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Abstract

We augment the Shapley–Shubik (1977) market game to include a spatial dimension à la Hotelling (1929). Taking firms' locations as given, we study and characterise, through several propositions, lemmata, and a theorem, the main equilibrium predictions of this new model. When both firms locate in the centre and there is no product differentiation at all, we derive a counterexample in which both firms charge a price that is greater than marginal cost. Intriguingly, we show that even when both firms are in the same location, it is possible for the Law of One Price (LOOP) to fail, i.e., the exact same good sells at different prices across two platforms that are a priori identical. We derive similar (equal- and unequal-price) counterexamples in the context where the firms locate at the extreme ends of the city. Now, it is well known that in the traditional Hotelling model, a pure-strategy Nash equilibrium (PSNE) fails to exist when the two firms are closely spaced and near the centre of the city. In our main result, we allow the firms to be arbitrarily close to each other, and propose two counterexamples in which a PSNE exists. In one, the LOOP holds, while in the other, it fails.

Keywords: Spatial Cournot oligopoly; Existence of pure-strategy equilibrium with closely spaced firms; Failure of Law of One Price; Strategic behaviour with a continuum of players

1 Introduction

> "[...] in spatial models, even in the limit of a continuum of firms, strategic interaction remains. In that case, firms interact locally, and neighbors count, no matter how large the economy is."
>
> (Mas-Colell et al., 1995: p. 400).

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Cournot's (1838) model of oligopoly is underpinned by two canons. The first is the thesis that agents[1] compete on producing/selling *quantities* of a homogeneous good, and the second presupposes the existence of an invisible auctioneer who engineers *market-clearing* prices such that whatever is produced is bought. In his review of Cournot's work, Bertrand justly criticises the Cournot machinery on the grounds that its price formation mechanism is lacking in two crucial respects: no such auctioneer exists, and firms choose prices, not quantities, as their strategic variables. Bertrand's (1883) postulate, and subsequent model, of price competition, however, does also not constitute much progress in terms of realism. In particular, while aesthetically appealing, it yields an unsatisfactory outcome: a unique equilibrium at which oligopolists, no matter their number, charge a perfectly competitive price, and make normal profits only—the so-called Bertrand paradox. A shortcoming common to both models is the assumption of a homogeneous good when in the real world, consumers perceive even physically identical goods are viewed as being different, due to transportation and search costs, time constraints, etc. Hotelling (1929) addressed this defect to some extent by advancing an intuitively simple yet powerful model of spatial product differentiation. Nonetheless, despite its allure, this model is also susceptible to awkward conclusions such as a pure-strategy equilibrium failing to exist when the firms locate close to the centre (d'Aspremont et al., 1979), or the Bertrand paradox resurfacing when both firms locate exactly in the centre of the city.

If, somehow, the aforementioned theories were genotyped and their desirable characteristics isolated from their shortcomings, then a marriage of the former could produce an optimal phenotype. In this work, we put forth a spatial model of quantity competition, and of price formation. We use a strategic market game of the Shapley and Shubik (1977) tradition, and we augment it to include a spatial element. There are many advantages to undertaking this enterprise. To start with, our model is genuinely decentralised. There is no need for an auctioneer or a referee to map quantities to prices as this is done automatically via agents' buy and sell decisions. The market game furnishes an endogenous and explicit price formation apparatus which is well-defined even out of equilibrium. Indeed, price is a continuous, surjective function of agents' strategies and it is derived in such a way as to always clear markets. This is meaningful because a common critique of Walrasian models is that they completely eschew price formation considerations outside of equilibrium. Additionally, our framework addresses the criticism of a unique, homogeneous good being traded. As in Hotelling (1929), the (physically identical) good is differentiated in the minds of consumers as they each incur a different transportation cost to go to the respective markets/trading platforms. Moreover, our formulation circumvents the well-known problems associated with Hotelling's (1929) model, such as the Bertrand paradox obtaining when the markets are both located exactly in the centre of the city, or even the inexistence of equilibrium if both platforms are located close to the centre but at different locations (d'Aspremont et al., 1979).

In this paper, we analyse three different exogenous locational setups, and we pro-

[1] Throughout this paper, the terms "agent(s)", "consumer(s)", "player(s)", and "trader(s)" will be used interchangeably.

pose two (counter)examples for each setup. In particular, we analyse the cases when: (i) both platforms are located in the exact same spot, irrespective of where they lie along the city; (ii) the two markets are stationed at the opposite extremes of the city, and; (iii) the trading posts are situated arbitrarily close to the centre, but may have different locations. For each parametrisation, we propose an example of a ***pure-strategy*** Nash equilibrium at which both posts charge the same ***positive*** price at equilibrium, and another example where the commodity in question trades at different prices across both platforms at equilibrium. This is even though agents still have plenty of money left to spend and goods to sell. The main takeaways from these scenarios are threefold. First, the Bertrand paradox, though a valid equilibrium in each counterexample (the trivial autarkic Nash equilibrium), is Pareto dominated by *at least* another equilibrium with positive prices. Second, even when both platforms are situated in the exact same location and are a priori identical to consumers, the LOOP can fail at equilibrium. Finally, as is now common knowledge, d'Aspremont et al. (1979) disproved Hotelling's (1929) so-called "Principle of Minimum Differentiation". In particular, they showed that if the sellers are near the centre and too closely spaced, then no equilibrium exists in pure strategies. If a Nash equilibrium price solution is to be reached, then it will only be through the use of mixed strategies. In §3.3, we allow the two platforms to begin business *anywhere* along the city (including near the centre), while being arbitrarily close to—or far from—each other. We exhibit, through two counterexamples, that a pure-strategy Nash equilibrium exists in such a scenario. Moreover, at every equilibrium we exhibit, agents place strictly positive bids, and are neither bindingly financially constrained nor offer-constrained, i.e., they have enough money and enough of the commodity being traded to freely modify their buy-and-sell decisions should they so wish. Thus, these equilibria are regular and ex-post stable—see Peck, Shell and Spear (1992), Spear (2003), and also Toraubally (2022a). This is arguably our most important contribution. Furthermore, the fact that firms can charge a positive price (greater than marginal cost, which is zero in our setup), let alone different prices at equilibrium, when they share the same location is particularly interesting given that in the extant literature, this is possible only when firms randomise. Indeed, Baye and Morgan (2002) and Xefteris (2013) prove that under mixed-strategy pricing, firms can earn positive expected profits when they share the same location. Nonetheless, as Byford (2015) shows, these mixed-strategy solutions are not always ex-post stable—see also Shy (2022).

Now, in our model prices are derived endogenously, and crucially, ***explicitly***, through the buy and sell strategies of every player. It is therefore natural to wonder how—if agents behave truly non-cooperatively and independently—one goes about building a suitable link between the strategic decisions of individually insignificant players and how these translate into the market as a whole. Hence, we must necessarily set up our model so as to tackle the requirements of the existence of measurable and integrable strategy mappings. Additionally, the cost of travel, while easy to incorporate in the description of the model, nonetheless induces disconvexities in players' budget constraints, making for ill-behaved optimisation programmes. This situation represents a major obstacle that needs to be overcome for our purposes since not only do we derive and characterise the equilibrium properties of our model, but we also present full-blown numerical (counter)examples which constitute Nash equilibria of our model. To attack this difficulty, we use a geometric approach which allows us transform a potentially intractable problem into a series of simple, well-behaved optimisation problems. In particular, by understanding the geometry of the problem, we are able to use simple Lagrange multiplier methods to derive conditions that completely characterise Nash equilibria of our market game.

The past few years have seen Hotelling-type models experience a revival of sorts, with new lines of investigation extending the original model in many directions—applied and theoretical. Balart (2022) provides a behavioural take on the subject matter by introducing semiorder lexicographic preferences in a Hotelling duopolistic model to incorporate the possibility of consumers being price-oriented when products are insufficiently differentiated yet unidentical. Kharbach and Chfadi (2022) consider a setup in which firms bear logistics-related costs in the form of support-only and transportations costs, while consumers incur a combination of transportation and search costs. Hinloopen and Martin (2017) consider a model in which location is costly, and more so as firms approach the centre of the linear city. Our paper is perhaps most closely related to Xefteris and Ziros's (2015) (henceforth, XZ) in which a market game with fiat money—see Postlewaite and Schmeidler (1978) and Peck et al. (1992)—is considered. However, our models are drastically different, not only in terms of scope and results, but also with respect to our formulations. XZ consider an economy with a continuum of small players only, whereas in our setup we use a mixed measure space of agents. In XZ, markets and their locations are formed endogenously, and various groups of agents can set up markets anywhere along the city, i.e., there can be more than two markets. Additionally, in their model, a one-market equilibrium generically exists. That is to say, all agents converge to a single location at which to carry out all their trades. In our case, the locations of the two platforms are exogenously fixed, and agents can only trade at these posts. Crucially, in each of our examples, trade always takes place at ***both*** platforms at equilibrium—i.e., every situation we construct showcases nontrivial ***two***-market equilibria. Moreover, in XZ, whenever a two-market equilibrium does exist, the LOOP fails. In our case, we establish through our counterexamples, for each parametrisation of our model, the existence of a two-market equilibrium at which the LOOP obtains. Last but not least, in XZ, there exists no two-market equilibrium when the trading posts are arbitrarily close to each other. In our case, as we demonstrate in §§3.1 and 3.3, no matter how closely spaced the posts are, we can find examples where trade takes place at both platforms at equilibrium, with the LOOP obtaining in some cases and failing in others. In very interesting work, Shy (2022) deploys the so-called undercut-proof equilibrium solution concept to derive equilibrium prices for all possible firm locations on the $[0, 1]$ interval. This is because, as Shy (2022) argues, in the traditional Hotelling model, a pure-strategy Nash equilibrium fails to exist in roughly 85 percent of all possible firm locations. This observation is very helpful for putting things into perspective since in the current paper, especially §3.3, we present two examples in which a pure-strategy Nash equilibrium exists for ***all*** possible firm locations.

2 The Hotelling–Shapley–Shubik market game, Γ

There is a linear city of length 1 along which there are two large players, and a continuum of uniformly distributed individually insignificant consumers with density 1.

Formally, we take the space of agents as being defined by the measure space $(N, \mathcal{N}, \mu)$, where $N = [0,1]^2 \cup \{L, R\}$, $\mathcal{N}$ is the collection of all μ-measurable sets of N, and μ is an extended real-valued, σ-additive measure defined on N. Let $\mathcal{N}_{[0,1]^2}$ denote the restriction of $\mathcal{N}$ to $[0,1]^2$, and $\mathcal{N}_{\{L,R\}}$ denote the restriction of $\mathcal{N}$ to $\{L, R\}$. Define μ to be the Lebesgue measure when restricted to $\mathcal{N}_{[0,1]^2}$, and the counting measure when restricted to $\mathcal{N}_{\{L,R\}}$, i.e., L and R are atoms.[2] The triple $(N, \mathcal{N}, \mu)$ as defined constitutes a complete, finite measure space of agents—see Toraubally (2018). Henceforth, the expressions *almost all* $(a.a.)$, *almost everywhere* $(a.e.)$, *μ-a.e.*, *every*, and *each*, will be taken to mean all traders except for a null set of players.

The city is the $[0,1]$ interval. The two large agents, L and R, own the markets (trading posts) at which all consumers across all locations in $[0,1]$ may come and trade amounts of a homogeneous physical good, with zero production cost, which we shall henceforth call k. L and R also engage in trade, similarly to the consumers in $[0,1]^2$, by presenting arrays of buy-and-sell strategies at their own, and/or at each other's post. However, since L and R own the trading platforms, they also levy a proportional service charge per unit of (monetary) net trade on all agents who transact at their post. This proportional service charge, $0 < c < 1$, is taken to be exogenously given,[3] and is the same at both trading posts. Hence, it is easy to deduce that prior to any trade being carried out, the *unique* difference between the markets owned by L and R lies in their location. Consequently, if L and R were to set up shop at the same location, both markets would be, a priori, ***exactly identical*** (and the goods traded, perfect substitutes) to $a.e.\ n \in [0,1]^2$.

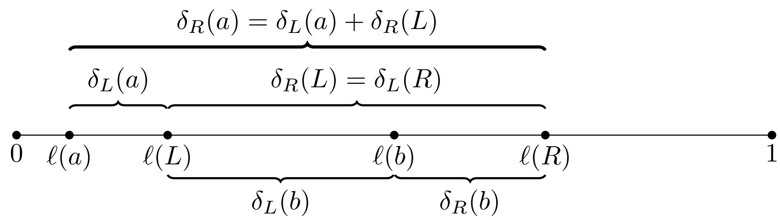
For any $n \in N$, $\delta_i(n)$ stands for the distance between n's location and trading platform i, $i = L, R$. Letting $t/2$ denote the transportation cost per unit of $v(\delta_i(n))$, where the twice-continuously differentiable function $v(\cdot)$ is such that $\frac{dv}{d\delta_i(n)} > 0$ and $\frac{d^2 v}{d(\delta_i(n))^2} \geq 0$, n incurs a transportation cost of $tv(\delta_i(n))$ by travelling to, and back from, trading platform i. Note that this specification is general enough to encompass the cases where transportation cost varies both linearly and non-linearly with distance. To illustrate, one may choose $v(\cdot)$ to be such that $v(\delta_i(n)) = \delta_i(n)$, or the oft-used quadratic variant $v(\delta_i(n)) = (\delta_i(n))^2$, amongst others—see Tirole (1988) for further details. For ease of exposition, it will also help to define a location function $\ell : N \to [0,1]$, where $\ell(n)$ is n's location in the linear city. We may thus define $\delta_i(n) = |\ell(n) - \ell(i)|$. Figure 1 exhibits $\ell(n)$ and $\delta_i(n)$ at work.

In addition to the homogeneous good k that is traded, there is also a numéraire good,

[2] An atom of $(N, \mathcal{N}, \mu)$ is a set X such that $\mu(X) > 0$, and for any $Y \subseteq X$, we either have $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$.

[3] One may think of this as, among other things, a value-added tax imposed by the government.

Figure 1: $\ell(n)$ and $\delta_i(n)$



m, which acts as money and yields utility in consumption.[4] The consumption set of each consumer is therefore identified with $X(n) = \mathbb{R}^2_+ = \{x(n) \in \mathbb{R}^2 : x_k(n) \geq 0, x_m(n) \geq 0\}$. In this light, each $n \in N$ may be described by a location $\ell(n) \in [0, 1]$, an initial endowment of commodities $e(n) \in \mathbb{R}^2_+$, and a preference relation representable by a utility function $U : N \times X \to \mathbb{R}$ given by $U(n, x) = U_n(x)$. Rationality is common knowledge.

In what follows, we will rely on the following assumptions:

Assumption 1. *$\ell(L) \leq \ell(R)$, and these are taken as given.*[5]

Assumption 2. *Each agent $n \in N$ is endowed with a strictly positive amount of both k and m, i.e., $e_k(n) \cdot e_m(n) > 0$ a.e. in N.*

Assumption 3. *$U_n(x(n))$, where $N \times \mathbb{R}^2_+$ is equipped with the σ-field generated by the product of $\mathcal{N}$ and the Borel sets of $\mathbb{R}^2_+$, is measurable a.e. in N.*

Assumption 4. *For each $n \in N$, U_n is concave, smooth, differentiably strictly monotone,*[6] *and indifference curves through the endowment do not intersect the axes.*

2.1 Trading Shapley–Shubik style in Hotelling's city

Trade in this linear city takes place at the trading platforms owned by L and R. On these markets, players offer commodity k for sale, and distribute amounts of money (bids) to purchase however much of k has been offered for sale. To be precise, bids (b) for commodity k are placed in terms of the numéraire m, while sales (q) are, obviously, made in terms of commodity k. The strategy sets of agents are described by a measurable correspondence $S : N \rightrightarrows 2^{\mathbb{R}^{2\times 2}_+}$ such that

$$S(n) = \Big\{ \big(b(n), q(n)\big) \in \mathbb{R}^4_+ : \sum_{i=L}^{R} (b_i(n) + \mathscr{P}_i(n)) + tv(\delta(n)) \leq e_m(n); \sum_{i=L}^{R} q_i(n) \leq e_k(n) \Big\}.$$

[4]See, e.g., Mas-Colell et al. (1995: p. 399).

[5]I.e., as in Hotelling (1929) and d'Aspremont et al. (1979), L and R do not choose their equilibrium locations. As we explain in §2, they compete only in bid and offer strategies. We will nonetheless analyse what happens when we exogenously vary $\ell(L)$ and $\ell(R)$.

[6]I.e., for all $x(n) \in \mathbb{R}^2_{++}$, $\frac{\partial U_n(x(n))}{\partial x_k(n)} > 0$ and $\frac{\partial U_n(x(n))}{\partial x_m(n)} > 0$.

$q_i(n)$ stands for the amount of commodity k offered for sale by player n on market i, $i = L, R$. Each $n \in N$ may simultaneously place bids for (make purchases of) commodity k by distributing amounts of money, m, across the two trading platforms, with $b_i(n)$ denoting the bid placed by n for good k on market i, $i = L, R$. $tv(\delta(n))$ is the total transportation cost incurred by n, and $\delta(n)$ is computed as follows (see also Figure 1):

$$\delta(n) = \begin{cases} \delta_i(n) & \text{if } n \text{ travels only to one platform, } i; \\ \sum_{i=L}^{R} \delta_i(n) & \text{if } \ell(L) \leq \ell(n) \leq \ell(R), \text{and } n \text{ travels to both platforms;} \\ \max\{\delta_L(n), \delta_R(n)\} & \text{if } \ell(n) \leq \ell(L) \text{ or } \ell(R) \leq \ell(n), \text{and } n \text{ travels to both posts.} \end{cases}$$

$\sum_{i=L}^{R} \mathscr{P}_i(n)$ is the total premium payable by n. We next explain how $\mathscr{P}_i(n)$, $i = L, R$, is computed, introducing some important concepts along the way.

A strategy profile consists of a pair of measurable mappings $b : N \to \mathbb{R}^2_+$ and $q : N \to \mathbb{R}^2_+$ such that $s(n) \equiv (b(n), q(n)) \in S(n)$ *a.e.* in N, i.e., a strategy profile is a measurable selection from the graph of the correspondence S, $Gr(S)$. Since $S : N \rightrightarrows 2^{\mathbb{R}^{2\times 2}_+}$ has measurable graph, b and q exist indeed, by Aumann's Measurable Choice Theorem. Consequently, for a given strategy profile $(b, q) \in Gr(S)$, we may then define $B_i = \int_N b_i(n)\mu(dn) < \infty$, and $Q_i = \int_N q_i(n)\mu(dn) < \infty$. We also define, for $i = L, R$, $B_{-i,i} = B_i - b_i(i)$, and $Q_{-i,i} = Q_i - q_i(i)$.[7]

Final consumption allocations of k and m for any $n \in [0, 1]^2$ are then determined as follows:

$$\begin{aligned} x_k(n) &= e_k(n) + \sum_{i=L}^{R} \frac{b_i(n)}{B_i} Q_i - \sum_{i=L}^{R} q_i(n); \\ x_m(n) &= e_m(n) + \sum_{i=L}^{R} \left(\frac{q_i(n)}{Q_i} B_i - b_i(n) \right) (1 + \vartheta_i(n)c) - tv(\delta(n)), \end{aligned} \tag{1}$$

where we use the market game convention that any division by zero, including $\frac{0}{0}$, is equal to zero if it appears in any of the above expressions above.

We explicate the meaning of the component parts of the allocation rule in (1). Q_i is the total amount of k offered for sale on market i, while B_i is the total amount of money placed (bid) at platform i to purchase k. When $B_i \cdot Q_i > 0$, trader n, having bid $b_i(n)$ at i, receives good k in proportion to his bids. In a similar vein, trader n, having offered $q_i(n)$ units of k for sale at i, receives money (m) in proportion to his offers. Notice that $\frac{\partial(B_i/Q_i)}{\partial B_i} > 0$, while $\frac{\partial(B_i/Q_i)}{\partial Q_i} < 0$. So, when $B_i \cdot Q_i > 0$, the fraction $B_i/Q_i \coloneqq p_i$ can be naturally construed as the *market-clearing* price of k at platform i. Hereinafter, the price of k on market i will be denoted by any of p_i and B_i/Q_i.

Now, from the second line in (1), let $q_i(n)p_i - b_i(n) \coloneqq z_i(n)$ denote the net (monetary) trade of player $n \in N$ at market i. The function $\vartheta_i(n)$ in (1) may then be defined as $\vartheta_i(n) : z_i(n) \to \{-1, 1\}$. More precisely, if $z_i(n) > 0$, then $\vartheta_i(n) = -1$, while $\vartheta_i(n) = 1$ if

[7] $B_{-i,i}$ and $Q_{-i,i}$ as described here are well-defined indeed: recall the definitions and dimensionalities of N and μ.

$z_i(n) < 0$. If $z_i(n) = 0$, following Toraubally (2018)—see ibid. for an interpretation—we use the following rule:

$$\vartheta_i(n) = \begin{cases} -1 & \text{if } \exists \xi \in \mathcal{N}, \text{ where } \mu(\xi \cap [0,1]^2) > 0, \text{ such that } z_i(n) \geq 0 \text{ } a.e. \text{ in } \xi \cap [0,1]^2; \\ 1 & \text{otherwise.} \end{cases}$$

Accordingly, the product $z_i(n)(1 + \vartheta_i(n)c)$ in (1) represents the monetary value of agent n's net trade in k at market i, $z_i(n)$, subtract the proportional premium payable, $\mathscr{P}_i(n) = cz_i(n)$, to platform owner i, $i = L, R$.

Final consumption allocations of k and m for any $h \in \{L, R\}$ are then determined as follows:

$$\begin{aligned} x_k(h) &= e_k(h) + \sum_{i=L}^{R} \frac{b_i(h)}{B_i} Q_i - \sum_{i=L}^{R} q_i(h); \\ x_m(h) &= e_m(h) + \sum_{i=L}^{R} \left(\frac{q_i(h)}{Q_i} B_i - b_i(h) \right) (1 + \vartheta_i(h)c) \\ &\quad - c \int_{n \in N} \left(\frac{q_i(n)}{Q_i} B_i - b_i(n) \right) \vartheta_i(n) \mu(dn) - tv(\delta(h)). \end{aligned} \tag{2}$$

In the present paper, we shall only concern ourselves with equilibria in bid and offer strategies, as opposed to bid-offer-location tuples. In this light, we have that:

Definition 1. A Nash equilibrium (NE) of Γ comprises players' bid and offer strategies such that

(i) μ-$a.e.$ in N, n's moves are best-responses given the expectations of other players' moves;

(ii) The best-responses are consistent with $a.a.$ players' expectations of other players' moves.

In other words, a strategy profile $s^* \equiv (b^*, q^*) \in Gr(S)$ is an NE iff:

$$U_n\Big(x\big(s^*(n), B_{-n}, Q_{-n}\big)\Big) \geq U_n\Big(x\big(s(n), B_{-n}, Q_{-n}\big)\Big) \text{ for all } s(n) \in S(n), \text{ } a.e. \text{ in } N. \tag{3}$$

Additionally, so we avoid trivial outcomes, we will assume, as is standard in the literature, that agents have enough money to travel to both trading posts (and back) should they so wish. We state this formally:

Assumption 5. *For each* $n \in N$, $e_m(n) > tv(\delta(n))$.

3 Locations, and equilibrium analysis

In this section, we will analyse the equilibrium properties of the model described in §2, and determine the extent to which different parametrisations of this model yield different conclusions. We begin with a few key results which we will use all through our analysis.

Proposition 1. *For L and R, the price(s) for good k across the markets they own should satisfy the following (no-arbitrage) condition at any NE:*

$$L: \quad \frac{p_L}{p_R} = \sqrt{\frac{B_{-L,L}Q_{-L,R}(1+\vartheta_R(L)c)}{B_{-L,R}\left(Q_{-L,L} + c\int_{n\in N\setminus\{L\}} q_L(n)\vartheta_L(n)\mu(dn)\right)}}.$$

$$R: \quad \frac{p_L}{p_R} = \sqrt{\frac{B_{-R,L}\left(Q_{-R,R} + c\int_{n\in N\setminus\{R\}} q_R(n)\vartheta_R(n)\mu(dn)\right)}{B_{-R,R}Q_{-R,L}(1+\vartheta_L(R)c)}}.$$

Proposition 2. *For μ-a.e., $n \in [0,1]^2$, the price(s) for good k at the markets owned by L and R should satisfy the following (no-arbitrage) condition at any NE:*

$$\frac{p_L}{p_R} = \frac{1+\vartheta_R(n)c}{1+\vartheta_L(n)c}.$$

The proofs for the above results derive effortlessly from the relevant first-order conditions which are laid out in §3.1.1. We next present and prove a few lemmata which will be useful for constructing our examples and counterexamples. Not only do these lemmata embody further equilibrium properties of our model, but they also hold irrespective of L and R's locations—and therefore, throughout the rest of this paper. Of course, varying the locations of the trading posts is not innocuous and has important repercussions. These will be analysed in the corresponding subsections and fleshed out in the examples.

Lemma 1. *Consider two arbitrary sets $\xi, \sigma \in \mathcal{N}$ with $\mu(\xi \cap [0,1]^2) > 0$ and $\mu(\sigma \cap [0,1]^2) > 0$. For $i, j \in \{L, R\}, i \neq j$, define: (**a**) $z_i(n) < 0$ a.e. in ξ and $z_i(n) > 0$ a.e. in σ, and: (**b**) $z_j(n) > 0$ a.e. in ξ and $z_j(n) < 0$ a.e. in σ. There is no NE at which (**a**) and (**b**) hold simultaneously.*

Proof. W.l.o.g., let $i = L$ and $j = R$. If $z_L(n) < 0$ and $z_R(n) > 0$ $a.e.$ in ξ, then by Proposition 2, for $a.a.$ $n \in \xi$, we have $p_L < p_R$. Similarly, if $z_L(n) > 0$ and $z_R(n) < 0$ $a.e.$ in σ, then for $a.a.$ $n \in \sigma$, we have $p_L > p_R$. □

Lemma 2. *If μ-a.e., $n \in [0,1]^2$, makes non-zero net trades of the same direction across markets L and R, then at equilibrium, $p_L = p_R$. On the other hand, if μ-a.e., $n \in [0,1]^2$ makes opposing non-zero net trades across markets L and R, then at equilibrium, $p_L \neq p_R$.*

Proof. If net trades are of the same direction, then $\vartheta_L(n) = \vartheta_R(n)$ $a.e.$ in $[0,1]^2$. From Proposition 2, $p_L = p_R$. If net trades go in opposite directions, then w.l.o.g., let $\vartheta_R(n) = -1$, such that $\vartheta_L(n) = 1$. From Proposition 2, we have that $p_L < p_R$, given that $c > 0$. □

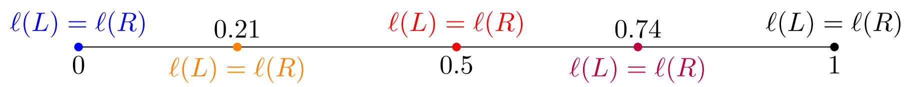
Lemma 3. *Consider two arbitrary sets $\xi, \sigma \in \mathcal{N}$ with $\mu(\xi \cap [0,1]^2) > 0$ and $\mu(\sigma \cap [0,1]^2) > 0$. For $i, j \in \{L, R\}, i \neq j$, if $z_i(n) < 0$ a.e. in ξ and $z_i(n) \geq 0$ a.e. in σ, then at any NE, we must have $z_j(n) < 0$ a.e. in ξ and $z_j(n) \geq 0$ a.e. in σ, such that $p_L = p_R$.*

Proof. Follows trivially from a combination of Lemmata 1 and 2. □

3.1 Parametrisation 1: L and R set up shop at the same location

In this section, we will assume that $\ell(L) = \ell(R)$ on the $[0, 1]$ interval. Where exactly on the $[0, 1]$ interval they are located is without import (see, e.g., Figure 2).

Figure 2: Examples of L and R at any, but the same, location



And indeed, in each of our examples that will follow, L and R's location will be left unspecified as this will not matter for the purposes of our analysis. Our examples are valid for when both L and R operate at 0 on the abscissa, when both operate at 1, or anywhere in between. Importantly, the transportation cost incurred by any $n \in N$ by travelling to trade at ***one*** post is the *same* as the travelling cost incurred to trade at ***both*** posts. This introduces a minimal number of points of disconvexity in agents' holdings surfaces, thus rendering our analysis more tractable. Nonetheless, this means that contrary to Toraubally (2018, 2019, 2022b), the first-order conditions that we derive here are only necessary but not sufficient. This is not to say that no further analysis is possible. We will show, by examining the geometry of the problem at hand,[8] how this technical obstacle can be circumvented to derive NE of Γ.

3.1.1 Good k trades at different prices across both platforms

The utility function for each $n \in [0, 1]^2$, is:

$$U_n(x(n)) = 2900 \ln x_k(n) + 2100 \ln x_m(n),$$

and the utility functions for L and R are as follows:

$$U_L(x(L)) = 302 \ln x_k(L) + 99 \ln x_m(L),$$
$$U_R(x(R)) = 1856 \ln x_k(R) + 2055 \ln x_m(R).$$

Put $c = \frac{1}{3}$. For μ-$a.e.$, $n \in N$, the endowments are $e_k(n) = 100$, and $e_m(n) = 100 + \frac{\delta_L(n)}{100}$. We take the function $v(\cdot)$ to be such that $v(\delta_i(n)) = \delta_i(n) = |\ell(n) - \ell(i)|$, $i = L, R$, and we let $t = \frac{1}{100}$.

At an NE, the first-order necessary (but not) sufficient conditions for μ-$a.e.$, $n \in [0, 1]^2$,

[8]This method is extremely helpful for simplifying complex-looking optimisation programmes. For another example where a geometric method has been employed, see Toraubally (2022a, 2023).

are:

$$\frac{\partial U_n(x(n))/\partial x_k(n)}{\partial U_n(x(n))/\partial x_m(n)} = \frac{B_L}{Q_L}(1+\vartheta_L(n)c).$$
$$\frac{\partial U_n(x(n))/\partial x_k(n)}{\partial U_n(x(n))/\partial x_m(n)} = \frac{B_R}{Q_R}(1+\vartheta_R(n)c).$$

For L, the first-order necessary conditions at an NE are:

$$\frac{\partial U_L(x(L))/\partial x_k(L)}{\partial U_L(x(L))/\partial x_m(L)} = \left(\frac{B_L}{Q_L}\right)^2 \cdot \frac{Q_{-L,L} + c\int_{n\in N\setminus\{L\}} q_L(n)\vartheta_L(n)\mu(dn)}{B_{-L,L}}.$$
$$\frac{\partial U_L(x(L))/\partial x_k(L)}{\partial U_L(x(L))/\partial x_m(L)} = \left(\frac{B_R}{Q_R}\right)^2 \cdot \frac{Q_{-L,R}(1+\vartheta_R(L)c)}{B_{-L,R}}.$$

Likewise, for R, the first-order necessary conditions at an NE are:

$$\frac{\partial U_R(x(R))/\partial x_k(R)}{\partial U_R(x(R))/\partial x_m(R)} = \left(\frac{B_L}{Q_L}\right)^2 \cdot \frac{Q_{-R,L}(1+\vartheta_L(R)c)}{B_{-R,L}}.$$
$$\frac{\partial U_R(x(R))/\partial x_k(R)}{\partial U_R(x(R))/\partial x_m(R)} = \left(\frac{B_R}{Q_R}\right)^2 \cdot \frac{Q_{-R,R} + c\int_{n\in N\setminus\{R\}} q_R(n)\vartheta_R(n)\mu(dn)}{B_{-R,R}}.$$

It can be verified, in light of the above, that the following profile of strategies constitutes a candidate NE of Γ:

$$\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) = (3,3,3,0) \ \ a.e. \text{ in } [0,1]^2;$$
$$\big(b_L(L), b_R(L), q_L(L), q_R(L)\big) = \left(7, 28, 1, \tfrac{124}{5}\right);$$
$$\big(b_L(R), b_R(R), q_L(R), q_R(R)\big) = \left(2, \tfrac{1}{5}, 2, \tfrac{32}{5}\right),$$

with $p_L = 2 \neq 1 = p_R$. The final allocations of k and m to each $n \in N$ are as follows:

$$\big(x_k(n), x_m(n)\big) = \left(\tfrac{203}{2}, 98\right) \ \ a.e. \text{ in } [0,1]^2;$$
$$\big(x_k(L), x_m(L)\big) = \left(\tfrac{1057}{10}, \tfrac{462}{5}\right);$$
$$\big(x_k(R), x_m(R)\big) = \left(\tfrac{464}{5}, \tfrac{548}{5}\right).$$

Discussion of example and the spatial market game mechanism

A few points are in order. First, to alleviate the skeptical reader's concern, a quick check shows that for each $n \in N$, $U_n(x_k(n), x_m(n)) > U_n(e_k(n), e_m(n))$, such that the strategy profile we have derived above constitutes a Pareto improvement upon autarky and is therefore an NE (see Appendix B). We remind the reader, in passing, that autarky is itself always a trivial NE of Γ (see, e.g., Shapley and Shubik, 1977). Prima facie, the fact that every agent can be made better off even after incurring a cost of travel, which effectively involves some of commodity m being wasted, may appear counterintuitive. However, this is reminiscent of the manifestation that in pure exchange economies, it is possible to Pareto-improve on the endowments by wasting some a good inasmuch as what is wasted is not too large and what remains of the endowments is allocated in a way that is close to efficient (see, e.g., Kreps, 2012). Second, both platforms for trade to take place at are located in one and the same spot. As previously remarked, this means they are a priori exactly identical, and their products perfect substitutes, to each $n \in [0,1]^2$. Third, note that no agent is financially constrained at equilibrium. Certainly, it is intriguing that even though agents still have plenty of money left to spend, and plenty of goods left to offer for sale, the situation derived above is still sustainable as an equilibrium. In other words, absolutely ***no*** agent, large or small, has any incentive to unilaterally deviate. Fourth, the market is covered: every agent engages in trade of some sort, with some being net sellers and others, of course, net buyers. Fifth, it is interesting to recall that in conventional spatial models, consumers strictly prefer to, and actually do, trade at one post only. The only outcome that never materialises is consumers trading at both platforms. Yet, in this example, *every* agent (including the post owners themselves), trades at both locations. Last but not least, even though both platforms are identical in every respect, the Law of One Price fails at equilibrium! While surprising in its own right, this also constitutes a substantial departure from traditional treatments, in which the only possible outcome if both firms were to locate at the same place is for both platforms to take prices down to marginal cost—i.e., the Bertrand paradox obtains. Hence, the model we put forth constitutes another solution to the Bertrand paradigm.

We now provide more intuition to some of the above-mentioned points. We start by noting that the market is covered because $a.a.$ agents $n \in [0,1]^2$ travel to the trading posts thanks to the cost of travel not being prohibitively high. What exactly counts as "prohibitively high" in our framework is a relative—rather than absolute—concept which depends on agents' preferences. Now, when both posts begin business in the same spot, their products are perfect substitutes for each other. Yet, the Bertrand paradox does not occur because it is not only the large players (atoms), but the small ones as well that affect prices—though not individually so. Above all, in our model, the only way for *any* trade at all to take place at a platform is for the price there to be positive. Next, to see why the failure of the LOOP is supportable as an equilibrium, it suffices to look at the allocation rules in (1) and (2) together with Propositions 1 and 2. The large players, should they unilaterally deviate, would change the prices unfavourably against them. For example, since L also buys and sells at R's platform, it is tempting to think that he could do better by only selling at his post, and buying only from platform R. Rejigging his bids and offers in this way increases (decreases) the aggregate bids at R's (L's) post, increases the aggregate offers of k at L's (R's) post, and therefore drives up (down) the price at R's (L's) post. The end result is L being able to buy less of k as its price soars on R's platform, and L's revenue from sales simultaneously falling due to k selling at a lower price on platform L. At best therefore, L can only break even. The small agents influence none of the aggregate bids, offers, and hence, price. Yet there exists no profitable deviation for them either. Consider any small agent n, whose equilibrium strategy is $s(n) = \big(b_L(n), b_R(n), q_L(n), q_R(n)\big) = (3,3,3,0)$. Reallocating, for example, all her bids to platform R and her offers to L's post seems to be the most intuitive strategy to adopt, i.e., $\tilde{s}(n) = \left(\tilde{b}_L(n), \tilde{b}_R(n), \tilde{q}_L(n), \tilde{q}_R(n)\right) = (0, 3+3, 3, 0)$. Ignoring all else for now, this indeed allows n to achieve a net trade of k of $(6/1) - 3 = 3$, and a net trade of m of $(3 \times 2) - (3+3) = 0$, as opposed to an initial net trade of k of $(3/1) + (3/2) - 3 = 3/2$, and a net trade of m of $(3 \times 2) - (3+3) = 0$. However, what this reallocation does is increase the total premium payable by n from $\frac{(3\times2-3)+3}{3} = 2$ previously, to $\frac{6+(3\times2)}{3} = 4$ now. Quickly plugging the new resulting allocations into the utility function for n shows that $\tilde{s}(n)$ in fact yields a lower level of utility than $s(n)$. Hence, no small agent moves.

While the different parametrisations we will analyse in the ensuing sections will have many distinct subtleties that will need to be addressed, the foregoing equilibrating mechanism operates in every other example which will follow in this paper. Below, we sketch a scenario in which the LOOP obtains at equilibrium.

3.1.2 Good k trades at the same price across both platforms

The utility function for each $n \in [0,1]^2$ is:

$$U_n(x(n)) = 361 \ln x_k(n) + 60 \ln x_m(n),$$

and the utility functions for L and R are as follows:

$$\begin{aligned} U_L(x(L)) &= 361 \ln x_k(L) + 50 \ln x_m(L), \\ U_R(x(R)) &= 10108 \ln x_k(R) + 375 \ln x_m(R). \end{aligned}$$

Put the service charge $c = \frac{1}{20}$. Let $v(\delta_i(n)) \coloneqq \delta_i(n) = \left|\ell(n) - \ell(i)\right|, i = L, R$, and $t = \frac{1}{100}$. Agents' endowments are as shown below:

$$\begin{aligned} \left(e_k(n), e_m(n)\right) &= \left(\tfrac{5869}{57}, \tfrac{11789}{150} + \tfrac{\delta_L(n)}{100}\right) \ a.e. \text{ in } [0,1]^2; \\ \left(e_k(L), e_m(L)\right) &= \left(\tfrac{1904}{19}, \tfrac{4817}{50}\right); \\ \left(e_k(R), e_m(R)\right) &= \left(\tfrac{5519}{57}, \tfrac{1876}{15}\right). \end{aligned}$$

Using the very same first-order conditions derived in §3.2.1, it can be verified that the following profile of strategies constitutes a candidate NE of Γ:

$$\begin{aligned} \left(b_L(n), b_R(n), q_L(n), q_R(n)\right) &= \left(3, 1, 3, \tfrac{28}{57}\right) \ a.e. \text{ in } [0,1]^2; \\ \left(b_L(L), b_R(L), q_L(L), q_R(L)\right) &= (6, 0, 1, 0); \\ \left(b_L(R), b_R(R), q_L(R), q_R(R)\right) &= \left(\tfrac{183}{5}, 3, 2, \tfrac{2}{57}\right), \end{aligned}$$

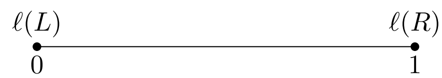
with $p_L = p_R = \frac{38}{5}$. The final allocations of k and m for μ-$a.e.$, $n \in N$, are $x_k(n) = x_m(n) = 100$.

Since $U_n(x_k(n), x_m(n)) > U_n(e_k(n), e_m(n))$ for each $n \in N$, the strategy profile we have derived above constitutes a Pareto improvement upon autarky and is an NE indeed.

3.2 Parametrisation 2: L and R at the opposite extremes of $[0, 1]$

In this subsection, we will assume that L is located at 0, and R at 1. In line with the traditional Hotelling (1929) model, this implies maximal product differentiation given $t > 0$. In Hotelling (1929), maximal product differentiation, alongside transportation cost and covered markets, is synonymous with the ***unique*** equilibrium outcome that prices, as well as demand functions, be the ***same*** across the two platforms at which good k is traded. Firms make supernormal profits because they compete less fiercely for the same market segment. Each firm has some local monopoly power and is able to extract more rent from neighbouring consumers who become more captive simply because it costs more to travel to the farther side of the city.

Figure 3: L and R at opposite extremes



Provided the LOOP prevails, perhaps unsurprisingly, a conclusion similar to the one in the conventional model holds true for Γ as well. And surely, as opposed to trading at both platforms, the individually negligible consumers, as do the post owners themselves, each trade on one platform only. Moreover, if prices are the same across both posts, then the indifferent consumer(s) will lie exactly in the middle of the city. We illustrate this with our first example.

3.2.1 Good k trades at the same price across both platforms

The utility function for each $n \in [0, 1]^2$ is:

$$U_n(x(n)) = 361 \ln x_k(n) + 60 \ln x_m(n),$$

and the utility functions for L and R are as follows:

$$U_L(x(L)) = 361 \ln x_k(L) + 50 \ln x_m(L),$$
$$U_R(x(R)) = 10108 \ln x_k(R) + 375 \ln x_m(R).$$

Put the service charge $c = \frac{1}{20}$. Let $v(\delta_i(n)) \coloneqq \delta_i(n) = |\ell(n) - \ell(i)|\,, i = L, R$, and $t = \frac{1}{100}$. Agents' endowments are as shown below:

$$\big(e_k(n), e_m(n)\big) = \left(\tfrac{1999}{19}, \tfrac{3119}{50} + \tfrac{\delta_L(n)}{100}\right) \ a.e. \text{ in } \left[0, \tfrac{1}{2}\right] \times [0, 1];$$
$$\big(e_k(n), e_m(n)\big) = \left(\tfrac{5741}{57}, \tfrac{14221}{150} + \tfrac{\delta_R(n)}{100}\right) \ a.e. \text{ in } \left(\tfrac{1}{2}, 1\right] \times [0, 1];$$
$$\big(e_k(L), e_m(L)\big) = \left(\tfrac{1904}{19}, \tfrac{4817}{50}\right);$$
$$\big(e_k(R), e_m(R)\big) = \left(\tfrac{5519}{57}, \tfrac{37523}{300}\right).$$

Using the very same first-order conditions derived in §3.2.1, it can be verified that the following profile of strategies constitutes a candidate NE of Γ:

$$\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) = (6,0,6,0) \;\; a.e. \text{ in } \left[0, \tfrac{1}{2}\right] \times [0,1];$$
$$\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) = \left(0,2,0,\tfrac{56}{57}\right) \;\; a.e. \text{ in } \left(\tfrac{1}{2}, 1\right] \times [0,1];$$
$$\big(b_L(L), b_R(L), q_L(L), q_R(L)\big) = (6,0,1,0)\,;$$
$$\big(b_L(R), b_R(R), q_L(R), q_R(R)\big) = \left(\tfrac{183}{5}, 3, 2, \tfrac{2}{57}\right),$$

with $p_L = p_R = \frac{38}{5}$. The final allocations of k and m for each $n \in N$, are $x_k(n) = x_m(n) = 100$.

We note that $U_n(x_k(n), x_m(n)) > U_n(e_k(n), e_m(n))$ for each $n \in N$, i.e., the strategy profile we have derived above constitutes a Pareto improvement upon autarky. However, satisfying this criterion alone is no longer sufficient for a strategy profile to constitute an NE of Γ. We must now also verify, for L and R, whether the allocations derived above are such that $U_i(x_k(i), x_m(i)) > U_i(\tilde{x}_k(i), \tilde{x}_m(i))$, $i = L, R$, where $\tilde{x}_k(i)$ and $\tilde{x}_m(i)$ denote the (optimal) allocations of k and m had i chosen to trade at his post only (see Appendix B). For L, the answer is obvious given L trades at his post only anyway. So, given the strategies of all players other than R, we have that $(\tilde{x}_k(R), \tilde{x}_m(R)) = (97.198, 122.052)$.[9] A quick calculation reveals that $U_i(x_k(R), x_m(R)) > U_i(\tilde{x}_k(R), \tilde{x}_m(R))$ indeed, such the strategy profile we have derived above is an NE.

Remark 1. It is interesting to note that in this scenario, while the individually insignificant consumers trade at one post only, the large trader R trades, not only at his own platform, but also at L's post. This occurrence is down to two factors. For R, given the strategies of every $n \in N \setminus \{R\}$, R incurs, at worst, "nothing" to travel and trade at L's platform as the cost of travel is more than offset by the premia receivable by R. Additionally, by trading at L's post, the market is made thicker. And as discussed in Goenka et al. (1998) and Toraubally (2022a), thick-market equilibria can Pareto dominate thin-market ones.

Naturally, one is led to wonder whether other patterns of trade may arise at equilibrium when the two platform owners are located at the opposite ends of the city. As we will demonstrate through our next example, this is certainly possible.

3.2.2 Good k trades at different prices across both platforms

The utility function for each $n \in [0,1]^2$ is:

$$U_n(x(n)) = 4000 \ln x_k(n) + 3000 \ln x_m(n),$$

[9] The exact figures are $(\tilde{x}_k(R), \tilde{x}_m(R)) = \left(\frac{2656417031 - 5\sqrt{398564585689}}{27297376}, \frac{2622806689 - 133\sqrt{398564585689}}{20801250}\right)$.

and the utility functions for L and R are as follows:

$$U_L(x(L)) = 302 \ln x_k(L) + 99 \ln x_m(L),$$
$$U_R(x(R)) = 1856 \ln x_k(R) + 2055 \ln x_m(R).$$

Put $c = \frac{1}{3}$. For each $n \in N$, the endowments are:

$$\big(e_k(n), e_m(n)\big) = \left(103, 96 + \tfrac{\delta_L(n)}{100}\right) \ a.e. \text{ in } \left[0, \tfrac{1}{2}\right] \times [0,1];$$
$$\big(e_k(n), e_m(n)\big) = \left(94, 108 + \tfrac{\delta_R(n)}{100}\right) \ a.e. \text{ in } \left(\tfrac{1}{2}, 1\right] \times [0,1];$$
$$\big(e_k(i), e_m(i)\big) = \left(100, \tfrac{10001}{100}\right), \ i = L, R.$$

We take the function $v(\cdot)$ to be such that $v(\delta_i(n)) = \delta_i(n) = |\ell(n) - \ell(i)|$, $i = L, R$, and we let $t = \frac{1}{100}$.

It can be verified, in light of the above, that the following profile of strategies constitutes a candidate NE of Γ:

$$\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) = (6, 0, 6, 0) \ a.e. \text{ in } \left[0, \tfrac{1}{2}\right] \times [0,1];$$
$$\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) = (0, 6, 0, 0) \ a.e. \text{ in } \left(\tfrac{1}{2}, 1\right] \times [0,1];$$
$$\big(b_L(L), b_R(L), q_L(L), q_R(L)\big) = \left(7, 28, 1, \tfrac{124}{5}\right);$$
$$\big(b_L(R), b_R(R), q_L(R), q_R(R)\big) = \left(2, \tfrac{1}{5}, 2, \tfrac{32}{5}\right),$$

with $p_L = 2 \neq 1 = p_R$. The final allocations of k and m to each $n \in N$ are as follows:

$$\big(x_k(n), x_m(n)\big) = (100, 100) \ a.e. \text{ in } [0,1]^2;$$
$$\big(x_k(L), x_m(L)\big) = \left(\tfrac{1057}{10}, \tfrac{462}{5}\right);$$
$$\big(x_k(R), x_m(R)\big) = \left(\tfrac{464}{5}, \tfrac{548}{5}\right).$$

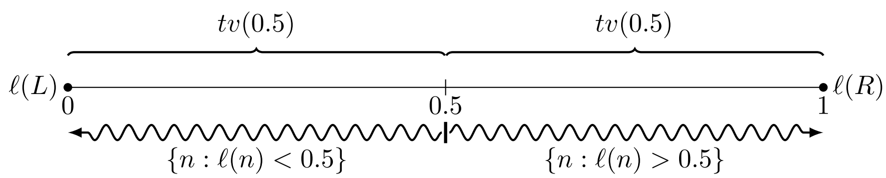
Observe that for each $n \in N$, $U_n(x_k(n), x_m(n)) > U_n(e_k(n), e_m(n))$. As before, we must also verify, for L and R, whether the allocations derived above are such that $U_i(x_k(i), x_m(i)) > U_i(\tilde{x}_k(i), \tilde{x}_m(i))$, $i = L, R$, where $\tilde{x}_k(i)$ and $\tilde{x}_m(i)$ denote the (optimal) allocations of k and m had i chosen to trade at his post only. Given the strategies of all players other than L, we have that $(\tilde{x}_k(L), \tilde{x}_m(L)) = (102.59, 96.43)$. Similarly, given the strategies of all players other than R, we have that $(\tilde{x}_k(R), \tilde{x}_m(R)) = (93.53, 108.56)$.[10] A quick calculation reveals that $U_i(x_k(i), x_m(i)) > U_i(\tilde{x}_k(i), \tilde{x}_m(i))$, $i = L, R$, indeed, such the strategy profile we have derived above is an NE.

The two preceding examples illustrate a very important fact, which we will next formalise. Before doing so, we must derive the following crucial intermediate lemma:

Lemma 4. *Let the* ***equilibrium*** *bids and offers of a.a.* $n \in N$ *be given, and assume that* $p_L < p_R$.

[10] The exact figures are $(\tilde{x}_k(L), \tilde{x}_m(L)) = \left(\frac{70185545 - 100\sqrt{248415610}}{668779}, \frac{9222391 - 40\sqrt{248415610}}{89100}\right)$ and $(\tilde{x}_k(R), \tilde{x}_m(R)) = \left(\frac{1539430351 - 155\sqrt{6308360349385}}{12296870}, \frac{677765297 - 62\sqrt{6308360349385}}{4808700}\right)$

Figure 4: L and R at opposite extremes, with $p_L = p_R$



If for any $n \in [0,1]^2$, $\sum_{i=L}^{R} (b_i(n) + q_i(n)) > 0$, *and* $\sum_{i=L}^{R} z_i(n)/p_i > 0$, *then excluding transportation costs, there exists* $q_R^*(n) > 0$ *(with* $b_R^*(n) = b_L^*(n) = q_L^*(n) = 0$*) which yields the same allocations of* k *and* m*. Conversely, if for any* $n \in [0,1]^2$, $\sum_{i=L}^{R} (b_i(n) + q_i(n)) > 0$, *and* $\sum_{i=L}^{R} z_i(n)/p_i < 0$, *then excluding transportation costs, there exists* $b_L^*(n) > 0$ *(with* $q_L^*(n) = q_R^*(n) = b_R^*(n) = 0$*) which yields the same allocations of* k *and* m*.*

Proof. See Appendix A. □

Thanks to the above lemma, we are now able to derive the following sharp result:

Proposition 3. *When* $\ell(L) = 0$ *and* $\ell(R) = 1$*, any* $n \in [0,1]^2$ *for whom* $\sum_{i=L}^{R} (b_i(n) + q_i(n)) > 0$ *trades at one post only at any NE.*

Proof. Suppose a contradiction: that $\bigtimes_{i=L}^{R} (b_i(n) + q_i(n)) > 0$, and consider the case when $p_L \neq p_R$. For clarity and w.l.o.g., let $p_L < p_R$. Then, by Lemmata 1, 2 and 3, and our claim, we see that at equilibrium, the only way for this to happen is to have μ-$a.e.$, $n \in [0,1]^2$, $z_L(n) < 0$ and $z_R(n) > 0$. But if this is the case, any $n \in [0,1]^2$ for whom $\sum_{i=L}^{R} (b_i(n)/p_i - q_i(n)) > 0$ can profitably deviate by choosing to trade at L only, while any $n \in [0,1]^2$ for whom $\sum_{i=L}^{R} (b_i(n)/p_i - q_i(n)) < 0$ has an incentive to deviate to trade at R only. This is because by Lemma 4, they can play $(b_i^*(n), q_i^*(n))_{i=L}^{R}$ to get the same allocations, but only incur $tv(\delta_i(n))$ instead of $tv(\delta(n)) = tv(\delta_i(n)) + tv(\delta_j(n))$, $i, j = L, R$. Hence, the initial situation cannot have been an NE.

Now, assume instead that $p_L = p_R$. In this case, there is nothing to be gained for μ-$a.e.$, $n \in [0,1]^2$, by trading at ***both*** posts and incurring $tv(\delta(n))$. To prove this claim, consider $a.a.$ $n \in [0,1]^2$ for whom $\ell(n) = 0.5$. The same bid and offer yield similar amounts of both goods k and m across either post, and cost the same in terms of $\mathscr{P}_i(n)$. In other words, the agents for whom $\ell(n) = 0.5$ are ***indifferent*** between trading at L and R. Thus, by travelling to either post L or post R but not both, they incur only $tv(0.5)$ as opposed to $tv(1)$—see Figure 4. It follows that $a.a.$ $n \in [0,1]^2$ for whom $\ell(n) < 0.5$ will therefore choose to go to market L, while $a.a.$ $n \in [0,1]^2$ for whom $\ell(n) > 0.5$ will go to market R. □

It would indubitably be more interesting and helpful if something could be said about which post agents $n \in [0,1]^2$ actually buy from and sell at when prices are unequal across platforms at equilibrium. The next theorem, which concisely captures four different cases, does precisely this.

Theorem 1. *Let $p_L < p_R$ at equilibrium. Consider any $n \in [0,1]^2$ for whom $\sum_{i=L}^{R}(b_i(n) + q_i(n)) > 0$. If $\sum_{i=L}^{R} z_i(n)/p_i \gtreqless 0$, then as long as $(p_R - p_L)\sum_{i=L}^{R} z_i(n)/p_i \gtreqless tv(\delta_R(n))$, n trades at R only. However, if $(p_R - p_L)\sum_{i=L}^{R} z_i(n)/p_i \lesseqgtr tv(\delta_R(n))$, then n trades at L only.*

Proof. n as hereby described trades at one post only at equilibrium by Proposition 3. Now, assume by way of contradiction, that at some equilibrium, $(p_R - p_L)\sum_{i=L}^{R} z_i(n)/p_i > tv(\delta_R(n))$, and n trades at L only such that $z_L(n)/p_L > 0$. As n sells more of k than s/he buys, n can simply choose $q_R^*(n) = q_L(n) - b_L(n)/p_L$ and $b_R^*(n) = b_L^*(n) = q_L^*(n) = 0$, which results in the same net trade of k. However, because $(p_R - p_L)\, z_L(n)/p_L > tv(\delta_R(n))$, we have that $q_R^*(n)\,(p_R - p_L) - tv(\delta_R(n)) > 0$. In other words, a profitable unilateral deviation exists. Hence, n in fact trades at R only at equilibrium. This same method can be used to prove the claim for the reverse inequality. □

Below we record a simple but very important point which shaped our unequal-price example in §3.2.2, and which will again come into play in the next section.

Corollary 1. *Let the bids and offers of a.a. $n \in N$ be given, and assume w.l.o.g., that $p_L < p_R$. For any $n \in [0,1]^2$ for whom $\ell(n) \leq \ell(L) < \ell(R)$, if at equilibrium $\sum_{i=L}^{R}(b_i(n) + q_i(n)) > 0$, and $\sum_{i=L}^{R} z_i(n)/p_i < 0$, then $b_R(n) = q_R(n) = 0$. Likewise, for any $n \in [0,1]^2$ for whom $\ell(L) < \ell(R) \leq \ell(n)$, if at equilibrium $\sum_{i=L}^{R}(b_i(n) + q_i(n)) > 0$, and $\sum_{i=L}^{R} z_i(n)/p_i > 0$, then $b_L(n) = q_L(n) = 0$.*

Proof. Suppose a contradiction: that for any $n \in [0,1]^2$ for whom $\ell(n) \leq \ell(L) < \ell(R)$ and $\sum_{i=L}^{R} z_i(n)/p_i < 0$, $\bigtimes_{i=L}^{R}(b_i(n) + q_i(n)) > 0$ at equilibrium. By Lemma 4, there exists $b_L^*(n) > 0$ (with $q_L^*(n) = q_R^*(n) = b_R^*(n) = 0$) which yields the same allocations of k and m, excluding transportation costs. But by trading at L only, n incurs $tv(\delta_L(n))$ only instead of $tv(\delta(n)) = tv(\delta_L(n)) + tv(\delta_R(L))$, i.e., a profitable deviation exists. A similar argument can be used to prove the second part of our claim. □

At this point, an extremely handy result by Koutsougeras (2003) comes to mind. This result holds in Toraubally (2018, 2019, 2022b), but not in the present paper. We explain why below.

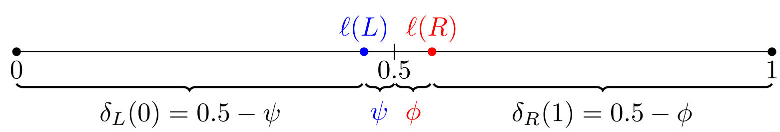
Remark 2. Fact 2 of Koutsougeras (2003) states the following, mutatis mutandis. Consider any feasible strategy profile $(b,q) \in Gr(S)$, where, say, $b_R(n) = q_R(n) = 0$ for some $n \in N$, and $p_R > 0$. Then, μ-$a.e.$ in N, there is a budget feasible $(\hat{b}(n), \hat{q}(n))$ with $\hat{b}_R(n) \cdot \hat{q}_R(n) > 0$, which results in the same net trades and clearing price. In our paper, with the exception of L and R at their own trading posts, this does not (always) hold: if, genuinely, $b_i(n) = q_i(n) = 0$ for some $n \in [0,1]^2$, then assuming that $b_i(n) \cdot q_i(n) > 0$ means that agent n must have travelled to post i, $i = L, R$ and also incurred $tv(\delta_i(n))$. The only time this would work is if $\ell(L) = \ell(R)$ **and** $b_j(n) + q_j(n) > 0$, $i \neq j$. Whenever $\ell(L) \neq \ell(R)$, this result cannot be applied—again, except for L and R at their own trading posts.

3.3 Parametrisation 3: L and R close to the centre

In this section, we analyse a setup à la d'Aspremont et al. (1979), where L and R are closely spaced, and near the centre. d'Aspremont et al.'s (1979) main contribution was the derivation of necessary and sufficient conditions on the two firms' locations for an equilibrium to exist—which, when it does exist, is unique for any *given* pair of locations. When both firms are located in the same spot, they make normal profits at the unique equilibrium. Amongst others, and of particular interest is their finding that a Cournot equilibrium fails to exist when the two firms are relatively close to each other. They show that this non-existence problem can be circumvented if instead of linear transportation costs, costs were assumed to vary quadratically per unit of distance.

We will show in our context, irrespective of the platforms' locations, i.e., no matter how closely spaced they are to each other, and regardless of the functional form of $v(\cdot)$—linear or quadratic—that we can derive an equilibrium at which trade takes place at both platforms, with price being positive. Moreover, even in this context, it is still possible for the LOOP to fail.

Figure 5: L and R close to the centre, at different locations



Proposition 4. *When $\ell(L) = \frac{1}{2} - \psi$ and $\ell(R) = \frac{1}{2} + \phi$, $0 \leq \psi, \phi \leq \frac{1}{2}$, any $n \in [0,1]^2$ for whom $\ell(L) \leq \ell(n) \leq \ell(R)$ and $\sum_{i=L}^{R} (b_i(n) + q_i(n)) > 0$ trades at one post only at any NE. Moreover, if $p_L = p_R$ at equilibrium, then a.a. $n \in [0,1]^2$ for whom $\ell(n) = \frac{1}{2} + \frac{\phi - \psi}{2}$ are indifferent between trading at L and R.*

Proof. Similar to the proof of Proposition 3. □

Remark 3. Each of Lemma 4, Propositions 1 and 2, Theorem 1, and Corollary 1 holds for the current parametrisation as well.

As mentioned in the Introduction, Shy (2022) uses the undercut-proof equilibrium solution concept to derive equilibrium prices for all possible firm locations on the $[0,1]$ interval. This is because a pure-strategy Nash equilibrium does not exist in roughly 85 percent of all possible firm locations. In §3.1, we parametrised our model such that L and R located in the same spot, no matter where along the linear city. In §3.2, we imposed that L and R set up shop at the extremes of the city. In the next two examples, we will present pure-strategy NE at which L locates anywhere along $\left[0, \frac{1}{2}\right]$ and R locates at any point in the $\left[\frac{1}{2}, 1\right]$ interval. We spell out three of the (uncountably) many possibilities which these NE admit: (i) L and R both locate exactly in the centre; (ii) L and R close to the centre but at different locations, and; (iii) only one of L or R at the centre, while the other begins business close by. Primarily, our next examples are general enough such that once

$\ell(L) = \frac{1}{2} - \psi$ and $\ell(R) = \frac{1}{2} + \phi$ have been chosen, agents' endowments can be straightforwardly modified accordingly such that a pure-strategy NE exists for ***all*** possible firm locations.

3.3.1 Good k trades at the same price across both platforms

In this example, to illustrate that an equal-price NE exists no matter how close (or how far) L and R are to the centre, we will take $\psi, \phi \in \left[0, \frac{1}{2}\right]$ but let these remain arbitrary. As with our previous examples, we will use linear transportation costs, which in the current context are notoriously problematic in traditional models.

Thanks to Propositions 3 and 4, and Theorem 1, we can just slightly modify the example in §3.2.1, and the conclusions will remain unaffected.
The utility function for each $n \in [0,1]^2$ is:

$$U_n(x(n)) = 361 \ln x_k(n) + 60 \ln x_m(n),$$

and the utility functions for L and R are as follows:

$$\begin{aligned} U_L(x(L)) &= 361 \ln x_k(L) + 50 \ln x_m(L), \\ U_R(x(R)) &= 10108 \ln x_k(R) + 375 \ln x_m(R). \end{aligned}$$

Put the service charge $c = \frac{1}{20}$. Let $v(\delta_i(n)) := \delta_i(n) = |\ell(n) - \ell(i)|\,, i = L, R$, and $t = \frac{1}{100}$. Agents' endowments are as shown below:

$$\begin{aligned} \big(e_k(n), e_m(n)\big) &= \left(\tfrac{1999}{19}, \tfrac{3119}{50} + \tfrac{\delta_L(n)}{100}\right) \ a.e. \text{ in } \left[0, \tfrac{1}{2} + \tfrac{\phi-\psi}{2}\right] \times [0,1]; \\ \big(e_k(n), e_m(n)\big) &= \left(\tfrac{5741}{57}, \tfrac{14221}{150} + \tfrac{\delta_R(n)}{100}\right) \ a.e. \text{ in } \left(\tfrac{1}{2} + \tfrac{\phi-\psi}{2}, 1\right] \times [0,1]; \\ \big(e_k(L), e_m(L)\big) &= \left(\tfrac{1904}{19}, \tfrac{4817}{50}\right); \\ \big(e_k(R), e_m(R)\big) &= \left(\tfrac{5519}{57}, \tfrac{1876}{15} + \tfrac{\delta_L(R)}{100}\right). \end{aligned}$$

Using the very same first-order conditions derived in §3.2.1, it can be verified that the following profile of strategies constitutes a candidate NE of Γ:

$$\begin{aligned} \big(b_L(n), b_R(n), q_L(n), q_R(n)\big) &= (6,0,6,0) \ a.e. \text{ in } \left[0, \tfrac{1}{2} + \tfrac{\phi-\psi}{2}\right] \times [0,1]; \\ \big(b_L(n), b_R(n), q_L(n), q_R(n)\big) &= \left(0,2,0,\tfrac{56}{57}\right) \ a.e. \text{ in } \left(\tfrac{1}{2} + \tfrac{\phi-\psi}{2}, 1\right] \times [0,1]; \\ \big(b_L(L), b_R(L), q_L(L), q_R(L)\big) &= (6,0,1,0); \\ \big(b_L(R), b_R(R), q_L(R), q_R(R)\big) &= \left(\tfrac{183}{5}, 3, 2, \tfrac{2}{57}\right), \end{aligned}$$

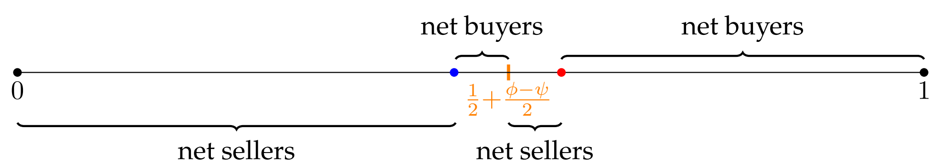
with $p_L = p_R = \frac{38}{5}$. The final allocations of k and m for μ-$a.e.$, $n \in N$, are $x_k(n) = x_m(n) = 100$.

Notice that for each $n \in N$, $U_n(x_k(n), x_m(n)) > U_n(e_k(n), e_m(n))$. We also have to verify, for L and R, whether the allocations derived above are such that $U_i(x_k(i), x_m(i)) >$

$U_i(\tilde{x}_k(i), \tilde{x}_m(i))$, $i = L, R$, where $\tilde{x}_k(i)$ and $\tilde{x}_m(i)$ are as defined previously. For L, no further analysis is needed given L trades at his post only at our candidate NE. So we may move on to R. A difficulty here is that the precise locations of L and R are unknown, such that we cannot derive explicit values for $\tilde{x}_k(i)$ and $\tilde{x}_m(i)$, $i = L, R$, as we did in §3.2.1. However, a workaround is at hand. If we take $e_m(R) = \frac{1876}{15}$, then we would have the optimal values of $\tilde{x}_k(R)$ and $\tilde{x}_m(R)$ to be $(\check{\tilde{x}}_k(R), \check{\tilde{x}}_m(R)) = (97.198, 122.042)$.[11] On the other hand, if $e_m(R) = \frac{37523}{300} = \frac{1876}{15} + \frac{1}{100}$, then given the strategies of all players other than R, we would have the optimal values of $\tilde{x}_k(R)$ and $\tilde{x}_m(R)$ to be $(\hat{\tilde{x}}_k(R), \hat{\tilde{x}}_m(R)) = (97.198, 122.052)$. Since both $\tilde{x}_k(R)$ and $\tilde{x}_m(R)$ are increasing in $e_m(R)$, it follows that $\check{\tilde{x}}_\alpha(R) < \tilde{x}_\alpha(R) < \hat{\tilde{x}}_\alpha(R)$, $\alpha = k, m$. As $U(\cdot)$ is increasing in both x_k and x_m, we thus know that $U_R(x_k(R), x_m(R)) > U_R(\hat{\tilde{x}}_k(R), \hat{\tilde{x}}_k(R)) > U_R(\tilde{x}_k(R), \tilde{x}_m(R))$ indeed, such that the strategy profile we have derived above is an NE.

3.3.2 Good k trades at different prices across both platforms

Figure 6: L (blue) and R (red) arbitrarily close to the centre



In this example, we will create an equilibrium situation where different segments of the linear city make net purchases or net sales as described in Figure 6 above.

Let $\psi, \phi \in \left[0, \frac{1}{2}\right]$. The utility function for each $n \in [0,1]^2$ is:

$$U_n(x(n)) = 4000 \ln x_k(n) + 3000 \ln x_m(n),$$

and the utility functions for L and R are as follows:

$$U_L(x(L)) = 302 \ln x_k(L) + 99 \ln x_m(L),$$
$$U_R(x(R)) = 1856 \ln x_k(R) + 2055 \ln x_m(R).$$

Put $c = \frac{1}{3}$. For each $n \in N$, the endowments are:

[11]The exact figures are $(\check{\tilde{x}}_k(R), \check{\tilde{x}}_m(R)) = \left(\frac{531241249-\sqrt{398532967789}}{5459042}, \frac{5245197353-266\sqrt{398532967789}}{41602500}\right)$. $\hat{\tilde{x}}_\alpha(R)$, $\alpha = k, m$, are as defined in §3.2.1.

$$
\begin{aligned}
\big(e_k(n), e_m(n)\big) &= \Big(100 + \tfrac{3}{1+\phi-\psi}, 100 - \tfrac{4}{1+\phi-\psi} + \tfrac{\delta_L(n)}{100}\Big) \ a.e. \text{ in } \big[0, \tfrac{1}{2} - \psi\big] \times [0,1];\\
\big(e_k(n), e_m(n)\big) &= \Big(100 - \tfrac{6}{1+\psi-\phi}, 100 + \tfrac{8}{1+\psi-\phi} + \tfrac{\delta_R(n)}{100}\Big) \ a.e. \text{ in } \Big(\tfrac{1}{2} - \psi, \tfrac{1}{2} + \tfrac{\phi-\psi}{2}\Big] \times [0,1];\\
\big(e_k(n), e_m(n)\big) &= \Big(100 + \tfrac{3}{1+\phi-\psi}, 100 - \tfrac{4}{1+\phi-\psi} + \tfrac{\delta_L(n)}{100}\Big) \ a.e. \text{ in } \Big(\tfrac{1}{2} + \tfrac{\phi-\psi}{2}, \tfrac{1}{2} + \phi\Big] \times [0,1];\\
\big(e_k(n), e_m(n)\big) &= \Big(100 - \tfrac{6}{1+\psi-\phi}, 100 + \tfrac{8}{1+\psi-\phi} + \tfrac{\delta_R(n)}{100}\Big) \ a.e. \text{ in } \big(\tfrac{1}{2} + \phi, 1\big] \times [0,1];\\
\big(e_k(L), e_m(L)\big) &= \Big(100, 100 + \tfrac{\delta_R(n)}{100}\Big);\\
\big(e_k(R), e_m(R)\big) &= \Big(100, 100 + \tfrac{\delta_L(n)}{100}\Big).
\end{aligned}
$$

We take the function $v(\cdot)$ to be such that $v(\delta_i(n)) = \delta_i(n) = |\ell(n) - \ell(i)|$, $i = L, R$, and we let $t = \frac{1}{100}$.

It can be verified, in light of the above, that the following profile of strategies constitutes a candidate NE of Γ:

$$
\begin{aligned}
\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) &= \Big(\tfrac{6}{1+\phi-\psi}, 0, \tfrac{6}{1+\phi-\psi}, 0\Big) \ a.e. \text{ in } \big[0, \tfrac{1}{2} - \psi\big] \times [0,1];\\
\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) &= \Big(0, \tfrac{6}{1+\psi-\phi}, 0, 0\Big) \ a.e. \text{ in } \Big(\tfrac{1}{2} - \psi, \tfrac{1}{2} + \tfrac{\phi-\psi}{2}\Big] \times [0,1];\\
\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) &= \Big(\tfrac{6}{1+\phi-\psi}, 0, \tfrac{6}{1+\phi-\psi}, 0\Big) \ a.e. \text{ in } \Big(\tfrac{1}{2} + \tfrac{\phi-\psi}{2}, \tfrac{1}{2} + \phi\Big] \times [0,1];\\
\big(b_L(n), b_R(n), q_L(n), q_R(n)\big) &= \Big(0, \tfrac{6}{1+\psi-\phi}, 0, 0\Big) \ a.e. \text{ in } \big(\tfrac{1}{2} + \phi, 1\big] \times [0,1];\\
\big(b_L(L), b_R(L), q_L(L), q_R(L)\big) &= \big(7, 28, 1, \tfrac{124}{5}\big);\\
\big(b_L(R), b_R(R), q_L(R), q_R(R)\big) &= \big(2, \tfrac{1}{5}, 2, \tfrac{32}{5}\big),
\end{aligned}
$$

with $p_L = 2 \neq 1 = p_R$. The final allocations of k and m to each $n \in N$ are as follows:

$$
\begin{aligned}
\big(x_k(n), x_m(n)\big) &= (100, 100) \ a.e. \text{ in } [0,1]^2;\\
\big(x_k(L), x_m(L)\big) &= \big(\tfrac{1057}{10}, \tfrac{462}{5}\big);\\
\big(x_k(R), x_m(R)\big) &= \big(\tfrac{464}{5}, \tfrac{548}{5}\big).
\end{aligned}
$$

Our strategy profile above is such that for each $n \in N, U_n(x_k(n), x_m(n)) > U_n(e_k(n), e_m(n))$. We next verify whether the allocations derived above are such that $U_i(x_k(i), x_m(i)) > U_i(\tilde{x}_k(i), \tilde{x}_m(i))$, $i = L, R$, where $\tilde{x}_k(i)$ and $\tilde{x}_m(i)$ are as previously defined. Since the precise locations of L and R are unknown, we cannot derive explicit values for $\tilde{x}_k(i)$ and $\tilde{x}_m(i)$, $i = L, R$. We will therefore employ the same approach as in our previous example. If we take $e_m(L) = 100$, then we would have the optimal values of $\tilde{x}_k(L)$ and $\tilde{x}_m(L)$ to be $(\check{\tilde{x}}_k(L), \check{\tilde{x}}_m(L)) = (102.589, 96.42)$.[12] On the other hand, if $e_m(L) = \frac{10001}{100}$, then given the strategies of all players other than L, we would have the optimal values of $\tilde{x}_k(L)$ and $\tilde{x}_m(L)$ to be $(\hat{\tilde{x}}_k(L), \hat{\tilde{x}}_m(L)) = (102.59, 96.43)$. Since both $\tilde{x}_k(i)$ and $\tilde{x}_m(i)$ are increasing

[12] The exact figures are $(\check{\tilde{x}}_k(L), \check{\tilde{x}}_m(L)) = \left(\frac{1965005-5\sqrt{77895601}}{18724}, \frac{645505-200\sqrt{77895601}}{6237}\right)$ and $(\check{\tilde{x}}_k(R), \check{\tilde{x}}_m(R)) = \left(\frac{61572871-31\sqrt{252316564081}}{491840}, \frac{67771721-31\sqrt{252316564081}}{480870}\right)$. $\hat{\tilde{x}}_\alpha(i)$, $i = L, R$, $\alpha = k, m$, are as defined in §3.2.2.

in $e_m(i)$, $i = L, R$, it follows that $\check{\tilde{x}}_\alpha(L) < \tilde{x}_\alpha(L) < \hat{\tilde{x}}_\alpha(L)$, $\alpha = k, m$. Likewise, given the strategies of all players other than R, we have that $(\check{\tilde{x}}_k(R), \check{\tilde{x}}_m(R)) = (93.5288, 108.553)$, while $(\hat{\tilde{x}}_k(R), \hat{\tilde{x}}_m(R)) = (93.5299, 108.562)$, such that $\check{\tilde{x}}_\alpha(R) < \tilde{x}_\alpha(R) < \hat{\tilde{x}}_\alpha(R)$, $\alpha = k, m$. As $U(\cdot)$ is increasing in both x_k and x_m, we thus know that $U_i(x_k(i), x_m(i)) > U_i(\hat{\tilde{x}}_k(i), \hat{\tilde{x}}_k(i)) > U_i(\tilde{x}_k(i), \tilde{x}_m(i))$, $i = L, R$, indeed, such the strategy profile we have derived above is an NE.

4 Conclusion

In this paper, we have presented a quantity-competition model of spatial product differentiation. Prices are determined endogenously and explicitly by agents' buy and sell decisions. Our model crytallises the fact that in spatial models, irrespective of the cardinality of the set of agents, strategic behaviour persists and is tremendously impactful. Verily, we have shown that even in the presence of individually insignificant agents, it is possible to find open sets of economies in which the LOOP obtains at equilibrium, and others in which it fails, no matter where the two trading posts are located. Amongst others, this finding makes our framework appealing because the latter thus constitutes an intuitive and simple-to-understand solution to the Bertrand paradox. Crucially, in what is perhaps our most important and intriguing result, we have demonstrated that even when the two markets are closely spaced, non-trivial pure-strategy Nash equilibria can be found. This represents a major departure from current models.

Admittedly, our model is cast in a static framework. In future research, it would be interesting to see how the setup hereby considered can be extended to a sequential game format, in which firms would choose their locations first, and then trade accordingly. Another compelling proposition would be to analyse if and how our conclusions would change if we were to consider more than two trading posts.

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Appendix A

Available from the author upon request

Appendix B: The holdings surface

Available from the author upon request

Appendix C: Robustness of examples

Available from the author upon request