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Department of Economics
University of Warwick, Coventry,
CV4 7AL, United Kingdom

A Difficulty in Characterising Mixed Nash Equilibria in a Strategic Market Game*

Ralph W. Bailey†, Maria Kozlovskaya‡ and Indrajit Ray§

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Abstract

We analyse the conditions for a strategy profile to be an equilibrium in a specific buy and sell strategic market game, with two goods, using best responses of a player against random bids from the opponents. The difficulty in characterising mixed Nash equilbria is that the expected utility is not quasiconcave in strategies. We still prove that any mixed strategy Nash equilibrium profile in which every player faces only two random bids is *trivial*, that is, is a convex combination of some pure strategy Nash equilibria; moreover, we show that the outcome (the price and the allocations) is deterministic in such an equilibrium.

Keywords: Mixed bids, Mixed strategy Nash equilibrium, strategic market games.

JEL Classification Numbers: C72.

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†Department of Economics, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK. E-mail: r.w.bailey@bham.ac.uk.

‡Economics, Finance and Entrepreneurship Department, Aston Business School, Aston University, Birmingham B4 7ET, UK. E-mail: m.kozlovskaya@aston.ac.uk.

§*Author for Correspondences.* Economics Section, Cardiff Business School, Cardiff University, Colum Drive, Cardiff CF10 3EU, UK. E-mail: rayi1@cardiff.ac.uk.

1 INTRODUCTION

Randomness in economic outcomes (prices and net allocations) has been extensively studied both in competitive and strategic models. Some of these studies are based on intrinsic uncertainty while many examine extrinsic uncertainty. In the competitive framework, the concept of sunspot equilibrium (Cass and Shell, 1983) models extrinsic uncertainty; in strategic models, the notion of correlated equilibrium (Aumann, 1974, 1987) has been widely used. Shapley and Shubik (Shapley, 1976; Shapley and Shubik 1977; Dubey and Shubik, 1978, 1980) introduced and developed strategic market games as a method of constructing a non-cooperative game from the general equilibrium framework within which there is a rich literature on randomness in terms of both correlated and sunspot equilibria.[1]

In a strategic market game, players buy and sell commodities by placing orders on trading posts. The orders are then executed at the unique market-clearing prices and added or subtracted from the players' initial endowments. Hence market prices and allocations are determined endogenously as the equilibrium of the game, unlike the Walrasian model where agents take prices as given. In strategic market games, the players can influence the prices through their buy and sell orders.[2] In this trading-post model, a specified monetary medium is used for buying and selling all the other commodities, where the price at each single-good trading post is simply the money / good ratio at that post.

The definition of non-cooperative equilibrium in this context is simply a Nash equilibrium in pure strategies, in which no trader, given the bids of others, can improve by deviating unilaterally. Surpri singly however, random moves of individuals in this game are still left to be analysed.[3]

Shapley and Shubik, in their paper (1977, p. 948, footnote 17) remarked: "The definition of Nash includes the possibility of mixed strategies. These have no plausible interpretation in our present model so we shall be searching only for Nash equilibrium in pure strategies." Shapley himself however, seemed to have a contrary view when he (1976, p. 158) claimed: "... a well-defined game for which other types of solution may be attempted (including the Nash equilibrium in mixed strategies)."

We agree with the view expressed in Shapley (1976) above. The strategic market game is indeed a well-defined non-cooperative game and therefore one should consider mixed strategies in these games. The purpose of this paper is to study the structure of possible mixed strategy Nash equilibria in such games.

To characterise the set of mixed Nash equilibria, we consider the simplest possible framework of a

[1] See for example, Aumann, Peck and Shell (1988), Davila (1999), Forges (1991), Forges and Peck (1995), Maskin and Tirole (1987), Peck (1994), Peck and Shell (1991) and Polemarchakis and Ray (2006).

[2] There are possible variations ("buy or sell" or "sell all") of this game.

[3] To the of best our knowledge, the first and perhaps only work that considers mixed strategies in market games is by Levando (2012); Levando and Sakharov (2018) analyses a market game with two goods and two players with a specific utility function however they do not explicitly charactrise the structure of the mixed equilibria in their game.

strategic market game, with *two* goods only (a commodity and a "money") and n players. We apply the concept of mixed strategies directly to the strategy sets of the players; the players are allowed to randomise over pure strategies in their respective strategy sets. The outcome (the prices and the allocations) is obtained on the basis of the realisations of the mixed strategies. The payoff for a player is the expected payoff over the outcomes from the realised strategies.

The intellectual appeal of strategic market games lies in their explicit pricing mechanism. A complete mapping from strategies to prices relaxes one of the least realistic assumption of the Walrasian model, that is, "the need to assume that entrepreneurs cannot single-handedly influence the price structure" (Shapley, 1976). We show that this analytical completeness comes at the cost of equilibrium under-determinacy. In particular, the explicit pricing mechanism means the players' payoff functions are not concave and their best responses are not unique. The intuition behind the result is as follows. Each player can affect the market price by varying the amount she sells and buys. Since these two variables affect the price, they do not linearly cancel out, as would be the case in fixed-price (Walrasian) trading. This means the player's payoff is not concave in strategies even if it is concave in final allocations of commodities.[4] Moreover, expected utility need not even be quasiconcave in the strategy, as we show in an example. Hence, the best response to a mixed profile of opponent's strategies need not be unique. This means strategic market games have a higher level of equilibrium under-determinacy than previously thought. Not only do they have multiple pure strategy equilibria; fully mixed-strategy equilibria also cannot be ruled out. The presence of fully mixed equilibria would have important implications in the real-life counterparts of strategic market games, suggesting price instability in commodity markets.[5]

We first analyse best responses for a player in our set-up. Our first result shows that the best response for a player is unique, apart from some degenerate cases. We prove that the outcome (price and allocation) from a mixed strategy equilibrium profile in such a market is deterministic. However, we can fully characterise the set of mixed Nash equilibria just for the special case in which the player who is playing a mixed strategy is mixing only over *two* pure strategies. Although this profile is very restrictive, we can prove a conclusive result and show that there is no effect of internal randomness in this game. We find that the mixed Nash equilibria are *trivial*, as they are simply the convexification of

[4]Concavity of a utility function in the final holdings of commodities is a standard assumption which reflects the decreasing marginal utility of those commodities.

[5]Payoff concavity can be restored by restricting the strategy space to only one dimension for each trading post ("buy or sell" or "sell all" variations of the game). However, the most natural formulation of a trading game is the unrestricted "buy and sell" variation studied in this paper. Indeed, strategy restrictions can be circumvented in real-life markets. In particular, requiring that all goods pass through the market ("sell all") is unrealistic wherever traders own their stocks, while limiting a trader's choice to either buying or selling can be sidestepped by splitting in two legal entities performing respective operations. Hence, the unrestricted buy-and-sell market game is the most realistic model of an economy with an explicit pricing mechanism, and characterising its equilibria can shed light on price stability in such economies.

pure equilibria. This perhaps justifies the concerns of Shapley and Shubik (1977), as mentioned above.

2 MODEL

Consider an exchange economy with n (≥ 2) agents (indexed by $i = 1, 2, ..n$) and two commodities, denoted by x and y and indexed by 1 and 2. Commodity 2 is money (the *numeraire* good). Each agent i is endowed with a positive vector of goods, $(w_{i1}, w_{i2}) >> 0$ and has a concave, strictly increasing and differentiable utility function over her final allocation of the two commodities, $u_i(x_i, y_i) : R^2 \to R$.

In a strategic market game, each agent is a player: $N = \{1, ...n\}$. Players announce the amount of good 1 they want to to sell, denoted by q_i, and also the amount of money they want to spend on buying back the same good (good 1), denoted by b_i. Hence, (q_i, b_i) is player i's pure strategy, or bid. Let $(q, b) = (q_i, b_i)_i$ denote the profile of pure strategies. A player cannot bid more than her endowment; hence, pure strategy set is given by $S_i = \{(q_i, b_i) : 0 \leq q_i \leq w_{i1}, 0 \leq b_i \leq w_{i2}\}$.

A player's final allocation is determined from her initial allocation (w_{i1}, w_{i2}), her bid (q_i, b_i) and the price p (of good 1):

$$\begin{cases} x_i & = w_{i1} - q_i + b_i/p \quad \text{if } p > 0, \\ & = w_{i1} - q_i \qquad\qquad \text{if } p = 0 \\ y_i & = w_{i2} - b_i + q_i p, \end{cases} \tag{1}$$

where the market-clearing price $p(q, b)$ is formed as a ratio of total bid to total supply (if positive):

$$\begin{aligned} p & = \textstyle\sum_{i=1}^n b_i / \sum_{i=1}^n q_i \quad \text{if } \sum_{i=1}^n q_i > 0; \\ & = 0 \qquad\qquad\qquad\quad\;\; \text{if } \sum_{i=1}^n q_i = 0. \end{aligned} \tag{2}$$

Player i's payoff from her final allocations of the two goods is given by $u_i\left(q, b\right) = u_i\left(x\left(q, b\right), y\left(q, b\right)\right)$, which is assumed to be concave, strictly increasing in each of its arguments and differentiable.

When it comes to mixed strategies in this game, we consider mixing over finitely many points only.

Definition 1 *A mixed strategy of player i is a probability distribution μ over finitely many pure strategies (q_i^k, b_i^k), $i = 1, 2, ..K$ with respective probabilities μ_k such that $\sum_{k=1}^K \mu_k = 1$.*

Player i's payoff from a mixed strategy profile is the usual expected payoff. The definition of equilibrium here is standard Nash equilibrium (either in pure or in mixed strategies). A (pure or mixed) strategy profile is said to be a (Nash) equilibrium if every player is playing a best response against opponents' strategies in the profile.

3 BEST RESPONSE ANALYSIS

Suppose all players apart from i are playing a pure strategy. Let (Q_{-i}, B_{-i}) denote the other players' total bid: $Q_{-i} \equiv \sum_j q_j,\ j \in N \setminus \{i\}$; $B_{-i} \equiv \sum_j b_j,\ j \in N \setminus \{i\}$. The set of achievable allocations by player i, given (Q_{-i}, B_{-i}), can be characterised by substituting one of the equations in system (1) into the other:

$$xy - (w_{12} + B_{-i})x - (w_{11} + Q_{-i})y + w_{11}w_{12} + w_{11}B_{-i} + w_{12}Q_{-i} = 0. \tag{3}$$

The set is closed, hence by the Extreme Value Theorem a continuous function $u(x, y)$ attains a maximum on the set. Moreover, the maximum (x^*, y^*) is unique.[6]

Player i can achieve (x^*, y^*) by a continuum of strategies (q_i, b_i) satisfying equation (4):

$$b_i = q_i \frac{B_{-i}}{w_{11} - x^* + Q_{-i}} + \frac{B_{-i}(x^* - w_{11})}{w_{11} - x^* + Q_{-i}} \tag{4}$$

Equation (4) characterises player i's best response to a pure strategy profile of her opponents, (Q_{-i}, B_{-i}). Note that, given $(Q_{-i}, B_{-i}) > 0$, the final allocation and payoff of player i depends only on the price:

$$\begin{cases} x_i &= w_{i1} + Q_{-i} - B_{-i}/p; \\ y_i &= w_{i2} + B_{-i} - Q_{-i}p. \end{cases} \tag{5}$$

Importantly, for a fixed (Q_{-i}, B_{-i}), player i's utility is strictly concave in price, which we show in the following lemma.

Lemma 1 *Let $u_i(x, y) : R_+^2 \to R$ be strictly increasing in each of its arguments, and concave. Let $x = x(p) : R_+ \to R_+$ be a strictly concave function and $y = y(p) : R_+ \to R_+$ be a weakly concave function. Then the composition $u = u(x(p), y(p)) : R_+ \to R$ is strictly concave.*

Proof. Strict concavity of $x(p)$, with weak concavity of $y(p)$, implies that $x(\lambda p + \lambda' p') > \lambda x(p) + \lambda' x(p')$ and $y(\lambda p + \lambda' p') \geq \lambda y(p) + \lambda' y(p')$. The monotonicity and the concavity of $u(x, y)$ imply that $u(x(\lambda p + \lambda' p'), y(\lambda p + \lambda' p')) > u(\lambda x(p) + \lambda' x(p'), \lambda y(p) + \lambda' y(p')) \geq \lambda u(x(p), y(p)) + \lambda' u(x(p'), y(p'))$. Thus, we have shown, with p and p' distinct, $\lambda \in (0, 1)$, $\lambda' = 1 - \lambda$, and $p'' = \lambda p + \lambda' p'$, $u(x(p''), y(p'')) > \lambda u(x(p), y(p)) + \lambda' u(x(p'), y(p'))$. ■

[6] To see why, observe that i's set of achievable allocations is the graph of the strictly concave function $y(x) = w_{12} + B_{-i} - (Q_{-i}B_{-i})(w_{11} + Q_{-i} - x)^{-1}$. Suppose two distinct points on the set (x, y) and (x', y') were maximising the concave $u(x, y)$, then so would their convex combination (x'', y''). However, the vertical projection of that point onto the graph of $y(x)$ would attain a strictly higher utility $u(x'', y(x'')) > u(x'', y'') = u(x, y) = u(x', y')$, by strict monotonicity of $u(x, y)$ and strict concavity of $y(x)$ – a contradiction; hence the maximum (x^*, y^*) is unique.

We now analyse best responses of a player against a mixed bid from others. First, we present an important negative result, found by numerical search. In a strategic market game with a concave (in fact, strictly concave) utility over allocations, the resulting expected utility over strategies need not be quasiconcave.

Consider a player i with the initial endowment of $(w_{i1}, w_{i2}) = (1, 1)$ and the following utility function over the allocation of the two commodities: $u_i(x_i, y_i) = x_i^\alpha + y_i^\alpha$, where $\alpha \in (0, 1]$. It is easy to verify that this utility function is concave, strictly increasing and differentiable.

Let the player face two different total bids $(Q_{-i}^1, B_{-i}^1) = (0.9, 0.37)$ and $(Q_{-i}^2, B_{-i}^2) = (0.01, 0.1)$, with probability 0.5 each.

Now consider the following two strategies of the player: $(q_i, b_i) = (0.05, 0.04)$ and $(q_i', b_i') = (1, 0.64)$, with respective price realisations as follows:

$(q_i, b_i) = (0.05, 0.04)$ with price realisations $p_1 = \frac{0.04+0.37}{0.05+0.9} = \frac{41}{95}$; $p_2 = \frac{0.04+0.1}{0.05+0.01} = \frac{7}{3}$,

$(q_i', b_i') = (1, 0.64)$ with price realisations $p_1' = \frac{0.64+0.37}{1+0.9} = \frac{101}{190}$; $p_2' = \frac{0.64+0.1}{1+0.01} = \frac{74}{101}$.

Now take the $\frac{1}{2}$-$\frac{1}{2}$ convex combination (the average) of these two strategies:

$(q_i'', b_i'') = (0.525, 0.34)$ with price realisations $p_1'' = \frac{0.34+0.37}{0.525+0.9} = \frac{142}{285}$; $p_2'' = \frac{0.34+0.1}{0.525+0.01} = \frac{88}{107}$.

With a linear utility function for this player: $u_i(x_i, y_i) = x_i + y_i$, the player's expected utility is not quasiconcave in her strategy, since the expected utility of a convex combination of strategies is less than the smaller of the two endpoint expected utilities, as shown below:

$E\left(u_i\left(q_i'', b_i''\right)\right) = 2.0296 < \min\{2.0340, 2.0309\} = \min\{E\left(u_i\left(q_i, b_i\right)\right), E\left(u_i\left(q_i', b_i'\right)\right)\}$.

The same is true for a strictly concave utility function $u_i(x_i, y_i) = x_i^{0.9} + y_i^{0.9}$ as well, as shown below[7]:

$E\left(u_i\left(q_i'', b_i''\right)\right) = 2.0255 < \min\{2.0304, 2.0261\} = \min\{E\left(u_i\left(q_i, b_i\right)\right), E\left(u_i\left(q_i', b_i'\right)\right)\}$.

It therefore follows that the best response for a player against a mixed strategy profile (of others) need not be unique. We are, nevertheless, able to characterise mixed strategy Nash equilibria in a special case in the next Section.

In particular, we deal with a specific mixed strategy profile, as below.

Definition 2 *(i) For a two-player game ($n = 2$), a **2-point mixed strategy profile** is a profile in which at least one player plays a mixed strategy (as in Definition 1) however no player mixes over more than two pure strategies (that is, for either player's mixed strategy, $K \leq 2$ as in Definition 1).*

[7] The full calculation is as follows: $E\left(u_i\right) = 0.5\left[\left(w_{i1} + Q1 - B1/p_1\right)^{0.9} + \left(w_{i2} + B1 - Q1 * p_1\right)^{0.9}\right] + 0.5[\left(w_{i1} + Q2 - B2/p_2\right)^{0.9} + \left(w_{i2} + B2 - Q2 * p_2\right)^{0.9}]$. Hence $E\left(u_i\left(q_i, b_i\right)\right) = 0.5\left[\left(1.9 - \frac{0.37*95}{41}\right)^{0.9} + \left(1.37 - \frac{0.9*41}{95}\right)^{0.9}\right] + 0.5[\left(1.01 - \frac{0.1*3}{7}\right)^{0.9} + \left(1.1 - \frac{0.01*7}{3}\right)^{0.9}] = 2.0304$, while $E\left(u_i\left(q_i', b_i'\right)\right) = 0.5\left[\left(1.9 - \frac{0.37*190}{101}\right)^{0.9} + \left(1.37 - \frac{0.9*101}{190}\right)^{0.9}\right] + 0.5[\left(1.01 - \frac{0.1*101}{74}\right)^{0.9} + \left(1.1 - \frac{0.01*74}{101}\right)^{0.9}] = 2.0261$ and $E\left(u_i\left(q_i', b_i'\right)\right) = 0.5\left[\left(1.9 - \frac{0.37*285}{142}\right)^{0.9} + \left(1.37 - \frac{0.9*142}{285}\right)^{0.9}\right] + 0.5[\left(1.01 - \frac{0.1*107}{88}\right)^{0.9} + \left(1.1 - \frac{0.01*88}{107}\right)^{0.9}] = 2.0255$.

(ii) For $n > 2$, a ***2-point mixed strategy profile*** *is a profile in which only one specific player j mixes over two pure strategies (that is, in player j's mixed strategy, $K = 2$ as in Definition* 1*), while others are playing a pure strategy.*

In accordance with Definition 2, in a 2-point mixed strategy profile, any player i, *either* faces a deterministic (pure) total bid by the other players *or* a probability distribution over two pure total bids by the other players (denoted by σ_{-i}), say a bid (Q^1_{-i}, B^1_{-i}) with probability μ and a bid (Q^2_{-i}, B^2_{-i}) with probability $1 - \mu$.

In other words, in such a profile, each player is facing at most one mixing opponent. In a game with more than two players, we are looking at strategy profiles where only one player is mixing (over two points), while in a game with two players, such profiles include both players mixing over two bids.

Our main theoretical contribution is the extension of the best response analysis of a player i who is facing a mixed total bid (denoted by σ_{-i}) to a 2-point mixed strategy profile. The best response here cannot be easily characterised by closed-form equations such as (3) and (4). However, the best response to a mixed total bid has an important property, which we derive below.

Proposition 1 *The best response to a mixed total bid is unique, apart from one degenerate case when the best response lines to the realisations of the mixed bid coincide. In that case, any point in that joint best response set is a best response to the mixed strategy.*

The detailed proof of Proposition 1 is relegated to Appendix. In the proof, we show that the best response falls into one of the three cases:

Case 1. Best response lines to realisations of the mixed bid coincide. In this case, any point on that line is a best response against the mixed bid (the degenerate case).

Case 2. Best response lines to realisations of the mixed bid have an intersection in the player's strategy set. In this case, this intersection is the unique best response against the mixed bid.

Case 3. Best response lines to realisations of the mixed bid do not have an intersection in the player's strategy set. In this case, a point on the boundary of the player's strategy set is the unique best response against the mixed bid.

We provide examples, one each, of these three cases in the Appendix.

4 MIXED STRATEGY NASH EQUILIBRIUM

The definition of Nash equilibrium (in pure or mixed strategies) is standard for such a game.

Definition 3 *A 2-point mixed strategy Nash equilibrium (hereafter, denoted by just* ***MSNE*** *in this paper) is a 2-point mixed strategy profile such that no player can strictly increase their expected utility by playing a different strategy.*

We now proceed to characterise the MSNE of our market game. We start by showing that the outcome of MSNE is deterministic.

Proposition 2 *If a 2-point mixed strategy profile is a MSNE, then the realised outcome (price and final allocations) is deterministic.*

Proof. First, note that any player i only mixes in response to a pure total bid of their opponents (Q_{-i}, B_{-i}) if all points in i's mixture belong to the best response line against that bid (Q_{-i}, B_{-i}), hence commanding the same price and a deterministic outcome for all players.

Second, we invoke the proof of Proposition 1 to show that the best response to a mixed strategy is either is a unique (pure) bid (Cases 2 and 3), or a continuum of bids generating the same price (Case 1).

Consider a MSNE in a game with more than two players (that is, $n > 2$). By Definition 2(ii), only one player is mixing, then that player is doing so in the manner described in the previous step, resulting in deterministic price and outcome for all players.

Now consider a MSNE in a game with two players (that is, $n = 2$); by Definition 2(i), two players are mixing, then each mixing player has to be best-responding to a mixed strategy of the opponent. A mixed strategy can only be a best response to a mixed total bid in Case 1 (coinciding best-response lines) described in the proof of Proposition 1. This case entails the player mixing between strategies collinear with the realisations of the opponents' total bids, hence entailing the same price, and the same outcome for both the players. ■

Before characterising MSNEs further, we introduce a useful concept.

Definition 4 *A MSNE is called* ***trivial*** *if any realisation of the players' strategies under that MSNE forms a pure strategy Nash equilibrium.*

Clearly, in a game with more than two players, any MSNE has exactly two pure strategy realisations while in a game with two players, there may be either two or four possible realisations. Triviality in Definition 4 implies a convex combination of pure strategy Nash Equilibria.
It turns out that non-trivial MSNE do not exist, as the theorem below demonstrates. In other words, in every such MSNE any of its constituent pure-strategy profiles is by itself a pure Nash equilibrium.

Theorem 1 *Consider a 2-good, n-player market game. Any MSNE of this game is trivial.*

Proof. First note that from the proof of Proposition 2, in any MSNE, the price is deterministic. This means that all players best-responding to a mixed total bid are playing a strategy giving rise to the same price under both total bids in the mix. Geometrically, their strategy thus is necessarily lying on the line connecting the two pure total bids of their opponents (implying also that this line intersects the strategy set). Since all strategies on this line give rise to the same expected utility, they are also best responses.

Recall that, according to Proposition 1, the best response to a mixed strategy is either a unique point in the pure strategy space, or, in the extreme case where the best response lines to constituent pure profiles in that mixed strategy coincide, this whole line is the best response. However we just argued above that a unique best response cannot be part of MSNE. The only remaining possibility is a continuum of best responses, which is only possible in Case 1, where any point in the best-response continuum is also a best response against either constituent pure bid of the opponents.

Theorem 1 now follows from this observation. It follows that in any MSNE, all players facing a mixed total bid are playing a strategy which would have been a best response against all constituent pure bids within this mixed bid, which proves the Theorem. ■

An important implication of Theorem 1 is that Cases 2 and 3 from the proof of Proposition 1 cannot be part of our MSNE (the cases where the best response is unique). The only remaining case is Case 1 (coinciding BR lines; hence any point on that line is a best response).

5 Conclusion

Our result shows that there are no "interesting" mixed strategy equilibria in a market game, where some players are made indifferent between their strategies by a specific mixing probability of their opponents (at least for the case where each player is facing a mixed bid over at most two profiles). In every realisation of a MSNE the outcome (price and allocations) of the game and utilities of all players are exactly the same. This result provides theoretical support to the intuitive dismissal by Shapley and Shubik (1977) of mixed equilibria case as uninteresting.

We admit that our characterisation of MSNE is indeed valid only in a very restrictive set-up. However, given the difficulty (posed by the fact that the expected utility may not be quasi-concave), this is the best result we could achieve. We have not managed to generalise these results, despite significant effort.

Payoff concavity can be restored by restricting the strategy space to only one dimension for each trading post ("buy or sell" or "sell all" variations of the game). However, we believe the most natural formulation of a trading game is the unrestricted "buy and sell" variation studied in this paper, and characterising its equilibria can shed light on price stability in such economies.

Lemma 1 for the 2-good case, that a convex combinations of actions leads to a convex combination of prices, seems not to generalise for more than two goods. Although the price of each good is a convex combination of its two prices, the same is not true of the price vector as a whole. So, it might be hard to show that a player's utility function, facing random strategies of the opponents, over actions is concave, for more than two goods; indeed, this result might not even be true. Therefore, the possibility that there are "interesting" MSNE, when $m \geq 3$, remains open.

In market game models, pure Nash equilibria are in general, (Pareto-) inefficient (Dubey 1980, Dubey and Shubik, 1980; Dubey and Rogawski, 1990). Our paper indicates that mixed strategies in strategic market games may not generate new equilibrium outcomes, even if we allow mixed strategies, in a restricted sense. It is now interesting to know whether mixed strategy Nash equilibria are efficient or not in a more general construct.

6 APPENDIX 1: PROOF OF PROPOSITION 1

Proof of Proposition 1. Consider player i and let one of his opponents ($j \neq i$) play a mixture over two pure strategy profiles: (Q, B) and (Q', B'). The best response sets to each of these profiles are denoted by BR and BR$'$ respectively. As shown in the main text of the paper, BR and BR$'$ are straight upward sloping lines in (q_i, b_i) space, characterised by equation (4). Finally, the player's payoff in a realisation of the mixed bid is denoted $u_{Q,B}(q_i, b_i)$ and $u_{Q',B'}(q_i, b_i)$ for the cases when the total bid is (Q, B) and (Q', B') respectively.

Player i's strategy set S_i is a rectangle in (q_i, b_i) space. Fix other players' total bid (Q, B) and observe that player i's best response line passes through the point $(-Q, -B)$, if extended to the third quadrant.[8] In the analysis below, we consider these extended best response lines defined on R^2, keeping in mind that only the line segments within S_i contains feasible strategies: $b_i = BR(q_i) : [0, w_{i1}] \to [0, w_{i2}]$.

Depending on the relative position of the two extended best response lines to realisations of the mixed bid, three cases can be considered:

Case 1. Best response lines coincide.

Case 2. Best response lines intersect in the player's strategy set.

Case 3. Best response lines intersect outside the player's strategy set.

We consider the three options one by one and prove that, in Case 1, the whole line is the best response, while in Cases 2 and 3 the best response is a unique point (q_i^*, b_i^*) in player i's strategy set. We also provide examples of best responses in each case for a player with the utility function $u_i = xy$ and an endowment $w_{i1} = w_{i2} = 3$, unless specified otherwise.[9]

For a player with the utility $u_i = xy$ the best response to total bid (Q, B) can be characterised as follows:

$$b_i = q_i \sqrt{\frac{B(w_{i2} + B)}{Q(w_{i1} + Q)}} + \sqrt{QB} \sqrt{\frac{w_{i2} + B}{w_{i1} + Q}} - B. \qquad (6)$$

6.1 Case 1. Coinciding Best Response Lines

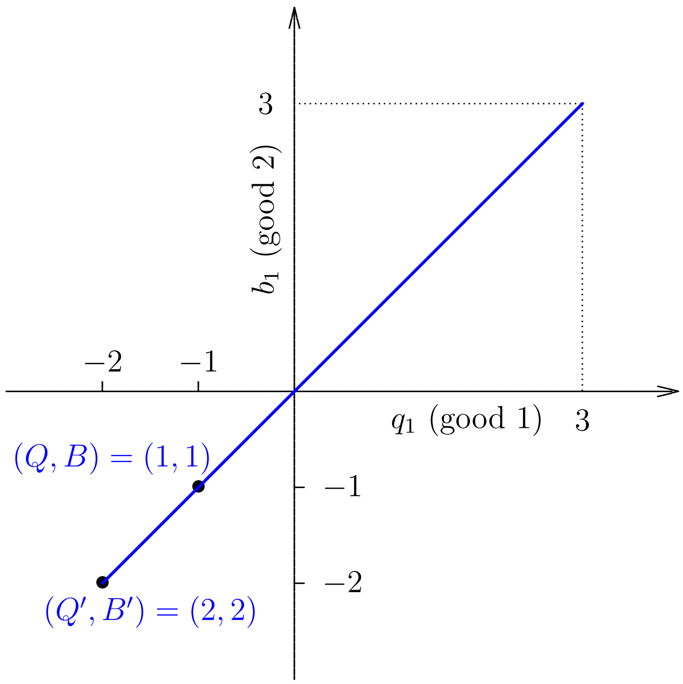
Let BR $=$ BR$'$. In this case, any point on the line is a best response, generating the same price and final allocation for player i. Indeed, a point on the line maximises player i's utility under either realisation of the opponents' mixed bid, and hence also maximises the expected utility.

Result: the best response coincides with the best response line to the realisations of the mixed bid.

[8] Indeed, note that $(-Q, -B)$ satisfies the best response equation (4).

[9] The results in this paper hold for all admissible utility functions and endowments, rather than that specific player only.

Figure 1: Example 1. Best Response Line to (1,1) and (2,2)



Example 1. Consider $(Q, B) = (1, 1)$ and $(Q', B') = (2, 2)$.

Using the formula (6) and substituting $w_{i1} = w_{i2} = 3$, we find that player i's best response lines to $(1, 1)$ and $(2, 2)$ coincide (Figure 1):

$$b_i = q_i. \tag{7}$$

Any point on the line $b_i = q_i$ maximises both $u_{Q,B}(q_i, b_i)$ and $u_{Q',B'}(q_i, b_i)$, and hence also their convex combination $U = \mu u_{Q,B}(q_i, b_i) + (1 - \mu)u_{Q',B'}(q_i, b_i)$.

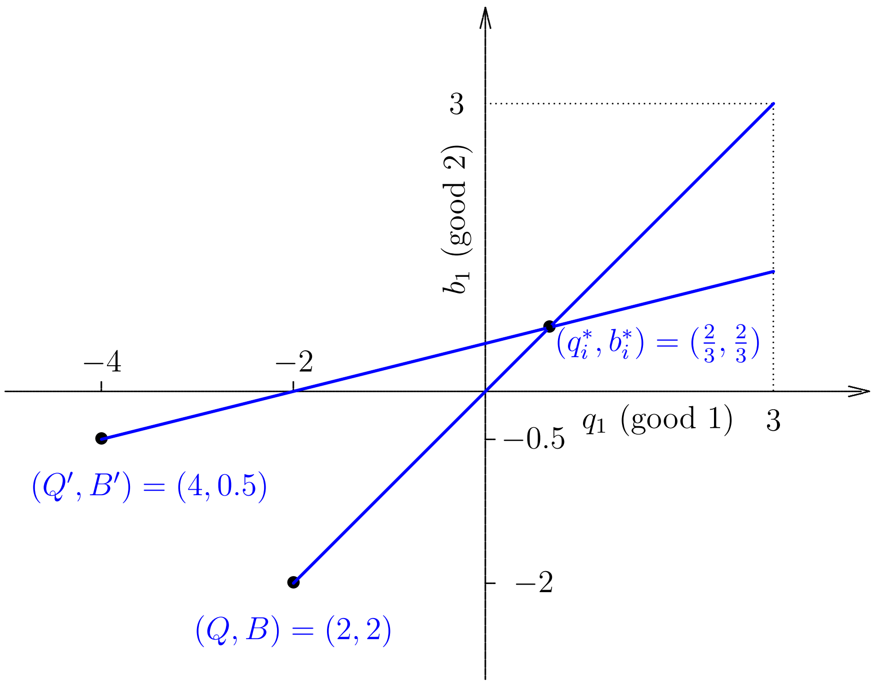
6.2 Case 2. Intersecting Best Response Lines

Let $\mathrm{BR} \cap \mathrm{BR}' = (q_i^*, b_i^*) \in S_i$ (that is, the best-response lines intersect within S_i). By the definition of the best response, $(q_i^*, b_i^*) = \arg\max u_{Q,B}(q_i, b_i) = \arg\max u_{Q',B'}(q_i, b_i)$ and hence $(q_i^*, b_i^*) = \arg\max \{\mu u_{Q,B}(q_i, b_i) + (1 - \mu)u_{Q',B'}(q_i, b_i)\}$. Moreover, (q_i^*, b_i^*) is the unique best response, since any other point in S_i lies outside either BR or BR$'$, hence generating a strictly lower utility in at least one realisation of the opponent's strategies.

Result: the best response is a unique point (q_i^*, b_i^*).

Example 2. Consider $(Q, B) = (2, 2)$ and $(Q', B') = (4, 0.5)$.

Figure 2: Example 2. Best Response Lines to (2,2) and (4,0.5)



It is easy to show using equation (6) that player i's best response to $(2, 2)$ is $b_i = q_i$, whereas her best response to $(4, 0.5)$ is $b_i = 0.25q_i + 0.5$. These lines intersect at the point $(q_i, b_i) = (\frac{2}{3}, \frac{2}{3})$, which is the best response to both $(2, 2)$ and $(4, 0.5)$ and hence also to any mixture between them (Figure 2).

In order to exhaust possible best response cases, we now consider a situation when best response lines to (Q, B) and (Q', B') do not intersect in player i's strategy set.

6.3 Case 3. Best Response Lines Intersecting Outside the Strategy Set

Let $\mathrm{BR} \cap \mathrm{BR}' = (q_i^*, b_i^*) \notin S_i$ (in words, BR lines intersect outside S_i). If the best response lines do not cross in S_i, one of them passes through S_i to the left of the other. Without loss of generality, let BR denote the left best response line and BR$'$ the right best response line. As we show below, the best response in this case is unique and lies on the boundary of S_i.

First, we show that a player's best response to such mixed bid lies on the boundary of her strategy set. Second, we show that it is unique.

Claim 1 *Consider a mixed total bid $(\mu(Q, B), (1 - \mu)(Q', B'))$ such that the best response lines to (Q, B) and (Q', B') are distinct and intersect outside S_i. A player's best response to the bid is either a unique point on the boundary of the strategy set S_i, or all the points within S_i on the line collinear with $(-Q, -B)$, $(-Q', -B')$.*

Proof: in Section 6.4.

Claim 1 asserts that a best response in Case 3 lies on the boundary of the strategy set. Moreover, it is either unique, or belongs to the best response set which is a line segment collinear with the realisations of a mixed total bid. In the next claim we rule out the latter possibility.

Claim 2 *Suppose the mixed total bid of player i's opponents is* $(\mu (Q, B) ; (1-\mu) (Q', B'))$*, such that the best response lines to* (Q, B) *and* (Q', B') *are distinct and intersect outside* S_i*. Then the points within* S_i *on the line collinear with* $(-Q, -B)$ *and* $(-Q', -B')$ *cannot all be best responses.*

Proof: in Section 6.6.

As shown above, the best response in Case 3 is a unique point at the boundary. It can also be shown that the best response lies strictly between BR and BR$'$ (proof available on demand).

Note that the best response can belong to either the inner or the outer boundary. Section 6.5 collects examples of boundary best responses to a mixed total bid (Case 3).

Result: the best response is a unique point (q_i^*, b_i^*).

6.4 Proof of Claim 1

First, observe that a function $U(q_i, b_i, \sigma_{-i}) \equiv \mu u_{Q,B(q_i,b_i)} + (1-\mu)u_{Q',B'}(q_i, b_i)$ is continuous. Hence, by the extreme value theorem, it attains a maximum on a closed bounded set S_i, which is a best response.

To show that a best response lies on the boundary, consider a point (q_i^*, b_i^*) which lies in the interior of S_i.

We show that such (q_i^*, b_i^*) cannot be a best response. For, consider contour lines of $u_{Q,B}$ and $u_{Q',B'}$ passing through (q_i^*, b_i^*), denoted L and L' respectively.

As demonstrated in the paper, given the others' total bid (Q, B), a player's utility is completely determined by the price p. Fixing (Q, B), player i's bids in S_i resulting in price p satisfy the following:

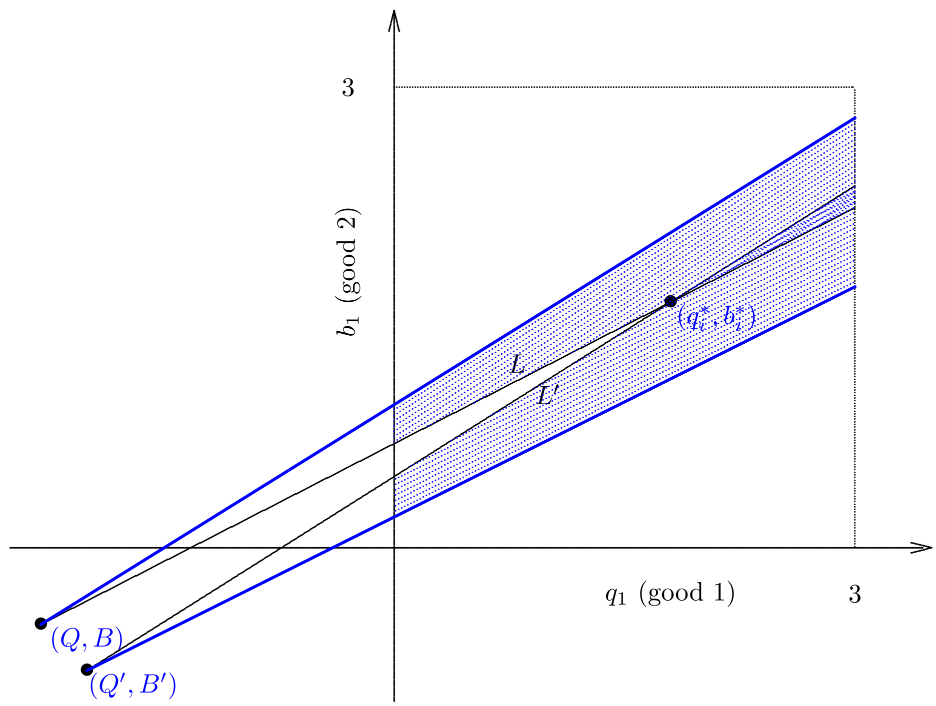
$$b_i = pq_i + pQ - B. \tag{8}$$

Hence, contour lines of $u_{Q,B}$ are straight lines passing through $(-Q, -B)$. Moreover, since utility is strictly concave in price, a contour line $u_{Q,B}$ separates the plane into lower and upper contour sets of the points on the line.

Note that, unless L and L' (i.e. the contour lines of $u_{Q,B}$ and $u_{Q',B'}$) coincide, they separate R_+^2 into four areas. One of these areas is the intersection of upper contour sets of (q_i^*, b_i^*) with respect to $u_{Q,B}$ and $u_{Q',B'}$ (a double shaded triangular area in Figure 3).

It follows that, unless (q_i^*, b_i^*) is on the boundary, the intersection of the strict upper contour sets of (q_i^*, b_i^*) with respect to $u_{Q,B}$ and $u_{Q',B'}$ in S_i is non-empty:

Figure 3: Upper Contour Sets of (q_i^*, b_i^*) w.r.t $u_{Q,B}$ and $u_{Q',B'}$



$$\{(q_i, b_i) : u_{Q,B}(q_i, b_i) > u_{Q,B}(q_i^*, b_i^*)\} \cup \{(q_i, b_i) : u_{Q',B'}(q_i, b_i) > u_{Q',B'}(q_i^*, b_i^*)\} \cup S_i \neq \emptyset \tag{9}$$

At any point in that intersection, the expected utility is strictly greater than at (q_i^*, b_i^*). Hence, (q_i^*, b_i^*) cannot be a best response.

Third, on any boundary, $U(q_i, b_i)$ becomes a one-dimensional function (either $U_{q_i}(b_i)$ or $U_{b_i}(q_i)$) of the strategic variable which is not fixed on that boundary. Moreover, $U(\cdot)$ is strictly concave in that variable, hence a best response which lies on the boundary is unique. To see why the one-dimensional restriction of expected utility is strictly concave, fix opponents' total bid (Q, B) and one dimension of player i's strategy (q_i). Observe that i's utility $u_{Q,B}(b_i) = x(b_i)y(b_i)$ is a strictly concave function. This follows from Lemma 1 and the fact that $x(b_i) = w_{i1} - q_i + b_i(q_i + Q)/(b_i + B)$ is strictly concave while $y(b_i) = w_{i1} - b_i + q_i(b_i + B)/(q_i + Q)$ is weakly concave. Similarly, $u_{Q',B'}(b_i) = x(b_i)y(b_i)$ is strictly concave. Hence, i's expected utility $U_{q_i}(b_i) = \mu u_{Q,B}(b_i) + (1 - \mu)u_{Q',B'}(b_i)$ is also strictly concave as a convex combination of strictly concave functions (the same holds for $U_{b_i}(q_i)$).

Fourth, if $L = L'$, we cannot rule out the case that a point on the line is the best response (the upper contour sets of that point with respect to the two realised utilities do not intersect). Moreover, by definition, both $(-Q, -B)$ and $(-Q', -B')$ lie on this line; hence, any point on the line generates the same price and the same outcome for player i under (Q, B) and (Q', B'). It follows that the whole line $L = L'$ is the best response.

Summing up the third and the fourth points, the best response is either unique and lies on the boundary, or is the whole line collinear with (Q, B) and (Q', B'), Q.E.D.

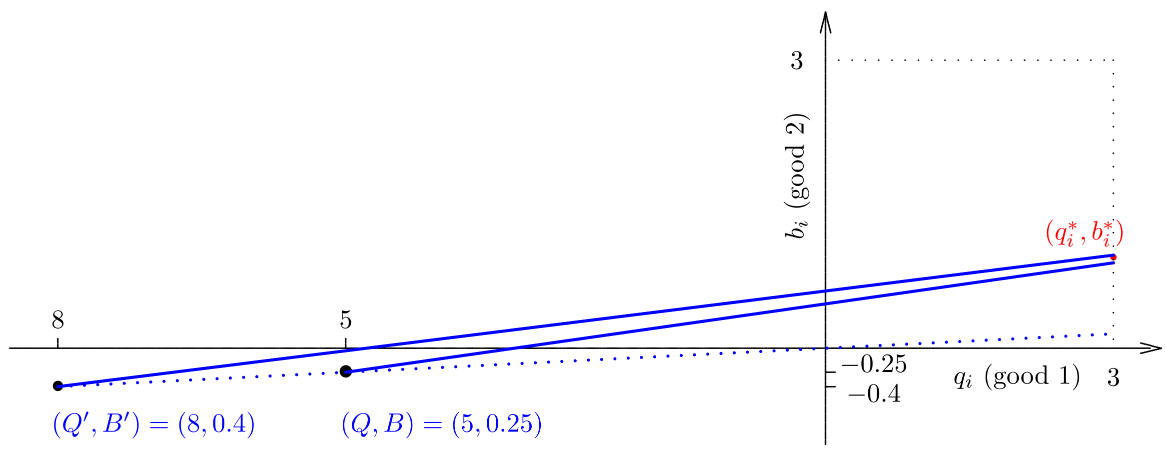
6.5 Examples of Case 3.

When best-response lines intersect outside S_i, there are three possible options for the unique best response to the mixed bid, illustrated by Examples 3, 4 and 5 below.

Example 3. Converging BR lines; unique best response on the outer boundary.

Let opponents' total bids be $(Q, B) = (5, 0.25)$ and $(Q', B') = (8, 0.4)$. The best response to these total bids are $b_i = q_i\sqrt{1.3}/8 + \sqrt{6.5}/8 - 0.25$. and $b_i = \sqrt{1.87}q_i + 8\sqrt{1.87} - 0.4$ respectively. Figure 4 shows that these BR lines cross in the first quadrant outside S_i, hence, they are converging.

Figure 4: Example 3. Best Response Lines to (5,0.25) and (8,0.4)



Also note that when the best-response lines are converging the best response always lies on the *outer* boundary. This is because, for any point on the inner boundary, the intersection of upper contour sets of $u_{Q,B}$ and $u_{Q',B'}$ lies *above and to the right* of this point (inside the strategy set) and hence be achievable.

In particular, in our example the unique best response to a mixed strategy can be determined from the following equation:

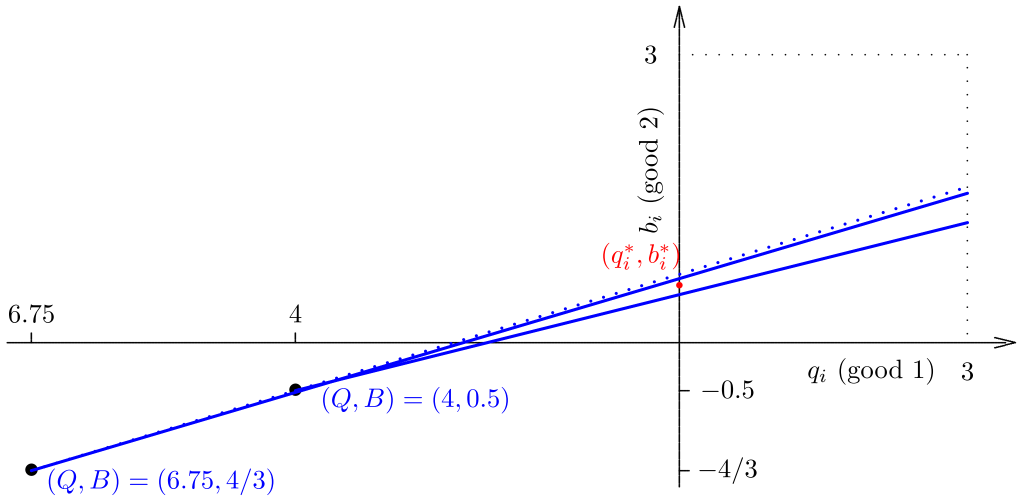
$$\mu\frac{13/2 - 5(b_i^* + 0.25)^2}{(b_i^* + 0.25)^2} = (1-\mu)\frac{8(b_i^* + 0.4)^2 - 374/25}{(b_i^* + 0.4)^2}, \qquad (10)$$

e.g. if the opponent is mixing w/p $\mu = 1216/4051$, then player i's best response is $(q_i^*, b_i^*) = (3, 0.95)$.

Example 4. Diverging BR lines; unique best response on the inner boundary.

Let the opponents' total bids be $(Q,B) = (4, 0.5)$ and $(Q', B') = (6.75, 4/3)$. The best responses to these total bids are $b_i = 0.25q_i + 0.5$ and $b_i = 8/27q_i + 2/35$ respectively. As shown in Figure 5, these best-response lines cross in the third quadrant; hence, they are diverging.

Figure 5: Example 4. Best Response Lines to $(4, 0.5)$ and $(6.75, 4/3)$



Note that the best response to this mixed strategy lies on the inner boundary. This is because, for any point on the outer boundary, the intersection of upper contour sets of $u_{Q,B}$ and $u_{Q'B'}$ lies *below and to the left* of this point (inside the strategy set) and hence be achievable.

In particular, in our example the unique best response to a mixed strategy can be determined from the following equation:

$$\mu \frac{-7(b_i^* + 1.5)(b_i^* - 0.5)}{(b_i^* + 0.5)^2} = (1-\mu)\frac{39(b_i^* + 10/3)(b_i^* - 2/3)}{4(b_i^* + 4/3)^2}. \tag{11}$$

For example, if $\mu = 92807/257643$, then player i's best response is $(q_i^*, b_i^*) = (0, 0.6)$.

Example 5. Diverging BR lines; unique best response on the outer boundary.

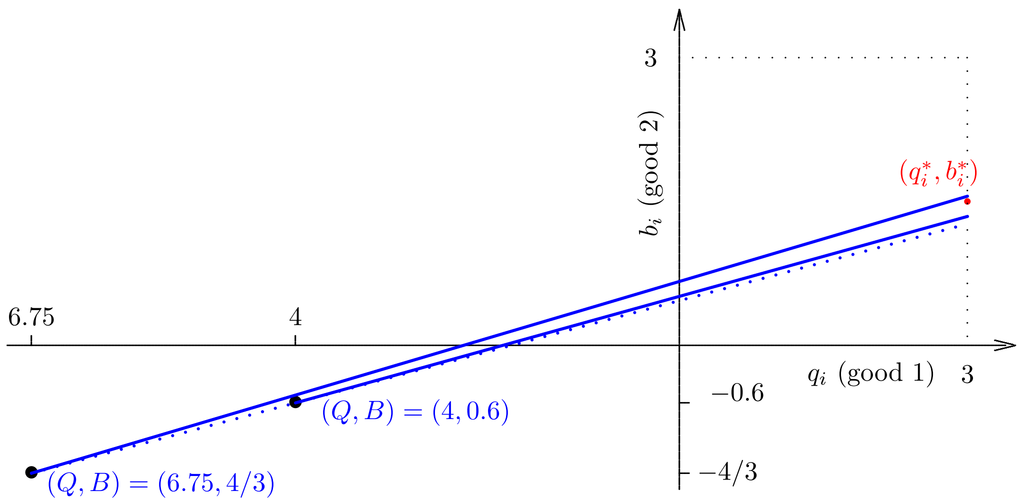
Let the opponents' total bids be $(Q,B) = (4, 0.6)$ and $(Q', B') = (6.75, 4/3)$. The best responses to these total bids are $b_i = 0.3q_i\sqrt{6/7} + 1.2\sqrt{6/7} - 0.6$ and $b_i = 8/27q_i + 2/35$ respectively. These best-response lines cross in the third quadrant; hence, they are diverging.

Unlike the previous example, the unique optimum lies on the outer boundary, by the logic similar to that of Example 3.

The unique best response to a mixed strategy can be determined from the following equation:

$$\mu \frac{4(b_i^* + 0.6)^2 - 15.12}{(b_i^* + 0.6)^2} = (1-\mu)\frac{169/3 - 27/4(b_i^* + 4/3)^2}{(b_i^* + 4/3)^2}. \tag{12}$$

Figure 6: Example 5. Best Response Lines to $(4, 0.6)$ and $(6.75, 4/3)$



For example, if $\mu = 2163/6787$, then player i's best response is $(q_i^*, b_i^*) = (3, 1.5)$.

6.6 Proof of Claim 2

Let $b_i^*(q_i)$ denote the function whose graph is a straight line connecting $(-Q, -B)$ and $(-Q', -B')$ in R^2. It is easy to derive the formula for $b_i^*(q_i)$:

$$b_i^* = q_i \frac{B' - B}{Q' - Q} + \frac{B'Q - BQ'}{Q' - Q}. \tag{13}$$

The set of i's strategies on the line is denoted $S_i^* \equiv \{(q_i, b_i) : b_i = b_i^*(q_i)\} \cup S_i$.

If all points in the set S_i^* were best responses, then any point $(q_i, b_i^*(q_i))$ in the set would need to be the maximiser of $U_{q_i}(b_i)$. Since $U_{q_i}(b_i)$ is strictly concave (as shown in the proof of Claim 1), any point $(q_i, b_i^*(q_i))$ on the line would need to satisfy the first-order condition:

$$\mu \frac{\partial u_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) + (1 - \mu)\frac{\partial u_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) = 0 \tag{14}$$

Applying the chain rule to (14) obtains:

$$\begin{aligned} &\mu \left[\tfrac{\partial u(x,y)}{\partial x}(q_i, b_i^*(q_i)) \tfrac{\partial x_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) + \tfrac{\partial u(x,y)}{\partial y}(q_i, b_i^*(q_i)) \tfrac{\partial y_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) \right] \\ &+(1-\mu) \left[\tfrac{\partial u(x,y)}{\partial x}(q_i, b_i^*(q_i)) \tfrac{\partial x_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) + \tfrac{\partial u(x,y)}{\partial y}(q_i, b_i^*(q_i)) \tfrac{\partial y_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) \right] \quad = \quad 0 \end{aligned} \tag{15}$$

Note that, at all points along S_i^* the player's final allocation of x and y is the same (under either realisation of mixed total bid). Hence, her utility $u(x, y)$ is the same, and, most importantly, partial derivatives of utility with respect to x and y are the same. Rewrite (15) denoting $u_x \equiv \frac{\partial u(x,y)}{\partial x}(q_i, b_i^*(q_i))$ and $u_y \equiv \frac{\partial u(x,y)}{\partial y}(q_i, b_i^*(q_i))$:

$$\mu\left[u_x \frac{\partial x_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) + u_y \frac{\partial y_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i))\right] + (1-\mu)\left[u_x \frac{\partial x_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) + u_y \frac{\partial y_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i))\right] = 0$$

The derivatives of x and y w.r.t. to b_i can be easily calculated, and rearranged using the formula for $b_i^*(q_i)$:

$$\frac{\partial x_{Q,B}}{\partial b_i} = \frac{B(q_1+Q)}{(b_1+B)^2} = \frac{B\left(Q'-Q\right)^2}{\left(B'-B\right)^2\left(q_1+Q\right)}; \tag{16}$$

$$\frac{\partial x_{Q',B'}}{\partial b_i} = \frac{B'(q_1+Q')}{(b_1+B')^2} = \frac{B'\left(Q'-Q\right)^2}{\left(B'-B\right)^2\left(q_1+Q'\right)}; \tag{17}$$

$$\frac{\partial y_{Q,B}}{\partial b_i} = \frac{-Q}{q_1+Q}; \tag{18}$$

$$\frac{\partial y_{Q',B'}}{\partial b_i} = \frac{-Q'}{q_1+Q'}. \tag{19}$$

The first-order condition can then be rewritten as follows:

$$\mu\left[u_x \frac{B\left(Q'-Q\right)^2}{\left(B'-B\right)^2\left(q_1+Q\right)} - u_y \frac{Q}{q_1+Q}\right] = (\mu-1)\left[u_x \frac{B'\left(Q'-Q\right)^2}{\left(B'-B\right)^2\left(q_1+Q'\right)} - u_y \frac{Q'}{q_1+Q'}\right], \tag{20}$$

Rearranging (20) obtains[10]

$$\frac{\left[u_x B\left(Q'-Q\right)^2 - u_y Q\left(B'-B\right)^2\right]\left(q_1+Q'\right)}{\left[u_x B'\left(Q'-Q\right)^2 - u_y Q'\left(B'-B\right)^2\right]\left(q_1+Q\right)} = \frac{\mu-1}{\mu}. \tag{21}$$

Denote $A = \frac{\left[u_x B\left(Q'-Q\right)^2 - u_y Q\left(B'-B\right)^2\right]\left(q_1+Q'\right)}{\left[u_x B'\left(Q'-Q\right)^2 - u_y Q'\left(B'-B\right)^2\right]\left(q_1+Q\right)}$. Observe that A is constant w.r.t. q_i (indeed, as argued above, u_x and u_y are constant w.r.t. q_i). Equation (21) becomes

[10] The expression in the square brackets on the right-hand-side of (20) is the value of the partial derivative of $u_{Q',B'}$ with respect to b_i at point $\left(q_i, b_i^*(q_i)\right)$. It equals zero iff $\left(q_i, b_i^*(q_i)\right)$ lies on player i's best response line to (Q', B'). However, recall that $\left(q_i, b_i^*(q_i)\right)$ is a point on the line connecting $(-Q', -B')$ and $(-Q, -B)$. This line intersects the best response to (Q', B') at $(-Q', -B')$ and not at $\left(q_i, b_i^*(q_i)\right)$. Hence the expression in the square bracket is non-zero, and we can divide both sides of (20) by it.

$$A\frac{q_1+Q'}{q_1+Q}=\frac{\mu-1}{\mu}. \tag{22}$$

Denote $f(q_i)=A\frac{q_1+Q'}{q_1+Q}$. The derivation above implies that all points in S_i^* are best responses iff $f(q_i)=(\mu-1)/\mu$ for all q_i. In other words, $f(q_i)$ needs to be constant with respect to q_i, *i.e.* $\partial f(q_i)/\partial q_i=0$:

$$A\frac{Q-Q'}{(q_1+Q)^2}=0. \tag{23}$$

Expression (23) holds if either (i) $Q=Q'$, in which case the line connecting $(-Q,-B)$ and $(-Q',-B')$ does not intersect S_i and hence the points on the line cannot be best responses; or (ii) $A=0$. However, if $A=0$, then $f(q_i)=0\neq(\mu-1)/\mu$, *i.e.* the first-order condition does not hold, implying that points in S_i^* are not best responses. In either case, the points in S_i collinear with $(-Q,-B)$ and $(-Q',-B')$ cannot all be best responses, Q.E.D.

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