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The Monte Carlo Integral
of a Continuum of Independent Random Variables

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Abstract

Consider a continuum of independent and identically distributed random variables corresponding to the points of the unit interval $[0, 1]$. Known technical difficulties are complemented by showing directly that the random sample path is almost surely not a Lebesgue measurable function. This refutes the common claim that, because of some version of the "law of large numbers", the integral of each sample path equals the common mean of each random variable. To obtain a valid and useful result, we apply to the continuum of random variables the Monte Carlo method of numerical integration based on limits as the sample size tends to infinity of empirical finite sample averages of the realized random values. The resulting "Monte Carlo integral" is almost surely a degenerate random variable concentrated on the mean. A suitably modified version works when the different indexed random variables are merely independent with cumulative distribution functions that are measurable w.r.t. the index. Further generalizations to Monte Carlo integrals of conditionally independent random variables result, under conditions discussed in Hammond and Sun (2008, 2021), in non-degenerate random integrals that are measurable w.r.t. the conditioning σ-algebra.

1 Introduction

1.1 Motivation

The paper by Judd (1985) opens with this sentence:

> In many models we find the following assertion: "Suppose that there is a continuum of agents each making a draw from a distribution F, such draws being independent; then the distribution of realized draws equals F." This appears to be a law of large numbers, but is there such a law of large numbers for a continuum of random variables?

The assertion that Judd questioned is close to one that appears in Lucas (1980, p. 206), and relates to the earlier claim by Lucas and Prescott (1974).

Similarly, here are the first two sentences of the closely related paper by Feldman and Gilles (1985):

> The analysis of equilibria in which individual agents bear some risk would often be easier if it were possible to assume that the risks are independent and disappear in the aggregate. Some authors have attempted to achieve this simplification by postulating a continuum of independent and identically distributed random variables, and informally invoked the strong law of large numbers in order to assert that the sample average will equal the mean of the random variables with probability one.

These two pioneering papers have prompted a few successors to consider whether there could be some version of a law of large numbers which might hold with a continuum of random variables. Yet, following the example of Lucas and of Lucas and Prescott, many other papers have simply assumed, without further discussion or elaboration, that some relevant version of the law of large numbers is valid. The issue is particularly pertinent for "macro" games of incomplete information with a continuum of players, as considered in widely cited work such as Diamond and Dybvig (1983), as well as Morris and Shin (1998, 2000, 2003).[1]

The present paper contributes to this literature by showing that:

[1] After a conversation with Stephen Morris, during his subsequent seminar presentation that I attended, he kindly acknowledged the integrability issue, as well as how it could potentially be resolved using the Monte Carlo integral.

- the sample path generated by a continuum of independent and identically distributed (IID) random variables is almost surely not Lebesgue measurable, so the kind of "law of large numbers" that is generally postulated, which involves an integral over the continuum of agents, is actually almost surely ill defined;

- nevertheless, under a weak measurability condition, one can define a "Monte Carlo" integral of the random variables that determine the sample path;

- also, in case the random variables are stochastically independent, Kolmogorov's strong law of large numbers implies that this Monte Carlo integral almost surely exists and equals a degenerate random variable whose value almost certainly matches what would be predicted if the generally postulated law of large numbers were to hold;

- furthermore, the empirical distribution of the random means of points on the random sample path almost surely converges weakly to the distribution that would be predicted if the generally postulated law of large numbers were to hold.

1.2 Further Background

Early work on pooling idiosyncratic risk such as the papers by Arrow and Lind (1970), Malinvaud (1972, 1973), and Arrow and Radner (1979) considered a countably infinite set of economic agents, as the limit of finite sets. That allowed them to apply Kolmogorov's classical strong law of large numbers.

Following the fundamental contributions of Aumann (1964, 1966) and Hildenbrand (1974), however, it became standard practice to model an economy with many agents or a game with many players using the unit interval $[0, 1]$ equipped with Lebesgue measure. When there is individual risk in a large economy, or when players in a large game choose mixed strategies, it seems to have become an article of faith among many macroeconomists and game theorists that, because of some version of a law of large numbers which is never explicitly stated, the average realized value of a continuum of independent random variables equals the integral of their expected values. As Lucas (1980, p. 206) wrote: " . . . with a continuum of agents, there is no aggregative uncertainty . . . ". Earlier, when considering a large number of risky markets, Lucas and Prescott (1974, p. 192, footnote 8) had written as follows:

> By large, we mean either a continuum of markets or a countable infinity. Economically, then, the assumption of independent demand shifts means that aggregate demand is taken to be constant through time.

Later Prescott and Townsend (1984a, b) used a similar idea in their model of an economy with a continuum of incompletely informed agents who trade lotteries over consumption vectors while satisfying incentive constraints.

1.3 The Problem

For the continuum of independent random variables that is formally described in Section 2, there is by definition a random process which generates, for each random state of the world, a *sample path* in the form of a function which is defined on the measurable space of agents' labels. Many of the difficulties concerning the measurability of each such sample path had already been discussed by Doob (1937). Economists had neglected these difficulties, however, until Judd (1985) and Feldman and Gilles (1985) appeared.

When the space of agents' labels is the Lebesgue unit interval, we prove in Section 3 that, unless the random sample path is a degenerate random function, it will almost surely be a function of the agent's numerical label that is not measurable, and so not Lebesgue integrable. Then, of course, any statement regarding the Lebesgue integral of the sample path, and whether it is equal to the integral of the means, is almost surely devoid of any content. These observations motivate the new concept of Monte Carlo integral that is introduced in Section 5. This integral is not calculated path by path as a pure number, but is instead a random variable defined over the whole random process that generates the continuum of random variables.

1.4 Outline of Paper

The next Section 2 begins the formal analysis of general random processes that produce a continuum of random variables, one for each point of the Lebesgue unit interval.

Thereafter Section 3 sets out a surprisingly simple yet important result, apparently new. This shows that with a continuum of independent non-degenerate random variables, almost every sample path will be a non-measurable function on the Lebesgue unit interval.

Section 4 offers a brief discussion of some previous work by Bewley (1986), Uhlig (1996), Sun (1998) and others which has set out, with varying degrees of success, to overcome the non-measurability issue.

The following Section 5 starts by discussing the well known Monte Carlo method of calculating numerical approximations to a Lebesgue integral. It then introduces the concept of a "Monte Carlo simulation" which, as in Hammond and Sun (2003, 2008), is a sequence of random variables that are independently and symmetrically selected from the continuum. This leads next to the concept of a "Monte Carlo integral", which applies the idea behind the Monto Carlo method of numerical integration to the random integrands which arise when considering a continuum of random variables.

The first two parts of Section 6 present the main results for the Monte Carlo integral of a continuum of: (i) IID random variables; (ii) independent but asymmetrically distributed random variables. Then the last part of Section 6 points to related joint work with Yeneng Sun that extends the key idea of this paper to the case of dependent random variables.

Hundreds of papers in macroeconomics appear to rely on misusing some implicit version of the law of large numbers in games of incomplete information with infinitely many players. The concluding Section 7 points out that, unlike some of the suggested remedies discussed in Section 4, using the Monte Carlo integral proposed here allows all this previous work to be given a new and valid interpretation. Indeed, it is enough to understand that the integral sign does not indicate a Lebesgue integral which, when there is a continuum of independent random variables, almost surely does not exist. Rather, it should be read as indicating a Monte Carlo integral, which is well-defined much more widely.

2 Modelling a Continuum of Random Variables

2.1 Basic Definitions

The basic definitions in this Section will be used to formalize the notion of a continuum of random variables.

Definition 1. *1. The* Borel unit interval *is the particular probability space* $([0,1],\mathcal{B},\ell)$ *where:*

- $\mathcal{B}$ *is the* Borel σ-algebra *of* Borel sets *generated by the family* $\mathcal{I}$ *of all open intervals* $I \subseteq [0,1]$;[2]
- $\mathcal{B} \ni E \mapsto \ell(E) \in [0,1]$ *is the unique measure on* $\mathcal{B}$ *with the property that, for each open interval* $(a,b) \subset [0,1]$*, the measure* $\ell((a,b)) \in [0,1)$ *equals its length* $b-a$.

[2] As explained in Billingsley (1995), this means that the σ-algebra $\mathcal{B}$ is the intersection of all the σ-algebras that each include the entire family $\mathcal{I}$.

2. *The set* $N \subset [0,1]$ *is* Lebesgue null *just in case there exists a set* $E \in \mathcal{B}$ *such that* $\ell(E) = 0$ *and* $N \subseteq E$. *Let* $\mathcal{N}$ *denote the family of Lebesgue null subsets of* $[0,1]$.

3. *The* Lebesgue unit interval *is the probability space* $(L, \mathcal{L}, \lambda)$ *where:*

   - L *is the unit interval* $[0,1]$*;*
   - $\mathcal{L}$ *is the* Lebesgue σ-algebra *on* $L = [0,1]$ *that is generated by the union* $\mathcal{B} \cup \mathcal{N}$ *of the Borel* σ*-algebra* $\mathcal{B}$ *with the family* $\mathcal{N}$ *of Lebesgue null sets;*
   - $\mathcal{L} \ni E \mapsto \lambda(E) \in [0,1]$ *is the unique extension to* $\mathcal{L}$ *of the Borel length measure* ℓ *on* $\mathcal{B}$ *which satisfies* $\lambda(N) = 0$ *for each Lebesgue null set* $N \in \mathcal{N}$.

Definition 2. *Given the Lebesgue unit interval* $(L, \mathcal{L}, \lambda)$ *together with a probability space* $(\Omega, \mathcal{F}, P)$, *a* random process *with a* continuum of random variables *is a mapping* $L \times \Omega \ni (t, \omega) \mapsto g_t(\omega) \in \mathbb{R}$ *which is defined on the Cartesian product* $L \times \Omega$, *and has the property that for each Borel set* $B \in \mathcal{B}(\mathbb{R})$*:*

1. *for each* $t \in L$, *the set* $g_t^{-1}(B) := \{\omega \in \Omega \mid g_t(\omega) \in B\}$ *is* $\mathcal{F}$*-measurable — i.e.,* $\omega \mapsto g_t(\omega)$ *is a random variable on* $(\Omega, \mathcal{F}, P)$*;*

2. *the mapping* $L \ni t \mapsto \pi_t(B) \in [0,1]$ *is* $\mathcal{L}$*-measurable, where the probability measure* π_t *on the measurable space* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is given by*

$$\pi_t(B) := (P \circ g_t^{-1})(B) := P\left(g_t^{-1}(B)\right) \tag{1}$$

Then, for each $\omega \in \Omega$, *the mapping* $L \ni t \mapsto g_t(\omega) \in \mathbb{R}$ *is the* sample path *of the process.*

2.2 The Case of Independent Random Variables

Definition 3. *Given a probability space* $(\Omega, \mathcal{F}, P)$, *the continuum of random variables* $\Omega \ni \omega \mapsto g_t(\omega) \in \mathbb{R}$ *indexed by* $t \in L = [0,1]$ *that are generated by the random process* $L \times \Omega \ni (t, \omega) \mapsto g_t(\omega) \in \mathbb{R}$ *are:*

1. independent *just in case for every set* $\{t_1, t_2, \ldots, t_k\}$ *of* k *disjoint elements of* L *and every corresponding collection* $B_1, B_2, \ldots, B_k \in \mathcal{B}(\mathbb{R})$ *of Borel sets, one has*

$$P\left(\bigcap\nolimits_{j=1}^k g_{t_j}^{-1}(B_j)\right) = \prod\nolimits_{j=1}^k (P \circ g_{t_j}^{-1})(B_j) = \prod\nolimits_{j=1}^k \pi_{t_j}(B_j) \tag{2}$$

2. *IID just in case the family of random variables $\omega \mapsto g_t(\omega)$ are independent, with a common probability measure π on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $P \circ g_t^{-1} = \pi_t = \pi$ for all $t \in L$.*

3 Non-Measurable Sample Paths in the IID Case

3.1 Simple Functions and Their Integrals

The definitions and results of Sections 3.1–3.3 are based on Royden (1988). Here they are applied to functions defined on the Lebesgue unit interval $(L, \mathcal{L}, \lambda)$ with $L = [0, 1]$.

Definition 4. 1. *Given any measurable set $E \in \mathcal{L}$, the* indicator function *of E is defined by*

$$L \ni t \mapsto 1_E(t) := \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{if } t \notin E \end{cases}$$

2. *A* finite measurable partition *of the measurable space $(L, \mathcal{L})$ is a finite collection $\{E_k | k \in K\}$ of pairwise disjoint measurable sets $E_k \in \mathcal{L}$ that satisfy $\cup_{k \in K} E_k = L$.*

3. *The function $f : L \mapsto \mathbb{R}$ is* simple *just in case $f(x) = \sum_{k \in K} a_k 1_{E_k}(x)$ where:*

   (a) *$\{E_k | k \in K\}$ is a finite measurable partition of L;*

   (b) *$\{a_k | k \in K\}$ is a corresponding finite collection of real constants.*

4. *Given the simple function $f(x) = \sum_{k \in K} a_k \, 1_{E_k}(x)$, its* integral *$I(f)$ w.r.t. the Lebesgue measure λ on $(L, \mathcal{L})$ is defined so that*

$$I(f) := \int_L f(x) \, \lambda(\mathrm{d}x) := \sum\nolimits_{k \in K} a_k \, \lambda(E_k) \tag{3}$$

3.2 The Upper and Lower Integrals of a Bounded Function

Let $\mathcal{S}(L, \mathcal{L})$ denote the set of all simple functions on the measurable space $(L, \mathcal{L})$

Definition 5. 1. *The function $L \ni t \mapsto \varphi(t) \in \mathbb{R}$ is* bounded *just in case there exist $a_*, a^* \in \mathbb{R}$ such that, for all $t \in L$, one has $a_* \leq \varphi(t) \leq a^*$.*

2. *Given any bounded function $L \ni t \mapsto \varphi(t) \in \mathbb{R}$, define the following two sets of simple functions in $\mathcal{S}(L, \mathcal{L})$:*

$$\begin{aligned} \mathcal{S}^*(\varphi; L, \mathcal{L}) &:= \{f^* \in \mathcal{S}(\varphi; L, \mathcal{L}) \mid \forall t \in L : f^*(t) \geq \varphi(t)\} \\ \mathcal{S}_*(\varphi; L, \mathcal{L}) &:= \{f_* \in \mathcal{S}(\varphi; L, \mathcal{L}) \mid \forall t \in L : f_*(t) \leq \varphi(t)\} \end{aligned}$$

3. *Given any bounded function $L \ni t \mapsto \varphi(t) \in [a_*, a^*]$, define its* upper integral $I^*(\varphi)$ *and* lower integral $I_*(\varphi)$ *as, respectively:*

$$\begin{aligned} I^*(\varphi) &:= \inf \{I(f^*) \mid f^* \in \mathcal{S}^*(\varphi; L, \mathcal{L})\} \\ \text{and} \quad I_*(\varphi) &:= \sup \{I(f_*) \mid f_* \in \mathcal{S}_*(\varphi; L, \mathcal{L})\} \end{aligned}$$

Proposition 1. *The upper and lower integrals of the simple function $L \ni t \mapsto \varphi(t) \in \mathbb{R}$ defined by $\varphi(t) = \sum_{k \in K} a_k \, 1_{E_k}(t)$ satisfy*

$$I^*(\varphi) = I_*(\varphi) = \sum\nolimits_{k \in K} a_k \, \lambda(E_k)$$

Proof. The result is a straightforward implication of Definitions 4 and 5. □

3.3 Integrating a Bounded Measurable Function

Definition 6. *Let $L \ni t \mapsto \varphi(t) \in [a_*, a^*]$ be any bounded function. The function is* Lebesgue integrable *just in case its upper integral $I^*(\varphi)$ and lower integral $I_*(\varphi)$ are equal, in which case the* Lebesgue integral *of φ is defined by their common value, so*

$$\int_L \varphi(t) \, \lambda(\mathrm{d}t) := I^*(\varphi) = I_*(\varphi)$$

Proposition 2. *The bounded function $L \ni t \mapsto \varphi(t) \in [a_*, a^*]$ is Lebesgue integrable if and only if it is Lebesgue measurable — i.e., for every interval $J \subseteq [a_*, a^*]$, one has $\varphi^{-1}(J) := \{t \in L \mid \varphi(t) \in J\} \in \mathcal{L}$.*

Proof. See Royden (1988, Ch. 4, Prop. 3). □

3.4 Almost Surely Non-Measurable Sample Paths

In the IID case we are considering, given the common probability measure $\pi = P \circ g_t^{-1}$ on the Borel subsets of $\mathbb{R}$, as specified in Definition 2, the corresponding cumulative distribution function $\mathbb{R} \ni \xi \mapsto F(\xi) \in [0, 1]$ has values defined by

$$F(\xi) := \pi\left((-\infty, \xi]\right) = P(\{\omega \in \Omega \mid g_t(\omega) \leq \xi\})$$

Then the essential infimum and essential supremum of the probability measure π will satisfy

$$\begin{array}{rcccl} \pi_* & := & \operatorname{ess\,inf} \pi & := & \sup \{a \in \mathbb{R} \mid F(a) = 0\} \\ \pi^* & := & \operatorname{ess\,sup} \pi & := & \inf \{a \in \mathbb{R} \mid F(a) = 1\} \end{array} \tag{4}$$

Of course one may have $\pi_* = -\infty$ and/or $\pi^* = +\infty$. But for simplicity we assume that both π_* and π^* are finite.

Next, define the two probabilities

$$\begin{array}{rcccl} p_* & := & P(\{\omega \in \Omega \mid g_t(\omega) < a\}) & := & \sup\{F(\xi) \mid \xi \in (a_*, a)\} \\ p^* & := & P(\{\omega \in \Omega \mid g_t(\omega) > a\}) & := & 1 - F(a) \end{array} \tag{5}$$

Lemma 1. *Suppose that $a \in \mathbb{R}$ satisfies $\pi_* < a < \pi^*$. Then the three probabilities $F(a)$, p_*, and p^* all belong to the open interval $(0, 1)$.*

Proof. The definitions (4) imply that:
(i) because $\pi_* < a$, one has $F(a) > 0$;
(ii) because $a < \pi^*$, one has $F(a) < 1$.

Next, the definitions (5) imply that:
(i) because $\pi_* < \frac{1}{2}\pi_* + \frac{1}{2}a < a$, one has $0 < F(\frac{1}{2}\pi_* + \frac{1}{2}a) \leq p_* \leq F(a) < 1$;
(ii) because $a < \pi^*$, one has $F(a) > 0$ and so $0 < p^* < 1$. □

Lemma 2. *Given any non-null Borel subset E of $[0, 1]$, one has*

1. *if $P(\{\omega \in \Omega \mid \operatorname{ess\,inf}_{t \in E} g_t(\omega) \geq a\}) = 1$, then $a \leq \pi_*$;*
2. *if $P(\{\omega \in \Omega \mid \operatorname{ess\,sup}_{t \in E} g_t(\omega) \leq a\}) = 1$, then $a \geq \pi^*$.*

Proof. Let $t^{\mathbb{N}} = \langle t_j \rangle_{j \in \mathbb{N}}$ be any infinite sequence of points in E.

1. Because Lemma 1 implies that $0 < p^* < 1$ and the random variables $\omega \mapsto g_t(\omega)$ are independent:
   - for any $m \in \mathbb{N}$, one has $P(\{\omega \in \Omega \mid \min_{j=1}^m g_{t_j}(\omega) > a\}) = (p^*)^m$;
   - as $m \to \infty$, one has

   $$P(\{\omega \in \Omega \mid \inf_{j=1}^\infty g_{t_j}(\omega) > a\}) = \lim_{m \to \infty} (p^*)^m = 0$$

   Hence, if there is probability 1 that the essential infimum over E of the sample path $\omega \mapsto g_t(\omega)$ is no less than a, then $a \leq \pi_*$.

2. Similarly, because Lemma 1 also implies that $0 < p_* < 1$:

- for any $m \in \mathbb{N}$, one has $P(\{\omega \in \Omega \mid \max_{j=1}^m g_{t_j}(\omega) < a\}) = (p_*)^m$;
- as $m \to \infty$, one has

$$P(\{\omega \in \Omega \mid \sup_{j=1}^\infty g_{t_j}(\omega) < a\}) = \lim_{m\to\infty}(p_*)^m = 0$$

Hence, if there is probability 1 that the essential supremum over E of the sample path $\omega \mapsto g_t(\omega)$ is no greater than a, then $a \geq \pi^*$. □

Proposition 3. *Suppose that the process $L \times \Omega \ni (t, \omega) \mapsto g_t(\omega) \to \mathbb{R}$ generates a continuum of IID random variables. Then, for each fixed $\omega \in \Omega$, the random upper and lower integrals w.r.t. $t \in [0,1]$ of the sample path $L \ni t \mapsto g_t(\omega) \to \mathbb{R}$ satisfy*

$$I^*(\omega) \underset{P-a.s.}{=} \pi^* \quad \text{and} \quad I_*(\omega) \underset{P-a.s.}{=} \pi_*$$

Proof. Consider any simple function $L \ni t \mapsto f(t) = \sum_{k\in K} c_k 1_{E_k}(t) \in \mathbb{R}$ where, after ignoring subsets E_k of $[0,1]$ that are λ-null and so do not contribute to the sum $\sum_{k\in K} c_k\,\lambda(E_k)$, we assume without loss of generality that $\lambda(E_k) > 0$ for each $k \in K$. Then, given any $k \in K$, if for P-a.e. $\omega \in \Omega$ one has $f(t) \leq g_t(\omega)$ for λ-a.e. $t \in E_k$, then $c_k \leq \pi_*$. Hence, if for P-a.e. $\omega \in \Omega$ one has $f(t) \leq g_t(\omega)$ for λ-a.e. $t \in [0,1]$, then $f(t) \leq \pi_*$. This is enough to prove that the lower integral $I_*(\omega) \leq \pi_*$. Moreover, because π_* is the essential infimum defined by (4), one has $I_*(\omega) = \pi_*$ for P-a.e. $\omega \in \Omega$.

A similar proof shows that the upper integral $I^*(\omega) = \pi^*$ for P-a.e. $\omega \in \Omega$. □

Finally, here is the main result of this section.

Theorem 1. *Whenever π_* and π^* defined by (4) satisfy $\pi_* < \pi^*$, then the sample path $L \ni t \mapsto g_t(\omega) \in \mathbb{R}$ is P-a.s. not Lebesgue measurable.*

Proof. By Proposition 3, there exist two Borel sets E_* and E^* with $P(E_*) = P(E^*) = 1$ such that: (i) $I_*(\omega) = \pi_*$ for all $\omega \in E_*$; (ii) $I^*(\omega) = \pi^*$ for all $\omega \in E^*$. But then, for all $\omega \in E_* \cap E^*$, one has $I_*(\omega) = \pi_* < \pi^* = I^*(\omega)$. So, by Proposition 2, for all $\omega \in E_* \cap E^*$ the sample path $L \ni t \mapsto g_t(\omega)$ is not Lebesgue measurable. But $P(E_*) = P(E^*) = 1$ implies that $P(\Omega \setminus E_*) = P(\Omega \setminus E^*) = 0$, and so

$$P\left(\Omega \setminus (E_* \cap E^*)\right) = P\left((\Omega \setminus E_*) \cup (\Omega \setminus E^*)\right) = 0$$

It follows that $P(E_* \cap E^*) = 1$, which proves the result. □

3.5 An Almost Surely Degenerate Law of Large Numbers

The following result shows that the non-measurability result of Theorem 1 implies that, for P-a.e. $\omega \in \Omega$, no relevant version of the law of large numbers can hold for the corresponding sample path $L \ni t \mapsto g_t(\omega) \in \mathbb{R}$.

Proposition 4. *Given any fixed $\omega \in \Omega$, suppose that in the IID case, there exists a limit $c \in \mathbb{R}$ such that $\frac{1}{n}\sum_{i=1}^{n} g_{t_i}(\omega) \to c$ for $\lambda^{\mathbb{N}}$-a.e. $t^{\mathbb{N}} \in L^{\mathbb{N}}$. Then the associated measure $\pi = P \circ g_t^{-1}$ on $\mathbb{R}$ specified in Definition 3 equals the* degenerate Dirac measure *δ_c that satisfies $\delta_c(\{c\}) = 1$.*

Proof. Theorem 2.4 in Hoffmann-Jørgensen (1985, p. 310), with a proof due to Talagrand, implies that for the given fixed $\omega \in \Omega$, the mapping $L \ni t \mapsto g_t(\omega) \in \mathbb{R}$ that describes the sample path must be integrable w.r.t. t, and so Lebesgue measurable. Then Theorem 1 implies that $\pi_* = \pi^*$, and so P-a.s. one has $g_t(\omega) = \pi_* = \pi^*$. The result follows from the definitions in (4) of π_* and π^*. □

Of course, the conclusion of Proposition 4 implies that, in an obvious sense, the process $(t, \omega) \mapsto g_t(\omega)$ is *essentially constant*.

4 Some Previously Suggested Remedies

4.1 Bewley Aggregation

Bewley (1986) was the first economist who, while citing Judd (1985), acknowledged the measurability issue in a model of a large economy with a continuum of agents who face independent risks. He circumvented this issue by *defining* both aggregate consumption and the aggregate endowment as the population means of agents' expected consumption and endowment levels, respectively. His chapter makes no attempt, however, to invoke any law of large numbers. His definition anticipates relevant results of Hammond and Sun (2003, 2008, 2021) on "one-way Fubini" processes, though without offering more than an intuitive justification. His work also inspired the analysis of large games by Acemoglu and Jensen (2010, 2013) and Jensen (2010), amongst others.

4.2 Uhlig's Mean Square Convergence

Uhlig (1996) proposes a remedy based on a logically valid law of large numbers which, however, has several limitations.

1. The law applies only when the continuum of IID random variables that induce the common probability measure π on the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ all have not only a common mean $m = \int_\Omega x\pi(\mathrm{d}x)$, but also a common variance $\sigma^2 = \int_\Omega (x-m)^2\,\pi(\mathrm{d}x)$. By contrast, the Monte Carlo integral proposed in Section 5 applies even when the integral $\int_\Omega (x-m)^2\,\pi(\mathrm{d}x)$ diverges.

2. Unlike Kolmogorov's strong law of large numbers, Uhlig's main result does not show convergence P-a.s. Instead, it demonstrates only *mean square convergence*. As Uhlig states, this is like Khinchin's *weak* law of large numbers showing convergence in probability.

3. Although it is relatively easy to generalize Uhlig's law to asymmetric independent random variables, it is not obvious how to extend it to consider dependent random variables, as is discussed later in Section 6.3 of this paper.

Similar limitations apply to the subsequent paper Al-Najjar (1995, 1998) in its final corrected version.

4.3 Beyond the Lebesgue Unit Interval

Sun (1998) pioneered a new class of results that can be described as "exact" laws of large numbers. These resolved the measurability issue by using a much richer space of economic agents than can be accommodated within the Lebesgue unit interval $(L, \mathcal{L}, \lambda)$. Initially, following the ideas of Loeb (1975), these results involved non-standard analysis and Loeb measures.

In later work summarized by He, Sun and Sun (2017) in particular, these Loeb measures were replaced by more general concepts such as nowhere equivalence and saturated measure spaces. Such concepts, however, by construction involve strict extensions of the Lebesgue unit interval that allow the random sample path to be integrated almost surely. By contrast, the Monte Carlo integral defined in Section 5.3 can, like the Bewley aggregates considered in Section 4.1, be calculated simply by Lebesgue integration of means — whether unconditional, or conditional where appropriate. In particular, all the standard results and techniques of the usual integral calculus remain relevant, including the fundamental theorem due to Leibnitz stating that, at any point where the integrand $L \ni t \mapsto g_t(\omega) \in \mathbb{R}$ is continuous, the definite integral $\int_a^b g_t(\omega)\lambda(\mathrm{d}\omega)$ can be differentiated partially with respect to its upper limit b, with partial derivative equal to the integrand evaluated at that point — that is,

$$\frac{\partial}{\partial b}\int_a^b g_t(\omega)\lambda(\mathrm{d}\omega) = g_b(\omega) \tag{6}$$

5 Definition and Basic Properties

5.1 Monte Carlo Integration as a Numerical Method

For each $k \in \mathbb{N}$, define the measure space $(\mathbb{R}^k, \mathcal{L}^k, \lambda^k)$ as the k-fold product of the Lebesgue real line $(\mathbb{R}, \mathcal{L}, \lambda)$. Let $(\mathbb{R}^{\mathbb{N}}, \mathcal{L}^{\mathbb{N}}, \lambda^{\mathbb{N}})$ be the measure space that results from taking the product of countably infinitely many copies of $(\mathbb{R}, \mathcal{L}, \lambda)$.

Given any fixed finite $k \in \mathbb{N}$, suppose that $D \subset \mathbb{R}^k$ is a $\mathcal{L}^k$-measurable domain of the $\mathcal{L}^k$-measurable function

$$D \ni (x_1, \ldots, x_k) = \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R} \tag{7}$$

Consider the Lebesgue integral

$$I := \int_D f(\mathbf{x})\, \lambda^k(\mathrm{d}\mathbf{x}) \tag{8}$$

of the function f with respect to the k-dimensional Lebesgue measure λ^k, evaluated over the domain D. For simplicity, we assume that $\lambda^k(D) < +\infty$ because there is a bounded k-dimensional rectangle

$$R := \prod\nolimits_{j=1}^{k} [a_j, b_j] \subset \mathbb{R}^k \quad \text{with} \quad \lambda^k(R) = \prod\nolimits_{j=1}^{k} (b_j - a_j) \tag{9}$$

such that $D \subset R$. Consider the indicator function $\mathbb{R}^k \ni \mathbf{x} \mapsto 1_D(\mathbf{x}) \in \{0, 1\}$ that is defined to satisfy $1_D(\mathbf{x}) = 1 \iff x \in D$. Because of the hypothesis that D is an $\mathcal{L}^k$-measurable set, this indicator function is $\mathcal{L}^k$-measurable. Then the integral (8) can be expressed as the k-fold integral

$$\begin{aligned} I &= \int_R 1_D(\mathbf{x})\, f(\mathbf{x})\, \lambda^k(\mathrm{d}\mathbf{x}) \\ &= \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} 1_D(\mathbf{x})\, f(\mathbf{x})\, \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_k) \end{aligned} \tag{10}$$

Trying to calculate either of the equivalent integrals (8) or (10) can be onerous when the dimension k is large and/or the domain D is awkward. Nevertheless, one robust numerical method that can work well even in such cases is the well known *Monte Carlo method* that has been discussed, *inter alia*, by Kloek and van Dijk (1978, 1980), van Dijk (1980), Bauwens (1984), and Geweke (1989, 1996). In its most basic form, the method involves:

1. taking a large sample of n points $\langle \mathbf{x}_r \rangle_{r=1}^n$ that are successive IID random draws from the uniform distribution over the rectangle R, whose density measure over the set R is $\lambda^k / \lambda^k(R)$;

2. computing the *Monte Carlo* approximation

$$I^n \left(\langle \mathbf{x}_r \rangle_{r=1}^n \right) := \lambda^k(R) \frac{1}{n} \sum\nolimits_{r=1}^n 1_D(\mathbf{x}_r) \, f(\mathbf{x}_r) \tag{11}$$

   based on the average value of the function $\mathbf{x} \mapsto 1_D(\mathbf{x}) \, f(\mathbf{x})$ for the sample $\langle \mathbf{x}_r \rangle_{r=1}^n$ of n points.

Proposition 5. *Suppose that the domain of integration D is an $\mathcal{L}^k$-measurable set and that $D \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R}$ is any $\mathcal{L}^k$-measurable function which is uniformly bounded over D. Let $(\lambda^k)^{\mathbb{N}}$ denote the product measure in the product of infinitely many copies of the k-dimensional measure space $\left(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), \lambda^k\right)$. Then:*

1. *the Lebesgue integral $I = \int_D f(\mathbf{x}) \, \lambda^k(\mathrm{d}\mathbf{x})$ exists;*

2. *for $(\lambda^k)^{\mathbb{N}}$-a.e. infinite sequence $\langle \mathbf{x}^r \rangle_{r=1}^\infty$, the Monte Carlo approximation $I^n \left(\langle \mathbf{x}_r \rangle_{r=1}^n \right)$ given by (11) converges as $n \to \infty$ to the integral limit I.*

Proof. The first part is a standard property of the Lebesgue integral. Then the second part is a routine application of Kolmogorov's strong law of large numbers. □

5.2 Monte Carlo Simulations

Let $(L^{\mathbb{N}}, \mathcal{L}^{\mathbb{N}}, \lambda^{\mathbb{N}})$ denote the *product probability space* of IID randomly drawn sequences where:

1. $L^{\mathbb{N}}$ is the Cartesian product space whose members are infinite sequences $t^{\mathbb{N}} = \langle t_i \rangle_{i \in \mathbb{N}}$;

2. $\mathcal{L}^{\mathbb{N}}$ is the σ-algebra generated by the Cartesian products $\prod_{i \in \mathbb{N}} E_i$ of any sequence $\langle E_i \rangle_{i \in \mathbb{N}}$ of measurable sets $E_i \in \mathcal{L}$;

3. $\lambda^{\mathbb{N}}$ is the product of a countably infinite set of copies of the Lebesgue measure λ.

Definition 7. *Given any random process $L \times \Omega \ni (t, \omega) \mapsto g_t(\omega) \in \mathbb{R}$ with a continuum of random variables, there is a corresponding:*

1. superprocess *in the form of a mapping*
$$L^{\mathbb{N}} \times \Omega \ni (t^{\mathbb{N}}, \omega) \mapsto G(t^{\mathbb{N}}, \omega) = \langle g_{t_i}(\omega) \rangle_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \tag{12}$$

2. Monte Carlo simulation *(or MCS) in the form of a superprocess, as defined in* (12)*, where the sequence $t^{\mathbb{N}} \in L^{\mathbb{N}}$ of labels is randomly selected from the product probability space $(L^{\mathbb{N}}, \mathcal{L}^{\mathbb{N}}, \lambda^{\mathbb{N}})$.*

5.3 Monte Carlo Integration of Random Variables

The main concern of this paper is with the implications of applying the Monte Carlo numerical method to the case when:

1. the general finite-dimensional measurable domain $D \subset \mathbb{R}^k$ is replaced by the one-dimensional unit interval $L := [0, 1] \in \mathbb{R}$;

2. the function value $f(\mathbf{x})$ at each point $\mathbf{x} \in D$ is replaced by the value $g_t(\omega)$ at $\omega \in \Omega$ of a random variable $\Omega \ni \omega \mapsto g_t(\omega) \in \mathbb{R}$ which, for each $t \in L$, is defined on a probability space $(\Omega, \mathcal{A}, P)$ that is rich enough to determine, as in Definition 2, the joint distribution of the entire continuum $\langle g_t(\omega) \rangle_{t \in L}$ of random variables.

This leads us to consider, for each infinite sequence $t^{\mathbb{N}} = \langle t_i \rangle_{i \in \mathbb{N}} \in L^{\mathbb{N}}$ that is randomly selected from the product probability space $(L^{\mathbb{N}}, \mathcal{L}^{\mathbb{N}}, \lambda^{\mathbb{N}})$, the limiting behaviour as $n \to \infty$ of the *Monte Carlo* sample average random variable

$$\Omega \ni \omega \mapsto \text{MCA}(t^{\mathbb{N}}, \omega) := \frac{1}{n} \sum\nolimits_{i=1}^{n} g_{t_i}(\omega) \tag{13}$$

Now, for any fixed $\omega \in \Omega$, if the sample path $L \ni t \mapsto g_t(\omega) \in \mathbb{R}$ were integrable, then the limit of (13) for that value of ω would be the mean given by the integral $\int_0^1 g_t(\omega)\lambda(\mathrm{d}t)$. By Proposition 3, however, the sample path is almost surely not measurable, so this integral almost surely does not exist. Accordingly, in order to derive a valid law of large numbers, instead of focusing on the sample path $L \ni t \mapsto g_t(\omega) \in \mathbb{R}$ for any fixed $\omega \in \Omega$, we consider the random variable that depends on $\omega \in \Omega$ as well as on $t^{\mathbb{N}}$.

Definition 8. *The process $L \times \Omega \ni (t, \omega) \mapsto g_t(\omega) \in \mathbb{R}$, together with the associated superprocess $L^{\mathbb{N}} \times \Omega \ni (t^{\mathbb{N}}, \omega) \mapsto G(t^{\mathbb{N}}, \omega) \in \mathbb{R}^{\mathbb{N}}$, are* Monte Carlo integrable *just in case, for $\lambda^{\mathbb{N}}$-a.e. $t^{\mathbb{N}} \in L^{\mathbb{N}}$, the Monte Carlo sample average given by* (13) *exists P-a.s. as a limit as $n \to \infty$. In this case the* Monte Carlo integral *of the process is the random variable $\Omega \ni \omega \mapsto {}_{MC}\!\int_0^1 g_t(\omega)\, \lambda(\mathrm{d}t) \to \mathbb{R}$ whose value is P-a.s. equal to that limit.*

6 Three Different Cases

6.1 The Case of IID Random Variables

The first important special case occurs when all the random variables $\omega \mapsto g_t(\omega)$ for different labels $t \in L$ are IID, as specified by Definition (3) in Section 2.2, with a common probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for all $t \in L$ given by

$$\mathcal{B}(\mathbb{R}) \ni B \mapsto \pi(B) = (P \circ g_t^{-1})(B) \in [0,1] \tag{14}$$

Equivalently, there should be a common cumulative distribution function $\mathbb{R} \ni x \mapsto F(x) \in [0,1]$, where

$$F(x) := \pi\left((-\infty, x]\right) = P\left(\{\omega \in \Omega \mid g_t(\omega) \leq x\}\right) \tag{15}$$

Suppose that for each $t \in L$ there exists a common mean, which must be independent of t, given by

$$m := \int_\Omega g_t(\omega) P(\mathrm{d}\omega) = \int_\mathbb{R} x\pi(\mathrm{d}x) = \int_\mathbb{R} xF(\mathrm{d}x) \tag{16}$$

Let δ_m denote the degenerate probability measure on $\mathbb{R}$ that satisfies

$$\delta_m(\{m\}) = 1 \tag{17}$$

Next, consider the set

$$U := \{t^\mathbb{N} = \langle t_i \rangle_{i\in\mathbb{N}} \in L^\mathbb{N} \mid h \neq i \implies t_h \neq t_i\} \tag{18}$$

of *unequal* sequences in the Cartesian product $L^\mathbb{N}$ of a countable infinity of copies of L. These unequal sequences are those that can result from a process of sampling without replacement from the Lebesgue unit interval $(L, \mathcal{L}, \lambda)$. But in the case of the probability space $(L^\mathbb{N}, \mathcal{L}^\mathbb{N}, \lambda^\mathbb{N})$ that is the product of countably many copies of $(L, \mathcal{L}, \lambda)$, it is evident that $\lambda^\mathbb{N}(U) = 1$, so there is essentially no difference between sampling with or without replacement.

Theorem 2. *For every fixed sequence $t^\mathbb{N} \in U$, the random variable $\Omega \ni \omega \mapsto \mathrm{MCA}(t^\mathbb{N}, \omega) \in \mathbb{R}$ that is defined as the limit as $n \to \infty$ of the Monte Carlo average defined by* (13)*:*

1. *exists P-a.s., and P-a.s. equals the common mean m given by* (16)*;*
2. *has a distribution that converges to the degenerate probability measure δ_m defined by* (17).

Proof. Given any fixed sequence $t^{\mathbb{N}} \in U$ of disjoint points in $[0,1]$, each corresponding set $\langle g_{t_i} \rangle_{i=1}^n$ of n random variables $\omega \mapsto g_{t_i}(\omega) \in \mathbb{R}$ are IID, with common mean m. Once again, therefore, a routine application of Kolmogorov's strong law of large numbers establishes that as $n \to \infty$, so the sample average $\frac{1}{n} \sum_{i=1}^n g_{t_i}(\omega)$ converges P-a.s. to m. This evidently implies that the distribution of this sample average converges P-a.s. to δ_m. □

6.2 Asymmetric Independent Random Variables

In a second special case, the continuum of random variables $\omega \mapsto g_t(\omega)$ are still independent for different $t \in L$. Unlike in the first special case, however, the following both depend on t:

1. the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$B \mapsto \pi_t(B) = (P \circ g_t^{-1})(B) := P\left(\{\omega \in \Omega \mid g_t(\omega) \in B\}\right) \in [0,1] \quad (19)$$

2. the associated cumulative distribution function

$$\mathbb{R} \ni \xi \mapsto F_t(\xi) := P\left(\{\omega \in \Omega \mid g_t(\omega) \leq \xi\}\right) \in [0,1] \quad (20)$$

Now we introduce the new assumption that, instead of (16), for each $t \in L$ the mean

$$m_t := \int_\Omega g_t(\omega) P(\mathrm{d}\omega) = \int_{\mathbb{R}} x \pi_t(\mathrm{d}x) = \int_{\mathbb{R}} \xi F_t(\mathrm{d}\xi) \quad (21)$$

exists, and that the mapping $L \ni t \mapsto m_t \in \mathbb{R}$ is also integrable w.r.t. t. This assumption implies existence of an average mean defined by

$$\begin{aligned} \bar{m} &:= \int_L m_t \lambda(\mathrm{d}t) = \int_0^1 \left[\int_\Omega g_t(\omega) P(\mathrm{d}\omega) \right] \lambda(\mathrm{d}t) \\ &= \int_0^1 \left[\int_{\mathbb{R}} \xi F_t(\mathrm{d}\xi) \right] \lambda(\mathrm{d}t) = \int_{\mathbb{R}} \xi \, \bar{F}(\mathrm{d}\xi) \end{aligned} \quad (22)$$

This is the case when the random process of Definition 2 is a "one-way Fubini process", as defined in Hammond and Sun (2003, 2006, 2008, 2021). The name is chosen because, unlike the usual Fubini property, reversing the order of integration in the double integral $\int_0^1 \left[\int_\Omega g_t(\omega) P(\mathrm{d}\omega)\right] \lambda(\mathrm{d}t)$ that determines $\bar{m}$ in (22) is not possible. This is because Theorem 1 implies that, for each state $\omega \in \Omega$, the mapping $[0,1] \ni t \mapsto g_t(\omega) \in \mathbb{R}$ that describes the sample path when the random state is ω is P-a.s. not measurable.

In this case of a one-way Fubini process, when t is randomly drawn from the uniform distribution on the Lebesgue unit interval, and then x_t is randomly drawn from the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \pi_t)$, the resulting pair (t, x_t) is randomly drawn from a common bivariate probability measure Q on the product measurable space $([0,1] \times \Omega, \mathcal{L} \otimes \mathcal{F})$ whose joint c.d.f. is

$$\begin{aligned} [0,1] \times \mathbb{R} \ni (\tau, \xi) \mapsto G(\tau, \xi) &:= Q\left(\{(t,x) \mid t \leq \tau, x \leq \xi\}\right) \\ &= \int_0^\tau F_t(\xi) \lambda(\mathrm{d}t) \end{aligned} \tag{23}$$

The marginal of this distribution on $\mathbb{R}$ is evidently found by integrating the relevant bounded measurable function of t in order to find:

1. the *mean probability measure* $\mathcal{B}(\mathbb{R}) \ni B \mapsto \bar{\pi}(B) \in [0,1]$ defined for each $B \in \mathcal{B}(\mathbb{R})$ by
$$\bar{\pi}(B) := \int_0^1 \pi_t(B) \lambda(\mathrm{d}t) \tag{24}$$

2. the associated *mean c.d.f* $\mathbb{R} \ni \xi \mapsto \bar{F}(\xi) \in [0,1]$ defined for each $\xi \in \mathbb{R}$ by
$$\bar{F}(\xi) := G(1, \xi) = \int_0^1 F_t(\xi) \lambda(\mathrm{d}t) \tag{25}$$

In this case, instead of the Monte Carlo average defined by (13), for each $n \in \mathbb{N}$ we consider the new Monte Carlo average defined for each (random) sequence $x^{\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ by

$$\mathbb{R}^{\mathbb{N}} \ni x^{\mathbb{N}} \mapsto M_n(x^{\mathbb{N}}) := \frac{1}{n} \sum\nolimits_{i=1}^n x_i \tag{26}$$

Theorem 3. *Suppose that: (i) for each $t \in L$, the mean m_t defined by* (21) *exists; (ii) the average mean $\bar{m}$ given by* (22) *exists. Suppose too that each random variable x_i in the infinite sequence $x^{\mathbb{N}} \in \mathbb{R}^N$ of IID random variables is randomly drawn from the common probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \bar{\pi})$, where $\bar{\pi}$ is the mean probability measure defined by* (24)*. Then:*

1. *the random variable $\mathbb{R}^{\mathbb{N}} \ni x^{\mathbb{N}} \mapsto M(x^{\mathbb{N}}) \in \mathbb{R}$ that is defined for each random $x^{\mathbb{N}}$ as the limit as $n \to \infty$ of the Monte Carlo average $M_n(x^{\mathbb{N}})$ defined by* (26) *exists $\bar{\pi}^{\mathbb{N}}$-a.s., and $\bar{\pi}^{\mathbb{N}}$-a.s. equals the average mean $\bar{m}$ given by* (22)*;*

2. *the limiting distribution as $n \to \infty$ of the Monte Carlo average $M_n(x^{\mathbb{N}})$ defined by* (26) *is the degenerate probability measure $\delta_{\bar{m}}$ that attaches probability 1 to the average mean $\bar{m}$ given by* (22)*.*

Proof. The result follows from a routine application of Kolmogorov's strong law of large numbers to the sequence $x^{\mathbb{N}} \in \mathbb{R}^N$ of IID random variables. $\square$

6.3 Dependent Random Variables

This paper has focused on the important special case of a random process with the property that, for different values of t in the Lebesgue unit interval $(L, \mathcal{L}, \lambda)$, the associated random variables $\Omega \ni \omega \mapsto g_t(\omega) \in \mathbb{R}$ are stochastically independent. Nevertheless, even outside this case, suppose that there exists a conditioning σ-algebra $\mathcal{C}$ on the measurable space $(\Omega, \mathcal{F})$ that underlies the basic probability space $(\Omega, \mathcal{F}, P)$ with the property that any pair of different random variables are conditionally independent given $\mathcal{C}$. Suppose too that, following Billingsley (1995), the σ-algebra $\mathcal{C}$ is countably generated in the sense that there is a countable subfamily $\mathcal{G}$ of $\mathcal{F}$ with the property that $\mathcal{C}$ is the smallest σ-algebra containing $\mathcal{G}$. Under these assumptions, the results of Hammond and Sun (2008, 2021) imply that the Monte Carlo average defined by (26) will still converge almost surely, but to a non-degenerate random variable on the basic probability space $(\Omega, \mathcal{F}, P)$ which is $\mathcal{C}$-measurable.

Furthermore, for this result the definition of "almost surely" has to be weakened somewhat. Specifically, the excluded sets $E \in \mathcal{L}^{\mathbb{N}}$ on which the Monte Carlo average defined by (26) does not converge are no longer restricted to null sets whose product measure in the infinite product probability space $(L^{\mathbb{N}}, \mathcal{L}^{\mathbb{N}}, \lambda^{\mathbb{N}})$ satisfies $\lambda^{\mathbb{N}}(E) = 0$. Instead, the excluded sets E can be merely "iteratively null" sets that, by definition, satisfy $\bar{\lambda}^{\mathbb{N}}(E) = 0$ relative to a new product measure $\bar{\lambda}^{\mathbb{N}}$ which extends the original product measure $\lambda^{\mathbb{N}}$ to a new domain $\bar{\mathcal{L}}^{\mathbb{N}}$ which is a new σ-algebra that slightly extends the original product σ-algebra $\mathcal{L}^{\mathbb{N}}$ in order to admit such iteratively null sets.

For more precise details, see Hammond and Sun (2008, 2021).

7 Concluding Remarks

The early economic models with a continuum of agents which appear in, inter alia, the work of Hotelling (1929), Vickrey (1945), Aumann (1964, 1966), Mirrlees (1971) and Hildenbrand (1974), involved no randomness. This allowed variables like consumers' endowments and consumption to be measurable functions that were Lebesgue integrable, so that their means are well-defined. However, the later work on random economies and games with incomplete information that was discussed in Section 1.2 did involve continua of random variables. For these, it was often claimed without justification that some version of the law of large numbers would remove all aggregate uncertainty from, if not every, then at least almost every sample path. Section 3, however, showed that almost every sample path is actually non-measurable. So representing aggregate uncertainty requires, at least, some non-standard definition.

Section 4 discussed some possible remedies to this non-measurability issue. One of these due to Uhlig (1996) applies only to square-integrable random variables, and instead of the almost sure convergence of Kolmogorov's strong law of large numbers, gives only the weak convergence in probability of Khinchin's weak law.

Another remedy that has received a great deal of recent attention, starting with Sun (1998), is to allow an index space of agents' labels that significantly enriches the Lebesgue unit interval. Such enrichments, however, may invalidate some of the claims that macroeconomists make for aggregative games of incomplete information where the index set of players is the Lebesgue unit interval. This troublesome possibility can be avoided provided that, for each random ω in the state space Ω, the integral $\int_0^1 f_t(\omega)\lambda(\mathrm{d}t)$ of each non-measurable sample path $[0,1] \ni t \mapsto f_t(\omega) \in \mathbb{R}$ is interpreted, not as a Lebesgue integral $\int_0^1 g_t(\omega)\,\lambda(\mathrm{d}t)$ which almost surely fails to exist, but as the corresponding Monte Carlo integral ${}_{\text{MC}}\!\int_0^1 g_t(\omega)\,\lambda(\mathrm{d}t)$ of random variables specified in Definition 8.

As an ideal that might help avoid any future misunderstanding, perhaps we should use some new notation like ${}_{\text{MC}}\!\int_0^1 f_t(\omega)\lambda(\mathrm{d}t)$ to indicate this new kind of integral. Note that it really is a generalization of the Lebesgue integral because it essentially reduces to that integral in any degenerate case when, for the particular random state $\omega \in \Omega$ being considered, the sample path $[0,1] \ni t \mapsto f_t(\omega) \in \mathbb{R}$ just happens to be a measurable function, and so integrable provided it is bounded.

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