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Pricing under asymmetry and ambiguity

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Abstract

Robust pricing models often suffer from being overly conservative. This is due to lack of asymmetry information within the set of possible valuation distributions. However, even when information on asymmetry is available incorporating it within pricing models makes the characterization of pricing policies very difficult. Our main results address this challenge by providing an explicit characterization of the worst-case prior under the extended information setting that includes semivariance as a measure of asymmetry on top of mean and variance. We illustrate the gain from having the asymmetry information captured via semivariance.

Keywords Pricing, Ambiguity, Distributionally Robust Mechanism, Asymmetry

1 Introduction

Price, argued by many, is the most effective control lever in increasing profitability, albeit a risky lever to pull. A bad price, whatever that means, can lead to an immediate and long standing decrease in profits. Undoubtedly, there are many dimensions to the question *"what is a good price"*, given the fact that buyers may have many options to choose from, and each buyer may value the same product differently. Clearly, the question of a good price presents a seller with a difficult conundrum because seller does not possess (completely) either of these information about the options available to the buyer or their valuations. The focus of this paper is to analyze the pricing problem in such contexts. Being at the heart of mechanism design, such problems with limited and/or minimal information about the prior of valuation distribution, motivated increase in literature in this direction of robust mechanism design. Within this literature, characterizing optimal pricing mechanism has been the

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primary focus, eg. [Carrasco et al., 2018], where worst-case revenue maximising price is found by solving a sequential game between seller and adversarial nature. In fact, optimal monopoly pricing forms the basis of profit maximizing mechanism design with incomplete information. Robust versions of classical pricing problem have gained a lot of attention in the last decade. Robust pricing literature mainly attempts to accommodate minimal information within models in providing revenue estimates and pricing decisions. The minimal information set often only includes information about first two moments of the buyer's valuation distribution making them prone to becoming overly conservative. This is due to worst case prior i.e., worst case valuation distributions are often very asymmetric. This highlights the need to accommodate asymmetry in robust models which is our main contribution in this work. More specifically, we study the pricing policy when ambiguity set also holds information about the asymmetry hence enriching the existing robust models. We start by describing the model when no information about asymmetry is given or considered which we then extend to incorporate asymmetry.

1.1 Model: robust pricing without asymmetry

A seller wants to sell a single unit of a good to a price sensitive buyer. Buyer's valuation (θ) of seller's product stems from her *reservation price or willingness-to-pay (WTP)* denoted as c. That is, $\theta = \min(c, r)$, where r is seller's price, and seller's product is sold only if r is less than c. Seller cannot observe the reservation price of buyer, in other words, valuation θ is unknown to seller. Moreover, the distribution from which c is drawn is not fully known to seller. In particular, seller only knows the mean (μ) and variance (σ^2), and the non-negative support ($\Omega = [m, M]$) of the distribution. Hence, the set of distributions he considers possible is given by:

$$\mathcal{P} = \mathcal{P}(\mu, \sigma, \Omega,) = \left\{ \mathbb{P} \;\middle|\; \begin{array}{r} \mathbb{E}_{\mathbb{P}}[c] = \mu, \\ \mathbb{E}_{\mathbb{P}}[(c-\mu)^2] = \sigma^2 \\ \mathbb{E}_{\mathbb{P}}(1) = 1 \end{array} \right\}. \tag{1}$$

(1) is commonly known as moment-ambiguity set in distributionally robust optimization [Rahimian and Mehrotra, 2019].

In this article we consider deterministic posted price mechanisms. That is, when the distribution, $\mathbb{P}$, of c is known to seller he sets a price r that maximizes his revenue:

$$R(\mathbb{P}) := \max_{r} R(r, \mathbb{P}), \tag{2}$$

where $R(r,\mathbb{P}) = \int_r^M r\mathbb{P}(c)dc$. Traditionally, seller's problem in the presence of ambiguity is modeled as

$$D := \max_r \min_{\mathbb{P}\in\mathcal{P}} R(r,\mathbb{P}). \tag{3}$$

As is common in literature, seller's revenue optimization problem under ambiguity is seen as a sequential bilevel optimization problem, often also referred to as a game, between seller and nature, with decision-making happening in that sequence:

$$\begin{aligned} \max_{r\in\Omega} \quad & R(r,\mathbb{P}) \\ s.t \quad & \mathbb{P} := \text{argmin}_{\mathbb{P}\in\mathcal{P}} N(r,\mathbb{P}) \end{aligned} \tag{4}$$

where $N() = R()$. When seller optimizes over worst prior, namely (3) is equivalent to playing against an adversarial nature. We first recall the following important property about optimal solution to nature's problem. A k-point distribution assigns all probability to exactly k points in the support.

Lemma 1. *Nature's optimal distributions are at most three-point.*

Proof. Follows from [Popescu, 2007]. □

Before stating our aims and contributions, we state 3 Lemmas that capture the characterization of (Monopolist's) optimal pricing policy under ambiguity.

Lemma 2. *For any price r chosen by seller, nature's optimal distributions are two-point.*

The two-point property of nature's solution implies the following lemma.

Lemma 3. *The demand under adversarial nature is: $1 - \frac{\sigma^2}{\sigma^2+(\mu-r)^{+2}}$, when $r \in [m, M]$.*

The concavity of the resulting revenue function will allows to solve for optimal price:

Lemma 4. *Optimal price characterization:*

$$r^* = \mu + \frac{\Delta^2 - \sigma^2}{\Delta}, \tag{5}$$

where $\Delta = \left(-\mu\sigma^2 + \sigma^2\sqrt{\mu^2+\sigma^2}\right)^{1/3}$.

1.2 Aims and contributions

Our two aims and the associated contributions in the rest of the paper are as follows:

- Our primary aim is to extend pricing model to account for asymmetry and characterize the pricing decision, just as in Lemmata (3)-(4). The key to achieving a similar result as in Lemma 3 in an extended model that takes asymmetry in to account is characterization of the optimal solution to the inner problem, that is, the nature's problem. Unlike the case of (3), achieving this characterization becomes highly involved and non-trivial task as this requires constructing various forms of dual feasible solutions to nature's primal problem. There is no result in literature which characterizes nature's solution for more than three moments. As will be seen in Theorem 1, nature's solution is quite complicated and hence an explicit expression for optimal price in this case is only bound to be very messy. Nevertheless, our analysis provides an easy bisection search type algorithm to compute optimal price. From a technical point of view analysis of nature's best response was previously only analyzed in a piece-wise linear case but not for piece-wise constant which is more common in robust models. We fill this gap as one of our main technical contribution.
- Our secondary aim is to illustrate the gain of incorporating the asymmetry information on price and revenue estimates. More specifically, we will show that even in the symmetric distributions (3) can end up suggesting highly conservative prices and this is because of lack of asymmetry knowledge within the model. In the process, we show that behavior of revenue curve is non-smooth and non-concave in general but illustrate that there exist three price ranges where we have the desirable property for bisection search.

To this end we motivate and formally extend (3) to incorporate asymmetry information in the following section.

2 Robust pricing with asymmetry

A drawback of mean-variance model is that $\mathcal{P}$ does not provide any information on the asymmetry of buyer's WTP distribution. Therefore $\mathcal{P}$ includes various distributions with distinct asymmetries. Though widely acknowledged, asymmetry is still not incorporated as it (generally) makes (3) intractable and hard to analyze. Following recent literature, we represent asymmetry using *semivariance*. Semivariance was originally proposed as a better metric than variance which captures downside risk by Markowitz. The semivariance of a random variable x for some target α is a special case of the lower partial moment $\mathbb{E}((\alpha - x)^{+2})$. Semivariance, as a measure of asymmetry, is at least 4 decades old when it is used to tighten Chebyshev inequality in [Berck and Hihn, 1982]. On the estimation side, several approaches exist for estimation of semivariance, [Sortino and Forsey, 1996, Bond and Satchell, 2002].

Following [Natarajan et al., 2018], we use normalized semi-variance,

$$s = \frac{\mathbb{E}_{\mathbb{P}}[(c-\mu)^{+2} - (\mu - c)^{+2}]}{\sigma^2}, \tag{6}$$

to characterize the solution to the pricing problem under ambituity and asymmetry information. Note that the value of s indicates whether random WTP (of buyer's) deviations from the mean are concentrated above or below the mean; and it takes a value between -1 and 1. Symmetric distributions, such as Normal and Uniform, will have $s = 0$ but the reverse is not necessarily true. Negative skewed (resp. positive skewed) distributions have negative s (resp. positive s) values.

We can now define the extension of $\mathcal{P}$, defined in (1), by incorporating asymmetry information via normalized semivariance:

$$\mathcal{P}(s) = \mathcal{P}(\mu, \sigma, s, \Omega,) = \left\{ \mathbb{P} \;\middle|\; \begin{array}{r} \mathbb{E}_{\mathbb{P}}[c] = \mu, \\ \mathbb{E}_{\mathbb{P}}[(c-\mu)^2] = \sigma^2 \\ \mathbb{E}_{\mathbb{P}}[(c-\mu)+^2] - \mathbb{E}_{\mathbb{P}}[(\mu-c)+^2] = s\sigma^2 \\ \mathbb{E}_{\mathbb{P}}(1) = 1 \end{array} \right\}. \tag{7}$$

The extension of (3) can now be written:

$$D := \max_r \min_{\mathbb{P} \in \mathcal{P}(s)} R(r, \mathbb{P}). \tag{8}$$

Remark 1. *Information on the knowledge of variance is often expressed as an upper bound for reasons mentioned in [Carrasco et al., 2018]. However, all results apply for equality and inequality cases of variance, as noted also in [Suzdaltsev, 2018]. Similarly, knowledge of higher moments is usually expressed as upper bound, in this case the results will be similar when only upper bound on semivariance is known.*

3 Related literature

The subject of this paper is setting prices of a product when seller faces ambiguity about options available to buyers and/or about valuations of buyers. Within this stream of literature, [Bergemann and Schlag, 2011], [Carrasco et al., 2018], [Pinar and Kizilkale, 2017] are closer to our work. [Bergemann and Schlag, 2011], [Y. Giannakopoulos and Tsigonias-Dimitriadis, 2020] consider a single product pricing under ambiguity and assume that distribution of buyer's valuations belong to a neighbourhood of a nominal distribution and analyze both max-min utility and min-max regret models. [Pinar and

Kizilkale, 2017] considers different cases of the seller possessing partial information about buyer's valuations by modeling the ambiguity about buyer's valuation in a Knightian uncertainty setting with discrete prices. Finally, [Carrasco et al., 2018] considers randomized mechanisms when the seller has information on the moments of distribution of the buyer's valuation.

Our work naturally relates to rapidly growing literature on (robust) mechanism design and auctions, see a survey [Carroll, 2019]. Within auctions literature, within distributionally robust setting, [Azar et al., , Azar and Micali, 2013, Suzdaltsev, 2018] also study partial information setting similar to ours with the seller possessing information on mean and variance within a multi-product multiple-buyers setting and searches for mechanisms that work well a given class of distributions. However, they also restrict to only mean and variance information.

Our work differs from the existing literature by considering the setting that incorporates asymmetry within the ambiguity set. To the best of our knowledge such a setting has not been studied in literature.

4 Pricing under asymmetry - Main result

Before stating our main result we need the following proposition which characterizes all possible feasible distributions for a given value of semivariance.

Proposition 1. *[Natarajan et al., 2018] Feasible distributions characterization] Consider a non-negative random variable with $\mu > 0$, $\sigma > 0$ and normalized semivariance s. Then necessary and sufficient condition for the moments to be feasible is*

$$\frac{\sigma^2 - \mu^2}{\sigma^2 + \mu^2} \le s \le 1. \tag{9}$$

The key idea in characterizing the nature's solution is by careful analysis of dual formulation. In fact, the constants h and y_1 correspond to dual prices. We are now ready to state our main result in the next theorem.

Theorem 1. *For a given (μ, σ, s) when seller chooses a price r, nature's solution is as follows: let $m_1 = \frac{1+s}{2}\sigma^2$ and $m_2 = \frac{1-s}{2}\sigma^2$ then,*

1. *when $0 < r \le \mu - \sqrt{\frac{m_2}{m_1}(m_1 + m_2)}$ nature's optimal solution is a three point distribution with support $(\omega_1^1, \omega_2^1, \omega_3^1)$ and respective probabilities (q_1^1, q_2^1, q_3^1) as follows:*

$$\omega_3^1 = \mu + \sqrt{\frac{m_1}{(1-\pi)\left(1 - \frac{m_2}{(\mu-r)^2}\right)}} \quad w.p. \quad q_3^1 = \quad (1-\pi)\left(1 - \frac{m_2}{(\mu-r)^2}\right)$$

$$\omega_2^1 = \mu \quad w.p. \quad q_2^1 = \quad \pi\left(1 - \frac{m_2}{(\mu - r)^2}\right)$$

$$\omega_1^1 = r \quad w.p. \quad q_1^1 = \frac{m_2}{(\mu - r)^2},$$

where $\pi = \frac{m_1\left((\mu-r)^2-m_2)\right)-m_2^2}{m_1((\mu-r)^2-m_2)}$.

2. *when* $\mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)} \le r < \mu$ *nature's optimal solution is a three point distribution with support* $(\omega_1^2, \omega_2^2, \omega_3^2)$ *and respective probabilities* (q_1^2, q_2^2, q_3^2) *as follows:*

$$\omega_3^2 = \frac{r\,(2r - \mu(h+4))}{h(r-2\mu)} \quad w.p. \quad q_3^2 = \quad \frac{m_1}{\left(\frac{r\cdot(2r-(h+4)\mu)}{h\cdot(r-2\mu)} - \mu\right)^2}$$

$$\omega_2^2 = r \quad w.p. \quad q_2^2$$

$$\omega_1^2 = 0 \quad w.p. \quad q_1^2 = 1 - (q_3^2 + q_2^2),$$

where $h = \frac{(r-2\mu)^2\sqrt{m_1\cdot(4\mu^2r^2+m_1r^2-8\mu^3r-4m_2\mu r-4m_1\mu r+4\mu^4+4m_2\mu^2+4m_1\mu^2)}r}{\mu\cdot(r-\mu)(4\mu^2r^2+m_1r^2-8\mu^3r-4m_2\mu r-4m_1\mu r+4\mu^4+4m_2\mu^2+4m_1\mu^2)} + \frac{r\cdot(r-2\mu)}{m\cdot(r-\mu)}$ *and* $q_2^2 = \frac{1}{r}\left(\mu - \frac{1}{r}\left(\frac{h\mu(\mu-r)}{2\mu-r} + \frac{h^2(2\mu-r)m_1}{4(h\mu(\mu-r)-r(2\mu-r))} + \frac{hm_2}{2\mu-r}\right)\right)$.

3. *when* $r \ge \mu$ *nature's optimal solution is a three point distribution with support* $(\omega_1^3, \omega_2^3, \omega_3^3)$ *with respective probabilities* (q_1^3, q_2^3, q_3^3) *as follows:*

$$\omega_3^3 = \frac{\left(2\mu y_1 - \left(-2y_1\left((r-\mu) + \sqrt{-\frac{r}{y_1}}\right)\right)\right)}{2y_1} \quad w.p. \quad q_3^3$$

$$\omega_2^3 = r \quad w.p. \quad q_2^3$$

$$\omega_1^3 = 0 \quad w.p. \quad q_1^3 = \frac{m_2}{\mu^2} = 1 - (q_2^3 + q_3^3),$$

where $y_1 = -\frac{\left(\frac{\left(\left(\mu^2-m_2\right)r^2-\mu^3 r\right)}{\left((m_2-\mu^2)r^2+2\mu^3r-\mu^2m_2-\mu^2m_1-\mu^4\right)}\right)^2}{r}$;
$q_3^3 = \left(\mu - r\left(1 - \frac{m_2}{\mu^2}\right)\right)\sqrt{-\frac{y_1}{r}}$;
$q_2^3 = \frac{\mu - w_3\left(\frac{\left(2\mu y_1-\left(-2y_1\left((r-\mu)+\sqrt{-\frac{r}{y_1}}\right)\right)\right)}{2y_1}\right)}{r}$.

Sketch of Proof We reformulate the inner problem (Nature's problem) in (8) to write the dual of the problem. We then exploit the nature of dual feasible region by differentiating different cases particularly when r is greater or less than μ. We show three price ranges and show the optimality using strong duality by extracting primal solution from dual solution. □

Proof of Theorem 1 can be found in Appendix.

Corollary 1. *Optimal price is obtained by solving:*

$$r^* := \underset{r}{argmax} \quad \{R_1, R_2, R_3\} \tag{10}$$

where $R_1 = r\left(1 - \frac{m_2}{(\mu-r)^2}\right)$*;* $R_2 = r\left(\frac{m_1}{\left(\frac{r\cdot(2r-(h+4)\mu)}{h\cdot(r-2\mu)}-\mu\right)^2}\right)$*,* $R_3 = r\left(\left(\mu - r\left(1-\frac{m_2}{\mu^2}\right)\right)\sqrt{-\frac{y_1}{r}}\right)$*;* h *and* y_1 *are as stated in Theorem 1.*

Remark 2. *From Corollary 1, it follows that we can obtain an expression for optimal price by solving three uni-dimensional optimization problems. In all three cases the revenue function is concave hence giving a unique maximizing price. Given that these functions are quite complex, obtaining an explicit expression for the optimal price is rather difficult, if not impossible, even under concavity. However, the concavity of revenue curves in three cases implies a simple bisection search to compute the optimal price. We emphasize that without explicit characterization of the nature's solution the only way to compute optimal price is by reformulating dual of the nature's problem, which is at best a conic program, which is computationally intensive to solve, see [Natarajan et al., 2018].*

4.1 Numerical illustration

The objective of this section is to illustrate the gain in pricing decision and estimated revenue by incorporating the asymmetry information. Figure 1 illustrates the revenue curves against changing price in three different semivariance settings: negative (Figure 1a), zero (Figure 1c), and positive (Figure 1b). Dashed curve represents the model that does not take semivariance in to account and solid curve represents revenue curve of the semivariance model. In all three cases mean and variance are kept constant and chosen arbitrarily at $\mu = 4$ and $\sigma = 2.45$, while semivariance values were chosen at -0.35, 0 and 0.35. Note that with these chosen mean and variance values, the feasible set of semivariance values is $[-0.46, 1]$.

The non-convexity and rather humpy looking shapes of revenue curves are due to different nature solutions arising in three different cases defined by the price (r) as detailed in Theorem 1. Note that the three cases (and the respective price ranges) are semivariance dependent and hence vary between the three chosen semivariance values, these are reported in Table 1 along with optimal price and estimated revenues.

Interestingly, even in the case when semivariance is equal to zero, which contains all symmetric distributions (and more), the impact of ignoring asymmetry information can be very high. Note that Figure 1b suggests a much higher price when semivariance information is considered. Moreover, this impact is seen to be even more significant in the negatively skewed case, see Figure 1a, where the price suggested, when taking semivariance in to account, can be double the price suggested otherwise. On the other hand, unsurprisingly, the price suggested, by semivariance model, while still very different from no semivariance model, is less drastically different to base model which does not include asymmetry information, see Figure 1c. However, in all three cases it can be clearly seen that revenue estimates can be seen to be overly conservative when asymmetry is ignored. Note that the revenue curve change for prices greater than μ is noticeable in the illustration due to the nature of the revenue functions in cases 2 and 3 of Theorem 1.

Semi-variance	Area 1	Area 2	Area 3	Optimal price	Estimated Revenue
-0.35	[0,0.47]	(0.47,4]	> 4	3.79	2.14
0.0	[0,1.55]	(1.55,4]	> 4	3.04	1.179
0.35	[0,2.29]	[2.29,4]	> 4	1.76	1.07
No Semivariance				1.87	0.805

Table 1: Comparison of optimal price and estimated revenue

5 Concluding Remarks

Our work provides impetus for a number of potential and interesting research areas. We characterize the optimal deterministic pricing under asymmetry information. Similar analysis for randomized mechanisms is a natural extension. Extensions to more general settings such as auctions is another direction, but challenging. Similarly, multi-product settings are yet another case for future study. None of these extensions are studied, to the best of our knowledge, in the information settings that consider beyond the knowledge of mean and variance.

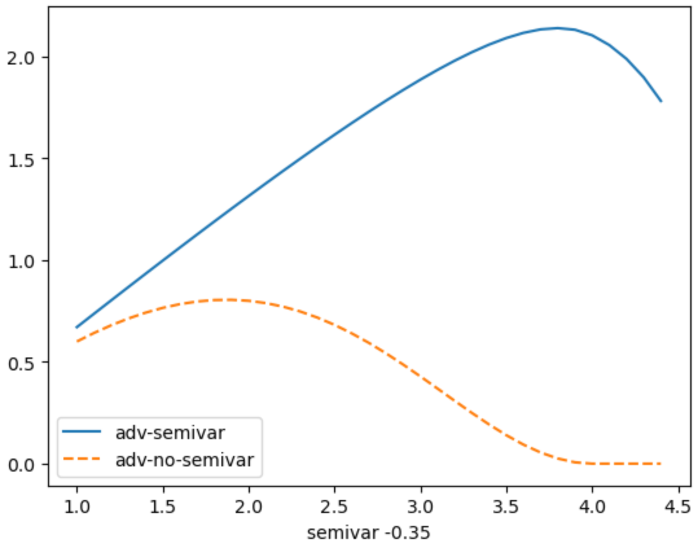
References

[Azar et al.,] Azar, P., Micali, S., Daskalakis, C., and Weinberg, S. M. *Optimal and Efficient Parametric Auctions*, pages 596–604.

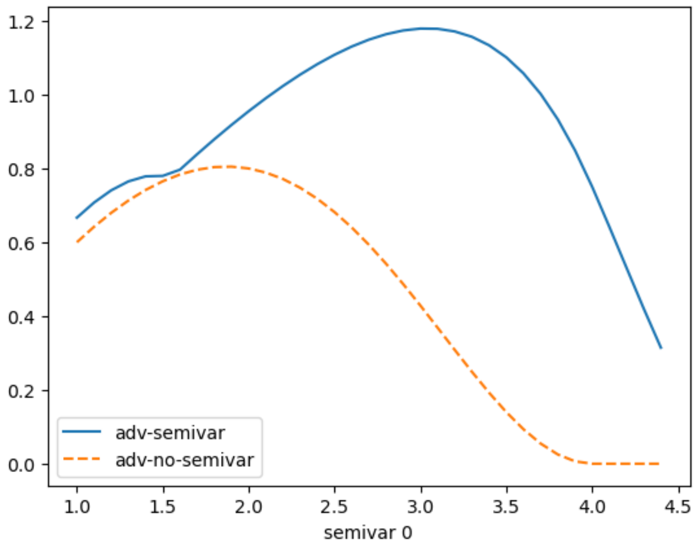
[Azar and Micali, 2013] Azar, P. D. and Micali, S. (2013). Parametric digital auctions. In *Proceedings of the 4th Conference on Innovations in Theoretical Computer Science*, ITCS '13, page 231–232, New York, NY, USA. Association for Computing Machinery.

[Berck and Hihn, 1982] Berck, P. and Hihn, J. M. (1982). Using the semivariance to estimate safety-first rules. *American Journal of Agricultural Economics*, 64(2):298–300.

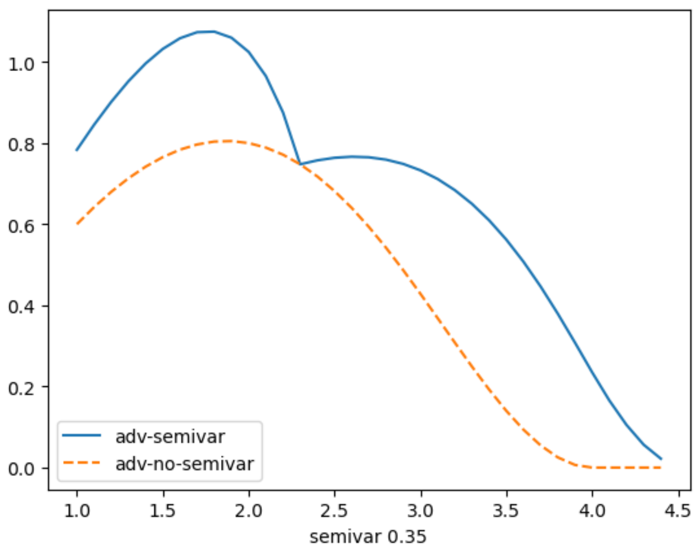
[Bergemann and Schlag, 2011] Bergemann, D. and Schlag, K. (2011). Robust monopoly pricing. *Journal of Economic Theory.*, 146(6):2527–2543.



(a) negative semivariance (−0.35): negative skewed



(b) Semivariance = 0: symmetric distribution



(c) Positive semivariance(0.35): positive skewed distribution

Figure 1: Illustration of revenue curves for different semivariance values: $\mu = 4$ and $\sigma = 2.45$

[Bond and Satchell, 2002] Bond, S. A. and Satchell, S. E. (2002). Statistical properties of the sample semi-variance. *Applied Mathematical Finance*, 9(4):219–239.

[Carrasco et al., 2018] Carrasco, V., Luz, V. F., Kos, N., Messner, M., Monteiro, P., and Moreira, H. (2018). Optimal selling mechanisms under moment conditions. *Journal of Economic Theory.*, 177:245–279.

[Carroll, 2019] Carroll, G. (2019). Robustness in mechanism design and contracting. *Annual Review of Economics*, 11(1):139–166.

[Isii, 1962] Isii, K. (1962). On the sharpness of chebyshev-type inequalities. *Annals of the Institute of Statistical Mathematics*, 14(1):185–197.

[Natarajan et al., 2018] Natarajan, K., Sim, M., and Uichanco, J. (2018). Asymmetry and ambiguity in newsvendor models. *Management Science*, 64(7):3146–3167.

[Pinar and Kizilkale, 2017] Pinar, M. and Kizilkale, C. (2017). Robust screening under ambiguity. *Mathematical Programming.*, 163:273–299.

[Popescu, 2007] Popescu, I. (2007). Robust mean-covariance solutions for stochastic optimization. *Operations Research.*, 55(1):98–112.

[Rahimian and Mehrotra, 2019] Rahimian, H. and Mehrotra, S. (2019). Distributionally robust optimization: A review.

[Sortino and Forsey, 1996] Sortino, F. A. and Forsey, H. J. (1996). On the use and misuse of downside risk. *The Journal of Portfolio Management*, 22(2):35–42.

[Suzdaltsev, 2018] Suzdaltsev, A. (2018). Distributionally robust pricing in independent private value auctions. *arXiv:2008.01618v2.*

[Y. Giannakopoulos and Tsigonias-Dimitriadis, 2020] Y. Giannakopoulos, D. P. and Tsigonias-Dimitriadis, A. (2020). Robust revenue maximization under minimal statistical information. *In proceedings of 16th Conference on Web and Internet Economics.*

Appendix - Proofs

Proof of Theorem 1. Recall the definition of $\mathcal{P}(s)$:

$$\mathcal{P}(s) = \mathcal{P}(\mu, \sigma, s, \Omega,) = \left\{ \mathbb{P} \;\middle|\; \begin{array}{r} \mathbb{E}_{\mathbb{P}}[c] = \mu, \\ \mathbb{E}_{\mathbb{P}}[(c-\mu)^2] = \sigma^2 \\ \mathbb{E}_{\mathbb{P}}[(c-\mu)+^2] - \mathbb{E}_{\mathbb{P}}[(\mu-c)+^2] = s\sigma^2 \\ \mathbb{E}_{\mathbb{P}}(1) = 1 \end{array} \right\}. \tag{11}$$

Primal feasibility of nature's problem can be equivalently written as

$$\mathbb{E}_{\mathbb{P}}(c_1) = \mathbb{E}(c_2) \tag{12}$$

$$\mathbb{E}_{\mathbb{P}}(c_1^2) = m_1 \tag{13}$$

$$\mathbb{E}_{\mathbb{P}}(c_2^2) = m_2 \tag{14}$$

$$c_1 \geq 0, \quad c_2 \in [0, \mu], \quad c_1 c_2 = 0 \tag{15}$$

where we let $c_1 = (c-\mu)^+$ and $c_2 = (\mu - c)^+$, $\sigma^2 = m_1 + m_2$ and $m_1 - m_2 = s\sigma^2$. Note that this implies $m_1 = \frac{1+s}{2}\sigma^2$ and $m_2 = \frac{1-s}{2}\sigma^2$.

Let $g_1(x) = t + h(x-\mu) + y_1(x-\mu)^2$, $g_2(x) = t - h(\mu - x) + y_2(\mu - x)^2$, $f_1(x) = r$, $f_2(x) = 0$.

Case 1

Using Isii's strong duality theorem [Isii, 1962], problem (8) is equivalent to the following dual problem:

$$\begin{aligned} \max \quad & t + y_1 m_1 + y_2 m_2 \\ s.t. \quad & t + h(x-\mu) + y_1(x-\mu)^2 \leq r \quad x \geq \mu \\ & t - h(\mu - x) + y_2(\mu - x)^2 \leq r \quad r \leq x \leq \mu \\ & t - h(\mu - x) + y_2(\mu - x)^2 \leq 0 \quad 0 \leq x \leq r \end{aligned}$$

When $0 < r \leq \mu - \sqrt{\frac{m_2}{m_1}(m_1 + m_2)}$ consider the dual solution where $g_2(x)$ intersects $f_2(x)$ at r and intersects $f_1(x)$ at μ and suppose $g_1(x) = f_1(x)$ (as depicted in Figure 2), then the dual solution is as follows:

$$h = 0$$

$$y_1 = 0$$

$$y_2 = -\frac{r}{(\mu - r)^2}$$

$$t = r$$

with a dual objective

$$r\left(1 - \frac{m_2}{(\mu - r)^2}\right).$$

We have $Pr(c_1 > 0) = Pr(c_2 = 0) = 1 - \frac{m_2}{(\mu-r)^2}$(note that to calculate this we note that there is only one point where $c_2 \neq 0$, that is, $c \leq \mu$ and $c \neq \mu$, this point is r. Also, probability $Pr(c_2 = r) = Pr(c = \mu - r)$ is nothing but one minus optimal dual objective value (which is $t - \frac{rm_2}{(\mu-r)^2}$) divided by r.); $\mathbb{E}[c_1] = \mathbb{E}[c_2] = \frac{m_2}{(\mu-r)^2} \cdot (\mu - r)$. Note by definition $\mathbb{E}[c_1^2] = m_1$. This implies conditional moments for c_1 are:

$$\mathbb{E}[c_1|c1 > 0] = \mathbb{E}[c - \mu|c > \mu] = \frac{\mathbb{E}[c_1]}{Pr(c_1 > 0)} = \frac{\frac{m_2}{(\mu-r)}}{\frac{(\mu-r)^2-m_2}{(\mu-r)^2}}$$

$$\mathbb{E}[c_1^2|c1 > 0] = \mathbb{E}[(c - \mu)^2|c > \mu] = \frac{\mathbb{E}[c_1^2]}{Pr(c_1 > 0)} = \frac{m_1}{\frac{(\mu-r)^2-m_2}{(\mu-r)^2}}$$

These conditional moments are only valid if and only if

$$\mathbb{E}[c_1^2|c1 > 0] - (\mathbb{E}[c_1|c1 > 0])^2 \geq 0$$

which is only possible when $r \leq \mu - \sqrt{\frac{m_2}{m_1}(m_1 + m_2)}$.

The primal solution with objective value $r(1 - \frac{m_2}{(\mu-r)^2})$ is given in Table 2, where

$$\pi = \frac{m_1 r^2 - 2mm_1 r - m_2^2 - m_1 m_2 + m^2 m_1}{m_1 \cdot \left((r - m)^2 - m_2\right)}.$$

Support point	probability
$\mu + \sqrt{\frac{m_1}{(1-\pi)\left(1-\frac{m_2}{(\mu-r)^2}\right)}}$	$w.p.\quad (1-\pi)\left(1 - \frac{m_2}{(\mu-r)^2}\right)$
μ	$w.p.\quad \pi\left(1 - \frac{m_2}{(\mu-r)^2}\right)$
r	$w.p.\quad \frac{m_2}{(\mu-r)^2}$

Table 2: Primal solution when $0 < r \leq \mu - \sqrt{\frac{m_2}{m_1}(m_1 + m_2)}$

Case 2

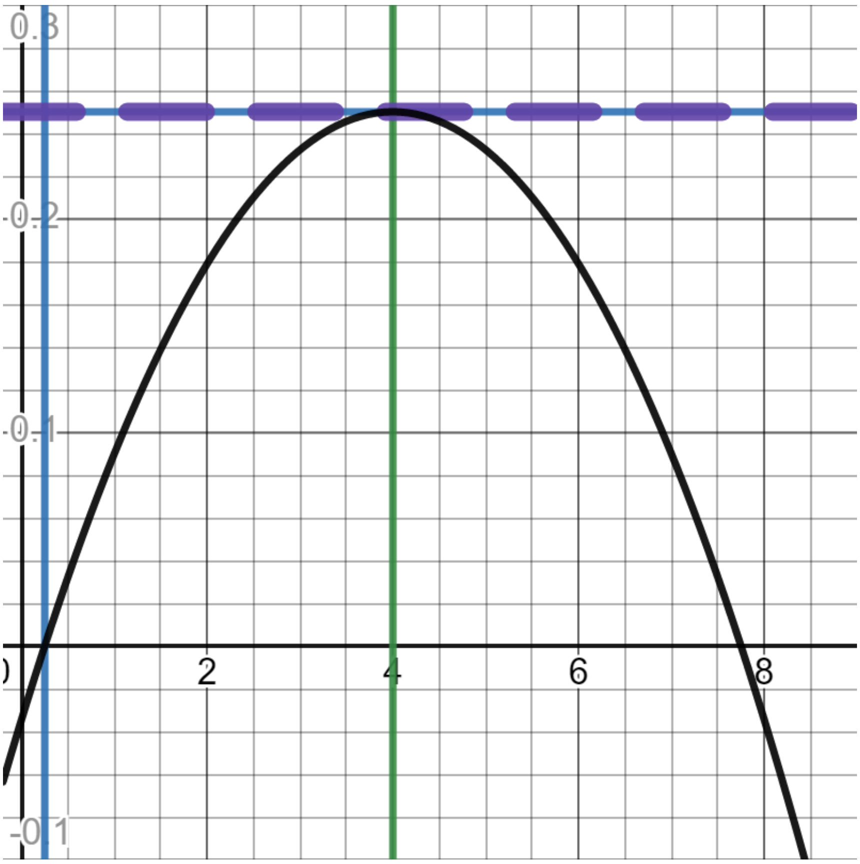


Figure 2: Illustration of dual optimal solution when when $0 < r \leq \mu - \sqrt{\frac{m_2}{m_1}(m_1 + m_2)}$, dual optimal solution where dashed correspond to $g_1(x)$ and solid line correspond to $g_2(x)$. In this illustration $r = 0.25$, hence solid blue lines correspond to $x = r$ and $f_1(x)$.

Dual formulation when $\mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)} \le r < \mu$ is same as in Case 1. To construct a dual feasible solution, suppose that, $g_1(x)$ intersects $f_1(x)$ exactly once at $x > \mu$ and $g_2(x)$ is convex and intersects $f_2(x)$ at $x = 0$ and $x = r$ (see Figure 3), therefore the following relationships hold which define dual feasible solution:

$$
\begin{aligned}
h &= \frac{(r-2\mu)^2\sqrt{m_1\cdot(4\mu^2r^2+m_1r^2-8\mu^3r-4m_2\mu r-4m_1\mu r+4\mu^4+4m_2\mu^2+4m_1\mu^2)}\,r}{\mu\cdot(r-\mu)\,(4\mu^2r^2+m_1r^2-8\mu^3r-4m_2\mu r-4m_1\mu r+4\mu^4+4m_2\mu^2+4m_1\mu^2)} \\
&\quad + \frac{r\cdot(r-2\mu)}{m\cdot(r-\mu)} \\
y_1 &= -\frac{h^2}{4\,(r-t)} \\
y_2 &= \frac{h}{2\mu-r} \\
t &= h\mu - y_2\mu^2
\end{aligned}
$$

Note that all variables are expressed as functions of h. The optimal value of h is obtained by solving the following

$$
\max_{h\in[0,r]}\left\{h\mu-\left(\frac{h}{2\mu-r}\right)\mu^2+\left(-\frac{h^2}{4\left(r-\left(h\mu-\left(\frac{h}{2\mu-r}\right)\mu^2\right)\right)}\right)m_1+\left(\frac{h}{2\mu-r}\right)m_2\right\}. \tag{16}
$$

The function in (16) is concave when $h \in [0, r]$ and convex elsewhere, hence the stationary point in the range $[0, r]$ gives the expression for optimal h.

This solution gives a dual objective value equal to

$$
\left(\frac{h\mu\,(\mu-r)}{2\mu-r}+\frac{h^2\,(2\mu-r)\,m_1}{4\,(h\mu\,(\mu-r)-r\,(2\mu-r))}+\frac{hm_2}{2\mu-r}\right),
$$

which is obtained by the primal feasible solution given in Table 3. The primal solution is obtained from the intersection points of dual constraints with right hand sides.

Support point	probability
$\frac{r(2r-\mu(h+4))}{h(r-2\mu)}$	$w.p.\quad w_3=\frac{1}{r}\left(\frac{h\mu(\mu-r)}{2\mu-r}+\frac{h^2(2\mu-r)m_1}{4(h\mu(\mu-r)-r(2\mu-r))}+\frac{hm_2}{2\mu-r}\right)$ $\iff \frac{m_1}{\left(\frac{r\cdot(2r-(h+4)\mu)}{h\cdot(r-2\mu)}-\mu\right)^2}$
r	$w.p.\quad w_2=\frac{1}{r}\left(\mu-\frac{1}{r}\left(\frac{h\mu(\mu-r)}{2\mu-r}+\frac{h^2(2\mu-r)m_1}{4(h\mu(\mu-r)-r(2\mu-r))}+\frac{hm_2}{2\mu-r}\right)\right)$
0	$w.p.\quad w_1=1-(w_2+w_3)$

Table 3: Primal solution when $\mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)} \le r < \mu$

To get the expression for upper support point note that $t + h\,(x-\mu) + y_1\,(x-\mu)^2$ is concave and intersects $y = r$ exactly at one point. Therefore, this has to be the maximizer, hence, this point can

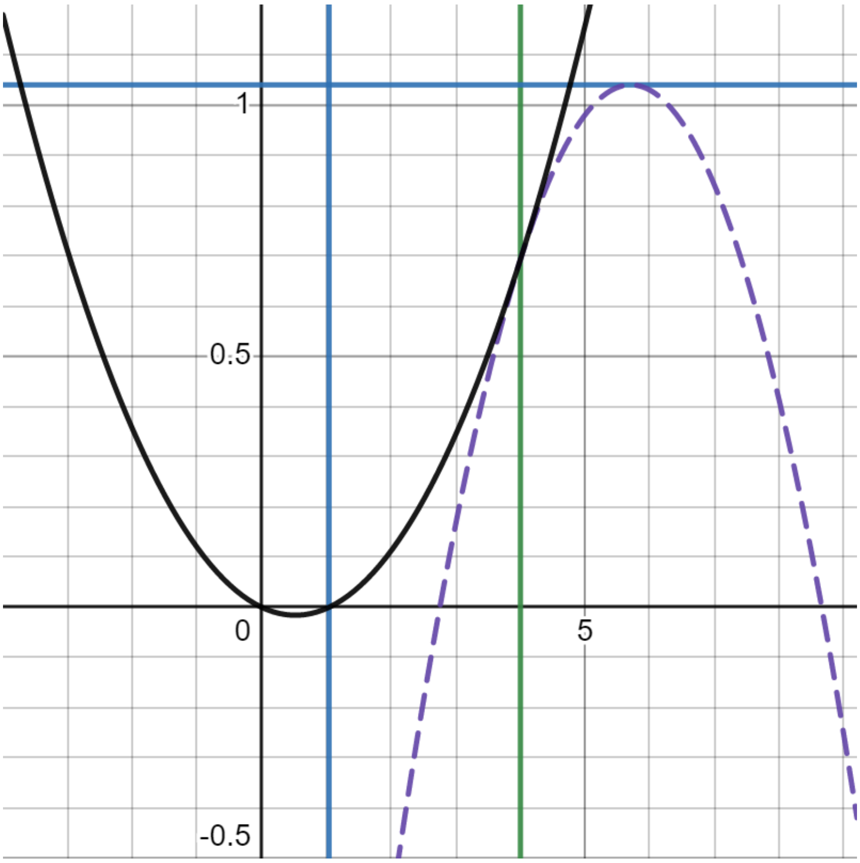


Figure 3: Illustration when $\mu - \sqrt{\frac{m_2}{m_1}(m_1+m_2)} \le r < \mu$ dual optimal solution where dashed correspond to $g_1(x)$ and solid line correspond to $g_2(x)$. In this illustration $r = 1.04$, hence solid blue lines correspond to $x = r$ and $f_1(x)$.

be obtained by taking derivatives and equating to zero,

$$\frac{\partial(t + h\,(x-\mu) + y_1\,(x-\mu)^2)}{\partial x} = h + 2xy_1 - 2\mu y_1 = 0 \longrightarrow x = \frac{2\mu y_1 - h}{2y_1},$$

substituting the values of y_1 and h from optimal dual solution will give the upper primal support point given in Table 3.

We have $Pr(c_2 > 0) = Pr(c_1 = 0) = 1 - \frac{m_1}{(B-\mu)^2}$(note that to calculate this we note that there is only one point where $c_1 \neq 0$, that is, $c \geq \mu$ and $c \neq \mu$, this point is $B = \frac{r(2r-\mu(h+4))}{h(r-2\mu)}$. Also, $\mathbb{E}[c_2] = \mathbb{E}[c_1] = \frac{m_1}{(B-\mu)^2} \cdot (B-\mu)$. Note by definition $\mathbb{E}[c_2^2] = m_2$. This implies conditional moments for c_2 are:

$$\mathbb{E}[c_2|c2 > 0] = \mathbb{E}[\mu - c|c < \mu] = \frac{\mathbb{E}[c_2]}{Pr(c_2 > 0)} = \frac{m_1(B-\mu)}{(B-\mu)^2 - m_1}$$
$$\mathbb{E}[c_2^2|c_2 > 0] = \frac{m_2(B-\mu)^2)}{(B-\mu)^2 - m_1}$$

These conditional moments are only valid iff

$$\mathbb{E}[c_1^2|c1 > 0] - (\mathbb{E}[c_1|c1 > 0])^2 \geq 0$$

which is only possible when $\mu + \frac{\mu m_1}{m_2} \geq B \geq \mu + \sqrt{\frac{m_1}{m_2}(m_1 + m_2)}$. Substituting the value of B we get

$$r \leq m,$$

and

$$r \geq \mu - \sqrt{\frac{m_2}{m_1}(m_1 + m_2)}$$

Case 3: $r \geq \mu$

Dual formulation in this case is:

$$\begin{aligned} \max \quad & t + y_1 m_1 + y_2 m_2 \\ s.t. \quad & t + h(x - \mu) + y_1(x - \mu)^2 \leq 0 \quad \mu \leq x \leq r \\ & t + h(x - \mu) + y_1(x - \mu)^2 \leq r \quad x > r \\ & t - h(\mu - x) + y_2(\mu - x)^2 \leq 0 \quad 0 \leq x \leq \mu \end{aligned}$$

Suppose that $g_1(x)$ intersects $f_2(x)$ and $f_1(x)$ exactly once at $x = r$ and $x > r$ respectively; and $g_2(x)$ intersects $f_2(x)$ once at $x = 0$, as shown in Figure 4, which gives dual feasible solution as follows:

$$\begin{aligned} y_1 &= -\frac{\left(\frac{((\mu^2 - m_2)r^2 - \mu^3 r)}{((m_2 - \mu^2)r^2 + 2\mu^3 r - \mu^2 m_2 - \mu^2 m_1 - \mu^4)}\right)^2}{r} \\ y_2 &= -\frac{2y_1 r}{\mu^2}\left((r - \mu) + \sqrt{-\frac{r}{y_1}}\right) + \frac{y_1}{\mu^2}(r - \mu)^2 \\ t &= y_1\left((r - \mu)^2 + 2(r - \mu)\sqrt{-\frac{r}{y_1}}\right) \\ h &= -2y_1\left((r - \mu) + \sqrt{-\frac{r}{y_1}}\right) \end{aligned}$$

Where y_1 is obtained solving

$$\max_{y_1} y_1\left((r - \mu)^2 + 2(r - \mu)\sqrt{-\frac{r}{y_1}}\right) + y_1 m_1 + m_2\left(-\frac{2y_1 r}{\mu^2}\left((r - \mu) + \sqrt{-\frac{r}{y_1}}\right) + \frac{y_1}{\mu^2}(r - \mu)^2\right)$$

This is a concave function in y_1, taking derivatives and equating to zero gives us:

$$-\frac{\left(\left(m_2-\mu^2\right)r^2+2\mu^3 r-\mu^2 m_2-\mu^2 m_1-\mu^4\right)\sqrt{-\frac{r}{x}}\,x+\left(\mu^2-m_2\right)r^2-\mu^3 r}{\mu^2\sqrt{-\frac{r}{x}}\,x}=0,$$

which gives us the required expression for y_1 in terms of r, m_1, m_2 and μ. Primal feasible distribution, that achieves same objective value as dual, can be constructed from intersection points of the dual solution as given in Table 4 The calculation of upper support point is similar to that in Case 2, that

Support point	probability
$\frac{\left(2\mu y_1-\left(-2y_1\left((r-\mu)+\sqrt{-\frac{r}{y_1}}\right)\right)\right)}{2y_1}$	$w.p.\quad w_3=\frac{1}{r}m_2\cdot\left(\frac{(r-\mu)^2 y_1}{\mu^2}-\frac{2r\cdot\left(\sqrt{-\frac{r}{y_1}}+r-\mu\right)y_1}{\mu^2}\right)$ $+\frac{1}{r}\left(2(r-\mu)\sqrt{-\frac{r}{y_1}}+(r-\mu)^2\right)y_1+m_1y_1$ $\iff\quad\left(\mu-r\left(1-\frac{m_2}{\mu^2}\right)\right)\sqrt{-\frac{y_1}{r}}$
r	$w.p.\quad w_2=\frac{\mu-w_3\left(\frac{\left(2\mu y_1-\left(-2y_1\left((r-\mu)+\sqrt{-\frac{r}{y_1}}\right)\right)\right)}{2y_1}\right)}{r}$
0	$w.p.\quad w_1=\frac{m_2}{\mu^2}=1-(w_2+w_3)$

Table 4: Primal Solution $r>\mu$

is using concavity of $g_1(x)$ and its intersection point with $y=r$ gives the required expression.

We have $Pr(c_2>0)=Pr(c_1=0)=\frac{m_2}{(\mu)^2}$. Also, $\mathbb{E}[c_2]=\mathbb{E}[c_1]=\frac{m_2}{(\mu)^2}\cdot(\mu-0)$. Note by definition $\mathbb{E}[c_2^2]=m_2$. This implies conditional moments for c_1 are:

$$\mathbb{E}[c_1|c1>0]=\mathbb{E}[c-\mu|c>\mu]=\frac{\mathbb{E}[c_1]}{Pr(c_1>0)}=\frac{m_2(\mu)}{(\mu)^2-m_2}$$
$$\mathbb{E}[c_1^2|c_1>0]=\frac{m_1(\mu)^2)}{(\mu)^2-m_2}$$

These conditional moments are only valid if

$$\mathbb{E}[c_1^2|c_1>0]-(\mathbb{E}[c_1|c1>0])^2\geq 0$$

These conditions are merely the conditions that define feasible distributions.

□

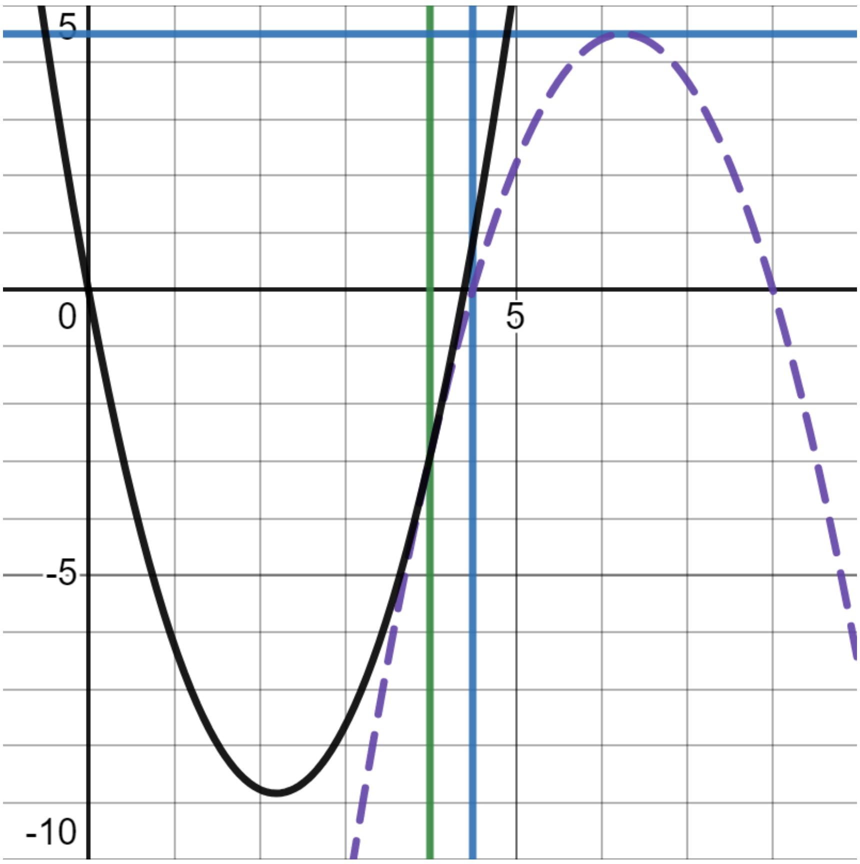


Figure 4: Illustration of $r > \mu$ dual optimal solution where dashed correspond to $g_1(x)$ and solid line correspond to $g_2(x)$. In this illustration $r = 4.5$, hence solid blue lines correspond to $x = r$ and $f_1(x)$.