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Two Preliminary Examples of Induced Contextual Boolean Algebras

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Quantum Measurement Trees, I: Two Preliminary Examples of Induced Contextual Boolean Algebras

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Abstract:

Quantum randomness evidently transcends the classical framework of random variables defined on a single comprehensive Kolmogorov probability space. One prominent example is the quantum double-slit experiment due to Feynman (1951, 1966). A related non-quantum example, inspired by Boole (1862) and Vorob′ev (1962), has three two-valued random variables X, Y and Z, where the pairs X, Y and X, Z are perfectly correlated, yet Y, Z are perfectly anti-correlated. Such examples can be accommodated using a "multi-measurable" space with several different σ-algebras of measurable events. This concept due to Vorob′ev (1962) allows construction of: 1) a measurable metaspace whose elements combine a point in the original sample space with a variable "contextual" Boolean algebra; 2) a parametric family of probability metaspaces, each of which is a Kolmogorov probability space that represents a two-stage stochastic process where a random choice from the original sample space is preceded by the random choice of a contextual Boolean algebra in the multi-measurable space. Subsequent work will explore how quantum experimental results can be described using a quantum measurement tree with one or more preparation nodes where an experimental configuration is determined that governs the probability distribution of relevant quantum observables. [197 words]

Keywords: Quantum measurement tree, quantum challenge, double-slit experiment, quantum contexts, multi-measurable space, measurable metaspace, multi-probability space, probability metaspace.

1 Introduction and Outline

1.1 Quantum Measurement Trees

Zermelo (1913) introduced a mathematical model of two-person zero-sum games like chess in which players choose a sequences of alternating moves. Von Neumann (1928) extended this model to general n-person games in extensive form, including those with incomplete information and even imperfect recall. Raiffa (1968) then considered decision trees, which are one-person games with complete information and perfect recall. His trees allowed, in addition to decision nodes, chance nodes with what Anscombe and Aumann (1963) described as "roulette lotteries" having specified "objective" or hypothetical probabilities.

The work on the normative decision theory set out in Hammond (1988, 2022) considers finite decision trees that have, in addition to decision and chance nodes, what Anscombe and Aumann (1963) described as "horse lotteries" where, as in Ramsey (1926), de Finetti (1937) and Savage (1954), any probabilities attached to different outcomes would have to be "subjective" or personal. Also, decision trees could end in consequences belonging to a general domain rather than the "pecuniary" consequences or payoffs that Raiffa (1968) considered.

This is the first paper describing a project devoted to quantum measurement trees. These have the same mathematical structure as decision trees. The difference is in how one interprets different nodes of the tree. Indeed, the early part of the project will consider trees with a special structure where each path through the tree has a series of three successive nodes: (i) first, an initial preparation node where an experimental configuration or context is determined; (ii) second, a measurement node where a roulette lottery, with specified classical probabilities that depend on the context, determines the random outcome of an experiment; (iii) third, a terminal node where whatever measurement emerges from the previous roulette lottery is determined, which can be observed provided that a suitable detector is set up.

In the quantum measurement trees that this project considers, all likelihoods correspond to probabilities in the classical sense due to Kolmogorov (1933), This is true whether, using again the terminology due to Anscombe and Aumann (1963), the likelihoods apply to roulette lotteries with objective probabilities, or to horse lotteries with subjective probabilities. Then question that this project addresses concerns the extent to which the mathematical structure of a quantum measurement tree can succeed in describing the distribution of possible results from actual laboratory experiments that can be observed, provided a suitable detector is made available.

1.2 The Quantum Challenge

There is, of course, a significant quantum challenge to this research programme. Indeed, it seems that a consensus view among both physicists and philosophers is that many quantum phenomena are so weird that they somehow transcend the usual laws of logic and probability which are embodied in Kolmogorov's (1933) standard definition of a probability space. Indeed, Birkhoff and von Neumann (1936) helped create a discipline of *quantum logic* that departs from the classical logic developed by Boole (1854) and various successors. In particular, the works by Suppes (1961, 1966, 1976), Jauch and Piron (1969), Suppes and Zanotti (1974, 1997), and many others, discuss how, even when the probabilities $\mathbb{P}(E)$ and $\mathbb{P}(E')$ of the two events E and E' are both well defined, the probability $\mathbb{P}(E \cup E')$ of their union may not be.

As for quantum probability, it may not satisfy the usual additivity condition because, even when the two events E and E' are disjoint, and the three probabilities $\mathbb{P}(E)$, $\mathbb{P}(E')$, and $\mathbb{P}(E \cup E')$ are all well defined, they may not satisfy $\mathbb{P}(E \cup E') = \mathbb{P}(E) + \mathbb{P}(E')$. It is in this sense that Feynman (1951, p. 533) is correct in writing:

> ...far more fundamental was the discovery that in nature the laws of combining probabilities were not those of the classical probability theory ...you may be delighted to learn that Nature with her infinite imagination has found another set of principles for determining probabilities; a set ...which nevertheless does not lead to logical inconsistencies.
>
> What is changed, and changed radically, is the method of calculating probabilities.

Indeed, it turns out that the right method of calculating quantum probabilities typically requires using matrix algebra, if not infinite-dimensional linear operators.

An associated difficulty in relating quantum probabilities to classical probabilities concerns Heisenberg's "uncertainty principle", which Feynman (1951, p. 538, footnote 1) relates to the double-slit experiment that is the main subject of his paper. In rather crude form, the uncertainty principle states that it is impossible to measure both the position and momentum of a single particle at the same time. Indeed, whereas it may be possible to model both position and momentum as random variables, each with its own well-defined distribution, the joint distribution of this or any other pair of quantum random variables or measurements may well not be meaningful.

1.3 The Mystery of Feynman's Double-Slit Experiment

In 1801 Thomas Young had used a celebrated double-slit experiment to demonstrate the wave nature of light.[1] Feynman (1951) wrote at length about a quantum version of this experiment. Later, in their famous series of published lectures, Feynman *et al.* (1964, pp. 1–2) justified this choice with the claim:

> We choose to examine a phenomenon which is impossible, *absolutely* impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the *only* mystery.

Like Young, the experiment involves a beam emanating from a point source. The beam impacts first a front plane surface which contains two pinholes or slits, here labelled L for left and R for right. Behind the first surface is a second parallel back plane surface which detects any part of the beam that was not stopped by the first surface. Let $f_{LR}(x, y)$ denote the intensity of the part of the beam which is detected at the point with co-ordinates (x, y) on this second surface. Young observed that $f_{LR}(x, y)$ can be represented as the result of an interference pattern between the two waves that pass through L and R.

Whereas Young worked with what appeared to be a constant and continuous light beam, the quantum version hypothesized by Feynman (1951) involves a beam of discrete electrons, though it could also be other subatomic particles. With both slits open, the arrival of individual particles detected at the second screen can be described as arising from a probability distribution whose density function is essentially the same function $f_{LR}(x, y)$ as in Young's experiment. There are also the two corresponding density functions $f_L(x, y)$ and $f_R(x, y)$ that arise when only one slit remains open, which is known to be either L or R. Alternatively, the experiment may be modified by adding a detector whose effect is to reveal which slit the particle has passed through.

In a classical world, when both slits are open, suppose there are conditional probabilities π_L and π_R that a particle will be detected at the back screen given that it has passed through slits L and R of the front screen

[1] See Young (1802) for his own account of the experiment.

respectively. Then the expected density function for particles reaching the back screen would be $\pi_L f_L(x,y) + \pi_R f_R(x,y)$, a convex combination of the two density functions $f_L(x,y)$ and $f_R(x,y)$.

Indeed, suppose that the equation

$$\pi_L f_L(x,y) + \pi_R f_R(x,y) = f_{LR}(x,y) \tag{1}$$

were observed to hold. Then the probabilities π_L and π_R could be inferred. But (**??**) is manifestly contradicted by, amongst other things, the empirical observation that there exists an open set of values of positions (x,y) on the back screen at which $f_{LR}(x,y) > \max\{f_L(x,y), f_R(x,y)\}$.[2] The inference generally drawn is that this observed interference effect contradicts the laws of probability. Indeed, the conditional probabilities π_L and π_R seem to be not even defined.

1.4 An Approach to Meeting the Quantum Challenge

This paper is the first of a series resulting from a project intended to set out in detail the argument that these and several other significant quantum challenges can be met without going beyond the Kolmogorov's (1933) classical laws of probability. The key idea is to recognize the relevance of ideas that, following the contributions of Khrennikov (2003, 2004), Spekkens (2005), and others, might be called "contextuality". Indeed, when restricted to data arising from particular important quantum experiments, the construction bears considerable resemblance to that used by Khrennikov (2014, 2015) and earlier by Avis et al. (2009), who write as follows in their introduction:[3]

> We recall that the use of a single probability space for statistical data collected with respect to a few different experimental contexts is *not a custom of probability theory.*
>
> ...
>
> If one wants to apply the classical probabilistic model, a *single Kolmogorov probability space*, then random experiments ... should be unified in a single random experiment in an intelligent way.

[2] See, for example, Tavabi *et al.* (2019) for descriptions of some recent implementations of Feynman's thought experiment.

[3] We also mention subsequent work by Dzhafarov and Kujala (2016) which recognizes the importance of contextual measurability when considering quantum random variables, but is less constructive than the preceding work by Khrennikov and his associates. For further discussion see also the interchange between Dzhafarov and Kon (2016, 2019) and Khrennikov (2019).

In an attempt to meet this aim as far as poosible, this project on quantum measurement trees adopts a "qualified" Kolmogorovian approach which is based on a principle of "contextuality", or even "contextuality by default".[4] Specifically, we recognize that

> any two random variables recorded under mutually exclusive conditions are stochastically unrelated, defined on different sample spaces (Dzhafarov and Kujala, 2016, p. 202)

Indeed, the view taken in this work is that such unrelated random variables are defined on different σ-algebras, which determine different measurable spaces, even though they may be defined on the same sample space. This opens the door for the approach used here based on quantum measurement trees, which allow one to construct a special extended kind of classical probability space that we call a probability "metaspace".[5]

This metaspace reflects the structure of an underlying quantum measurement tree. It has an expanded sample space whose members each include not just the usual random state of the world, but also a random context, in the form of a Boolean algebra, or more generally a σ-algebra, of measurable events which is determined at the initial preparation node of the tree. Indeed, my argument will be that the quantum challenge only arises because of insufficient recognition that in order for classical probabilities to describe quantum phenomena, they can only apply in the context of a specific Boolean algebra — or in an infinite-dimensional space, a specific σ-algebra. This context, moreover, typically depends on key details of the quantum experiment whose random results are being described. For example, in the case of Heisenberg's uncertainty principle, an experimental configuration that allows the position of a particle at any time t to be measured is inconsistent with a configuration that allows momentum to be measured. In particular, there is no experimental configuration associated with just one σ-algebra that makes both position and momentum simultaneously measurable functions of the relevant quantum state. Equivalently, there is no physically feasible quantum measurement tree with a measurement node at which both position and momentum get measured simultaneously.

[4] See also Dzhafarov and Kujala (2014a).

[5] Note that both Khrennikov (2014) and Dzhafarov and Kujala (2014b) have used the term "Kolmogorovization", though for a somewhat different construction.

1.5 Outline of Paper

After this introductory section, Section **??** introduces a key example based on one introduced by Vorob′ev (1962, p. 147) that was motivated by his work on what correlated strategies may be available in a coalition game with 3 or more players. The example has three dichotomous random variables X, Y, Z, of which the two pairs X, Y and X, Z are both perfectly correlated, yet the third pair Y, Z is perfectly anti-correlated. Evidently these weird correlations are logically impossible in case all three random variables are defined on whatever single probability space is used in a vain attempt to describe the outcomes of all three random variables simultaneously. Nevertheless the weird correlations can be modelled in a measurement tree with paths that are selected from a probability "metaspace". This metaspace includes each of the three different pairs of random variables, along with an associated Boolean algebra, as a random context that is selected at the initial "preparation node" of the tree.

Next, Section **??** revisits the example of Feynman's quantum double-slit experiment which was briefly introduced in Section **??**. It turns out that the results of this particular experiment can be described by using an especially simple measurement tree. The associated probability metaspace is built up to include three different contextual σ-algebras, each corresponding to a different non-empty subset of open slits.

The final Section **??** offers a brief concluding summary and disclaimer.

2 A Simple Example of Weird Correlations

2.1 Vorob′ev's Example

Given an arbitrary three-element set S, Boole (1854, 1862) gives inequalities that must be satisfied in order that probabilities defined on singleton and pair subsets of S allow consistency with a single probability distribution defined on all the eight subsets of S — see also Pitowsky (1994). This raises the possibility that examples could violate these inequalities.

The example we are about to present, which has the same mathematical structure as that in the opening section of Vorob′ev (1962), is somewhat more extreme than such violations of the Boolean inequalities. We give a homely version of this example involving three siblings Xavier, Yvonne, and Zoë, indicated by X, Y and Z. All three are keen supporters of the same local sports team. An unfortunate shortage of season tickets, however, leaves them unable to buy more than two for adjacent seats. This makes it impossible for all three to watch any home game sitting together. So the three of them take it in turn at any home game either for one of them to sit far away from the other two, or to miss the game altogether.

Whichever two siblings sit together are observed to wear clearly identifiable colours that are either Red (R) or Blue (B). Yvonne and Zoë are identical twins. If Xavier sits with one of his sisters, he will wear whatever colour that sister has chosen that day. But if the twins sit together, they will wear different colours that allow them to be told apart.[6]

2.2 Stochastic Representation

Observations (x, y, z) of the three sibling's choices of red or blue clothing can all be accommodated within a single sample space which, using obvious notation, can be expressed as the triple Cartesian product

$$\Omega = \{R_X, B_X\} \times \{R_Y, B_Y\} \times \{R_Z, B_Z\} \tag{2}$$

Not all components of Ω can be observed simultaneously, however. Indeed, we cannot observe what colour the excluded sibling would have chosen if it had been possible to attend a particular match and sit next to the other two. So, to represent what can be observed, define

$$C := \{XY, XZ, YZ\} \tag{3}$$

as the set of all possible *contexts*, each of which takes the form of a pair of siblings selected from $\{X, Y, Z\}$. Then, for each context $c \in C$, the observable part of the sample space is given by the relevant contextual subproduct space Ω_c in the collection

$$\begin{aligned} \Omega_{XY} &= \{R_X, B_X\} \times \{R_Y, B_Y\} \\ \Omega_{XZ} &= \{R_X, B_X\} \times \{R_Z, B_Z\} \\ \Omega_{YZ} &= \{R_Y, B_Y\} \times \{R_Z, B_Z\} \end{aligned} \tag{4}$$

Suppose that when the context is $c \in C$, the probability of the observed pair of colours in the relevant space Ω_c specified by (**??**) is given by a contextual probability mass function π_c. Then the correlations reported at the end of Section **??** occur if and only if there exist three constants $\alpha, \beta, \gamma \in [0, 1]$ such that, for each $c \in C$, the relevant contextual probability mass function $\Omega_c \ni \omega_c \mapsto \pi_c(\omega_c) \in [0, 1]$ is as specified in Table **??**.

[6] This is not necessarily realistic because it is often said that identical twins enjoy wearing matching clothing intended to make it hard to tell them apart.

Note that the two pairs of random variables specified by the two contextual mappings $\omega_{XY} \mapsto (x(\omega_{XY}), y(\omega_{XY}))$ and $\omega_{XZ} \mapsto (x(\omega_{XZ}), y(\omega_{XZ}))$ are both perfectly correlated. If there were a single probability mass function π_{XYZ} on the space Ω of triples (x, y, z) defined by (**??**), these two perfect correlations would imply that the pair (y, z) is also perfectly correlated. Yet this would contradict the specification of π_{YZ} in the third part of Table **??**. This contradiction highlights the need for an enriched probability model if one is to describe the entire pattern of observations specified in Table **??**.

π_{XY}	R_Y	B_Y
R_X	α	0
B_X	0	$1-\alpha$

π_{XZ}	R_Z	B_Z
R_X	β	0
B_X	0	$1-\beta$

π_{YZ}	R_Z	B_Z
R_Y	0	γ
B_Y	$1-\gamma$	0

Table 1: Table of three contextual joint probability mass functions

2.3 Classical Probability

Recall that, following Kolmogorov (1933), a *probability measure* on the sample space Ω is a function $\mathcal{A} \ni E \mapsto \mathbb{P}(E) \in [0, 1]$ for which:

- the domain of definition is a *σ-algebra* $\mathcal{A}$ on Ω, which is a family of subsets of Ω having the three properties: (i) $\Omega \in \mathcal{A}$; (ii) if $E \in \mathcal{A}$, then $\Omega \setminus E \in \mathcal{A}$; (iii) the union of any countable indexed family $\{E_i \mid i \in I\}$ of sets in $\mathcal{A}$ satisfies $\cup_{i \in I} E_i \in \mathcal{A}$.

- the function $\mathcal{A} \ni E \mapsto \mathbb{P}(E) \in [0, 1]$ satisfies: (i) $\mathbb{P}(\Omega) = 1$; and (ii) the *countable additivity* condition stating that, for every countable indexed family of sets $\{E_i \mid i \in I\}$ in $\mathcal{A}$ that is pairwise disjoint, one has $\mathbb{P}(\cup_{i \in I} E_i) = \sum_{i \in I} \mathbb{P}(E_i)$.

Also, if $\mathcal{A}$ is a σ-algebra on Ω, then the pair $(\Omega, \mathcal{A})$ is a *measurable space.* And if $\mathcal{A} \ni E \mapsto \mathbb{P}(E) \in [0, 1]$ is a probability measure on the measurable space $(\Omega, \mathcal{A})$, then the triple $(\Omega, \mathcal{A}, \mathbb{P})$ is a *probability space.*

The following well known result is invoked later:

Lemma 2.1. *If $\mathcal{A}$ is a σ-algebra on Ω, then the intersection of any countable family $\{E_i \mid i \in I\}$ of sets in $\mathcal{A}$ satisfies $\cap_{i \in I} E_i \in \mathcal{A}$.*

Proof. Suppose that $E_i \in \mathcal{A}$ for each $i \in I$. By definition of σ-algebra, it follows that $\Omega \setminus E_i \in \mathcal{A}$ for each $i \in I$, and then that $\cup_{i \in I}(\Omega \setminus E_i) \in \mathcal{A}$. But de Morgan's Law implies that $\Omega \setminus \cap_{i \in I} E_i = \cup_{i \in I}(\Omega \setminus E_i)$, and so $\Omega \setminus \cap_{i \in I} E_i \in \mathcal{A}$.

Finally, because $\cap_{i\in I} E_i = \Omega \setminus (\Omega \setminus \cap_{i\in I} E_i)$, the definition of σ-algebra implies that $\cap_{i\in I} E_i \in \mathcal{A}$. □

2.4 Contextual σ-Algebras in a Multi-Measurable Space

Kolmogorov's classical definitions set out in Section **??** were extended by Vorob′ev (1962, p. 154) to allow a *generalized measurable space* in which the unique σ-algebra $\mathcal{A}$ is replaced by "some system Σ of σ-algebras". Motivated by the discussion of Section **??**, we introduce the following definition:

Definition 2.2. *Given the sample space* Ω *and the arbitrary set* C *of* contexts*:*

1. *the collection* $(\Omega, (\mathcal{A}_c)_{c\in C})$ *is a* multi-measurable space *just in case, for each context* $c \in C$*, the family* $\mathcal{A}_c$ *of events is a* contextual *σ-algebra on* Ω*.*

2. *the collection* $(\Omega, (\mathcal{A}_c, \mathbb{P}_c)_{c\in C})$ *is a* multi-probability space *just in case, for each context* $c \in C$*, the triple* $(\Omega, \mathcal{A}_c, \mathbb{P}_c)$ *is a* contextual *probability space.*

In the formulation of Vorob′ev's example set out in Section **??**, the set C consists of three possible contexts described by (**??**). We will now specify explicitly the three associated contextual σ-algebras $\mathcal{A}_c$ on which the joint probabilities set out in Table **??** are defined. Then we will also give obvious specifications of the three contextual probability measures $\mathbb{P}_c$.

Consider first the case when the context c is the pair XY. Note that the Cartesian product set Ω_{XY} defined in (**??**) has 4 members. Now we postulate that each non-empty set in the σ-algebra $\mathcal{A}_{XY}$ takes the form $E_{XY} \times \{R_Z, B_Z\}$, where E_{XY} is any one of the 9 non-empty subsets of Ω_{XY}. In particular, note that for all pairs $(x, y) \in \Omega_{XY}$ and all events or measurable sets $E \in \mathcal{A}_{XY}$, one has

$$(x, y, R_Z) \in E \Longleftrightarrow (x, y, B_Z) \in E$$

This is because observing the colour choices of only Xavier and Yvonne allows nothing to be inferred about what Zoë's choice would have been if she had also been able to attend the game. As for the probability measure $\mathbb{P}_{XY}$ on $\mathcal{A}_{XY}$, given any non-empty subset E_{XY} of the Cartesian product set Ω_{XY}, we specify that

$$\mathbb{P}_{XY}(E_{XY} \times \{R_Z, B_Z\}) = \pi_{XY}(E_{XY})$$

where $\pi_{XY}(E_{XY})$ is calculated in the obvious way from the leftmost part of Table **??**.

Similarly, in the case when the context c is the pair XZ, each non-empty set in the σ-algebra $\mathcal{A}_{XZ}$ takes the form $E_{XZ} \times \{R_Y, B_Y\}$, where E_{XZ} is any non-empty subset of the set Ω_{XZ} defined in (**??**). The probability of this set is then given by

$$\mathbb{P}_{XZ}(E_{XZ} \times \{R_Y, B_Y\}) = \pi_{XZ}(E_{XZ})$$

where $\pi_{XZ}(E_{XZ})$ is calculated in the obvious way from the middle part of Table **??**.

Finally, when $c = YZ$, each non-empty set in the σ-algebra $\mathcal{A}_{YZ}$ takes the form $E_{YZ} \times \{R_X, B_X\}$, where E_{YZ} is any non-empty subset of the set Ω_{YZ} defined in (**??**). The probability of this set is then given by

$$\mathbb{P}_{YZ}(E_{YZ} \times \{R_X, B_X\}) = \pi_{YZ}(E_{YZ})$$

where $\pi_{YZ}(E_{YZ})$ is calculated in the obvious way from the rightmost part of Table **??**.

2.5 From Multi-Probability Space to Measurement Tree

The main claim to be examined in this research project is that some probabilistic phenomena, such as those that occur in mathematical models of quantum experiments can be accommodated after all within a classical probability model, even though that space may have to include within it multiple contextual probability spaces. So far, we have only established that the multi-probability space $(\Omega, (\mathcal{A}_c, \mathbb{P}_c)_{c \in C})$ may offer an adequate probabilistic description of Vorob′ev's example. It remains to show how the family of different contextual probability spaces $(\Omega, \mathcal{A}_c, \mathbb{P}_c)$ in the multi-probability space model can all be assembled into one classical probability space of paths through a "measurement tree". The resulting space will be called a "probability metaspace", denoted by $(\Omega^M, \mathcal{A}^M, \mathbb{P}^M)$. Or rather, we will construct a parametric family $(\Omega^M, \mathcal{A}^M, \mathbb{P}^M_q)$ of probability metaspaces, where the parameter q indicates a probability distribution over set $\{\mathcal{A}_c\}_{c \in C}$ of three contextual σ-algebras.

The construction leads to the measurement tree illustrated in Figure **??**. Each path through this tree results from a two-stage stochastic process. The first stage involves a preparation node where a chance move determines which of the three possible contexts $c \in C$ occurs with the specified probability q_c. Then, at each subsequent measurement node, depending on the

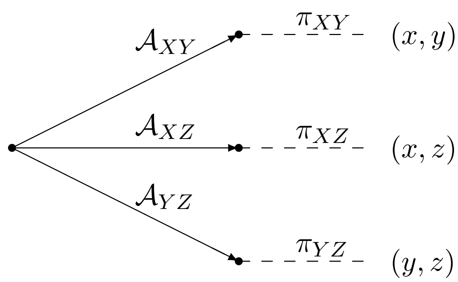

Figure 1: The process implied by the multi-probability space

context c that emerged randomly at the initial preparation node, a second stage lottery determines randomly one pair of colours according to the probability mass function π_c. The overall result is described by the probability metaspace $(\Omega^M, \mathcal{A}^M, \mathbb{P}_q^M)$ consisting of:

1. the augmented sample space defined by

$$\Omega^M := \Omega \times \{\mathcal{A}_c\}_{c \in C} \tag{5}$$

whose typical member $(x, y, z, \mathcal{A})$ combines a triple of colours $(x, y, z) \in \Omega$ with one contextual σ-algebra $\mathcal{A}$ chosen from the collection $\{\mathcal{A}_c\}_{c \in C}$;

2. the augmented σ-algebra $\mathcal{A}^M$ on Ω^M defined by

$$\mathcal{A}^M := \left\{ \bigcup\nolimits_{c \in C} (E_c \times \{\mathcal{A}_c\}) \mid (E_c)_{c \in C} \in \prod\nolimits_{c \in C} \mathcal{A}_c \right\} \tag{6}$$

whose typical member takes the form of the union $\bigcup_{c \in C} (E_c \times \{\mathcal{A}_c\})$ of three possibly empty sets $E_c \times \{\mathcal{A}_c\}$ that, for each context $c \in C$, is the Cartesian product of an $\mathcal{A}_c$-measurable set $E_c \subseteq \Omega$ with the singleton set $\{\mathcal{A}_c\}$ whose only member is the contextual σ-algebra $\mathcal{A}_c$;

3. for each probability distribution $q \in \Delta(C)$ over the set C of three possible contexts, or equivalently, over the corresponding collection $\{\mathcal{A}_c\}_{c \in C}$ of three possible contextual σ-algebras, the probability measure $\mathbb{P}_q^M$ on the measurable space $(\Omega^M, \mathcal{A}^M)$ that is defined for each set $\bigcup_{c \in C} (E_c \times \{\mathcal{A}_c\}) \in \mathcal{A}^M$, as specified in (**??**), by

$$\mathbb{P}_q^M \left(\cup_{c \in C} (E_c \times \{\mathcal{A}_c\}) \right) = \sum\nolimits_{c \in C} q_c \, \pi_c(E_c) \tag{7}$$

Note that the lottery given by (**??**) is the result of compounding the lottery q over contexts with, for each fixed context $c \in C$, the contextual lottery π_c over the measurable space $(\Omega, \mathcal{A}_c)$.

2.6 A Consistent Multi-Probability Space

Following Vorob′ev (1962, p. 154), say that the multi-probability space $(\Omega, (\mathcal{A}_c, \mathbb{P}_c)_{c \in C})$ defined in Section **??** is *consistent* just in case, for every pair c, c' of contexts in C, one has

$$E \in \mathcal{A}_c \cap \mathcal{A}_{c'} \Longrightarrow \pi_c(E) = \pi_{c'}(E) \tag{8}$$

In this case there is a single function

$$\cup_{c \in C} \mathcal{A}_c \ni E \mapsto \pi(E) \in [0,1] \tag{9}$$

such that

$$E \in \mathcal{A}_c \Longrightarrow \pi_c(E) = \pi(E) \tag{10}$$

In general, however, the function π specified in (**??**) is not a probability measure. This is because only in case the two disjoint sets $E, E' \subseteq \Omega$ are both members of the same σ-algebra $\mathcal{A}_c$ does the definition of consistent multi-probability space guarantee that $\pi(E \cup E') = \pi(E) + \pi(E')$.

Consider the three contextual probability measures π_c $(c \in C)$ specified in Table **??** of Section **??**. For these, given any pair $c, c' \in C$, the consistency conditions (**??**) have force only for the those "marginal" subsets of the three-dimensional sample space Ω which are measurable w.r.t. both contextual σ-algebras $\mathcal{A}_c$ and $\mathcal{A}_{c'}$. Of these, the first two marginal subsets, which belong to both $\mathcal{A}_{XY}$ and $\mathcal{A}_{XZ}$, are

$$\begin{aligned} M_X(R_X) &:= \{R_X\} \times \{R_Y, B_Y\} \times \{R_Z, B_Z\} \\ M_X(B_X) &:= \{B_X\} \times \{R_Y, B_Y\} \times \{R_Z, B_Z\} \end{aligned}$$

For these two particular marginal sets and the joint probabilities specified in Table **??**, the consistency conditions (**??**) imply that

$$\begin{aligned} \pi_{XY}(M_X(R_X)) = \alpha &= \pi_{XZ}(M_X(R_X)) = \beta \\ \pi_{XY}(M_X(B_X)) = 1 - \alpha &= \pi_{XZ}(M_X(B_X)) = 1 - \beta \end{aligned}$$

Obviously these two equations are satisfied if and only if $\alpha = \beta$.

The corresponding equalities for the two other corresponding pairs of marginal subsets are

$$\begin{aligned} \pi_{XY}(M_Y(R_Y)) = \alpha &= \pi_{YZ}(M_Y(R_Y)) = \gamma \\ \pi_{XY}(M_Y(B_Y)) = 1 - \alpha &= \pi_{YZ}(M_Y(B_Y)) = 1 - \gamma \\ \pi_{XZ}(M_Z(R_Z)) = \beta &= \pi_{YZ}(M_Z(R_Z)) = 1 - \gamma \\ \pi_{XZ}(M_Z(B_Z)) = 1 - \beta &= \pi_{YZ}(M_Z(B_Z)) = \gamma \end{aligned}$$

The first four of these six equalities reduce to $\alpha = \beta = \gamma$ and the last two reduce to $\beta = 1 - \gamma$. Hence $\alpha = \beta = \gamma = \frac{1}{2}$. So consistency implies that the probabilities in Table **??** become those in Table **??**, as in the example on the first page of Vorob′ev (1962).

π_{XY}	R_Y	B_Y
R_X	$\frac{1}{2}$	0
B_X	0	$\frac{1}{2}$

π_{XZ}	R_Z	B_Z
R_X	$\frac{1}{2}$	0
B_X	0	$\frac{1}{2}$

π_{YZ}	R_Z	B_Z
R_Y	0	$\frac{1}{2}$
B_Y	$\frac{1}{2}$	0

Table 2: Three consistent contextual joint probability measures over pairs

2.7 Meta-random Variables

Consider the probability meta-space $(\Omega^M, \mathcal{A}^M, \mathbb{P}_q^M)$ defined by (**??**), (**??**), and (**??**). Given the domain Ω^M, which is the augmented sample space of this probability meta-space, consider the function

$$\Omega^M = \Omega \times \{\mathcal{A}_c\}_{c \in C} \ni (x, y, Z, \mathcal{A}) \mapsto \xi(x, y, Z, \mathcal{A}) = x \in \{R_X, B_X\} \quad (11)$$

whose value indicates Xavier's chosen colour x. When this colour is R_X, for example, the pre-image set satisfies

$$\xi^{-1}(\{R_X\}) = E_X(R_X) \times \{\mathcal{A}_c\}_{c \in C} = \cup_{c \in C}(E_X(R_X)) \times \{\mathcal{A}_c\}) \quad (12)$$

where

$$E_X(R_X) := \{R_X\} \times \{R_Y, B_Y\} \times \{R_Z, B_Z\} \quad (13)$$

Now, the σ-algebra $\mathcal{A}_{YZ}$ was defined in Section **??** so that each of its non-empty member sets takes the form of the Cartesian product $\{R_X, B_X\} \times \hat{E}_{YZ}$ for some non-empty set $\hat{E}_{YZ}$ of $\{R_Y, B_Y\} \times \{R_Z, B_Z\}$. But then (**??**) evidently implies that $E_X(R_X) \notin \mathcal{A}_{YZ}$. From this it follows that $\xi^{-1}(\{R_X\}) \notin \mathcal{A}^M$. The function $(x, y, Z, \mathcal{A}) \mapsto \xi(x, y, Z, \mathcal{A})$ specified in (**??**) is therefore not measurable. It follows that Xavier's chosen colour does not determine a properly defined meta-random variable on the probability meta-space.

To arrive at a more tractable model of the measurement process in which the colour choices are random variables, given any sibling $s \in \{X, Y, Z\}$, let us extend the set $\{R_s, B_s\}$ of possible colours to include the extra outcome U_s, which signifies that the sibling s's chosen colour is unobserved or even irrelevant. In the case of Xavier, for example, when $s = X$, this allows consideration of a modified function

$$\Omega^M = \Omega \times \{\mathcal{A}_c\}_{c \in C} \ni (x, y, z, \mathcal{A}) \mapsto \xi^M(x, y, z, \mathcal{A}) \in \{R_X, B_X, U_X\} \quad (14)$$

An obvious definition of this function is

$$\xi^M(x,y,z,\mathcal{A}) = \begin{cases} \xi(x,y,z,\mathcal{A}) & \text{if } \mathcal{A} = \mathcal{A}_{XY} \text{ or } \mathcal{A} = \mathcal{A}_{XZ} \\ U_X & \text{if } \mathcal{A} = \mathcal{A}_{YZ} \end{cases} \tag{15}$$

For Yvonne and Zoë, the corresponding modified functions

$$\Omega^M \ni (x,y,z,\mathcal{A}) \mapsto \begin{cases} \eta^M(x,y,z,\mathcal{A}) \in \{R_Y, B_Y, U_Y\} \\ \zeta^M(x,y,z,\mathcal{A}) \in \{R_Z, B_Z, U_Z\} \end{cases} \tag{16}$$

are defined by

$$\eta^M(x,y,z,\mathcal{A}) = \begin{cases} \eta(x,y,z,\mathcal{A}) & \text{if } \mathcal{A} = \mathcal{A}_{XY} \text{ or } \mathcal{A} = \mathcal{A}_{YZ} \\ U_Y & \text{if } \mathcal{A} = \mathcal{A}_{XZ} \end{cases} \tag{17}$$

$$\zeta^M(x,y,z,\mathcal{A}) = \begin{cases} \zeta(x,y,z,\mathcal{A}) & \text{if } \mathcal{A} = \mathcal{A}_{XZ} \text{ or } \mathcal{A} = \mathcal{A}_{YZ} \\ U_Z & \text{if } \mathcal{A} = \mathcal{A}_{XY} \end{cases} \tag{18}$$

Define the Cartesian product co-domain

$$\hat{\Omega} := \prod\nolimits_{s \in \{X,Y,Z\}} \{R_s, B_s, U_s\} \tag{19}$$

Then the three functions defined by (**??**), (**??**), and (**??**) determine one consolidated function

$$\Omega^M \ni (x,y,z,\mathcal{A}) \mapsto (\xi^M, \eta^M, \zeta^M)(x,y,z,\mathcal{A}) \in \hat{\Omega} \tag{20}$$

Proposition 2.3. *Given the the σ-algebra $\mathcal{A}^M$ of the probability metaspace $(\Omega^M, \mathcal{A}^M, \mathbb{P}_q^M)$ and the power set $2^{\hat{\Omega}}$ that consists of all subsets of the finite co-domain $\hat{\Omega}$, the function defined by* (**??**) *that maps the measurable space $(\Omega^M, \mathcal{A}^M)$ to the measurable space $(\hat{\Omega}, 2^{\hat{\Omega}})$ is measurable.*

Proof. For Xavier, instead of (**??**), and with $E_X(R_X)$ defined by (**??**), the relevant pre-image set becomes

$$\begin{aligned} (\xi^M)^{-1}(\{R_X\}) &= \cup_{c \in \{XY,XZ\}} (E_X(R_X) \times \{\mathcal{A}_c\}) \cup (\emptyset \times \{\mathcal{A}_{YZ}\}) \\ &= E_X(R_X) \times \{\mathcal{A}_{XY}, \mathcal{A}_{XZ}\} \\ &= \{R_X\} \times \{R_Y, B_Y\} \times \{R_Z, B_Z\} \times \{\mathcal{A}_{XY}, \mathcal{A}_{XZ}\} \end{aligned} \tag{21}$$

This preimage set is $\mathcal{A}^M$-measurable, as is

$$\begin{aligned} (\xi^M)^{-1}(\{B_X\}) &= E_X(B_X) \times \{\mathcal{A}_{XY}, \mathcal{A}_{XZ}\} \\ &= \{B_X\} \times \{R_Y, B_Y\} \times \{R_Z, B_Z\} \times \{\mathcal{A}_{XY}, \mathcal{A}_{XZ}\} \end{aligned} \tag{22}$$

and also

$$(\xi^M)^{-1}(\{U_X\}) = \cup_{c\in\{XY,XZ\}}(\emptyset \times \{\mathcal{A}_c\}) \cup (\Omega \times \{\mathcal{A}_{YZ}\} = \Omega \times \{\mathcal{A}_{YZ}\} \quad (23)$$

It follows that the function $(x, y, z, \mathcal{A}) \mapsto \xi^M(x, y, z, \mathcal{A}) \in \{R_X, B_X, U_X\}$ from the measurable space $(\Omega^M, \mathcal{A}^M)$ to the measurable space $(\hat{\Omega}, 2^{\hat{\Omega}})$ is measurable. Similar arguments apply to the two functions

$$(x, y, z, \mathcal{A}) \mapsto \eta^M(x, y, z, \mathcal{A}) \in \{R_Y, B_Y, U_Y\}$$
$$(x, y, z, \mathcal{A}) \mapsto \zeta^M(x, y, z, \mathcal{A}) \in \{R_Z, B_Z, U_Z\}$$

Now, for each point ω of the Cartesian product co-domain $\hat{\Omega}$ specified by (**??**), define the pre-image set

$$(\xi^M, \eta^M, \zeta^M)^{-1}(\{\omega\}) \\ := \{(x, y, z, \mathcal{A}) \in \Omega^M \mid (\xi^M, \eta^M, \zeta^M)(x, y, z, \mathcal{A}) = \omega\} \quad (24)$$

Note that any point $\omega \in \hat{\Omega}$ can be expressed as $\omega = (\hat{x}, \hat{y}, \hat{z})$ where

$$\hat{x} \in \{R_X, B_X, U_X\}, \quad \hat{y} \in \{R_Y, B_Y, U_Y\}, \quad \hat{z} \in \{R_Z, B_Z, U_Z\} \quad (25)$$

But then (**??**) and (**??**) together imply that

$$(\xi^M, \eta^M, \zeta^M)^{-1}(\{\omega\}) = (\xi^M)^{-1}(\{\hat{x}\}) \cap (\eta^M)^{-1}(\{\hat{y}\}) \cap (\zeta^M)^{-1}(\{\hat{z}\}) \quad (26)$$

Because the three functions ξ^M, η^M, and ζ^M are all measurable, the right-hand side of (**??**) is the intersection of three measurable sets. By Lemma **??**, this intersection is itself measurable. Since $\hat{\Omega}$ is a finite set, this establishes that the function defined by (**??**) is measurable. □

An obvious but important implication of Proposition **??** is that the three random variables ξ^M, η^M and ζ^M specified by (**??**), (**??**), and (**??**) have a well defined joint probability distribution over the Cartesian product co-domain $\hat{\Omega}$ defined by (**??**). This distribution is easily calculated by multiplying the probabilities specified in Table **??**, or in the consistent case, in Table **??**, by the appropriate probabilities q_c for $c \in C$. Provided the first-stage probabilities satisfy $q_c > 0$ for all $c \in C$, this construction accounts for the weird correlations between all three different observed pairs of random variables determined by the colours chosen by the two siblings who are observed at the match.

3 The Double-Slit Experiment Revisited

3.1 Three Contextual Probability Spaces

The double-slit experiment that was briefly described in Section **??** involves the sample space $\Omega = S \times D$ where:

1. $S = \{L, R\}$ is the set of two slits in the front screen through either of which, if it is open, any particle could pass;

2. a bounded rectangular subset $D \subset \mathbb{R}^2$ is the domain of possible points of observed impact on the back screen.

An obvious way to try to make this a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, according to the classical definition in Section **??** would be to define the σ-algebra $\mathcal{A}$ as the family of subsets of $2^S \times D$ whose members take the form

$$(\{L\} \times D_L) \cup (\{R\} \times D_R) \cup (\{L, R\} \times D_{LR})$$

where D_L, D_R, and D_{LR} are three Borel subsets of D.[7] But then in Section **??** it was shown that no single probability mass function $\mathbb{P}$ on $(\Omega, \mathcal{A})$ can account for all the observations in the different contexts where either or both slits are open.

The remedy proposed here involves a quantum measurement tree. This starts with an initial which is a preparation node. There a first-stage process selects one of the three different experimental contexts $c \in C$ which belong to the set $C := \{L, R, LR\}$ whose members correspond in an obvious way to the non-empty set $O_c \subseteq \{L, R\}$ of one or two open slits. At the end of this first-stage process is a measurement node where a second-stage process determines at what point $(x, y) \in D$ of the back screen the particle is observed to make an impact. Together, the context $c \in C$ and observed impact point $(x, y) \in D$ determine a path through the tree given by the point (c, x, y) in the sample space $\Omega = C \times D$. Then, as in Section **??**, the relevant multi-probability space of possible paths takes the form $(\Omega, (\mathcal{A}_c, \mathbb{P}_c)_{c \in C})$, where Ω is the common sample space, which here is $C \times D$, and each of the three triples $(\Omega, \mathcal{A}_c, \mathbb{P}_c)_{c \in C}$ is a probability space in its own right.

Before giving details of the construction, for each context $c \in C$, let

$$D \ni (x, y) \mapsto f_c(x, y) \in \mathbb{R}_+ \tag{27}$$

denote the continuous probability density function on D that is relevant in the context c.

[7] Recall that the Borel σ-algebra of any topological space such as D is defined as the smallest σ-algebra that includes all open subsets.

- In the context where $c = LR$, so both slits are open, and therefore nothing is known *a priori* about which slit the particle could have passed through, the probability space $(C \times D, \mathcal{A}_{LR}, \mathbb{P}_{LR})$ has:

  1. the σ-algebra $\mathcal{A}_{LR}$ on $C \times D$ whose only non-empty sets take the form $D_{LR} \times \{L, R\}$ for some Borel set $D_{LR} \subseteq D$;
  2. the probability measure $\mathbb{P}_{LR}$ that, for each Borel set $D_{LR} \subseteq D$ and so for each $D_{LR} \times \{L, R\} \in \mathcal{A}_{LR}$, satisfies

$$\mathbb{P}_{LR}(D_{LR} \times \{L, R\}) = \int_{D_{LR}} f_{LR}(x, y)(\mathrm{d}\, x \times \mathrm{d}\, y) \tag{28}$$

- In either of the two contexts where $c = L$ or $c = R$, so only one known slit is open, the probability space $(C \times D, \mathcal{A}_c, \mathbb{P}_c)$ has:

  1. the σ-algebra $\mathcal{A}_c$ on $C \times D$ whose only non-empty sets take the form $D_c \times \{c\}$ for some Borel set $D_c \subseteq D$;
  2. the probability measure $\mathbb{P}_c$ that, for each $D_c \subseteq D$ and so for each $D_c \times \{c\} \in \mathcal{A}_c$, satisfies

$$\mathbb{P}_c(D_c \times \{c\}) = \int_{D_c} f_c(x, y)(\mathrm{d}\, x \times \mathrm{d}\, y) \tag{29}$$

3.2 Constructing a Probability Metaspace

As in Section **??**, the construction of an overall probability metaspace over paths through the two-stage tree requires a randomization which determines the context $c \in C := \{L, R, LR\}$. Specifically, for each $c \in C$, let $q_c \in [0, 1]$ denote the probability that context is c. Then the metaspace construction requires us to assemble the three probability spaces $(S \times D, \mathcal{A}_c, \mathbb{P}_c)_{c \in C}$ into the one probability metaspace $(\Omega^M, \mathcal{A}^M, \mathbb{P}^M_q)$, defined as the triple where:

1. The sample meta-space Ω^M is the Cartesian product

$$C \times D \times \cup_{c \in C}\{\mathcal{A}_c\} \tag{30}$$

   of the basic sample space $C \times D$ with the range of possible contextual σ-algebras. Its typical member takes the form $(c, x, y, \mathcal{A})$ in which one of the three non-empty subsets of open slits is combined with both a point $(x, y) \in D$ in the plane of the second screen and a σ-algebra $\mathcal{A}$ that belongs to the family $\{\mathcal{A}_c\}_{c \in C} = \{\mathcal{A}_L, \mathcal{A}_R, \mathcal{A}_{LR}\}$ of three possible contextual σ-algebras.

2. The σ-algebra $\mathcal{A}^M$ on Ω^M is the family of all sets which, for some triple (B_L, B_R, B_{LR}) of arbitrary Borel subsets of D, take the form

$$E^M(B_L, B_R, B_{LR}) := (\{L\} \times B_L \times \{\mathcal{A}_L\}) \cup (\{R\} \times B_R \times \{\mathcal{A}_R\}) \cup (\{L, R\} \times B_{LR} \times \{\mathcal{A}_{LR}\}) \quad (31)$$

It is straightforward to verify that this definition makes $\mathcal{A}^M$ the smallest σ-algebra which contains all the basic Cartesian product sets that, for some Borel set $B \subseteq D$, take one of the three forms

$$\{L\} \times B \times \{\mathcal{A}_L\}, \quad \{R\} \times B \times \{\mathcal{A}_R\}, \quad \{L, R\} \times B \times \{\mathcal{A}_{LR}\} \quad (32)$$

3. For each non-empty set $O \subseteq S$ of open slits and each Borel set $B \subseteq D$, the conditional probability, given O, that the observed impact on the back screen occurs within the set B is

$$\beta_O(B) := \int_B f_O(x, y)(\mathrm{d}\, x \times \mathrm{d}\, y) \quad (33)$$

Using (**??**), the probability of each set $E^M(B_L, B_R, B_{LR}) \in \mathcal{A}^M$ specified by equation (**??**) is then given by

$$\mathbb{P}^M_q(E^M(B_L, B_R, B_{LR})) = q_L \beta_L(B_L) + q_R \beta_R(B_R) + q_{LR} \beta_{LR}(B_{LR}) \quad (34)$$

Because the set O of open slits in the double-slit experiment is assumed to be observed, the probability metaspace $(\Omega^M, \mathcal{A}^M, \mathbb{P}^M_q)$ can evidently be simplified by omitting the unique contextual σ-algebra $\mathcal{A}_O$ that corresponds to each known non-empty set $O \in C$ of open slits. The result is the probability space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}}_q)$ with:

1. sample space $\hat{\Omega} := C \times D$ whose members (c, x, y) include the context $c \in C$ that corresponds to the non-empty set $O_c \subseteq \{\{L\}, \{R\}, \{L, R\}\}$ of open slits;

2. the σ-algebra $\hat{\mathcal{A}}$ on $C \times D$ whose non-empty members take the form

$$\hat{E}(B_L, B_R, B_{LR}) := (\{L\} \times B_L) \cup (\{R\} \times B_R) \cup (\{L, R\} \times B_{LR}) \quad (35)$$

where B_L, B_R, and B_{LR} are arbitrary Borel sets of D;

3. probability measure $\hat{\mathbb{P}}_q$ whose value, for each set given by (**??**), is

$$\hat{\mathbb{P}}_q(\hat{E}(B_L, B_R, B_{LR})) = q_L \beta_L(B_L) + q_R \beta_R(B_R) + q_{LR} \beta_{LR}(B_{LR}) \quad (36)$$

4 Concluding Remarks

4.1 Representing Quantum Contexts as σ-Algebras

At least some part of the apparent weirdness of the experimental results which arise in quantum mechanics can be attributed to the fact that the probability distribution of those experimental observations typically depends on a variable context. Moreover, this context typically depends in turn on what experimental configuration was used to generate those observations. In particular, it is generally impossible to describe properly the random measurements in different contexts without resorting to a family of different contextual probability spaces. This need for different contextual probability spaces is what underlies the common assertion that the random observations in different quantum contexts cannot be described within a single classical probability space.

This paper begins a series concerned with a project intended to contest this common assertion by constructing quantum measurement trees whose only randomness can be described using classical probability spaces. Any such tree is typically associated with one member of a parametric family of probability "meta-spaces". Each meta-space may have a different sample space whose members are different possible paths through the tree. Each such path corresponds to one possible combination of a context which depends on the experimental configuration, followed by an observed outcome that results randomly from a contextual measurement process. Following the important contribution of Vorob′ev (1962), the key idea of this project is to identify each possible context with a distinctive σ-algebra of events in a fixed sample space of possible measurement outcomes.[8]

This initial paper has illustrated this construction with two simple examples. One of these is the noted two-slit experiment that Feynman (1951) famously used as a canonical example to illustrate quantum weirdness. The second is a homely example inspired by Vorob′ev (1962) but based on Boole (1862). It involves three random dichotomous variables X, Y, Z in which the two pairs (X, Y) and (X, Z) are perfectly correlated, yet the pair (Y, Z) is perfectly anti-correlated.

[8] The importance of context in quantum theory has been widely recognized, notably in the work on "contextuality". See especially the "theme issue" published by the Royal Society whose preface appears as Dzhafarov (2019). As far as I am aware, however, there is no previous work, either in quantum theory or more generally, which explicitly identifies each context with a unique corresponding σ-algebra of measurable events.

4.2 Disclaimer

Let me emphasize that all parts of this research project are entirely about abstract mathematical concepts developed from Vorob′ev's (1962) key extension of classical Kolmogorov probability theory to allow multiple probability measures over different σ-algebras. I am not a physicist, and I make no attempt to offer any physical interpretation or explanation of quantum weirdness. Instead, my only aim is show how the abstract device of a probability metaspace derived from a quantum measurement tree can encompass multiple contexts, especially multiple quantum contexts. This construction allows an alternative mathematical representation of quantum weirdness which some may find easier to understand, especially anybody who is already familiar with the classical concepts due to Kolmogorov (1933) of σ-algebra and probability measure.

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This paper and its successors that discuss quantum measurement trees are dedicated to the memory of the noted philosopher Patrick Suppes. His accomplishments were recognized by the award in 1990 of a National Medal of Science of the U.S.A. for his contributions to Behavioral and Social Science. Apart from his papers cited here, his contributions to the foundations of quantum mechanics included the intensive seminar at Stanford during the academic years 1972–1973 and 1973–1974, whose results include many published in a double issue of *Synthese* with an introduction that appeared as Suppes (1974). Indeed, the current project was initially inspired by Patrick's repeated and patient attempts to arouse my interest in the topic during discussions we held over the several decades when I was fortunate enough to be his colleague at Stanford.

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