post is a mistaken decision. They will not be the unlucky bad consequences that, following Savage's (1954) minimax regret criterion and an extensive subsequent literature, can cause regret even when a consequence lottery has been rationally chosen.

Regrettable consequences are also different from the kind of regret that appears in the work of Loomes and Sugden (1982, 1987) as well as Bell (1982), Sugden (1985, 1993), Hayashi (2008) and Sarver (2008). Indeed, their concept of regret is included, in effect, as part of the description of each relevant consequence, and so as an extra argument of any utility function. This is the kind of psychological "regret consequence" that arises because of the DM's awareness that a decision has had to be made ex ante, before knowing ex post what deterministic consequence would result after any risk or uncertainty has been resolved. Instead, the "regrettable consequences" considered here concern ordinary consequences, possibly risky or uncertain, that arise because of mistakes that should have been apparent ex ante.

Predictably regrettable consequences will also not be the unpredictable consequences that may arise because even a rational DM, in order to analyse any decision that is about to be made, is necessarily restricted to using a bounded decision tree. This raises questions such as how to devise a normative decision theory that builds on Simon's (1955, 1957) concept of satisficing decisions made by a boundedly rational agent, as well as accommodating the kind of "transformative" experience discussed by Paul (2014) and various successors. The extension of rationality to "enlivened" decision trees, which allow regrettable consequences to arise that could not be predicted because they were not in the DM's original bounded model, is left for a later paper that builds on the preliminary discussion in Hammond (2007).

2.2 Choosing to Avoid Regrettable Consequences

Let X denote a domain of consequences that should be relevant to the DM's behaviour.

Definition 2.1. *Let $\mathcal{F}(X)$ denote the family of non-empty finite subsets of X. A* regret function *is a mapping*

$$\mathcal{F}(X) \ni F \mapsto R(F) \in \mathcal{F}(X) \tag{1}$$

whose value, for each finite feasible set $F \in \mathcal{F}(X)$ of possible consequences, is a proper subset $R(F) \subsetneq F$ of regrettable *consequences.*

Note that the set $R(F) \subsetneq F$ could be empty. Indeed, it must be in case F is a singleton subset $\{x\}$ for some $x \in X$. Also, more generally, it should be empty in case the DM has reason to regard all members of F as indifferent.

On the other hand one must have $R(F) \neq F$ because there should be at least one feasible consequence which is not regrettable.[5]

The following result is an immediate implication of Definition 2.1.

Proposition 2.2. *Corresponding to any regret function specified as in Definition 2.1, there exists an associated* regret avoiding *consequence choice function*

$$\mathcal{F}(X) \ni F \mapsto C(F) \in \mathcal{F}(X) \tag{2}$$

such that, for all $F \in \mathcal{F}(X)$*, one has* $C(F) = F \setminus R(F) \subseteq F$.

Conversely, given any choice function as specified in (2) *that for all* $F \in \mathcal{F}(X)$ *satisfies* $\emptyset \neq C(F) \subseteq F$*, there is a unique corresponding regret function satisfying Definition 2.1 that is defined for all* $F \in \mathcal{F}(X)$ *by* $R(F) = F \setminus C(F) \subsetneq F$.

Non-emptiness of $C(F)$ is equivalent to the requirement that $R(F) \neq F$. To quote Arrow (1963, p. 118), "Abstention from a decision cannot exist." This can be made tautological by including abstention among the feasible set of decisions whenever it is possible.

2.3 Three Base Preference Relations

Though the concept of a preference relation had become standard in economics much earlier, the following definition uses terminology that is due to Herzberger (1973).

Definition 2.3. *Given any regret function* $F \mapsto R(F)$ *as specified in Definition 2.1, and the associated choice function* $F \mapsto C(F)$ *as specified in* (2)*, there are three* base preference *binary relations* $\succ$, $\succsim$*, and* $\sim$ *defined over* X *so that for each pair* $y, z \in X$ *with* $y \neq z$*, one has:*

1. y *is (strictly)* preferred *to* z*, denoted by* $y \succ z$*, just in case one has* $R(\{y, z\}) = \{z\}$*, or equivalently* $C(\{y, z\}) = \{y\}$*;*

2. y *is* weakly preferred *to* z*, denoted by* $y \succsim z$*, just in case one has* $y \notin R(\{y, z\})$*, or equivalently* $y \in C(\{y, z\})$*;*

[5] This is a direct contradiction of the quotation from Søren Kierkegaard's *Either/Or* at the head of Sarver (2008): " ... do it or do not do it — you will regret both". But presumably Kierkegaard was concerned with psychological feelings of regret that may be unavoidable, rather than with regrettable consequences that should be avoided because there is a more suitable alternative.

3. y is indifferent *to z, denoted by $y \sim z$, just in case $R(\{y,z\}) = \emptyset$, or equivalently $C(\{y,z\}) = \{y,z\}$.*

Most of our subsequent analysis focuses on the weak base relations $\succsim$ that determine both regret and choice when the feasible set is a pair of consequences. An immediate implication of Definition 2.3 is that the base relation $\succsim$ is always complete in the sense that, for each pair $y, z \in X$ with $y \neq z$, one has either $x \succsim y$, or $y \succsim x$, or both.

2.4 Ordinal Regret and Choice Functions

The following definition owes much to Arrow (1959). It characterizes a choice function which selects all those consequences that maximize its base relation, as well as the associated regret function.

Definition 2.4. *The regret function $\mathcal{F}(X) \ni F \mapsto R(F) \subsetneq F$ with base relation $\succsim$ defined on X is* ordinal *just in case $\succsim$ is transitive, with the property that for all $F \in F(X)$, one has*

$$\begin{aligned} & R(F) = \{x \in F \mid \exists y \in F : y \succ x\} = F \setminus C(F) \\ \textit{where} \quad & C(F) = \{x \in F \mid y \in F \Longrightarrow x \succsim y\} \end{aligned} \tag{3}$$

3 A Domain of Finite Decision Trees

3.1 Trees as Rooted Acyclic Directed Graphs

The main subject of this paper is rational behaviour when a DM has to make one or more decisions in a sequence. Each of these successive decisions is represented mathematically as the choice of a move at one decision node of a decision tree, which will now be defined.

Definition 3.1. *1. A* directed graph *is a pair (N, O) where:*

(a) N is a set of nodes *or vertices;*

(b) $O \subset N \times N$ is a set of ordered pairs (n, n^+) representing directed edges *that each join one node $n \in N$ to an* immediate successor *node $n^+ \in N \setminus \{n\}$.*

2. *Given any $n, n^+ \in N$ with $n \neq n^+$, say that n^+* immediately succeeds *n, and write $n^+ >^1 n$, just in case (n, n^+) belongs to O, the set of directed edges of the graph (N, O).*

   Next, define the sequence $>^{k+1}$ ($k \in \mathbb{N}$) of binary relations recursively so that, for each $k \in \mathbb{N}$ and for all $n, n^+ \in N$ with $n \neq n^+$, one has $n^+ >^{k+1} n$ just in case there exists $n' \in N$ such that $n^+ >^1 n'$ and $n' >^k n$.

   Finally, define the eventual successor *binary relation $>^*$ so that one has $n^+ >^* n$ just in case there exists $k \in \mathbb{N}$ such that $n^+ >^k n$.*

3. *Given any node n in a directed graph (N, O), let*

$$N_{+1}(n) := \{n^+ \in N \mid (n, n^+) \in O\} = \{n^+ \in N \mid n^+ >^1 n\} \quad (4)$$
$$\text{and } N_{-1}(n) := \{n^- \in N \mid (n^-, n) \in O\} = \{n^- \in N \mid n >^1 n^-\} \quad (5)$$

   denote respectively the sets of nodes that immediately succeed *and* immediately precede *n.*

4. *The directed graph (N, O) is* acyclic *just in case the eventual successor relation $>^*$ is* irreflexive *in the sense that no $n \in N$ satisfies $n >^* n$.*

5. *The particular node $n_0 \in N$ is the* root *or* initial node *of the directed graph (N, O) just in case it is the unique node $n \in N$ with the property that $N_{-1}(n) = \emptyset$.*

6. *A* tree *$T = (N, O)$ is an acyclic directed graph that is* rooted *in the sense that it has a root node n_0.*

3.2 Finite Decision Trees

Following Savage (1954), Anscombe and Aumann (1963), and many successors, we postulate:

Assumption 3.2. *There exists a fixed finite domain $S \neq \emptyset$ of possible uncertain* states of the world.

An event *E in S is any non-empty subset of S.*

Definition 3.3. *Given the tree T with graph (N, O), an* event correspondence *is a mapping $N \ni n \mapsto E(n) \in 2^S \setminus \{\emptyset\}$.*

The tree T and event correspondence together form a finite decision tree *just in case the set N of nodes is finite and can be partitioned into the following four pairwise disjoint subsets, of which any of the first three may be empty:*

1. *a set N^d of* decision *nodes n at each of which $E(n^+) = E(n)$ for all $n^+ \in N_{+1}(n)$;*

2. *a set N^c of* chance *or* roulette lottery *nodes n at each of which one has $E(n^+) = E(n)$ for all $n^+ \in N_{+1}(n)$, and also there exists a positive-valued* transition probability $N_{+1}(n) \ni n^+ \mapsto \pi(n^+|n) \in (0, 1]$ *which satisfies $\sum_{n^+ \in N_{+1}(n)} \pi(n^+|n) = 1$;*[6]

3. *a set N^e of* event *or* horse lottery *nodes n at each of which the family of sets $\{E(n^+) \mid n^+ \in N_{+1}(n)\}$ at all the nodes n^+ that immediately succeed n form a partition of $E(n)$ into pairwise disjoint events in S;*

4. *a non-empty set N^t of* terminal *nodes (or "leaves") n at each of which one has $N_{+1}(n) = \emptyset$.*

Remark 3.4. *In the case of a decision tree with no event nodes, both the set S of states and the event correspondence $N \ni n \mapsto E(n)$ essentially play no role and so could be eliminated.*

3.3 Anscombe–Aumann Consequence Lotteries

In normative decision theory under uncertainty, again following the example of Savage (1954), Anscombe and Aumann (1963), and many successors, it has become standard to postulate that there exists a fixed state-independent domain $Y \neq \emptyset$ of normatively relevant *consequences* which determine the acceptability of any possible decision. Following Hammond (1998b, 1999), we relax this postulate as follows:

Assumption 3.5. *For each state of the world $s \in S$, there exists a fixed* state-dependent *consequence domain $Y_s \neq \emptyset$.*

Given any event E in S, let Y^E denote the Cartesian product set $\prod_{s \in E} Y_s$ of state contingent consequences *that can arise in the event E.*

[6] As explained in Section 6 of Hammond (1988), the restriction to positive probabilities is required in order to avoid the trivial implication that no consequences are regrettable.

Remark 3.6. *Each element* $\langle y_s \rangle_{s \in E} \in Y^E$ *meets Savage's (1954) concept of an* act *in the sense of a mapping* $E \ni s \mapsto y_s \in Y_s$ *from states to consequences.*

Definition 3.7. *Given any non-empty set* Z*, let* $\Delta(Z)$ *denote the family of* simple lotteries $Z \ni z \mapsto \lambda(z) \in [0,1]$ *for which there exists a* finite support $\Lambda \subseteq Z$ *such that: (i)* $\lambda(z) > 0 \Longleftrightarrow z \in \Lambda$*; (ii)* $\sum_{z \in \Lambda} \lambda(z) = 1$*.*

Also, for each $z \in Z$*, let* $\delta_z \in \Delta(Z)$ *denote the* degenerate lottery *satisfying* $\delta_z(z) = 1$*.*

Definition 3.8. *Given any event* E *in* S*, define the associated domain* L^E *of* Anscombe–Aumann consequence lotteries *(or* AA lotteries*) as the Cartesian product set* $\prod_{s \in E} \Delta(Y_s)$ *whose members* $\lambda^E = \langle \lambda_s \rangle_{s \in E}$ *each correspond to a mapping* $E \ni s \mapsto \lambda_s \in \Delta(Y_s)$ *from states of the world* $s \in E$ *to the relevant domain* $\Delta(Y_s)$ *of consequence roulette lotteries.*[7]

3.4 Compound Lotteries

We consider two different ways of compounding a pair of AA lotteries. The first way applies when a roulette lottery precedes an AA lottery; the second way applies when a horse lottery precedes an AA lottery.

Definition 3.9.

1. *Given any fixed event* E *in* S*, any pair of lotteries* $\lambda^E, \mu^E \in L^E$*, and any* $\alpha \in [0,1]$*, let* $\alpha\lambda^E + (1-\alpha)\mu^E$ *denote the* compound AA lottery $\nu^E = \langle \nu_s \rangle \in L^E$ *whose component* $\nu_s \in \Delta(Y_s)$*, for each* $s \in S$*, is the compound roulette lottery in which the probability of each* $y \in Y_s$ *satisfies* $\nu_s(y) = \alpha\lambda_s(y) + (1-\alpha)\mu_s(y)$*.*

2. *Given any pair of disjoint events* E_1, E_2 *in* S*, and any pair* $\lambda^{E_1} \in L^{E_1}$ *and* $\mu^{E_2} \in L^{E_2}$ *of lotteries, let* $\langle \lambda^{E_1}, \mu^{E_2} \rangle$ *denote the* compound AA lottery $\nu^{E_1 \cup E_2} = \langle \nu_s \rangle_{s \in E_1 \cup E_2} \in L^{E_1 \cup E_2} = L^{E_1} \times L^{E_2}$ *whose component* $\nu_s \in \Delta(Y_s)$*, for each* $s \in S$*, is the roulette lottery that satisfies* $\nu_s = \lambda_s$ *if* $s \in E_1$ *and* $\nu_s = \mu_s$ *if* $s \in E_2$*.*

[7] Section 4 of Hammond (1998b) considers the corresponding set $\Delta(Y^E)$ of all finitely supported joint distributions over the Cartesian product set Y^E. Section 4.4 of that chapter includes a discussion of how to use an assumption similar to Anscombe and Aumann's (1963) "reversal of order" axiom in order to reduce $\Delta(Y^E)$ to the Cartesian product set $\prod_{s \in E} \Delta(Y_s)$. Indeed, this definition treats as irrelevant any possible dependence between different random variables in the family $y^E = \langle y_s \rangle_{s \in E} \in Y^E$. See also Hammond (1997, Section X).

3.5 Finite Decision Trees with Consequences

Starting with any finite decision tree T with graph (N, O) as specified in Definition 3.3, along with the *event correspondence* $N \ni n \mapsto E(n) \in 2^S \setminus \{\emptyset\}$, one obtains a finite decision tree with consequences by attaching to each terminal node $n \in N^t$ of T an Anscombe–Aumann consequence lottery $\lambda^{E(n)}$ in the appropriate Cartesian product set $L^{E(n)}$. Formally:

Definition 3.10. *Given any event E in S, let $\mathcal{T}^E$ denote the domain of all* finite decision trees with consequences *T in the form of a decision tree (N, O), as in Definition 3.3, whose event correspondence satisfies $E(n_0) = E$ at the initial node n_0, supplemented by a* consequence mapping

$$N^t \ni n \mapsto \gamma^t(n) = \langle \gamma^t_s(n) \rangle_{s \in E(n)} \in L^{E(n)} \tag{6}$$

that attaches to each terminal node $n \in N^t$ of the tree T an AA consequence lottery $\gamma^t(n)$ which, given the relevant event $E(n)$, lies in the corresponding co-domain $L^{E(n)}$ specified in Definition 3.8.

Let $\hat{\mathcal{T}} := \cup_{E \subseteq 2^S \setminus \{\emptyset\}} \mathcal{T}^E$ denote the domain of all possible finite decision trees that are associated with some event E in S.

3.6 Continuation Subtrees

Given any finite decision tree with consequences, as specified in Definition 3.10, along with any node $\bar{n}$ in that tree, the following definition recognizes that the DM, upon reaching node $\bar{n}$, is confronted in effect with a fresh "continuation" decision tree whose initial node is $\bar{n}$.

Definition 3.11. *Consider any decision tree $T \in \hat{\mathcal{T}}$ with graph (N, O), event correspondence $N \ni n \mapsto E(n) \in 2^S \setminus \{\emptyset\}$, and consequence mapping $N^t \ni n \mapsto \gamma(n) \in L^{E(n)}$, as specified in Definition 3.10. Given any node $\bar{n} \in N$, the* continuation subtree *$T_{\geq \bar{n}} \in \mathcal{T}^{E(\bar{n})}$ has:*

1. *the set of nodes $N_{\geq \bar{n}} := \{\bar{n}\} \cup \{n \in N \mid n >^* \bar{n}\}$ that either ultimately succeed or equal $\bar{n}$;*
2. *the set of edges $O_{\geq \bar{n}} := O \cap (N_{\geq \bar{n}} \times N_{\geq \bar{n}})$, which equals the restriction of O to the nodes of $T_{\geq \bar{n}}$;*
3. *the set $N^t_{\geq \bar{n}} = N^t \cap N_{\geq \bar{n}}$ of terminal nodes that ultimately succeed $\bar{n}$;*

4. *the event correspondence* $N_{\geq \bar{n}} \ni n \mapsto E(n)$*, which is the restriction to* $N_{\geq \bar{n}}$ *of* $N \ni n \mapsto E(n)$*;*

5. *the consequence mapping* $N^t_{\geq \bar{n}} \ni n \mapsto \gamma^t(n)$*, which is the restriction to* $N^t_{\geq \bar{n}}$ *of* $N^t \ni n \mapsto \gamma^t(n)$*.*

4 Decision Strategies, Plans, and Behaviour

4.1 Decision Strategies

The usual definition of pure strategy in extensive form games needs to be modified slightly so that it can apply to decision trees, including continuation subtrees, when viewed as one-person games in which real-valued payoffs are replaced by AA consequence lotteries, as specified in Definition 3.8.

Definition 4.1. *Given any finite decision tree* T *in the domain* $\hat{\mathcal{T}}$ *with graph* (N, O) *and* N^d *as its set of decision nodes, a* decision strategy *is a list*

$$\mathbf{d} = \langle d(n) \rangle_{n \in N^d} \in \mathbf{D}(T) := \prod\nolimits_{n \in N^d} N_{+1}(n) \tag{7}$$

of decisions $d(n) \in N_{+1}(n)$ *which, at each decision node* $n \in N^d$*, specify which succeeding node* $n^+ \in N_{+1}(n)$ *is chosen.*

Given any node $\bar{n}$ *of the tree* T *and associated continuation subtree* $T_{\geq \bar{n}}$*, a* continuation decision strategy *is the restriction*

$$\mathbf{d}_{\geq \bar{n}} = \langle d(n) \rangle_{n \in N^d_{\geq \bar{n}}} \in \mathbf{D}(T_{\geq \bar{n}}) := \prod\nolimits_{n \in N^d_{\geq \bar{n}}} N_{+1}(n) \tag{8}$$

of a decision strategy $\mathbf{d} \in \mathbf{D}(T)$ *to decision nodes in the subtree* $T_{\geq \bar{n}}$*.*

4.2 Planned Behaviour

At the initial node n_0 of any finite decision tree T in the domain $\hat{\mathcal{T}}$, the DM's planned behaviour in the tree T will be represented by a non-empty set $P(T)$ of decision strategies $\mathbf{d} = \langle d(n) \rangle_{n \in N^d} \in \mathbf{D}(T)$ that, following Definition 4.1, specify a move $d(n) \in N_{+1}(n)$ at each decision node $n \in N^d$ of the tree T.

This planned behaviour, however, is in principle subject to change as the DM moves through the tree T from each node n to an immediate successor $n^+ \in N_{+1}(n)$. Indeed, in some work in progress that was briefly discussed in

Section 2.1, the theory presented here is extended to consider decision trees T which might become unpredictably "enlivened". Then, in order to avoid any unnecessarily regrettable consequences, it is important to allow departures from any originally planned behaviour.

Accordingly, we postulate that at each node $\bar{n} \in N$ of T, the DM formulates what may be a revised plan for the continuation subtree $T_{\geq \bar{n}}$. Like the original plan at the initial node n_0 of T, this revised plan takes the form of a new non-empty set $P(T_{\geq \bar{n}})$ of decision strategies $\mathbf{d}_{\geq \bar{n}} \in \mathbf{D}(T_{\geq \bar{n}})$ that, following Definition 4.1 and Equation (8), are defined at each decision node $n \in N^d_{\geq \bar{n}}$ of the continuation subtree $T_{\geq \bar{n}}$.

Planned behaviour is *dynamically consistent* just in case, for each node $\bar{n} \in N$ of T, the set $P(T_{\geq \bar{n}})$ consists of the restrictions to the decision nodes $n \in N^d_{\geq \bar{n}}$ in the continuation subtree $T_{\geq \bar{n}}$ of those decision strategies in the set $P(T)$ which originally, at node n_0, were intended to apply at all decision nodes $n \in N^d$ in the whole tree T.[8] Typically planned behaviour is dynamically inconsistent. Yet this is really irrelevant because the actual behaviour that is about to be defined will be dynamically consistent by construction.

4.3 Actual Behaviour

In the original decision tree T, any planned decision strategy $\mathbf{d} = \langle d(n) \rangle_{n \in N^d}$ by itself really determines only the first move $d(n_0)$ from the initial node n_0 to one of its immediate successors in $N_{+1}(n_0)$, and then only in case n_0 is a decision node. Similarly for any continuation subtree $T_{\geq n}$. By the time any later decision node $\bar{n} \in N^d$ of T or $T_{\geq n}$ has been reached, the original plan is subject to revision. In determining actual behaviour, therefore, all that really counts at any decision node $\bar{n} \in N^d$ is the move $d(\bar{n})$ at the initial node $\bar{n}$ of the continuation subtree $T_{\geq \bar{n}}$ which is specified by the DM's planned decision strategy $\mathbf{d}_{\geq \bar{n}} = \langle d(n) \rangle_{n \in N^d_{\geq \bar{n}}}$ for that subtree. This is true whether or not the DM's actual behaviour at decision node $\bar{n}$ coincides with what had been planned before reaching $\bar{n}$.

With this in mind, at each decision node $\bar{n} \in N^d$ of T, define

$$\beta(T_{\geq \bar{n}}, \bar{n}) := \{n^+ \in N_{+1}(\bar{n}) \mid \exists\, \mathbf{d}_{\geq \bar{n}} \in P(T_{\geq \bar{n}}) : d(\bar{n}) = n^+\} \tag{9}$$

[8] For a philosophical discussion of intention, see in particular Anscombe (2000) and Bratman (1987).

as the set of moves at $\bar{n}$ that can be extended to an entire decision strategy $\mathbf{d}_{\geq \bar{n}}$ which belongs to the set $P(T_{\geq \bar{n}})$ of planned decision strategies in the continuation subtree $T_{\geq \bar{n}}$. This motivates the following:

Definition 4.2. *An* actual behaviour rule *is a mapping* $(T, \bar{n}) \mapsto \beta(T_{\geq \bar{n}}, \bar{n})$ *which is defined at every decision node* $\bar{n} \in N^d$ *of every tree* $T \in \hat{\mathcal{T}}$ *and satisfies* $\emptyset \neq \beta(T_{\geq \bar{n}}, \bar{n}) \subseteq N_{+1}(\bar{n})$.

The following is immediately implied by Definition 4.2 and Equation (9).

Proposition 4.3. *Given any possible collection* $\langle P(T_{\geq \bar{n}}) \rangle_{\bar{n} \in N}$ *of planned sets of decision strategies* $\mathbf{d}_{\geq \bar{n}} = \langle d(n) \rangle_{n \in N^d_{\geq \bar{n}}}$ *in the respective continuation subtrees* $\langle T_{\geq \bar{n}} \rangle_{\bar{n} \in N}$*, there is a unique associated actual behaviour rule defined by Equation* (9)*.*

5 Consequences of Behaviour

5.1 Consequences of Decision Strategies

We are interested in behaviour that avoids regrettable consequences or, equivalently, results in consequences that should be chosen. To specify such behaviour, we must first specify what consequences result from different decision strategies, and from actual behaviour. The following proposition uses the structure set out in Definition 3.3 of a decision tree with consequences. This structure allows backward recursion to be applied in order to construct the unique consequence lottery that results from any specified decision strategy.

Proposition 5.1. *Let* $T \in \hat{\mathcal{T}}$ *be any finite decision tree with consequences, in which the AA consequence lottery attached to each terminal node* $n \in N^t$ *is* $\gamma^t(n) = \langle \gamma_s(n) \rangle_{s \in E(n)} \in L^{E(n)}$*, as in Definition 3.10. Given any node* $\bar{n} \in N$ *and any fixed decision strategy* $\mathbf{d}_{\geq \bar{n}} \in \mathbf{D}(T_{\geq \bar{n}})$ *in the continuation subtree* $T_{\geq \bar{n}}$*, the hierarchy of successive AA lotteries* $\gamma^{E(n)}(\mathbf{d}_{\geq \bar{n}}; n) \in L^{E(n)}$ *at each node* $n \in N_{\geq \bar{n}}$ *that result from using the decision strategy* $\mathbf{d}_{\geq \bar{n}}$ *throughout the subtree is the unique solution to the backward recurrence relation defined by the following four-part rule:*

1. *in case* $n \in N^t_{\geq \bar{n}}$ *is a terminal node, the consequence is*

$$\gamma^{E(n)}(\mathbf{d}_{\geq \bar{n}}; n) = \gamma^t(n) \in L^{E(n)} \tag{10}$$

2. in case $n \in N^d_{\geq \bar{n}}$ is a decision node, and so $E(d(n)) = E(n)$, the consequence is

$$\gamma^{E(n)}(\mathbf{d}_{\geq \bar{n}}; n) = \gamma^{E(n)}(\mathbf{d}_{\geq \bar{n}}; d(n)) \in L^{E(d(n))} = L^{E(n)} \tag{11}$$

3. in case $n \in N^c_{\geq \bar{n}}$ is a chance node at which the probability of moving to each immediately succeeding node $n^+ \in N_{+1}(n)$ is $\pi(n^+|n)$, the consequence is the compound AA lottery

$$\gamma^{E(n)}(\mathbf{d}_{\geq \bar{n}}; n) = \textstyle\sum_{n^+ \in N_{+1}(n)} \pi(n^+|n)\, \gamma(\mathbf{d}_{\geq \bar{n}}; n^+) \in L^{E(n)} \tag{12}$$

4. in case $n \in N^e_{\geq \bar{n}}$ is an event node at which the move from n to an immediately succeeding node $n^+ \in N_{+1}(n)$ determines the partition of $E(n)$ into the family $\{E(n^+) \mid n^+ \in N_{+1}(n)\}$ of pairwise disjoint events $E(n^+)$, the consequence is the compound AA lottery

$$\gamma^{E(n)}(\mathbf{d}_{\geq \bar{n}}; n) = \langle \gamma^{E(n^+)}(\mathbf{d}_{\geq \bar{n}}; n^+) \rangle_{n^+ \in N_{+1}(n)} \in L^{E(n)} \tag{13}$$

Proof. The result is evident from combining the various definitions, including those of compound lotteries in Definition 3.9, with a backward induction argument. See Hammond (1988, 1998b) for a similar argument in more detail. □

5.2 Feasible Sets of Consequence Lotteries

Next, we consider the feasible sets of all consequence lotteries that can arise from pursuing one of the possible decision strategies in a given decision tree, or continuation decision tree, with consequences.

Definition 5.2. *Given any finite decision tree T with consequences, as specified in Definition 3.10, along with any node $\bar{n} \in N$, the set $F(T_{\geq \bar{n}})$ of consequence lotteries that are* feasible *in the continuation subtree $T_{\geq \bar{n}}$ is the range of the mapping*

$$\mathbf{D}(T_{\geq \bar{n}}) = \textstyle\prod_{n \in N^d_{\geq \bar{n}}} N_{+1}(n) \ni \mathbf{d}_{\geq \bar{n}} \mapsto \gamma^{E(\bar{n})}(\mathbf{d}_{\geq \bar{n}}) \in L^{E(\bar{n})} \tag{14}$$

from the domain $\mathbf{D}(T_{\geq \bar{n}})$ defined by (8) *in Definition 4.1 of all decision strategies $\mathbf{d}_{\geq \bar{n}}$ in the subtree $T_{\geq \bar{n}}$ to their consequences in the form of AA lotteries. That is, for each $\bar{n} \in N$, one has $F(T_{\geq \bar{n}}) = \gamma^{E(\bar{n})}(\mathbf{D}(T_{\geq \bar{n}}))$.*

The following definition simplifies the notation introduced in Section 2.2.

Definition 5.3. *For each event E in S and associated space L^E specified in Definition 3.8 of all conditional AA lotteries that are possible if the event E occurs, let $\mathcal{F}^E$ denote the domain of all finite non-empty subsets F of L^E.*

Now, an obvious implication of Proposition 5.1 is that the family of feasible sets $F(T_{\geq \bar{n}})$ at different nodes $\bar{n} \in N$ of any tree T can be constructed by backward recursion. The resulting appropriate modifications of Equations (10)–(13) obviously rely on each set $F(T_{\geq \bar{n}})$ being a mixture set, as defined in Herstein and Milnor (1953) and Mongin (2001). But it also requires each set $\mathcal{F}^E$ whose members are finite subsets of L^E to be a mixture set, even though it is not a subset of a vector space. See Hammond (1988, 1998b) for details of the relevant construction.

5.3 Consequences of Actual Behaviour

The consequences of following any decision strategy were set out in Proposition 5.1. A variation of Definition 5.2 can be used to find the set of all possible consequences of following any actual behaviour rule as specified in Definition 4.2.

Definition 5.4. *Given any finite decision tree T with consequences as specified in Definition 3.10, any actual behaviour rule $(T, n) \mapsto \beta(T_{\geq n}, n)$ as specified in Definition 4.2, and any node $\bar{n} \in N$, let*

$$\mathbf{D}_\beta(T_{\geq \bar{n}}) := \prod\nolimits_{n \in N^d_{\geq \bar{n}}} \beta(T_{\geq n}, n) \tag{15}$$

denote the subset of the domain $\mathbf{D}(T_{\geq \bar{n}})$ of all possible decision strategies $\mathbf{d}_{\geq \bar{n}} = \langle d(n) \rangle_{n \in N^d_{\geq \bar{n}}}$ in the continuation subtree $T_{\geq \bar{n}}$ which satisfy the restriction that $d(n) \in \beta(T_{\geq n}, n)$ for all $n \in N^d_{\geq \bar{n}}$. Then the set $\Phi_\beta(T_{\geq \bar{n}})$ of all possible AA lotteries that can result from following the actual behaviour rule $(T, n) \mapsto \beta(T_{\geq n}, n)$ at successive initial nodes n of continuation subtrees $T_{\geq n}$ of the continuation subtree $T_{\geq \bar{n}}$ is the range of the mapping

$$\mathbf{D}_\beta(T_{\geq \bar{n}}) \ni \mathbf{d}_{\geq \bar{n}} \mapsto \gamma^{E(\bar{n})}(\mathbf{d}_{\geq \bar{n}}) \in L^{E(\bar{n})} \tag{16}$$

That is, for each $\bar{n} \in N$, one has $\Phi_\beta(T_{\geq \bar{n}}) = \gamma^{E(\bar{n})}(\mathbf{D}_\beta(T_{\geq \bar{n}}))$.

Another straightforward implication of Proposition 5.1 is that, like the feasible sets $F(T_{\geq \bar{n}})$, the four-part rule set out in Proposition 5.1 allows the hierarchy of sets $\Phi_\beta(T_{\geq \bar{n}})$ at different nodes $\bar{n} \in N$ of any tree T to be constructed by backward recursion. Once again, see Hammond (1988, 1998b) for details.

5.4 Formal Definition of Prerationality

As already stated, our main idea is to adapt the prior results of Hammond (1988, 1997, 1998a, b) in order to characterize prerational complete families of conditional base relations. By definition, these complete families allow actual behaviour throughout the domain $\hat{\mathcal{T}}$ of finite decision trees with consequences, as specified in Definition 3.10, to be explained as the avoidance of regrettable consequences, or equivalently, as the choice of desirable consequences. The concept of prerationality is formalized in the following definition:

Definition 5.5. *The complete family* $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ *of conditional base relations is* prerational *just in case there exist:*

1. *a corresponding complete family of conditional regret functions* $\mathcal{F}^E \ni F \mapsto R^E(F) \subsetneq F$*, or equivalently, of conditional choice functions* $\mathcal{F}^E \ni F \mapsto C^E(F) = F \setminus R^E(F) \in \mathcal{F}^E$*, both defined for all events* E *in* S*, such that for each pair* λ^E, μ^E *of AA lotteries in* L^E*, one has*

$$\mu^E \in R^E(\{\lambda^E, \mu^E\}) \Longleftrightarrow \mu^E \notin C^E(\{\lambda^E, \mu^E\}) \Longleftrightarrow \lambda^E \succ^E \mu^E \quad (17)$$

2. *a behaviour rule* $(T, n) \mapsto \beta(T_{\geq n}, n)$ *defined on the unrestricted domain* $\hat{\mathcal{T}}$ *of finite decision trees, as specified in Definition 3.10, with the property that, for any decision tree* T *in* $\hat{\mathcal{T}}$ *with continuation subtrees* $T_{\geq n}$ *for all* $n \in N$*, as specified in Definition 3.11, the successive sets* $\Phi_\beta(T_{\geq n})$ *of AA lotteries which result from* $(T, \bar{n}) \mapsto \beta(T_{\geq \bar{n}}, \bar{n})$*, as specified in Definition 5.4, satisfy*

$$\Phi_\beta(T_{\geq n}) = C^{E(n)}(F(T_{\geq n})) = F(T_{\geq n}) \setminus R^{E(n)}(F(T_{\geq n})) \quad (18)$$

Remark 5.6. *A rational DM should be expected to follow a behaviour rule that, as in* (18)*, avoids regrettable consequences or, equivalently, seeks desirable ones. Yet according to part 2 of Definition 5.5, prerationality makes no* *such presumption; the behaviour rule $(T, n) \mapsto \beta(T_{\geq n}, n)$ can be any feasible rule that happens to allow* (18) *to be satisfied. The striking implication of Theorem 6.3 is that when prerationality alone is imposed on the entire unrestricted domain $\hat{\mathcal{T}}$ of finite decision trees, it implies conditions on base preferences and on behaviour that are often assumed as rationality hypotheses.*

6 Characterizing Prerationality

6.1 Conditions on Base Relations

We are now ready to apply previous results in consequentialist decision theory that were derived in Hammond (1988) in order to characterize those complete families of conditional base relations that satisfy prerationality, as specified in Definition 5.5. The characterization, like that of consequentialism in decision theory, adds completeness and transitivity to the conditions stated in the following two definitions.

The first extra condition involves Samuelson's (1952) key independence axiom of expected utility theory, formally stated as follows:

Definition 6.1. *Given any event E in S, the conditional base relation $\succsim^E$ on L^E satisfies the* independence axiom *just in case, for any three lotteries λ^E, μ^E, and ν^E in L^E and any $0 < \alpha < 1$, the two compound lotteries $\alpha\lambda^E + (1-\alpha)\,\nu^E$ and $\alpha\mu^E + (1-\alpha)\,\nu^E$ in L^E that are as specified in Definition 3.9 satisfy*

$$\alpha\lambda^E + (1-\alpha)\,\nu^E \succsim^E \alpha\mu^E + (1-\alpha)\,\nu^E \Longleftrightarrow \lambda^E \succsim^E \mu^E \tag{19}$$

The second extra condition is that the family of conditional base relations $\succsim^E$ for different events E in S must satisfy the following extension and strengthening of Savage's sure-thing principle to our setting with AA lotteries:

Definition 6.2. *The complete family $\langle \succsim^E \rangle_{\emptyset \neq E \subseteq S}$ of conditional base relations $\succsim^E$ defined on the corresponding domains L^E of AA lotteries satisfies the* sure thing principle *just in case whenever $\{\lambda^{E_1}, \mu^{E_1}\} \subset L^{E_1}$ and $\nu^{E_2} \in L^{E_2}$, where E_1 and E_2 partition E into two non-empty pairwise disjoint subsets, then the two compound AA lotteries $\langle \lambda^{E_1}, \nu^{E_2} \rangle$ and $\langle \mu^{E_1}, \nu^{E_2} \rangle$ in $\Delta(Y^{E_1 \cup E_2})$ that are as specified in Definition 3.9 satisfy*

$$\langle \lambda^{E_1}, \nu^{E_2} \rangle \succsim^{E_1 \cup E_2} \langle \mu^{E_1}, \nu^{E_2} \rangle \Longleftrightarrow \lambda^{E_1} \succsim^{E_1} \mu^{E_1} \tag{20}$$

Definition 6.2 strengthens Savage's sure-thing principle because, as will be discussed in Section 7.3, it excludes null events.

6.2 First Characterization Theorem

The following result, which is the first main theorem of the paper, characterizes complete families of conditional base relations that are prerational.

Theorem 6.3. *The complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations is prerational, as specified in Definition 5.5, if and only if the following three conditions are all satisfied:*

1. *for each event E in S, the relation $\succsim^E$ is complete and transitive;*

2. *for each event E in S, the relation $\succsim^E$ satisfies the independence axiom stated in Definition 6.1;*

3. *for each pair of pairwise disjoint events E_1, E_2 in S, the two relations $\succsim^{E_1}$ and $\succsim^{E_2}$ satisfy the sure thing principle stated in Definition 6.2.*

Proof. Suppose that the complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations is prerational. By Definition 5.5, there exists a behaviour rule $(T, n) \mapsto \beta(T, n)$ on the domain $\hat{\mathcal{T}}$ and, for each event E in S, both a choice function $\mathcal{F}^E \ni F \mapsto C^E(F) \subseteq F$ and an associated regret function $\mathcal{F}^E \ni F \mapsto R^E(F) \subsetneqq F \setminus C^E(F)$ such that, at every node n of every decision tree $T \in \hat{\mathcal{T}}$, one has

$$C^{E(n)}(F(T_{\geq n})) = F(T_{\geq n}) \setminus R^{E(n)}(F(T_{\geq n})) = \Phi_\beta(T_{\geq n}) \subseteq F(T_{\geq n}) \qquad (21)$$

The argument used to prove Theorem 5.4 via Lemmas 5.1, 5.2 and 5.3 in Hammond (1988) can now be adapted to the present setting. Note in particular that the decision tree in Figure 5.1 on p. 40 of that paper involves a continuation decision tree where the feasible set is a pair. The argument shows that, for each event E in S:

1. for each non-empty finite set F in L^E and each pair λ^E, μ^E in F, if $\lambda^E \in C^E(F)$, then $\lambda^E \succsim^E \mu^E$;

2. for each non-empty finite set F in L^E and each pair λ^E, μ^E in F, if $\lambda^E \in C^E(F)$ and $\mu^E \succsim^E \lambda^E$, then $\mu^E \in C^E(F)$;

3. both the choice function $\mathcal{F}^E \ni F \mapsto C^E(F) \subseteq F$ and associated regret function $\mathcal{F}^E \ni F \mapsto R^E(F) = F \setminus C^E(F)$ are ordinal;

4. the relation $\succsim^E$ is transitive.

Thereafter, note that the decision trees in Figure 6.1 on p. 43 and Figure 6.1 on p. 45, like that in Figure 5.1 on p. 40 of Hammond (1988), both involve a continuation decision tree where the feasible set consists of a pair of consequence lotteries. It follows that the arguments used to prove Theorems 6 and 7 in Hammond (1988) can also be adapted to the present setting. They imply the independence condition and sure thing principle respectively.

Conversely, to prove the sufficiency part of the theorem, suppose that the complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of complete and transitive conditional base relations all satisfy the independence axiom stated in Definition 6.1, as well as the sure thing principle stated in Definition 6.2. For each event E in S, consider the ordinal regret function $\mathcal{F}^E \ni F \mapsto R^E(F) \subsetneq F$ generated by the base relation $\succsim^E$, as specified in Definition 2.4. Consider too the behaviour rule $(T, n) \mapsto \beta(T, n)$ that is defined for each decision node $n \in N^d$ in any tree $T \in \hat{\mathcal{T}}$ by

$$\beta(T, n) = \{n^+ \in N_{+1}(n) \mid F(T_{\geq n^+}) \setminus R^{E(n)}(T_{\geq n}) \neq \emptyset\} \tag{22}$$

By Definition 5.5, it follows from (22) that

$$\beta(T, n) = \{n^+ \in N_{+1}(n) \mid F(T_{\geq n^+}) \cap C^{E(n)}(F(T_{\geq n})) \neq \emptyset\} \tag{23}$$

With slightly revised notation, this is the equation in part (3) of the proof of Theorem 8 in Hammond (1988, p. 47). That proof by backward recursion shows that, throughout the domain of the mapping $(T, n) \mapsto \Phi_\beta(T_{\geq n})$, the complete family of choice functions $\mathcal{F}^E \ni F \mapsto C^E(F) = F \setminus R^E(F)$ satisfies (18) in Definition 5.5. It follows that the family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of base relations is prerational. □

7 Refined Bayesian Rationality

7.1 Complete Conditional Probabilities

The Bayesian paradigm involves expected utility based on conditional personal or subjective probabilities. The following definition, based on ideas discussed in Rényi (1955), Császár (1955), Myerson (1986), Hammond (1994), and elsewhere, requires that conditional probabilities be specified even when the conditioning event has probability zero.

Definition 7.1. *Given the fixed non-empty finite set S of possible states of the world, define the domain*

$$\mathcal{E} := \{(E, E_0) \in 2^S \times 2^S \mid E \subseteq E_0 \neq \emptyset\} \tag{24}$$

of conditioned events*. Then a* complete conditional probability system *(or CCPS) is a mapping*

$$\mathcal{E} \ni (E, E_0) \mapsto \mathbb{P}(E|E_0) \in [0, 1] \tag{25}$$

such that:

1. *for all $E \in 2^S \setminus \{\emptyset\}$, one has $\mathbb{P}(E|E) = 1$;*
2. *for all $E, E' \in 2^S$ satisfying $E, E' \subseteq S \neq \emptyset$ and $E \cap E' = \emptyset$, one has $\mathbb{P}(E \cup E'|S) = \mathbb{P}(E|S) + \mathbb{P}(E'|S)$;*
3. *for all $E, E', S \in 2^S$ with $E \subseteq E' \subseteq S$ and $E' \neq \emptyset$, one has*

$$\mathbb{P}(E|S) = \mathbb{P}(E|E')\,\mathbb{P}(E'|S) \tag{26}$$

The CCPS defined by (25) *is* refined *just in case $\mathbb{P}(E|E_0) > 0$ for all $(E, E_0) \in \mathcal{E}$ with $E \neq \emptyset$.*

Note that (26) is the version of Bayes' rule that is relevant for these conditional probabilities. Note too that in any finite decision tree $T \in \hat{\mathcal{T}}$, following the concept of "probabilistic sophistication" due to Machina and Schmeidler (1992), the existence of a CCPS allows each event node $n \in N^e$ and associated horse lottery with outcomes $\langle E(n^+)\rangle_{n^+ \in N_{+1}(n)}$ to be replaced by a chance node with the associated roulette lottery whose transition probabilities satisfy $\pi(n^+|n) = \mathbb{P}(E(n^+)|E(n))$ for all $n^+ \in N_{+1}(n)$. Furthermore, the AA lottery $\gamma(n) \in \Delta(Y^{E(n)})$ attached to each terminal node $n \in N^t$ is replaced by a single roulette lottery $\lambda(n) \in \Delta(\cup_{s \in E(n)} Y_s)$. The resulting decision tree, of course, like those discussed in Remark 3.4, has only roulette lotteries but no horse lotteries.

7.2 Bayesian Rationality and a Refinement

The hypothesis of Bayesian rationality, by definition, requires that for each event E in S, the conditional base relation $\succsim^E$ can be represented by a subjective expected utility (SEU) function, with personal probabilities that form a CCPS. In general this CCPS may include zero probability events. But it will not when, as in Proposition 7.4, null events are excluded.

Definition 7.2. *The complete family* $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ *of conditional base relations over the respective domains* L^E *of AA lotteries is* Bayesian rational *just in case there exist both a CCPS* $\mathcal{E} \ni (E, E_0) \mapsto \mathbb{P}(E|E_0) \in [0,1]$*, as in Definition 7.1, and a* von Neumann–Morgenstern utility function *(or NMUF)* $\cup_{s \in S} Y_s \ni y \mapsto u(y) \in \mathbb{R}$*, such that for each event* E *in* S *the conditional base relation* $\succsim^E$ *is represented by the conditional subjective expected utility (or SEU) function*

$$L^E \ni \lambda^E = \langle \lambda_s \rangle_{s \in E} \mapsto U^E(\lambda^E) \in \mathbb{R} \tag{27}$$

which is defined, for all $\lambda^E \in L^E$*, by*

$$U^E(\lambda^E) := \sum\nolimits_{s \in E} \mathbb{P}(\{s\}|E) \sum\nolimits_{y \in Y_s} \lambda_s(y)\, u(y) \tag{28}$$

That is, for all pairs of AA lotteries $\lambda^E, \mu^E \in L^E$*, one has the equivalence* $\lambda^E \succsim^E \mu^E \iff U^E(\lambda^E) \geq U^E(\mu^E)$*.*

The family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ *is* refined Bayesian rational *just in case it is Bayesian rational with a CCPS that is refined, according to Definition 7.1.*

7.3 Excluding Null Events

The following definition of a null event involves a violation of the sure thing principle which is enunciated in (20) of Definition 6.2. That definition extends to AA lotteries the original definition due to Savage (1954).

Definition 7.3. *Given the complete family* $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ *of conditional base relations, the event* E *in* S *is* null *relative to the event* E_0 *in* S *with* $E \subsetneqq E_0$ *just in case there exist AA lotteries* $\lambda^E, \mu^E \in L^E$ *and* $\nu^{E_0 \setminus E} \in L^{E_0 \setminus E}$ *such that* $\lambda^E \succ^E \mu^E$ *and yet* $\langle \lambda^E, \nu^{E_0 \setminus E} \rangle \sim^{E_0} \langle \mu^E, \nu^{E_0 \setminus E} \rangle$*.*

For any family of base relations that is Bayesian rational, the next result identifies null events with those that have zero conditional probability.

Proposition 7.4. *Suppose that the complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations is Bayesian rational. Then the event E in S is null relative to the event E_0 in S with $E \subsetneqq E_0$ if and only if $\mathbb{P}(E|E_0) = 0$.*

Proof. Given any $\lambda^E, \mu^E \in L^E$ and $\nu^{E_0 \setminus E} \in L^{E_0 \setminus E}$, it follows from (28) that $\langle \lambda^E, \nu^{E_0 \setminus E} \rangle \sim^{E_0} \langle \mu^E, \nu^{E_0 \setminus E} \rangle$ if and only if

$$U^{E_0}(\langle \lambda^E, \nu^{E_0 \setminus E} \rangle) - U^{E_0}(\langle \mu^E, \nu^{E_0 \setminus E} \rangle) = \mathbb{P}(E|E_0) \left[U^E(\lambda^E) - U^E(\mu^E) \right] = 0$$

Yet $\lambda^E \succ^E \mu^E$ if and only if $U^E(\lambda^E) - U^E(\mu^E) > 0$. So $\lambda^E \succ^E \mu^E$ is consistent with $\langle \lambda^E, \nu^{E_0 \setminus E} \rangle \sim^{E_0} \langle \mu^E, \nu^{E_0 \setminus E} \rangle$ if and only if $\mathbb{P}(E|E_0) = 0$. □

The next result shows that if any complete family of conditional base relations is Bayesian rational and also satisfies the sure thing principle, then the family must be refined Bayesian rational.

Proposition 7.5. *Suppose that the complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations is Bayesian rational, and satisfies the sure thing principle, as stated in Definition 6.2. Then the family is refined Bayesian rational.*

Proof. Given any pair of events E, E_0 in S with $E \subsetneqq E_0$, suppose that $\lambda^E, \mu^E \in L^E$ with $\lambda^E \succ^E \mu^E$, and that $\nu^{E_0 \setminus E} \in L^{E_0 \setminus E}$. Then the sure thing principle implies that $\langle \lambda^E, \nu^{E_0 \setminus E} \rangle \succ^{E_0} \langle \mu^E, \nu^{E_0 \setminus E} \rangle$. It follows from Definition 7.3 that there can be no null event and so, by Proposition 7.4, that all conditional probabilities $\mathbb{P}(E|E_0)$ are positive. So the Bayesian rational family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ is refined Bayesian rational. □

7.4 Discontinuous Preferences

Theorem 6.3 implies that if a family of conditional base relations is prerational, then those relations must satisfy both independence and the sure thing principle. The latter, of course, are two of the more contentious conditions that are treated as axioms in the standard theory of Bayesian rationality. Without additional restrictions, however, the following example illustrates how prerationality by itself does not imply Bayesian rationality.

Example 7.6. *Suppose that $S = \{\bar{s}\}$ and that Y consists of the three distinct consequences a, b, c. Suppose that the preference ordering $\succsim$ over lotteries in $\Delta(Y)$ satisfies*

$$\lambda \succ \mu \iff [\textit{either } \lambda(c) < \mu(c) \textit{ or both } \lambda(c) = \mu(c) \textit{ and } \lambda(a) > \mu(a)\ ] \quad (29)$$

This is a lexicographic *preference ordering of the kind discussed in, for example, Fishburn (1971) and Martinez-Legaz (1998). As is well known, there is no von Neumann–Morgenstern (NMUF) $Y \ni y \mapsto u(y)$ such that*

$$\lambda \succsim \mu \Longleftrightarrow \sum_{y \in Y} [\lambda(y) - \mu(y)]\, u(y) \geq 0$$

Instead, one needs two distinct NMUFs $Y \ni y \mapsto u_j(y)$ ($j = 1, 2$) such as those defined by

$$u_1(a) = u_1(b) = 1; \quad u_1(c) = 0; \quad u_2(a) = 1; \quad u_2(b) = u_2(c) = 0 \tag{30}$$

Then one has a lexicographic expected utility ordering which, using the notation $\mathbb{E}_\pi u$ for the expected utility $\sum_{y \in Y} \pi(y)\, u(y)$ w.r.t. $\pi \in \Delta(Y)$, is defined by

$$\lambda \succ \mu \Longleftrightarrow \mathbb{E}_\lambda u_1 > \mathbb{E}_\mu u_1 \quad \textit{or} \quad \mathbb{E}_\lambda u_1 = \mathbb{E}_\mu u_1 \quad \textit{and} \quad \mathbb{E}_\lambda u_2 > \mathbb{E}_\mu u_2 \tag{31}$$

7.5 Continuous Preferences

The lexicographic preferences of Example 7.6 can be excluded by the following continuity condition, which is the key Axiom 2 of Herstein and Milnor (1953, p. 293). It involves the concept of "Marschak triangle", due to Marschak (1950).

Definition 7.7. *Fix any event E in S.*

1. *The* Marschak triangle *generated by any set of three distinct lotteries $\lambda^E, \mu^E, \nu^E \in \Delta(Y^E)$ is the set*

   $$\Delta(\{\lambda^E, \mu^E, \nu^E\}) := \{p_1\lambda^E + p_2\mu^E + p_3\nu^E \mid p_1, p_2, p_3 \geq 0; p_1 + p_2 + p_3 = 1\}$$

   obtained as the triple (p_1, p_2, p_3) varies over the unit simplex in $\mathbb{R}^3$.

2. *The base relation $\succsim^E$ on L^E is* continuous on triangles *just in case, for any $\lambda^E, \mu^E, \nu^E \in \Delta(Y^E)$ with $\lambda^E \succ^E \mu^E$ and $\mu^E \succ^E \nu^E$, the two sets*

   $$\begin{aligned} &\{\alpha \in [0,1] \mid \alpha\lambda^E + (1-\alpha)\nu^E \succsim^E \mu^E\} \\ \textit{and} \quad &\{\alpha \in [0,1] \mid \mu^E \succsim^E \alpha\lambda^E + (1-\alpha)\nu^E\} \end{aligned}$$

   are both closed subsets of the closed interval $[0, 1]$.

7.6 Expected Evaluation

The following definition of evaluation function is based on Wilson (1968) and Myerson (1979). It has the same mathematical structure as the kind of state-dependent utility function introduced by Drèze (1961, 1962, 1987).[9]

Definition 7.8. *Given the fixed state space S and family $\langle Y_s \rangle_{s \in S}$ of state-dependent consequence domains, define the domain*

$$\hat{Y} := \cup_{s \in S}(\{s\} \times Y_s) \tag{32}$$

of state–consequence pairs. Then the complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations over the respective domains L^E has an expected evaluation representation *just in case there exists an* evaluation function

$$\hat{Y} \ni (s, y) \mapsto v(s, y) \in \mathbb{R} \tag{33}$$

such that, for each event E in S, the conditional expected evaluation function *defined by*

$$L^E \ni \lambda^E \mapsto V^E(\lambda^E) := \sum_{s \in E} \sum_{y \in Y_s} \lambda_s(y)\, v(s, y) \tag{34}$$

has the property that, for each pair $\lambda^E, \mu^E \in L^E$, one has

$$\lambda^E \succsim^E \mu^E \Longleftrightarrow V^E(\lambda^E) \geq V^E(\mu^E) \tag{35}$$

The following definition extends to evaluation functions the standard notion of a "cardinal" utility function.

Definition 7.9. *The two evaluation functions $\hat{Y} \ni (s, y) \mapsto v(s, y) \in \mathbb{R}$ and $\hat{Y} \ni (s, y) \mapsto \tilde{v}(s, y) \in \mathbb{R}$ are* co-cardinally equivalent *just in case there exist an additive constant α_s for each $s \in S$, as well as a single positive multiplicative constant ρ that is independent of s, such that for all $(s, y) \in \hat{Y}$ one has $\tilde{v}(s, y) = \alpha_s + \rho\, v(s, y)$.*

[9] See also Drèze (1987, pp. 76–81) for a 1971 exchange of letters between Aumann and Savage on the issue of state-dependence. State-dependent utility functions were subsequently considered by, amongst others, Jones-Lee (1974), Karni et al. (1983), Karni (1985), Mongin (1998), Hammond (1998b, 1999), as well as Karni and Mongin (2000). For an extensive survey, see Drèze and Rustichini (2004). Note that in the textbook by Mas-Colell et al. (1995, pp. 200–1), the expected evaluation function of Definition 7.8 is described as an "extended expected utility representation".

The next result relies on base relations being non-trivial, as defined below.

Definition 7.10. *Given any fixed state $s \in S$, the base relation $\succsim^{\{s\}}$ on $\Delta(Y_s)$ is* non-trivial *just in case there exist three consequences $a, b, c \in Y_s$ such that the corresponding degenerate lotteries $\delta_a, \delta_b, \delta_c \in \Delta(Y_s)$ satisfy $\delta_a \succ^{\{s\}} \delta_b$ and $\delta_b \succ^{\{s\}} \delta_c$.*

Proposition 7.11. *Suppose that for each state $s \in S$, the base relation $\succsim^{\{s\}}$ is non-trivial on the state-dependent consequence domain Y_s. Then the complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations $\succsim^E$ is prerational and continuous on triangles if and only if, given the domain $\hat{Y}$ defined in* (32)*, there exists a unique co-cardinal equivalence class of evaluation functions $\hat{Y} \ni (s, y) \mapsto v(s, y) \in \mathbb{R}$ such that each base relation $\succsim^E$ over the AA lottery domain L^E can be represented by the appropriate conditionally expected evaluation specified by* (34)*.*

Proof. Necessity of the representation of the particular base relation $\succsim^S$ by the expected evaluation function

$$L^S \ni \lambda^S \mapsto V^S(\lambda^S) := \sum\nolimits_{s \in S} \sum\nolimits_{y \in Y_s} \lambda_s(y)\, v(s, y) \tag{36}$$

follows from Lemma 6.1 of Hammond (1998b).

Next, because each base relation $\succsim^{\{s\}}$ is non-trivial on the state-dependent consequence domain Y_s, uniqueness of the co-cardinal equivalence class of evaluation functions follows from uniqueness of the ratios of evaluation differences, which represent marginal rates of substitution between probability shifts. This can be shown using arguments such as those set out, for example, in Hammond (1998b, pp. 245–246).

Similar properties for each member $\succsim^E$ of the complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations follow from applying the previous arguments to the domain L^E of AA lotteries, for each event E in S.

Conversely, consider any complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations $\succsim^E$ that can each be represented by the appropriate conditional expected evaluation defined by (34). Then it is easy to prove by a direct argument that this family must satisfy continuity on triangles, completeness, transitivity, independence, and the sure thing principle. In particular, Theorem 6.3 implies that the complete family of conditional base relations $\succsim^E$ must be prerational. Once again, details are omitted. □

Example 7.12. *Consider the special case when the state space S consists of only the single state s. Then it is as if every decision tree T in $\hat{\mathcal{T}}$ has no event nodes, so all lotteries are roulette lotteries. Suppose that the base relation $\succsim^{\{s\}}$ over the domain $\Delta(Y_{\{s\}})$ of roulette lotteries satisfies non-triviality. In this case Proposition 7.11 reduces to the statement that $\succsim^{\{s\}}$ is prerational and continuous on triangles if and only if there exists a unique cardinal equivalence class of NMUFs $Y_{\{s\}} \ni y \mapsto u(y) \in \mathbb{R}$ such that $\succsim^{\{s\}}$ can be represented by the expected utility function $\Delta(Y_{\{s\}}) \ni \lambda \mapsto U(\lambda) = \sum_{y \in Y} \lambda(y)\, u(y)$.*

7.7 State Independence and Bayesian Rationality

The following definition extends two key assumptions in Savage (1954) to the Anscombe and Aumann (1963) framework being used in this paper.

Definition 7.13. *The collection $\langle \succsim^{\{s\}} \rangle_{s \in S}$ of conditional base relations over the respective domains $L^{\{s\}}$ of roulette lotteries is* state-independent *just in case there exist a single consequence domain Y^* and a single base preference relation $\succsim^*$ defined on the domain $\Delta(Y^*)$ of roulette lotteries on Y^* such that, for each state $s \in S$, the consequence domain Y_s and conditional base relation $\succsim^{\{s\}}$ satisfy $Y_s = Y^*$ and $\succsim^{\{s\}} = \succsim^*$.*

Assuming state independence allows the conclusion of Proposition 7.11 to be strengthened as follows:

Proposition 7.14. *Suppose that the collection $\langle \succsim^{\{s\}} \rangle_{s \in S}$ of conditional base relations satisfies non-triviality and state independence. Then the complete family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of conditional base relations $\succsim^E$ is prerational and continuous on triangles if and only if it is refined Bayesian rational.*

The result will emerge as an obvious corollary of Theorem 7.16 below, so its proof is omitted.

7.8 State-Dependent Consequence Domains

The state-dependent utility functions mentioned at the start of Section 7.6 were motivated by the evident unrealism of the assumption that the domain of relevant consequences is always independent of the state of the world. This is especially true in the case of potentially fatal accidents that was addressed by, amongst others, Drèze (1962) and Jones-Lee (1974). Here we consider utility functions that, as in Hammond (1998b, 1999), are state-independent even though the consequence domains Y_s are state-dependent.

Definition 7.15. *Given the fixed state space S and fixed family $\langle Y_s \rangle_{s\in S}$ of state-dependent consequence domains, define the* union domain $Y^* := \cup_{s\in S} Y_s$ *of all possible consequences. Say that the complete family $\langle \succsim^E \rangle_{E\in 2^S\setminus\{\emptyset\}}$ of conditional base relations satisfies* generalized state independence *just in case there exists a* state-independent base relation $\succsim^*$ *on $\Delta(Y^*)$ whose restriction to $\Delta(Y_s)$, for each state $s \in S$, equals the relation $\succsim^{\{s\}}$.*

The following is the second main theorem of the paper.

Theorem 7.16. *Suppose that the collection $\langle \succsim^{\{s\}} \rangle_{s\in S}$ of conditional base relations satisfies non-triviality and generalized state independence. Then the complete family $\langle \succsim^E \rangle_{E\in 2^S\setminus\{\emptyset\}}$ of conditional base relations $\succsim^E$ is continuous on triangles and prerational if and only if it is refined Bayesian rational.*

Proof. Suppose that the collection $\langle \succsim^{\{s\}} \rangle_{s\in S}$ satisfies non-triviality and generalized state independence. By Proposition 7.11, if the complete family $\langle \succsim^E \rangle_{E\in 2^S\setminus\{\emptyset\}}$ satisfies prerationality and continuity on triangles, then there exists a unique co-cardinal equivalence class of evaluation functions $\hat{Y} \ni (s, y) \mapsto v(s, y) \in \mathbb{R}$ such that, for each event E in S, the associated conditional relation $\succsim^E$ over the domain L^E is represented by the expected evaluation function (34). In particular, when $E = \{s\}$ for any state $s \in S$, the conditional base relation $\succsim^{\{s\}}$ over $\Delta(Y_s)$ is represented by the expected evaluation function

$$\Delta(Y_s) \ni \lambda \mapsto \sum\nolimits_{y\in Y_s} \lambda(y)\, v(s, y) \in \mathbb{R} \tag{37}$$

Because of Example 7.12, however, generalized state independence combined with prerationality and continuity on triangles implies the existence of an NMUF $Y^* \ni y \mapsto u(y) \in \mathbb{R}$ defined on the whole of Y^* such that, for each $s \in S$, the conditional base relation $\succsim^{\{s\}}$ over $L^{\{s\}}$ is also represented by the expected utility function

$$\Delta(Y_s) \ni \lambda \mapsto \sum\nolimits_{y\in Y_s} \lambda(y)\, u(y) \in \mathbb{R} \tag{38}$$

Now, for each state $s \in S$, because the conditional base relation $\succsim^{\{s\}}$ over the domain $\Delta(Y_s)$ of roulette lotteries is non-trivial, a necessary condition for both (37) and (38) to represent $\succsim^{\{s\}}$ is that the two functions $Y_s \ni y \mapsto$

$v(s,y) \in \mathbb{R}$ and $Y_s \ni y \mapsto u(y) \in \mathbb{R}$ are cardinally equivalent NMUFs. That is, there must exist a unique additive constant $\alpha_s \in \mathbb{R}$ and a unique positive multiplicative constant $\rho_s \in \mathbb{R}$ such that $v(s,y) \equiv \alpha_s + \rho_s u(y)$ throughout the domain Y_s. It follows that the conditional expected evaluation function defined by (34) takes the particular form

$$L^E \ni \lambda^E \mapsto V^E(\lambda^E) = \sum_{s \in E} \sum_{y \in Y_s} \lambda_s(y) \left[\alpha_s + \rho_s u(y)\right] \tag{39}$$

For each state $s \in S$, note that $\lambda_s \in \Delta(Y_s)$ and so $\sum_{y \in Y_s} \lambda_s(y) = 1$. By routine algebra, therefore, (39) implies that

$$V^E(\lambda^E) = \sum_{s \in E} \alpha_s + \sum_{s \in E} \rho_s \sum_{y \in Y_s} \lambda_s(y) u(y) \tag{40}$$

Finally, because each $\rho_s > 0$, we can define a unique refined CCPS on the domain specified in (24) of Definition 7.1 by the equation

$$\mathcal{E}(S) \ni (E, E_0) \mapsto \mathbb{P}(E|E_0) := \frac{\sum_{s \in E} \rho_s}{\sum_{s \in E_0} \rho_s} \in [0,1] \tag{41}$$

Then the function $L^E \ni \lambda^E \mapsto V^E(\lambda^E)$ given by (40) is cardinally equivalent to the function

$$\tilde{V}^E(\lambda^E) = \sum_{s \in E} \mathbb{P}(\{s\}|E) \sum_{y \in Y_s} \lambda_s(y) u(y) \tag{42}$$

whose right-hand side is the same as that of Equation (28). So for each event E in S, the associated conditional base relation $\succsim^E$ over the domain L^E of AA lotteries is represented by the conditional subjective expected utility function (42). Since all conditional probabilities defined by (41) are positive, this confirms refined Bayesian rationality.

The converse result follows from the fact that refined Bayesian rationality evidently implies the existence of a family $\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$ of induced conditional base relations which are complete, transitive, and continuous on triangles while also satisfying both independence and the sure-thing principle. It follows that the family of base relations is continuous on triangles and, by Theorem 6.3, prerational. □

8 Concluding Remarks

8.1 Summary of Main Results

Recently Gilboa (2010, 2011, 2015) has argued that one should construct a decision theory that avoids regrettable axioms. It is entirely understandable that his argument can be made entirely consistent with the work by Gilboa and Schmeidler (1995, 2001) and Gilboa et al. (2009, 2010), amongst many others, on some prominent alternatives to the SEU hypothesis of Bayesian rationality.

Now, there is much to commend an approach to rational decision making that is founded on avoiding regret. This follows a basic idea that appears, for example, in Savage's (1954) discussion of the Allais (1953) paradox. Instead of Gilboa's focus on regrettable axioms, however, this paper follows what seems Savage's much more compelling idea that rationality requires a decision maker's behaviour in finite decision trees to be explicable as the avoidance of regrettable consequences. This idea led to the key hypothesis of prerationality that was introduced informally in Section 1.3, and formally in Definition 5.5. Our first main result, which is Theorem 6.3, characterizes a complete family of conditional base preference relations as prerational if and only if they are complete, transitive, and satisfy both independence and a strengthened form of Savage's sure thing principle. In particular, several prominent alternatives to the SEU hypothesis are excluded.

Restricting attention to base relations that satisfy a familiar continuity condition leads to Theorem 7.16, our second main result. It assumes both a non-triviality condition and a generalized form of the state-independence axiom that usually plays a key in the theory of subjective probability. Under these assumptions, Theorem 7.16 characterizes a complete family of conditional base relations as both prerational and continuous if and only if each conditional base relation is refined Bayesian rational in the sense that it corresponds to a conditional subjective expected utility function which excludes zero probabilities.

8.2 Possible Extensions

This paper follows previous work on consequentialism and expected utility theory in Hammond (1988, 1997, 1998a, b, 1999) by considering finite decision trees in which an AA consequence lottery is assigned to each terminal node. The work of Siniscalchi (2011) in particular suggests an interesting class of enriched "recursive" decision trees in which a continuation decision tree is attached to each terminal node. Work such as that surveyed by Streufert (1998) may also be useful for analysing such trees.

A second possible extension that was briefly discussed in Section 2.1 concerns the "enlivened" decision trees that arise whenever practical considerations force the DM to use a bounded decision model. Finally, a third possible extension involves the kind of discontinuous lexicographic preferences that were briefly mentioned in Section 7.4.

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List of Notation in Order of Appearance

Notation	meaning	first occurrence
X	consequence domain	Section 2.2
$\mathcal{F}(X)$	family of non-empty finite subsets of X	Definition 2.1
$\mathcal{F}(X) \ni F \mapsto R(F)$	regret function	Definition 2.1
$\mathcal{F}(X) \ni F \mapsto C(F)$	choice function	Proposition 2.2
$\succ$	strict preference base relation	Definition 2.3
$\succsim$	weak preference base relation	Definition 2.3
$\sim$	indifference base relation	Definition 2.3
N	finite set of nodes	Definition 3.1
$(n, n^+) \in N \times N$	directed edge	Definition 3.1
O	finite set of directed edges	Definition 3.1
(N, O)	directed graph	Definition 3.1
$>^1$	immediate successor relation on N	Definition 3.1
$\mathbb{N}$	set $\{1, 2, 3, \ldots\}$ of natural numbers	Definition 3.1
$>^k$	k step successor relation on N	Definition 3.1
$>^*$	eventual successor relation on N	Definition 3.1
$N_{+1}(n)$	the set of immediate successors of n	Definition 3.1
$N_{-1}(n)$	the set of immediate predecessors of n	Definition 3.1
n_0	initial or root node	Definition 3.1
S	fixed finite set of states of the world	Assumption 3.2
E	(non-empty) event in S	Assumption 3.2
T	finite decision tree	Definition 3.3
$N \ni n \mapsto E(n)$	event correspondence	Definition 3.3
N^d	set of decision nodes	Definition 3.3
N^c	set of chance nodes	Definition 3.3
N^e	set of event nodes	Definition 3.3
N^t	set of terminal nodes	Definition 3.3
$\pi(n^+ \vert n)$	transition probability	Definition 3.3
s	state of the world	Section 3.3
Y_s	state-dependent consequence domain	Section 3.3
$Y^E = \prod_{s \in E} Y_s$	space of state-contingent consequences	Section 3.3
$Z \ni z \mapsto \lambda(z) \in [0, 1]$	simple lottery on Z	Definition 3.7
$\Delta(Z)$	domain of simple lotteries on Z	Definition 3.7
Λ	finite support of simple lottery	Definition 3.7
δ_z	degenerate lottery concentrated at z	Definition 3.7
$\lambda^E = \langle \lambda_s \rangle_{s \in E}$	Anscombe–Aumann consequence lottery	Definition 3.8

$L^E = \prod_{s\in E} \Delta(Y_s)$	Anscombe–Aumann lottery domain	Definition 3.8
$\alpha\lambda^E + (1-\alpha)\mu^E$	compounded roulette and AA lotteries	Definition 3.9
$\langle \lambda^{E_1}, \mu^{E_2} \rangle$	compounded horse and AA lotteries	Definition 3.9
$\mathcal{T}^E$	decision trees with $E(n_0) = E$	Definition 3.10
$\hat{\mathcal{T}}$	domain of finite decision trees	Definition 3.10
$n \mapsto \gamma^t(n) \in L^{E(n)}$	consequence mapping	Definition 3.10
$T_{\geq \bar{n}}$	continuation subtree of tree T starting at $\bar{n}$	Definition 3.11
$N_{\geq \bar{n}}$	set of nodes in $T_{\geq \bar{n}}$	Definition 3.11
$O_{\geq \bar{n}}$	set of edges in $T_{\geq \bar{n}}$	Definition 3.11
$N^t_{\geq \bar{n}}$	set of terminal nodes in $T_{\geq \bar{n}}$	Definition 3.11
$\mathbf{d} = \langle d(n) \rangle_{n \in N^d}$	decision strategy in T	Definition 4.1
$\mathbf{D}(T)$	set of decision strategies in T	Definition 4.1
$\mathbf{d}_{\geq \bar{n}} = \langle d(n) \rangle_{n \in N^d_{\geq \bar{n}}}$	decision strategy in $T_{\geq \bar{n}}$	Definition 4.1
$\mathbf{D}(T_{\geq \bar{n}})$	set of decision strategies in $T_{\geq \bar{n}}$	Definition 4.1
$P(T) \subseteq \mathbf{D}(T)$	planned set of decision strategies in T	Section 4.2
$\beta(T_{\geq \bar{n}}, \bar{n})$	actual behaviour set at node $\bar{n}$	Definition 4.2
$\gamma^{E(n)}(\mathbf{d}_{\geq \bar{n}}; n)$	consequence at n of $\mathbf{d}_{\geq \bar{n}}$	Proposition 5.1
$F(T_{\geq \bar{n}}) \subseteq L^{E(\bar{n})}$	feasible set of lotteries	Definition 5.2
$\mathcal{F}^E = \mathcal{F}(L^E)$	domain of feasible sets	Section 5.2
$\mathbf{D}_\beta(T_{\geq \bar{n}})$	decision strategies consistent with rule β	Definition 5.4
$\Phi_\beta(T_{\geq \bar{n}})$	consequences of strategies in $\mathbf{D}_\beta(T_{\geq \bar{n}})$	Definition 5.4
$\langle \succsim^E \rangle_{E \in 2^S \setminus \{\emptyset\}}$	complete family of conditional base relations	Definition 5.5
$F \mapsto R^E(F) \subsetneq F$	conditional regret function given event E	Definition 5.5
$F \mapsto C^E(F) \subseteq F$	conditional choice function given event E	Definition 5.5
$\mathcal{E} \in 2^S \times 2^S$	conditioned events in S	Definition 7.1
$(E, E_0) \mapsto \mathbb{P}(E \mid E_0)$	complete conditional probability system	Definition 7.1
$\cup_{s\in S} Y_s \ni y \mapsto u(y)$	von Neumann–Morgenstern utility function	Definition 7.2
$L^E \ni \lambda^E \mapsto U^E(\lambda^E)$	subjective expected utility function	Definition 7.2
$\mathbb{E}_\pi u$	expectation of u w.r.t. π	Example 7.6
$\Delta(\{\lambda^E, \mu^E, \nu^E\})$	Marschak triangle generated by $\{\lambda^E, \mu^E, \nu^E\}$	Definition 7.7
$\hat{Y} = \cup_{s\in S}(\{s\} \times Y_s)$	domain of state–consequence pairs	Definition 7.8
$\hat{Y} \ni (s, y) \mapsto v(s, y)$	evaluation function	Definition 7.8
$L^E \ni \lambda^E \mapsto V^E(\lambda^E)$	expected evaluation function	Definition 7.8
Y^*	state-independent consequence domain	Definition 7.13
$\succsim^*$	state-independent base relation on $\Delta(Y^*)$	Definition 7.13