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Political Competition and Strategic Voting in Multi-Candidate Elections*

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Abstract

We develop a model of strategic voting in a spatial setting with multiple candidates when voters have both expressive and instrumental concerns. The model endogenizes the strategic coordination of voters, yet is flexible enough to allow the analysis of political platform competition by policy-motivated candidates. We characterize all strategic voting equilibria in a three-candidate setting. Highlighting the utility of our approach, we analyze a setting with two mainstream and a spoiler candidate, showing that the spoiler can gain from entering, even though she has no chance of winning the election and reduces the winning probability of her preferred mainstream candidate.

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1 Introduction

There is manifest evidence that voters have both expressive and instrumental voting considerations, that is, they both receive a direct payoff from voting for a particular candidate, and they care about who wins the election both in real world election data (e.g., Fujiwara (2011); Pons and Tricaud (2018); Spenkuch (2018)) and in laboratory experiments (e.g., Forsythe et al. (1996); Duffy and Tavits (2008); Esponda and Vespa (2014); Bouton et al. (2015)). In a two-candidate election there is no conflict between these two objectives: voting *for* a preferred candidate is equivalent to voting *against* a disfavored candidate, so there is no reason to vote tactically. This calculus changes when there are more than two candidates because expressive and instrumental concerns can now easily be mis-aligned, giving rise to strategic voting.

The contribution of this paper is to develop a tractable model of strategic voting within a canonical spatial environment. That is, the model allows for voters with both expressive and instrumental concerns who can coordinate strategically in multi-candidate plurality elections. Further, we characterize how the primitives of the political environment affect the possible natures of strategic coalition formations. In particular, we describe how changes in candidate characteristics and policy platforms affect the incidence of strategic voting and electoral outcomes. Finally, by allowing for a continuum of citizens, we also analyze political competition between policy-motivated candidates in the presence of strategic voting.

The standard approach to studying multi-candidate elections in the strategic voting literature (e.g., Myatt (2007); Bouton and Castanheira (2012); Bouton (2013)) is to focus on a finite number of strategic voter types who individually *only* care about how their voting decision affects a candidate's probability of winning. Such pivotal voting models have a long tradition (e.g., Palfrey and Rosenthal (1985); Myerson and Weber (1993)), but they do not apply to large elections when the strategic voters have any expressive considerations at all; such considerations swamp the tiny probabilities that one's vote is pivotal. To skirt this issue, some authors assume that voters' perceived pivot probabilities are exogenous and large (e.g., Chapter 1, Aldrich et al. (2018)). While this approach can accommodate voters with both instrumental and expressive concerns, it cannot provide causal links between policy platforms and pivot probabilities.

To circumvent these issues, we formulate a new model of group coordination of voters and propose a new solution concept. We borrow concepts from the ethical voting literature (e.g., Feddersen and Sandroni (2006); Coate and Conlin (2004)), which formalizes the idea that some voters internalize the welfare of similarly-situated citizens and hence do not want to free-ride on the voting efforts of other "group" members. Our construction shares the feature of ethical voting models that there is a benefit of voting for a certain candidate if others in the group also do so, and that a psychic cost of letting the group down prevents free riding. However, we need to modify the ethical voting approach to account for one fundamental difference: Ethical voting models are designed to explain voter participation in large elections when participation is costly and pivot probabilities are small, so they suppose group members only differ in their voting costs. Thus, citizens only need to decide whether or not to vote for their most preferred candidate. In contrast, in our canonical spatial model, a voter needs to decide whom to vote for by comparing the distances between her bliss point and the policy platforms of the different candidates weighing her expressive and strategic concerns. That is, our equilibrium concept must detail *how* voting coalitions for different candidates form, i.e., who joins a group of citizens to vote for a less preferred candidate to raise the probability of defeating an even less preferred candidate.

Our model of group voting is formulated in a plurality election with three candidates. There are two types of voters, partisans who always vote on party lines, and non-partisans whose votes hinge on candidate policies and what they think other voters will do. A nonpartisan's utility is given by the weighted sum of expressive and instrumental payoffs, i.e., of the expressive payoff from voting for a given candidate and the instrumental payoff derived from the winning candidate's policy.

A realized strategic coalition for a candidate includes all citizens who expressively prefer that candidate plus some citizens who are willing to vote for that particular candidate rather than for their top-rated choice. Such coalitions must satisfy two properties. First, each coalition member must be better off in expectation if the coalition forms than if all members voted expressively. Second, a new or larger coalition cannot form that would make all members better off relative to having no coalition. Analogous to a Nash equilibrium, each realized coalition calculates its payoff given the actual probability distributions over the formation of strategic coalitions for rival candidates. We can have "pure" or "mixed" coalitional voting equilibria according to whether or not there is stochastic coalition formation.

We characterize all equilibria to this strategic voting game in a setting with three candidates who have arbitrary policy platforms. We first establish ways in which the centrist has an electoral advantage. If the centrist is top ranked by a plurality of partisan and expressive voters, then there is a unique pure coalitional voting equilibrium. In this equilibrium, everyone votes expressively and the centrist wins. Intuitively, expressive centrist supporters get their preferred candidate and hence do not vote strategically, and strategic voting by expressive non-centrist supporters only makes sense if it can improve electoral outcomes.

This is not true for non-centrist with a leading advantage among expressive and partisan voters. Whether or not such a candidate wins the election depends on the strategic support that the other two candidates may get, i.e., it depends on the second choices of voters who do not rank her at the top, and on the balance of expressive and instrumental weights in voter utilities. First, the leading non-centrist may lose because a centrist candidate who is not too far behind may be able to win by attracting enough tactical support from voters on the other side of the political spectrum. When this is so, the leading non-centrist cannot draw strategic votes from expressive supporters of the centrist, and hence cannot defeat the centrist.[1] Second, the rival "un-

[1]Relatedly, Callander (2005a) observes that strategic voters may coalesce around a centrist, albeit in a setting where strategic

derdog" non-centrist candidate, say on the right, may be able to win by attracting enough strategic support from center-right voters. However, this strategic support has to be large enough that the underdog's victory could not be overturned by strategic voting of center-left citizens—there must be enough more potential center-right than center-left strategic voters. Thus, if the non-centrist underdog wins in a pure coalitional voting equilibrium, she must win by a non-trivial margin.

We show there cannot be strategic voting in a pure coalitional voting equilibrium in which the leading non-centrist wins. When the underdog (on the right) can attract enough strategic support to win absent tactical voting for the leading candidate (on the left), but not enough if sufficiently many centre-left citizens vote strategically, then the equilibrium must involve randomization. Either of the two non-centrist candidates can win the election, and the winner is the candidate who, by chance, attracts the largest voting coalition. For each non-centrist candidate, the realized coalition forms randomly according to the equilibrium distribution over coalition formation. The coalition includes partisan and expressive voters, as well as possibly strategic voters who expressively prefer the centrist. For each realized coalition, the most centrist voter called to vote for a non-centrist is just indifferent between doing so and having the coalition break up, given the distribution of voting coalitions that form to support the opposing non-centrist candidate.

Stochastic coordination for both non-centrist candidates is required because a citizen will only vote against her expressive preferences if her coalition has a positive probability of influencing the electoral outcome. The mixed coalitional voting equilibrium shares similarities with equilibria of all-pay auctions. For example, we show that with strictly positive probability there is no strategic voting on each side, and both non-centrists win with strictly positive probability. These equilibria are well-behaved. For example, when one non-centrist has an initial plurality advantage, the rival "underdog" non-centrist's maximum potential strategic support is always greater. Further, a larger lead for a non-centrist due to greater partisan support raises her chances of winning and reduces the strategic voting on both sides. The equilibrium also behaves continuously as we approach the boundary cases of purely instrumental or purely expressive voting.

Strategic voting can sharply alter the logic and comparative statics of voting. Consider an election with two main candidates, symmetrically situated around the median, and a spoiler candidate on the far right who has no chance of winning. With purely expressive voting, by moving toward the spoiler, the center-right candidate would always reduce her vote share and increase her center-left rival's vote share. With strategic voting this need not be true. In particular, as the center-right candidate moves toward the spoiler, she (i) better differentiates herself from her center-left rival, raising the benefits of strategic coordination by voters; and (ii) is politically closer to the spoiler, reducing the costs of strategic coordination. By facilitating strategic voting in this way, the center-right candidate can increase her vote share by more than it increases her

voters only have instrumental concerns. Thus, the free-riding issues and tradeoffs between instrumental and expressive concerns that are central to our analysis do not arise in Callander (2005a).

center-left rival's vote share.

To illustrate the consequences of this finding for electoral competition, we endogenize the positions of the center-left and center-right candidates as a function of the spoiler's policy platform. We embed our model of coalitional voting in a two-stage game. In the first stage, policy-motivated candidates choose their platforms. In the second stage, citizens decide which strategic voting coalitions (if any) to form. When candidates choose policies they are uncertain about the level of each candidate's partisan support. Partisan support becomes public information before nonpartisans choose whether and how to vote strategically. Hence, the electoral game played by candidates is one of probabilistic competition à la Wittman (1983) or Calvert (1985). When voters place little weight on expressive concerns, the spoiler's entry has no impact on equilibrium outcomes, as even extreme expressive supporters vote strategically. However, when weights on expressive motives are increased further, the spoiler can attract votes, causing both of the main candidates to move their policy positions towards the spoiler—the spoiler's preferred candidate does so to differentiate herself, thereby increasing strategic voting, while her mainstream rival has the opposite incentive. This makes the spoiler strictly better off: While the spoiler's preferred mainstream candidate is less likely to win due to the spoiler's entry (consistent with Pons and Tricaud (2018)), this loss is outweighed from the spoiler's perspective by the benefits of the policy shifts of the mainstream candidates' platforms. Further increases of the weight on expressive motives eventually harm the spoiler, because the spoiler takes too many votes away from her preferred mainstream candidate.

Our model can reconcile the entry of a spoiler candidate such as Ralph Nader who has no chance of winning, even though the spoiler does not receive ego rents. We show that with strategic voting, spoilers can gain by having mainstream candidates adopt some of their policies. Consistent with this, in an interview in 2019 in the Washingtonian[2], Ralph Nader indicated that he had hoped to "push the Democrats toward a more progressive agenda," understanding that his campaign could end up costing Al Gore the presidency.[3] Of course, if the preferred mainstream candidate loses due to the spoiler's entry, then the spoiler has ex post regret. The entry of Nigel Farage's UKIP party in the UK illustrates an outcome where the ex-post outcome benefitted the spoiler. Farage's entry forced the Conservatives to adopt UKIP-like positions on Brexit (and induced Labour to moderate its opposition). It also resulted in a Conservative victory and the subsequent implementation of Brexit. Similarly, the entry of populist parties in Europe has encouraged mainstream parties to move to the right, with center-right parties moving further.

[2]see `https://www.washingtonian.com/2019/11/03/ralph-nader-is-opening-up-about-his-regrets/`, Ralph Nader Is Opening Up About His Regrets, Rob Brunner, November 3, 2019

[3]Hillygus (2007) studies the dynamics of Nader support in the United States during the 2000 campaign, and documents evidence of strategic voting. The Nader supporters likely to switch to vote for Gore were more politically aware, and more concerned about policy outcomes (such as, paradoxically, the environmental policy) and respondents in competitive states.

2 Related Literature

Beginning with Duverger (1959), researchers have noted that plurality rule systems generate strong forces for only two parties to be competitive, as voters who care about electoral outcomes do not want to "waste" votes on candidates who are sure to lose. Cox (1997) documents consistent patterns of strategic voting across different electoral systems, and observes that the "reduction of parties" in single member districts reflects the coordination of voters on parties. Applied to the United Kingdom, supporters of the LDP and Labour may have incentives to coordinate to defeat the Conservatives (c.f., Aldrich et al. (2018), Figure 2.2). Similar coordination is found in Canada (Merolla and Stephenson, 2007), and elsewhere (Blais et al., 2019). Fujiwara (2011) uses a RDD on population thresholds to show that the top two candidates in Brazilian elections secure more votes under simple plurality than runoff or proportional elections, and Pons and Tricaud (2018) find the presence of a third candidate reduces vote shares for the two leading candidates in French elections. Structural estimates based on models where some voters only have instrumental concerns while the others are purely expressive also suggest that many voters behave strategically. Kawai and Watanabe (2013) find this in Japan and Fisher and Myatt (2017) do so for Britain. Conversely, Degan and Merlo (2006) conclude that in U.S. national elections, which are dominated by two parties, the hypothesis that voters vote sincerely cannot typically be rejected. Ujhelyi et al. (2021) document extensive none-of-the-above voting in India, indicating that expressive voting considerations dominate for some voters.

The extent of strategic voting can be substantial: Abramson et al. (2018) finds that the incidence of strategic voting—voting for a second-choice candidate—was almost 40% in some constituencies for the 2010 British general election, while Daoust (2018) finds that 22.6% of voters selected their second-choice in the 2015 Canadian general election. Daoust (2018) also illustrates what Cox describes as the challenge of voter strategic coordination, and "the rapidity with which vote intentions change when coordination takes off", with the NDP's polling share falling from 37% to 20% in the last two months of the campaign, with two-thirds of the drop occurring in the last month and the Liberals winning a majority due to this shift. Spenkuch (2018) provides evidence that "voters cannot be neatly categorized into sincere and strategic "types", and that voters, instead, weigh both expressive and instrumental voting considerations in their choices of whether or not to vote strategically.[4] Consistent with this, Abramson et al. (2018) find that the extent of strategic voting varies with the perceived ability to sway the outcome: while only 1.4% of voters with minimal strategic incentives reported an intention to vote for their second-choice party, it was 27.1% of those with the strongest incentives.

These empirical findings suggest the following key features of voting in multi-candidate elections:

[4] The observation that voters have both expressive and instrumental voting considerations dates back to at least Riker and Ordeshook (1968).

1. Voters sometimes coordinate strategically to try to defeat a less preferred candidate.

2. Voters trade off expressive and instrumental voting considerations, and are more likely to vote strategically if the chance of changing the electoral outcome is higher.

3. Coordination can be difficult and can quickly generate large shifts in candidate support, suggesting that there can be multiple ways to coordinate, and that small changes in candidate strengths may shift coordination from one candidate onto another.

4. Duverger's law says that only two candidates are competitive, but does not imply that only two candidates receive meaningful vote shares.

We develop a formal theory of strategic voting that is rich enough to generate these salient features, and yet is sufficiently tractable to permit the analysis of political competition by policy-motivated candidates who anticipate how their how platform choices affect strategic voting.

The earliest study of multi-candidate elections with the standard strategic voting approach dates back to Palfrey (1989), who considers three candidate races and assumes that voters know precisely the fraction of the electorate who ranks candidates in any given order. He shows that generically all equilibria are Duvergerian: only two candidates receive votes.[5] Fey (1997) examines how public opinion polls can coordinate voters on particular Duvergerian equilibria in the case of "divided majority:" more than half of the voters ranks third the candidate who is ranked first by the largest share of voters. Myatt (2007) also studies a model of divided majority in which candidates are not perfectly informed of each other preferences. He finds that the equilibrium is not perfectly Duvergerian: Some voters vote strategically, but others do not. The amount of strategic voting depends on the intensity of voters' preferences and how precise their information is about each other's preferences. Bouton and Castanheira (2012) introduce interdependent preferences. They find that when the common motive is sufficiently strong, approval voting ensures the election of the full information Condorcet winner, whereas such a result is not guaranteed under plurality and run-off rules.[6]

The basic idea of expressive voting is formulated by Brennan and Lomasky (1993). Within the context of multi-candidate election models, Fisher and Myatt (2017) extend the model of Myatt (2007) to incorporate a fraction of sincere, expressive voters. Further, Osborne and Slivinski (1996) model citizen candidate elections with sincere voting, whereas Besley and Coate (1997) consider strategic voting in which candidates do not waste votes on sure losing candidates.

Our formulation of strategic voting builds on the ethical voting framework. An early formalization is Harsanyi (1977), who postulates that some voters are "rule utilitarians," receiving utility for choosing a

[5]Feddersen (1992) finds that all equilibria are Duvergerian when voting is costly, even if voters may potentially vote for any position in the policy space.

[6]Bouton (2013) studies 3-candidate elections under the run-off rule in details.

strategy that maximizes social welfare when a socially inferior candidate receives a fixed number of votes. Feddersen and Sandroni (2006) remove this assumption, endogenizing support for both candidates by introducing preference diversity into Harsanyi's framework while preserving his Kantian calculus of duty. Coate and Conlin (2004) consider rules that maximize the group's welfare, if adopted by all ethical voters in the group. Levine and Mattozzi (2020) model group voting that complements ethical voting by stipulating that social norms of voting participation must be enforced through costly peer monitoring and punishment. In contrast to our model, in these settings, groups are ex-ante clearly defined and within-group differences are only non-ideological (e.g., voting costs).

Our framework does not provide an explicit non-cooperative formulation of the group formation process. Uhlaner (1989) propose that foundations for group formation can stem from leaders who mobilize groups such as unions, churches, or environmental organizations. Morton (1987) and Morton (1991) formalize these ideas, while Herrera and Martinelli (2006) provide foundations for group mobilization when both leaders and groups are endogenous.

To our knowledge, the only model of group voting in multi-candidate elections is by Bouton and Ogden (2021). Voter groups are defined by the different possible strict preference orders over three candidates. They characterize pure strategy Nash equilibria in the game in which each group acts as a single decisional unit. Their focus is on the case of divided majority preference orders. They identify when equilibrium displays either sincere voting or coordination on two candidates in plurality elections. Our approach is complementary: We consider endogenous group formation in a canonical spatial model. Because our model features both expressive and strategic group voting, all candidates receive votes in equilibrium, and yet tactical voting may determine the election outcome. The spatial setting allows us to refine predictions with respect to the case of divided majority preferences, and to determine election outcomes for other preference profiles. We determine when the divided majority coordinates on a particular candidate.

Our analysis of strategic candidate platform selection finds that a fixed-platform extremist third party candidate may gain from entering an election between policy-motivated candidates. A related literature has modeled platform choices by existing "mainstream" office-motivated parties that face the threat of entry by a third party (e.g., Palfrey (1984); Weber (1992); Callander (2005b)). There are several differences between these models and ours. First, the mainstream parties separate to induce the entrant to locate at the center and hence steal votes equally from both parties. Second, voters in these models are not strategic. In particular, applying our model, as long as voters placed enough weight on instrumental preferences, central entry would result in a pure coalitional voting equilibrium in which the centrist wins with strategic support from either the right or the left, an observation first made with purely instrumental voters in Callander (2005a). Of note, our spoiler does not care about vote shares or winning, but instead benefits by moving mainstream party policies in her direction.

3 Model

We first develop a model of strategic voting when candidate positions are fixed, and citizens choose which candidates to support. This model can be easily embedded into a model of political competition in platforms à la Wittman (1983). We do this in Section 6. We formulate the model for an arbitrary number of candidates to illustrate our general formulation of strategic voting. We then focus on the three-candidate case to provide an exhaustive characterization of coalitional voting equilibria.

There are $i = 1, \ldots, n$ parties/candidates and a continuum of citizens. The voting game features two types of citizen voters—partisans and non-partisans. Non-partisan voters have both expressive and instrumental voting motives. The utility of a non-partisan voter with ideology $\theta \in \mathbb{R}$ who votes for a candidate with policy x when the candidate with policy x_w wins is given by

$$u_\theta(x, x_w) = \beta v_\theta(x) + (1 - \beta) v_\theta(x_w),$$

where $0 \leq \beta < 1$ measures the weight placed by non-partisan voters on expressive relative to instrumental considerations. The ideal policies θ of non-partisans are distributed according to $\Phi(\cdot)$, with associated strictly positive density ϕ on a connected support.

A partisan for candidate i always votes for i regardless of the policy positions taken. Thus, the partisan support for a candidate i is summarized by the number $\rho_i \geq 0$ of i's partisan supporters. Partisans have two roles in our model: First, they provide a simple way to parametrize heterogeneity in a party's base support, facilitating comparative static analyses. Second, they provide a source of uncertainty required when we endogenize candidate policy choices. Before citizens vote, the levels ρ_i of partisan support for each party i are realized and observed by all citizens. That is, to highlight how strategic voting can generate endogenous uncertainty about who wins, we assume away all extrinsic sources of uncertainty at the voting stage.

4 Strategic Voter Coalition Formation

4.1 Motivation

Non-partisan voters with an expressive preference for a candidate i can coordinate their voting behavior and commit to vote for some candidate $j \neq i$ if it is beneficial for them to do so. We say that such non-partisan voters are "strategic voters". There are three conceptual issues with describing strategic voting in a large electorate. First, each individual voter has a negligible impact on the electoral outcome, so absent other considerations, voters would always support their expressively-preferred candidates. Thus, when describing strategic voting in such an environment one must consider groups or coalitions of strategic voters who coordinate in some way to increase a candidate's chance of being elected. Second, given that coalitions rather than individuals matter in such an environment, voters may have incentives to convince others to vote strategically, but then free ride and vote for their expressively-preferred candidates. Third, we must describe which voter coalitions can reasonably form and would be robust to both defections and solicitation of additional members.

We build on the idea that voters incur costs for not doing their civic duty formalized in Feddersen and Sandroni (2006). They recognize that some individuals have higher voting costs than others, and hence should not incur a utility penalty from not voting, while citizens with low voting costs should participate and feel an ethical cost from free-riding. Individual voters whose group benefits from strategic voting should feel guilty if they free ride by voting expressively rather than in the interest of their (endogenously-determined) fellow coalition members. This is the feature of Feddersen and Sandroni (2006) that we adopt. In contrast to their model where group identity is exogenous, in our setting voter *choose* whether or not to join a group, and ethical concerns only matter after joining. A second distinction is that, instead of incurring exogenous costs of participating in elections, our strategic voters incur endogenously-determined "costs" from voting against their expressive preferences.

The remaining issue is to characterize the strategic coalitions that can form in equilibrium. While we do not explicitly characterize the process, in practice coalition formation occurs via candidates' get-out-the-vote efforts, via social media, or leaders coordinating particular voter blocs. An example of the latter is the ongoing effort of the Strategic Voting Project developed in 2008 by Hisham Abdel-Rhaman, a software engineer who sought to coordinate progressive voters in Canada in each electoral riding on either the NDP or the Liberal candidate with the best chance of defeating the Conservative candidate (see http://www.strategicvoting.ca). This coordination can also happen from voters evaluating candidates after debates, or their earlier primary performances for US presidential primaries. Our analysis describes the possible equilibrium outcomes of such coordination processes.

Before providing formal definitions, we note that one should not expect a concept of coordination to always yield a unique prediction. This feature is well known, as it arises in standard coordination games such as the battle of the sexes. In some coordination games, equilibrium refinements can be used to reduce the number of equilibria. However, in our analysis we choose to remain agnostic as to which equilibrium may arise. Importantly, most equilibria yield the same candidate winning probabilities, in which case candidate location does not vary with the equilibrium in the voting subgame.

4.2 Coalition Formation and Equilibrium

When analyzing strategic voter behavior it is sufficient to describe the behavior of non-partisans and add the votes of partisans to determine total candidate votes. In the remainder of the paper any reference to a set or coalition of voters therefore refers exclusively to non-partisans. Let E_i be the set of (non-partisan) voters with an expressive preference for candidate i, i.e.,

$$E_i = \left\{ \theta \,\middle|\, v_\theta(x_i) > v_\theta(x_j), \text{ for all } j \neq i \right\}. \tag{1}$$

For simplicity of exposition we assume that all policies differ, i.e., $x_i \neq x_j$ for $i \neq j$, in which case $E_i \neq \emptyset$. We discuss the special case where $x_i = x_j$ for some $i \neq j$ below.

Unlike in Feddersen and Sandroni (2006), the preference intensities of voters in E_i differ. For example, voters in E_i close to the boundary with E_j have weaker incentives to vote expressively than voters closer to x_i. Our solution concept determines whether or not all members of E_j vote for candidate j or whether some vote strategically for a different candidate. Our coalition formation concept defines sets S_j that either include E_j or are empty. If S_j strictly contains E_j then there is strategic voting in the sense that some members of S_j vote against their expressive interests. The case $S_j = E_j$ is the degenerate case where all members of the realized coalition vote according to their expressive interest. This can happen when strategic voting only occurs with some probability that is strictly between 0 and 1.

The fundamental idea of coalition formation is that citizens who support a particular candidate have the ability to coordinate their votes. This means that expressive supporters of a candidate i who want to vote strategically for some candidate j should be able to reach out to expressive j supporters to get them to also vote for j. Our assumption that S_j contains E_j captures this. It eliminates coordination failures in which expressive supporters of candidate i vote for j solely because some j supporters vote for i—neither of these groups wants to stop this offsetting strategic voting unilaterally to prevent a third candidate from winning.

Once a strategic coalition is formed, all citizens in S_i vote according to their common interests, similar to the ethical voters in Feddersen and Sandroni (2006). However, in their setting the supporters of a candidate are pre-determined, and their choice is whether or not to vote. In our case, instrumental voters come together to form coalitions, and we must allow for randomized coordination of coalition members. This means that, in equilibrium, a particular coalition S_i may have only a probabilistic understanding of rival coalition formations. We now describe this possibly stochastic coalition formation.

Voting coalitions are determined, possibly stochastically, for some strict subset I of the set of all candidates $\{1, \ldots, n\}$. In particular, there cannot exist a strategic voter coalition for every candidate, because the sets S_i would have to overlap. Citizens who are not in strategic coalitions vote expressively. For example, if S_1 and S_4 are the only strategic voter coalitions, then $S_1 \cap S_4 = \emptyset$, because a citizen cannot simultaneously vote for both candidate 1 and 4. Thus, $E_i \backslash (S_1 \cup S_4)$ is the set of non-partisans who vote for candidate $i \neq 1, 4$.

Coalitions formation is independent across candidates $i \in I$. Formally, let $(\Omega, \mathcal{A}, \lambda)$ be a probability space, where $\Omega = \prod_{i \in I} \Omega_i$, and $\lambda = \prod_{i \in I} \lambda_i$, reflecting independence. A realized coalition for candidate $i \in I$ is a set of voters $S_i(\omega_i)$. Let $\bar{S}_i = \bigcup_{\omega_i \in \Omega_i} S_i(\omega_i)$. The sets $\bar{S}_i$, $i \in I$ must satisfy Assumption 1

Assumption 1 *The sets $\bar{S}_i$, $i \in I$ are pairwise disjoint with $\Phi(\bar{S}_i) > \Phi(E_i)$ and $E_i \subset S_i(\omega_i)$ for all $\omega_i \in \Omega_i$.*

The sets $\bar{S}_i$ have to be disjoint since there is no coordination across groups. Otherwise, if $\bar{S}_i \cap \bar{S}_j \neq \emptyset$, for $i \neq j$, then by independence there would exist realized coalitions $S_i(\omega_i)$ and $S_j(\omega_j)$ that intersect, i.e., contain some citizens who would be voting for both candidate i and candidate j. As mentioned above, if there is strategic voting for candidate i then all expressive i supporters coordinate on candidate i, i.e., $E_i \subset S_i(\omega_i)$ for all $\omega_i \in \Omega_i$. This, together with $\Phi(\bar{S}_i) > \Phi(E_i)$, means that strategic voting occurs.

Consider a realized collection S of strategic voter coalitions $S_i, i \in I$. Adding partisan votes, the total number of votes for a candidate i is

$$V_i(S) = \begin{cases} \rho_i + \Phi(S_i) & \text{if } i \in I; \\ \rho_i + \Phi\left(E_i \setminus \bigcup_{j \in I} S_j\right) & \text{if } i \notin I, \end{cases} \tag{2}$$

where, abusing notation, we use Φ to describe both the probability and the cdf of non-partisan voter support for each candidate. The candidate with the plurality of votes wins. In case of a tie we assume each candidate wins with strictly positive probability as the exact split does not affect our results. Let $W_i(S)$ be an indicator function that assumes the value 1 if and only if party i wins. Candidate i's winning probability is given by

$$P_i(\lambda) = \int W_i(S(\omega))\, d\lambda(\omega). \tag{3}$$

If the distribution λ_i places probability one on a single coalition S_i, we denote it by δ_{S_i}. Note that if all λ_i take this form then there is no randomization. The expected instrumental payoff for a voter θ who is a member of a realized coalition S_i is obtained by integrating the voter's instrumental utility over all possible realization of coalitions for rival candidates:

$$U_\theta(\lambda_{-i}, S_i) = \sum_{k=1}^{n} P_k(\lambda_{-i}, \delta_{S_i}) v_\theta(x_k). \tag{4}$$

We next introduce our notion of a *coalitional voting equilibrium* in which groups of citizens coordinate their votes to make their group better off. Individual members of a strategic coalition do not cheat on other members by secretly changing their votes without telling other coalition members due to the large negative psychic payoffs incurred from being "unethical" in this way. However, it would not be unethical for a member to tell other coalition members that they are not willing to follow the coalition's recommendation. In such a case a coalition would not form. Thus, a minimal requirement for a strategic coalition is that no member should be worse off in expectation if the coalition forms.

Definition 1 (Common Interest) *Let S_i be a realized coalition of (non-partisan) voters for candidate i, and let λ_{-i} be the distribution over rival coalitions. S_i satisfies common interest if none of its members is worse off in expectation if the coalition forms: There does not exist $j \neq i$ and $\theta \in S_i \cap E_j$ such that $\beta v_\theta(x_j) + (1-\beta)U_\theta(\lambda_{-i}, E_i) > \beta v_\theta(x_i) + (1-\beta)U_\theta(\lambda_{-i}, S_i)$.*

Members of each coalition S_i take the probabilities that other coalitions forms as given when calculating expected utilities associated with coalition formation. If there is no randomization, we can replace λ_{-i} by the collection of known coalitions other than S_i.

Equilibrium also demands that coalition S_i cannot gain by reaching out to attract new members who are not currently in a coalition and who would all be strictly better off from joining compared to having the strategic coalition break-up.

Definition 2 (Inclusivity) *Let $A_i \supset E_i$ be a set of (non-partisan) voters for candidate i, and let λ_{-i} be the distribution over rival coalitions. A_i satisfies inclusivity if there does not exist an $\epsilon > 0$ and a set T of citizens with $T \cap A_i = \emptyset$ and $T \cap \bigcup_{j \neq i} \bar{S}_j = \emptyset$, such that common interest is satisfied for all members of $A_i \cup T$ and everyone in T is uniformly strictly better off:*

$$\beta v_\theta(x_i) + (1-\beta) U_\theta(\lambda_{-i}, A_i \cup T) \geq \beta v_\theta(x_j) + (1-\beta) U_\theta(\lambda_{-i}, E_i) + \epsilon, \textit{ for all } \theta \in T,\ j \neq i. \tag{5}$$

Definition 3 (Coalitional Voting Equilibrium) *A probability distribution $\lambda = \prod_{i \in I} \lambda_i$ on $\Omega = \prod_{i \in I} \Omega_i$ is a coalitional voting equilibrium if and only if*

1. *All realized strategic coalitions $S_j(\omega_j)$, $\omega_j \in \Omega_j$ satisfy common interest and inclusivity.*
2. *If $E_j \subset \bigcup_{i \in I} \bar{S}_i^c$ then E_j satisfies inclusivity.*

The role of the second condition is to allow for the possibility of strategic voting for some candidate j in a case where candidate j has currently no strategic voting support. The condition also ensures that such strategic voting would include all expressive candidate j supporters.

5 Coalitional Voting Equilibrium

We now characterize all coalitional voting equilibria when there are three candidates and voters have quadratic preferences, i.e., $v_\theta(x) = -(x-\theta)^2$. We focus on the interesting case where platforms are distinct, i.e., $x_1 < x_2 < x_3$.[7] Coalitional voting equilibria also exist when some platforms are identical.

Proposition 1 *Given any set of candidate platforms and all possible levels of partisan supports ρ_1, ρ_2, and ρ_3, at least one equilibrium exists.*

[7] If two candidates have exactly the same platform and collectively a majority supports these candidates, then enough non-partisans will coordinate to ensure that one of these candidates wins.

We prove this result in a series of propositions below that exhaustively characterize all voting equilibria that emerge given the different possible parameterizations.

In the following, let $\theta_{ij} = (x_i + x_j)/2$. The sets of expressive supporters for the three candidates are $E_1 = (-\infty, \theta_{12})$, $E_2 = (\theta_{12}, \theta_{23})$, and $E_3 = (\theta_{23}, \infty)$. Let $N_i = \Phi(E_i) + \rho_i$ be the number of votes for party i if all citizens vote according to their expressive preferences. We call N_i the base support for candidate i; that is, N_i corresponds to candidate i's vote total if all citizens vote according to their expressive preferences (i.e., as if $\beta = 1$). Without loss of generality, we can assume that $N_1 \geq N_3$.

In order to characterize coalitional voting equilibria, it is useful to determine the largest set of expressive candidate i supporters who would be willing to vote for candidate j. This is a set $S_{ij} \subset E_i$ that would form to defeat the least favored candidate k, by strategically voting for j. Formally, the set S_{ij} is an interval, with one endpoint given by θ_{ij} and the other by the voter type y_{ij} who is indifferent between having the coalition S_{ij} form to make j win and not having the coalition form so that k wins. The only possible such sets are S_{12}, S_{21}, S_{23}, and S_{32}. In particular, S_{13} and S_{31} do not exist, because no citizen would vote strategically for their least favored candidate. Thus, y_{ij} solves

$$-\beta(y_{ij} - x_i)^2 - (1-\beta)(y_{ij} - x_k)^2 = -(y_{ij} - x_j)^2, \tag{6}$$

when such a solution exists, and it is $-\infty$ (for S_{12}) or ∞ (for S_{32}), otherwise. Let $M_{ij} = \Phi(S_{ij})$ be the associated maximal number of expressive i supporters who may vote strategically for candidate j.

The following is immediate.

Lemma 1 *There exist $\bar{\beta}_{ij}$ such that:*

1. *M_{ij} is strictly decreasing in β for $\beta \geq \bar{\beta}_{ij}$ and $M_{ij} = 0$ for $\beta = 1$.*
2. *If $\beta \leq \bar{\beta}_{ij}$ then $M_{12} = \Phi(E_1)$, $M_{32} = \Phi(E_3)$, $M_{21} = \Phi([\theta_{12}, \theta_{13}])$ and $M_{23} = \Phi([\theta_{13}, \theta_{23}])$.*

5.1 The Centrist's Strategic Advantage

We first show in Proposition 2 that if the centrist candidate 2's base support is larger than that of the non-centrists i.e., if $N_2 > N_1$, then in the unique equilibrium all citizens vote for their top-ranked candidate and the centrist wins. This result reflects that (i) no voter views the centrist as their least favorite candidate, (ii) no voter will ever vote for their least favored candidate, and (iii) expressive centrist (candidate 2) supporters get their most-preferred candidate if everyone votes for their top-ranked choice. While this result might seem to suggest that the candidate with the greatest ex-ante support always wins, this is not true in general. Indeed, Proposition 3 shows that the vote advantage of a leading non-centrist candidate must be sufficiently large to win in a pure coalitional voting equilibrium.

Proposition 2 *If $N_2 \geq N_1 \geq N_3$, then a pure coalitional voting equilibrium exists in which citizens vote according to their expressive preferences and the centrist candidate 2 always wins. If $N_2 > N_1$, then this equilibrium is unique.*

The result reflects that in this equilibrium, expressive supporters of the centrist candidate 2 always get their most preferred outcome—they vote for candidate 2 and 2 wins. Thus, they have no incentive to vote strategically. Further, it does not make sense for expressive supporters of non-centrist candidates 1 and 3 to vote strategically for the centrist because the centrist already wins. So, too, supporters of say candidate 1 would not vote strategically for candidate 3, because they are better off if the centrist wins.

Proposition 2 also rules out the existence of mixed coalitional equilibria in which a candidate other than the centrist wins with strictly positive probability. In particular, we rule out coordination-failure equilibria among candidate 2's expressive supporters in which candidate 2 loses because some of her supporters vote for candidate 1 to keep candidate 3 from winning, while others vote for candidate 3 to prevent candidate 1 from winning.

The result of Proposition 2 does not extend to a non-centrist who has the most base support, i.e., when $N_1 > N_2$; this non-centrist must have a large base support advantage to ensure victory:

Proposition 3 *There exists a pure coalitional voting equilibrium in which candidate 1 wins if and only if $N_1 \geq \max\{N_2 + M_{32}, N_3 + M_{23}\}$. When this equilibrium exists then all citizens vote expressively.*

Recall from Lemma 1 that the maximal strategic vote by expressive 2 and 3 supporters for candidates 3 and 2, respectively, M_{23} and M_{32}, are strictly decreasing in β unless β is too small, and are strictly positive for $\beta > 0$. Thus, $N_1 > \max\{N_2, N_3\}$ only guarantees candidate 1's victory if voters place enough weight on expressive payoffs. If that weight is decreased, then a non-centrist candidate with an expressive plurality can only be assured of victory if she has a sufficiently large vote advantage.

For candidate 1 to win, two types of strategic voting have to be ruled out. First, more rightist supporters of candidate 2 cannot have an incentive to vote strategically for candidate 3, i.e., M_{23} cannot be too large, where we recall that M_{23} is decreasing in β. Second, enough candidate 3 supporters must not have an incentive to support the centrist candidate that they would overcome candidate 1's initial advantage. Indeed, when $\beta \leq (x_3 - x_2)/(x_3 - x_1)$, all candidate 3 supporters are better off supporting the centrist, and hence candidate 1 can only win if he has an expressive majority.

It follows that if $N_1 > N_2$ but the base support vote gap is not large enough (given β), then any equilibrium must involve strategic voting in which some citizens vote with positive probability against their expressive preferences. We next characterize when such coalitional voting equilibria arise and their properties.

5.2 Pure coalitional voting equilibria

We first show that it is "easy" for the centrist candidate 2 to draw strategic support from candidate 3 supporters when non-centrist candidate 1's base advantage over 2 is small enough. In particular, this pure coalitional voting equilibrium only requires that at least $N_1 - N_2$ moderate expressive supporters of candidate 3 be willing to coordinate on candidate 2 in order to defeat candidate 1, whom they least prefer. Importantly, when this is so, candidate 1 cannot win any strategic support—since expressive supporters of candidate 2 get their preferred winner in this equilibrium, they do not want to deviate from voting expressively.

Proposition 4 *Suppose that $N_1 \geq N_3$ and $N_1 > N_2$. Then:*

1. *There exists a pure coalitional equilibrium in which candidate 2 wins if and only if $N_2 + M_{32} > N_1$.*
2. *Candidate 2 wins with probability 1 in all coalitional voting equilibria if and only if $N_2 + M_{32} > N_1$ and $N_3 + M_{23} \leq N_1$.*

The proposition shows that candidate 2 can win even without a plurality of expressive votes. The reason is that candidate 2 is the second choice for all expressive candidate 3 supporters, and hence some of them are willing to vote strategically to prevent candidate 1 from winning. This result is a mirror image of Proposition 3 that the leading non-centrist needs a vote cushion to win.

While all voters who expressively prefer candidate 2 will vote for 2, the set of voters who expressively prefer candidate 3, but strategically coordinate on candidate 2 can be given by any set of expressive candidate 3 voters with measure at least $N_1 - N_2$. However, the size of the coalition is indeterminate and not pinned down by inclusivity. The key feature is that in all such equilibria, the strategic support for candidate 2 is enough to ensure 2's victory—all equilibria take the same qualitative form, so there is no need to further refine the set.

When $N_3 + M_{23} \leq N_1$, the set of expressive candidate 2 supporters willing to vote for candidate 3 is not large enough for strategic voting to deliver victory for candidate 3. If equality holds, candidate 3 would only win half the time, and the maximal coalition M_{23} would not form, as common interest would be violated.

We next consider what happens when this condition is violated so that there are, in principle, enough strategic voters for candidate 3 to win. We show that there is either a pure coalitional voting equilibrium in which candidate 3 wins due to strategic voting by expressive supporters of the centrist, or there is a mixed coalitional voting equilibrium in which leftist supporters of centrist candidate 2 vote strategically for candidate 1, and rightist supporters for candidate 3. The (pure or mixed) form of such equilibria depends on the relative numbers of expressive centrist supporters who are prepared to vote strategically for candidate 3 vs. 1.

We first determine when a pure strategy equilibrium exists in which enough rightist candidate 2 expressive supporters coordinate on candidate 3 to defeat candidate 1. For this equilibrium to exist, there must not be a coalition of left-of-center voters that could flip the election back to candidate 1 by voting strategically.

Proposition 5 *Suppose that $N_1 \geq N_3$ and $N_1 > N_2$. Then a pure coalitional voting equilibrium exists in which candidate 3 wins with probability 1 if and only if $N_3 + M_{23} \geq N_1 + M_{21}$. Further, the margin of victory over candidate 1 must be at least M_{21}.*

The condition $N_3 + M_{23} \geq N_1 + M_{21}$ means that candidate 3 draws enough strategic support to defeat candidate 1, and the latter cannot gather sufficient strategic vote to flip back the election. In particular, candidate 3 must be able to differentially attract enough more strategic support than candidate 1. This implies that when $\rho_1 > \rho_3$, this condition cannot hold if candidate platforms are symmetrically located, i.e., $x_2 = 0$, $x_1 = -x_3$ and the distribution Φ of non-partisans is symmetric, because then $M_{23} = M_{21}$.

The set of voters who expressively prefer candidate 2, but strategically coordinate on candidate 3 is again not uniquely pinned down. What is pinned down is that (i) they must all prefer to vote for candidate 3 in order to deliver candidate 3's victory rather than vote expressively and have candidate 1 win, and (ii) together with the expressive supporters of candidate 3, they must comprise a measure of at least $N_1 + M_{21}$.

A casual observer of this equilibrium outcome might conclude that there is "excessive strategic coordination" by right-of-center voters on candidate 3 because candidate 3 receives at least M_{21} more votes than candidate 1. However, this conclusion is misplaced because if fewer right-of-center voters coordinated on candidate 3 (and instead voted expressively for candidate 2), then left-of-center voters would have an incentive to coordinate on candidate 1 in sufficient numbers to defeat candidate 3, making those right-of-center voters worse off and breaking the equilibrium.

5.3 Mixed Coalitional Voting Equilibria

When enough expressive supporters of candidate 2 on both sides are willing to vote strategically to defeat the non-centrist candidate whom they like least, the equilibrium must involve randomization, as each non-centrist must have a chance of winning in order to draw strategic support from expressive candidate 2 supporters. Proposition 6 identifies conditions under which a mixed coalitional voting equilibrium exists. These conditions comprise a superset of the set of parameter values for which pure strategy equilibria do not exist. Hence, Proposition 6 ensures existence of equilibria for all parameters.

Proposition 6 *Suppose that $N_1 \geq \max\{N_2, N_3\}$. Then:*

1. *There exists a mixed coalitional voting equilibrium if and only if $N_1 + M_{21} \geq N_3 + M_{23} > N_1$.*

2. *If the mixed coalitional voting equilibrium exists then it is unique.*

3. *In the mixed coalitional voting equilibrium some expressive candidate 2 supporters vote for candidate 1 and some vote for candidate 3 with positive probability, and the identity of the winning candidate depends on the realized coalitions.*

The proof in the Appendix provides the explicit construction of the equilibrium with stochastic coalition formation. To understand the key ingredients, first observe that candidate 3 must gain enough strategic votes to overcome candidate 1's base advantage of $N_1 - N_3$ to have a chance of victory. Thus, when non-zero, candidate 3's strategic support must be at least $[\theta_{23} - Z, \theta_{23}]$, where Z is the vote gap, implicitly defined by $N_3 - N_1 = \Phi(\theta_{23}) - \Phi(\theta_{23} - Z)$. When non-zero, the sets of strategic voters for candidates 1 and 3 are given by intervals of the form $[\theta_{12}, \theta_{12} + z_1]$ with $z_1 \geq 0$, and $[\theta_{23} - Z - z_3, \theta_{23}]$ with $z_3 \geq 0$. Candidate 3 wins if $\Phi(\theta_{12} + z_1) - \Phi(\theta_{12}) < \Phi(\theta_{23} - Z) - \Phi(\theta_{23} - Z - z_3)$, and candidate 1 wins if the inequality is reversed.

The key to the construction of the equilibrium distribution is that when a strategic coalition forms, the marginal coalition member, say $\theta_{12} + z_1$, must have the same expected payoff when coalition $[\theta_{12}, \theta_{12} + z_1]$ forms as when it does not. Were $\theta_{12} + z_1$ to strictly prefer that the coalition form, it would violate inclusivity, because there would then be a set T of expressive candidate 2 supporters close to $\theta_{12} + z_1$, who are currently outside the strategic voter coalition, but would all receive a uniformly higher expected payoff (by at least $\epsilon > 0$ for all of them) by joining versus not having the coalition form. So too, type $\theta_{12} + z_1$ cannot be strictly worse off if the coalition forms, else it would violate common interest.

These indifference conditions pin down the distributions over strategic coalitions. Equilibrium is described by probabilities q_1 and q_3 that no coalition forms, and distributions $F_1(z_1)$ and $F_3(z_3)$ over the marginal coalition member when a coalition forms. F_1 and F_3 are continuous without mass points, reflecting the discontinuity of payoffs at a tie. Further, there must be a tie when the largest equilibrium coalitions form.

To see what pins down q_1, observe that when $Z > 0$, a voter at $\theta_{23} - Z$ incurs an expressive cost from strategic voting that is bounded away from zero. To make $\theta_{23} - Z$ indifferent, when a coalition forms the winning probability for candidate 3 must exactly offset the expressive costs. That winning probability is the probability, q_1, that no coalition forms for candidate 1.

To see what pins down q_3, first observe that when the largest coalition for candidate 3 forms, candidate 3 is sure to win, but if no strategic coalition forms then candidate 3 loses because $Z > 0$. Thus, the marginal voter of the largest coalition for candidate 3 is y_{23} (defined in (6)). In contrast, the marginal voter $\bar{z}_1 \leq y_{12}$ because candidate 1 wins with probability q_3. In particular, $\bar{z}_1$ is a decreasing function of q_3, and q_3 is pinned down by the requirement that there has to be a tie when the largest coalitions are realized on both sides.

Mixed coalitional voting equilibria can take two qualitatively different forms reflecting that there are two possibly opposing effects at play:

1. One non-centrist candidate may have a greater base support, N_i. Without loss of generality we assume that this is the case for candidate 1.

2. One non-centrist candidate's platform may be closer to more expressive supporters of the centrist candidate than her rival's platform, in which case she can differentially attract strategic support.

As a result, there are two different types of mixed coalitional equilibria, one where the two effects go in opposite directions, and one where they work in the same direction. The two different cases are shown in Figures 1 and 2. We now illustrate these cases in more detail, starting with Example 1 where candidate 1 has more base support, but candidate 3 is better able to attract strategic votes. The example also illustrates the contents of Propositions 2–6 by numerically solving for the different pure and mixed coalitional equilibria. Examples 1 and 2 also introduce the comparative statics that we formally analyze in Section 5.4.

Example 1 Suppose that the candidate locations are $x_1 = -1$, $x_2 = 0$, $x_3 = 0.4$. Let $\rho_1 = 0.2$ and $\rho_2 = \rho_3 = 0$. Suppose that Φ is a uniform distribution on $[-1, 1]$. Then the cutoffs between the parties are $\theta_{12} = -0.5$, $\theta_{13} = -0.3$, and $\theta_{23} = 0.2$. As a consequence, $N_1 = 0.45$, $N_2 = 0.35$, and $N_3 = 0.4$.

Solving for the maximal sizes of the sets of strategic voters yields:

$$M_{21} = \frac{7}{10} \cdot \frac{1-\beta}{7-2\beta}, \qquad M_{23} = \frac{7}{4} \cdot \frac{1-\beta}{7-5\beta}, \text{ and } M_{32} = \begin{cases} \frac{7}{4} \cdot \frac{1-\beta}{7\beta-5} & \text{if } \beta \geq \frac{75}{91}; \\ 0.4 & \text{otherwise.} \end{cases} \tag{7}$$

Proposition 3 shows that a pure coalitional equilibrium in which candidate 1 wins exists if $N_1 \geq \max\{N_2 + M_{32}, N_3 + M_{23}\}$. $N_1 \geq N_2 + M_{32}$ if and only if $\beta \geq 45/49 \approx 0.9184$, and $N_1 \geq N_3 + M_{23}$ if and only if $\beta \geq 14/15 \approx 0.9333$. Thus, the condition of Proposition 3 holds if and only if $\beta \geq 14/15$. Next, Proposition 4 show that candidate 2 wins in a pure coalition equilibrium if and only if $N_2 + M_{32} > N_1$. This condition holds if $\beta < 45/49$. The last type of pure strategy equilibrium is characterized by Proposition 5. In particular, $N_1 > N_2$. and thus it remains to verify when $N_3 + M_{23} \geq N_1 + M_{21}$. This inequality holds if and only if $\beta < 0.7(\sqrt{69} - 7) \approx 0.9146$, and a pure coalition equilibrium where candidate 3 wins exists for these values of β.

Thus, no pure coalitional equilibrium exists when $0.7(\sqrt{69} - 7) < \beta < 14/15$. Instead, there exists a mixed coalitional voting equilibrium in which leftist candidate 2 supporters sometimes vote for candidate 1, and rightist candidate 2 supporters sometimes vote for candidate 3. For $0.7(\sqrt{69} - 7) \leq \beta < 45/49$ this mixed coalitional voting equilibrium co-exists with the pure coalitional voting equilibrium in which candidate 2 wins due to strategic support by expressive candidate 3 supporters. In the mixed coalitional equilibrium candidate 1's winning probability increases in β, as base support advantage matters more when β is larger.

The left panel of Figure 1 illustrates the coexistence of equilibria. The right panel of Figure 1 shows the probabilities with which some coalition forms that includes candidate 2 expressive supporters who vote

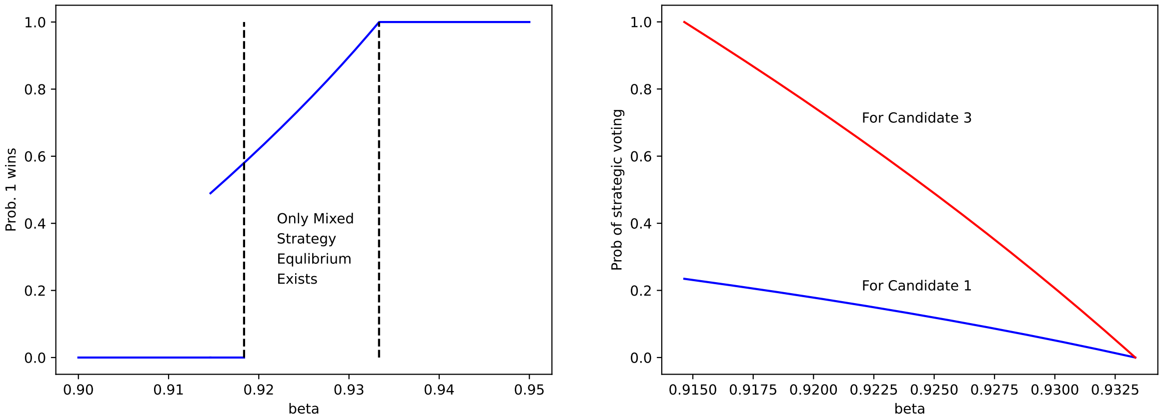

Figure 1: Left: Candidate 1's winning probabilities in all equilibria as a function of β. Right: Probability expressive candidate 2 supporters strategically coordinate on candidates 1 and 3 as a function of β.

strategically for extreme candidates 1 and 3. Posed differently, one minus these probabilities yields the probabilities q_1 and q_3 that strategic coalitions do not form for the respective extreme candidates. ■

Example 2 illustrates the nature of the mixed coalitional voting equilibrium when candidate 1 has both the base and strategic support advantage.

Example 2 Figure 2 presents outcomes when candidate positions are $x_1 = -0.6$, $x_2 = 0$, $x_3 = 0.64$ and, $\rho_1 = \rho + 0.01, \rho_2 = 0, \rho_3 = \rho$. Now $\theta_{12} = -0.3$, $\theta_{23} = 0.32$, and hence $N_1 = 0.36 + \rho$, $N_2 = 0.31$, $N_3 = 0.32 + \rho$. Because $x_2 = 0$ is further from x_3 than from x_1, and zero is also the position of the median voter, candidate 1 has an advantage over candidate 3 in attracting strategic voters, as the figure on the right illustrates.

When ρ is sufficiently large, e.g., $\rho \geq 0.3$, then a pure coalitional voting equilibrium does not exist in which candidate 2 wins, because even if all expressive candidate 3 supporters voted for candidate 2, candidate 1 would still win. This means that there is a unique equilibrium. For sufficiently large β, candidate 1 always wins. For lower values there is only the mixed coalitional voting equilibrium.

When, as in this example, the same candidate has both the ex-ante vote advantage and also appeals more strongly to potential strategic voters that candidate must always win when β is very large or very small. This delivers the ∪-shaped relationship between β and candidate 1's probability of winning. In the figure on the right this is reflected by the fact that candidate 3 has no strategic support both for $\beta = 0$ and for β sufficiently large. For large β there is no strategic voting, but candidate 1's base vote advantage ensures victory. ■

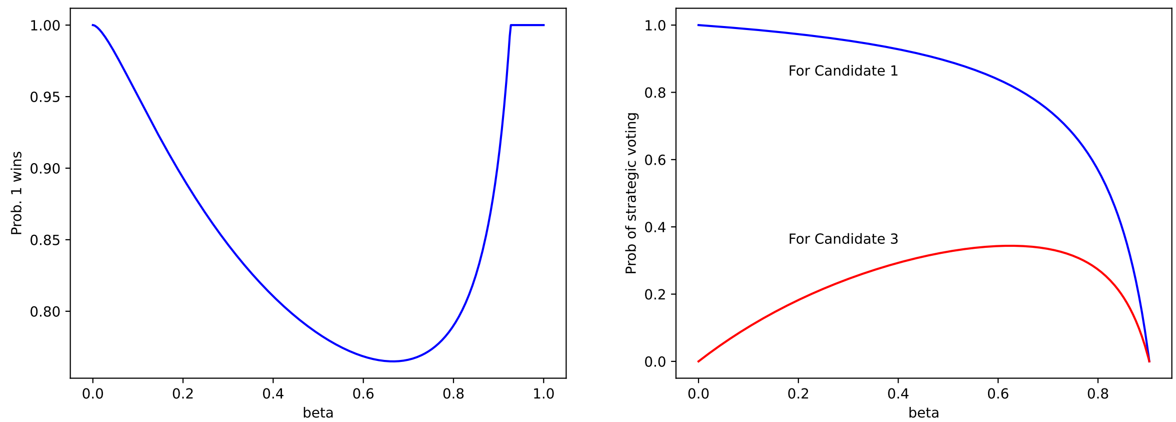

Figure 2: Left: Candidate 1's winning probabilities in the mixed coalitional voting equilibrium as a function of β. Right: Probability expressive candidate 2 supporters strategically coordinate on candidates 1 and 3 as a function of β.

5.4 Continuity of Mixed Coalitional Equilibria and Comparative Statics

We now show that the mixed coalitional equilibria are well behaved with natural comparative statics. It is immediate that as $\beta \to 1$ mixed coalitional equilibria disappear, converging to the outcome with purely sincere voting. In fact, unless $N_1 = N_3$ the mixed coalitional voting equilibrium disappears before reaching $\beta = 1$ as Figure 2 illustrates. When $\beta \to 0$ such continuous behavior is less immediate because the mixed coalitional voting equilibrium does not disappear.

Recall in Example 2 that candidate 1's winning probability converges to 1 as $\beta \to 0$, which corresponds to the pure coalitional voting equilibrium at $\beta = 0$ in which enough center-left voters strategically support candidate 1 that there are not enough rightist candidate 2 supporters who could swing the election to candidate 3. We now show that this result is general in nature.

Proposition 7 *Let $N_1 > \max\{N_2, N_3\}$ and suppose that the mixed coalitional voting equilibrium exists for all $\beta > 0$ in the neighborhood of $\beta = 0$. Then the winning probabilities in the mixed coalitional equilibrium converge to those with purely instrumental voting, i.e., when $\beta = 0$. That is, candidate 1 wins with probability approaching 1 if $N_1 + \Phi([\theta_{12}, \theta_{13}]) > N_3 + \Phi([\theta_{13}, \theta_{23}])$, and candidate 3 wins with probability approaching 1 if the opposite strict inequality holds.*

Proposition 7 shows that the mixed coalitional voting equilibrium is well-behaved, converging continuously in electoral outcomes to those at $\beta = 0$, where the equilibrium is in pure strategies with the non-centrist candidate who is preferred by the majority of voters winning. We next show that the mixed coalitional voting equilibrium also has the natural comparative static features that when a non-centrist's ex ante advantage is greater, that candidate is more likely to win and voters are less likely to engage in strategic behavior by voting against their expressive interests.

Proposition 8 *Suppose $N_1 > \max\{N_2, N_3\}$, and that the mixed coalitional voting equilibrium exists. In the mixed coalitional voting equilibrium, an increase in ρ_1 to ρ_1' that increases candidate 1's base support:*

1. *reduces the probability of strategic voting for both candidates 1 and 3, i.e., $q_1' > q_1$ and $q_3' > q_3$.*
2. *reduces the measure of the largest coalition for candidate 1 by $\rho_1' - \rho_1$ and shifts the distribution over the measure of realized strategic coalitions for candidate 1 to the left by $\rho_1' - \rho_1$.*
3. *does not affect the measure of the largest coalition for candidate 3, but increases the measure of the smallest (non-trivial) strategic coalition that forms for candidate 1 by $\rho_1' - \rho_1$.*
4. *increases the probability that candidate 1 wins.*

5.5 Non-monotonicities of Winning Probabilities in Candidate Positions

We next illustrate how candidate 2's winning probability can be a non-monotone function of its platform x_2 as a precursor to our analysis of a spoiler's entry decision. We first show that with strategic voting, a centrist candidate may be able to increase its probability of winning by moving further away from the stronger candidate even though this increases the stronger candidate's expressive support, and does not increase its own.

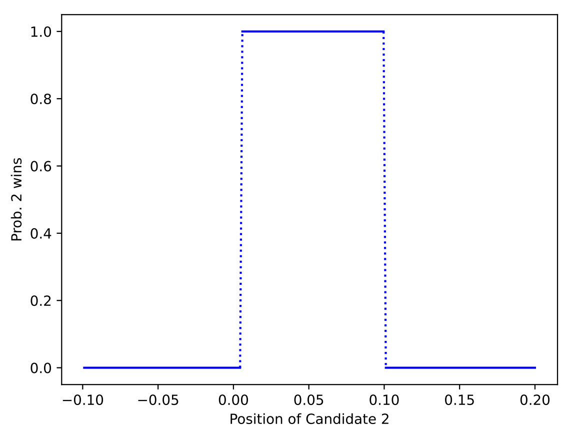

Figure 3: Candidate 2's winning probability as a function of x_2.

Figure 3 illustrates this in a setting where voters are uniformly distributed on $[-1, 1]$. The candidate positions are $x_1 = -0.1$, $x_3 = 1$, and the weight on expressive preferences is $\beta = 0.2$ and there are no partisans, i.e., all $\rho_i = 0$. Now as x_2 is shifted to the right, away from candidate 1, N_1 rises, N_2 stays unchanged, and N_3 falls. However, with strategic voting, this does not imply that candidate 2's winning probability falls.

With expressive voting, candidate 1 would always win whenever $x_1 < x_2 < x_3$. Now suppose that instrumental preferences matter. As candidate 2 shifts x_2 to the right, it has three effects. First it increases N_1, increasing the amount of strategic voting needed to defeat candidate 1. Second, x_2 increasingly differentiates itself from x_1 by locating closer to expressive candidate 3 supporters. This starker contrast with candidate 1 makes strategic voting more attractive if it can deliver a victory for candidate 2. Third, locating closer to candidate 3 also facilitates strategic voting by reducing the expressive loss to strategic voters. At $x_2 < 0$, candidate 2 fails to attract strategic voters, both due to insufficient differentiation from candidate 1 and because the expressive voting cost is too high. For somewhat larger x_2, strategic voting occurs, until $x_2 = 0.1$. For $x_2 > 0.1$, candidate 1's expressive vote share is large enough to render forming a sufficiently large strategic coalition infeasible, implying that candidate 1 always wins.

6 Political Competition with an Extreme Spoiler Party

6.1 Overview

In this section, we illustrate how one can endogenize the platform choices of two policy-motivated parties. We consider an otherwise symmetric setting in which we consider the entry choice by a third, extreme spoiler "citizen-candidate" party. We contrast the equilibrium platform choices with strategic and purely-expressive voting. We focus on a setting with two mainstream parties that have symmetrically opposing ideal policies $\theta_1 = -\theta_2$. The position of party 3 is fixed at its ideal policy $x_3 = \theta_3 > \theta_2$. We assume that ρ_3 is sufficiently small relative to ρ_1 and ρ_2 so that candidate 3 cannot win. We use a symmetric setting to ensure existence of a local equilibrium.[8] For similar reasons we assume that the spoiler is a citizen candidate who only chooses whether or not to enter, before the other candidates choose their platforms.

We now assume that when candidates 1 and 2 simultaneously choose policy positions x_1 and x_2 there is uncertainty about the extent of partisan support ρ_i for each party i. In a standard two-candidate model with policy motivation, candidates face a basic tradeoff between moving away from the other candidate's policy by locating closer to their own ideal point versus increasing their chance of winning by moving closer to their rival. This calculus can change when voters are strategic. In particular, consider again the example depicted in Figure 3. The spoiler now siphons off voters on the far right, disadvantaging candidate 2. By moving towards the spoiler and away from candidate 1, candidate 2 increases candidate 1's expressive support, hurting herself. However, this shift and resulting differentiation from candidate 1 raises candidate 2's appeal to

[8]For general non-symmetric settings of Wittman (1983) there are no known existence results.

the spoiler's supporters, more than offsetting the first effect. In response, candidate 1 also moves to the right to reduce differentiation with candidate 2, and hence the likelihood of strategic support for candidate 2.

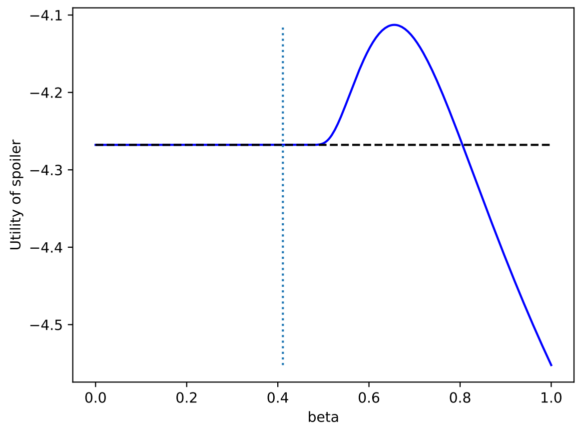

Figure 4: The spoiler's expected utility as a function of β for $-\theta_1 = \theta_2 = 1$, Φ is $N(0,1)$ distributed, and G is a $N(0,1)$ distribution truncated to the interval $[-1,1]$.

To summarize, the possibility of strategic voting induces candidates 1 and 2 to move toward the spoiler: Candidate 1 does it to reduce strategic voting, while candidate 2 does it to increase strategic voting. We will show that in equilibrium candidate 2 moves further to the right than candidate 1, implying that candidate 1 becomes more likely to win. Thus, whether the spoiler benefits from these shifted positions depends on the relative magnitudes of the rightward shift versus the change in winning probabilities. We show that the first effect dominates the second whenever voters have both sufficient expressive and strategic concerns.

Figure 4 illustrates the spoiler's utility as a function of β when $-\theta_1 = \theta_2 = 1$, $\theta_3 = x_3 = 2$, Φ is a standard normal distribution and G is a standard normal distribution truncated to the interval $[-1,1]$. The dotted vertical line indicates the critical value $\bar{\beta}$ at which the spoiler begins to draw votes. For $\beta \leq \bar{\beta}$, non-partisan voters care so much about who wins the election that do not ever vote for the spoiler—even if their ideal position is arbitrarily far to the right. Increasing β past $\bar{\beta}$, first causes candidates 1 and 2 to move their platforms to the right in order to alter the incidence of strategic voting for candidate 2. As a result, the spoiler's utility first rises as voters care more and more about their expressive component of preferences before falling sharply as β approaches 1. That is, candidate 3 is best off when voters' instrumental preferences are intermediate, neither too low nor too high.

Figure 5 provides intuition. For intermediate values of β the spoiler's entry induces both mainstream candidates to move toward the spoiler, and this "closer location" effect more than offsets the reduction in

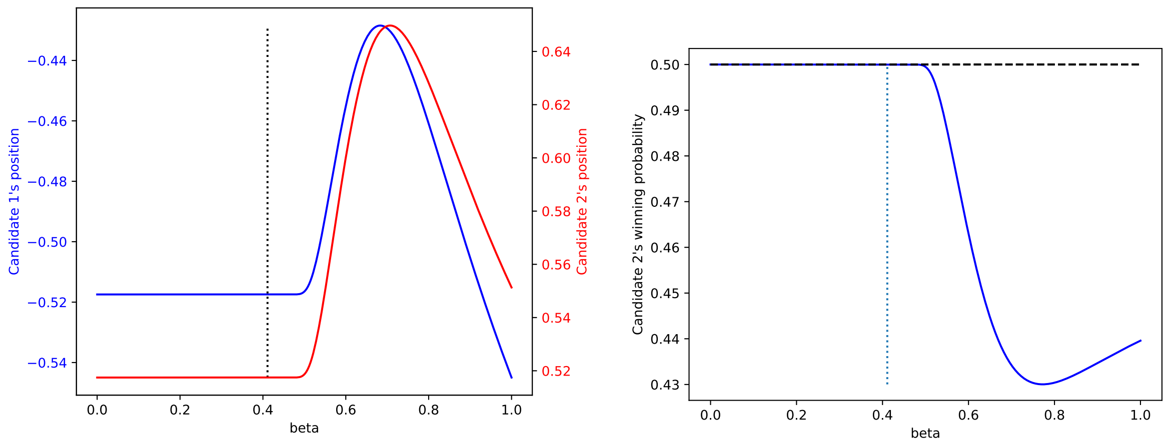

Figure 5: Candidate positions and candidate 2's winning probability as a function of β for $-\theta_1 = \theta_2 = 1$, Φ is $N(0,1)$ distributed, and G is a $N(0,1)$ distribution truncated to the interval $[-1,1]$.

candidate 2's winning probability,[9] in the spoiler's expected utility. In contrast, when instrumental motives are stronger, the mainstream parties ignore the spoiler; and when expressive considerations are very strong, the spoiler is hurt in multiple ways. First, once β is sufficiently high, further increases in β increasingly disadvantage candidate 2, because right wing citizens are more likely to vote for the spoiler and less likely to vote strategically to defeat candidate 1. In turn, reflecting this difficulty, candidate 2 starts to retreat away from the spoiler to reduce candidate 1's base support. Further, candidate 1 also moves away from the spoiler due to the reduced risk of strategic voting. This also underlies the non-monotonicity of candidate 2's winning probability in Figure 5. In particular, once β becomes sufficiently large, reducing strategic voting matters less to candidate 1, causing her to shift her platform to the left by enough that 2's winning probability actually rises.

6.2 General Analysis

We next formally describe the model. Because candidate 3 is a spoiler with zero probability of winning, only the net-stalwart difference $\rho = \rho_2 - \rho_1$ matters. First candidates choose policy positions x_1 and x_2, and then ρ is realized. Let G be the cdf of ρ.

Assumption 2

1. *G is twice continuously differentiable, with a density g that is symmetric around 0.*

[9] Although this example is just illustrative, Pons and Tricaud (2018) use a regression discontinuity design to show that the presence of a spoiler in French parliamentary and local elections reduces the chances of the ideologically-closest candidate by about one-fifth. Our model can reconcile why the spoiler may want to enter despite the impact on winning probabilities.

2. *The distribution of voter types Φ is twice continuously differentiable and symmetric around zero.*

3. *The fourth moment of Φ is finite.*

Recall that y_{32}, defined in (6), is the most extreme voter type that would be prepared to vote for candidate 2 in order to defeat candidate 1. The value of y_{23} is finite if $\beta \leq \bar{\beta}$, where $\bar{\beta} = \frac{x_2 - x_1}{x_3 - x_1}$. In this case

$$y_{32} = \frac{1}{2}\frac{(1-\beta)x_1^2 + \beta x_3^2 - x_2^2}{(1-\beta)x_1 + \beta x_3 - x_2}. \tag{8}$$

Note that y_{32} strictly decreases in x_1, i.e., M_{32} decreases: By shifting x_1 to the right, candidate 1 can reduce the ex-ante probability that voters strategically coordinate on candidate 2 to defeat 1. Conversely, increasing x_2 causes y_{32} and M_{32} to increase. That is, just as candidate 1 can reduce strategic voting for candidate 2 by making her policy more attractive to right-wing voters, candidate 2 can increase her strategic support from right-wing voters by making her policy more attractive to them. Thus, $\partial y_{32}/\partial x_1 < 0$ and $\partial y_{32}/\partial x_2 > 0$.

The votes for candidates 1 and 2 if there is strategic voting are $V_1 = \Phi((x_1 + x_2)/2)$, and $V_2 = \Phi(y_{32}) - \Phi((x_1 + x_2)/2) + \rho$, respectively. Strategic voting will occur if and only if the coalition $[(x_1 + x_2)/2, y_{32}]$ suffices to deliver victory to candidate 2. Let $\bar{\rho}$ be the value of ρ at which $V_1 = V_2$, i.e.,

$$\bar{\rho} = 2\Phi((x_1 + x_2)/2) - \Phi(y_{32}). \tag{9}$$

Candidate 1's winning probability is $G(\bar{\rho})$.

The candidates' optimization problems are therefore given by

$$\max_{x_1} -G(\bar{\rho})(x_1 - \theta_1)^2 - (1 - G(\bar{\rho}))(x_2 - \theta_1)^2, \text{ and} \tag{10}$$

$$\max_{x_2} -G(\bar{\rho})(x_1 - \theta_2)^2 - (1 - G(\bar{\rho}))(x_2 - \theta_2)^2. \tag{11}$$

Proposition 9 *Let $\theta_2 = -\theta_1$, $x_3 \geq \theta_2$, and Assumption 2 hold. Then there exists a $\bar{\beta}$ such that in equilibrium:*

- *For $\beta \leq \bar{\beta}$, instrumental considerations of voters dominate. The outcome is same as when candidate 3 is not present. The equilibrium platforms of candidates 1 and 2 are*

$$x_2 = -x_1 = \frac{\theta_2}{1 + 4\theta_2 g(0)\phi(0)}. \tag{12}$$

- *A sufficiently small increase in β above $\bar{\beta}$ causes both candidates 1 and 2 to shift x_1 and x_2 to the right, with x_2 shifting by more than x_1, reducing candidate 2's chance of winning but raising candidate 3's expected utility.*

In the proof we show that $\bar{\beta} = (x_2 - x_1)/(x_3 - x_1)$, where x_1 and x_2 are given by (12). When β is small enough—where small enough depends on the spoiler's location—nonpartisans care so much about who wins that they all vote for either candidate 1 or 2. As a result, political competition reduces to the classical two-candidate Wittman (1983) setting, with associated symmetric locations. A slight increase in β above $\bar{\beta}$ now means that the spoiler can steal votes from extreme right-wing voters away from candidate 2. In the proof, we use the implicit function theorem at $\bar{\beta}$ to show that this induces both mainstream candidates to shift their policies to the right, with candidate 2 moving further because rightward shifts move toward candidate 2's ideal policy and away from candidate 1's. It follows that candidate 1's probability of winning rises. However, a second application of the implicit function theorem reveals that the spoiler gains more from the rightward policy shifts of the two mainstream candidates than the spoiler loses from the increased probability that the spoiler's least preferred candidate wins. As a result, even though the spoiler steals votes away from her preferred mainstream candidate, she still gains from entry.

7 Conclusion

There is extensive evidence that voters care both about which candidate they vote for, and which candidate wins. In a two candidate setting, this distinction is irrelevant because expressive and instrumental concerns coincide. However, as recent polling data over potential Republican presidential primary candidates illustrates, this distinction matters with more than two candidates—51 percent preferred a candidate with the best chance of winning versus 44 percent who wanted to agree with the candidate on everything even if the candidate would have a tougher time winning in November.[10]

We develop a model of strategic voting in a spatial model with multiple candidates when voters have both expressive and instrumental concerns. The model endogenizes the strategic coordination of citizens on a less-preferred candidate in order to raise the chances of defeating an even less-preferred candidate. We fully characterize all coalitional voting equilibria in a three-candidate setting. We provide several important insights. First, even though elections may be close, one candidate may be systematically more likely to win, indicating that close elections may not be a good natural experiment.[11] Second, strategic voting does not only have to occur in close elections. Third, strategic voting can generate endogenous uncertainty about who wins. To highlight this, we assume away all extrinsic sources of uncertainty at the voting stage. The presence of endogenous uncertainty, in turn, may add to the difficulty of forecasting electoral outcomes even with accurate polling data, as voters efforts to coordinate strategically may necessarily be unpredictable.

A virtue of our formulation of strategic voting is that it is simple enough to incorporate into a standard

[10]See, FiveThirtyEight, "Which Republican Candidate Should Biden Be Most Afraid Of?" `https://fivethirtyeight.com/features/which-republican-candidate-should-biden-be-most-afraid-of/`

[11]See Levine and Martinelli (2022) who also make this point in a setting with campaign spending.

model of political competition with policy motivated candidates. To illustrate this, we endogenize candidate policy choices with the two mainstream candidate and a spoiler who understand that voters may coordinate strategically. We show that the spoiler can gain from entering, even though she has no chance of winning the election and reduces the winning probability of her preferred mainstream candidate. This occurs because both mainstream candidates partially incorporate the spoiler's platform by moving toward the spoiler.

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8 Appendix

Lemma 2 *Suppose that $N_2 > N_1, N_3$. Then there does not exist a mixed coalitional voting equilibrium in which leftist expressive supporters of candidate 2 vote stochastically for candidate 1, while rightist expressive supporters of candidate 2 vote stochastically for candidate 3.*

Proof of Lemma 2. We proceed by way of contradiction, assuming that there exists a mixed coalitional equilibrium. First note that all realized coalitions $S_i(\omega_i)$ must be intervals: Suppose by way of contradiction that $S_i(\omega_i)$ is not an interval. Then there exist $\theta_1 < \theta_2 < \theta_3$ with $\theta_1, \theta_3 \in S_i(\omega_i)$ and $\theta_2 \notin S_i(\omega_i)$. By common interest both θ_1 and θ_3 are at least as well off if the coalition is formed. Next, either θ_1 or θ_3 must be further away from x_2 than θ_2. Thus, citizen θ_2 is strictly better off if the coalition forms, violating inclusivity.

Note that $S_1(\omega_1) \supset E_1$ and hence must be unbounded from below. Let $S_1(\omega_1) = (-\infty, y_1(\omega_1))$ be an open interval. By common interest the net benefit of $y_1(\omega_1)$ from having a coalition must be non-negative. If the net-benefit is strictly positive, then inclusivity would be violated. Therefore the net-benefit of $y_1(\omega_1)$ from having a coalition is zero. The same argument applies if $S_1(\omega_1)$ is a closed interval, and the argument is analogous for $S_3(\omega_3)$. Further, because the distribution of voter types is continuous, we can restrict attention to closed intervals without loss of generality.

Let $z_1(\omega_1)$ be defined by $S_1(\omega_1) = (-\infty, z_1(\omega_1)]$. Let $\bar{\Omega}_1$ be the support of λ_1. Define $\underline{z}_1 = \inf_{\omega_1 \in \bar{\Omega}_1} z_1(\omega_1)$ s.t. $z_1(\omega_1) > \theta_{12}$. We show that $\underline{z}_1 > \theta_{12}$. We proceed by way of contradiction, i.e., we assume that $\underline{z}_1 = \theta_{12}$.

We consider different cases for the realized coalition for candidate 3 $\Phi(S_3(\omega_3))$, and show that conditional on each case occurring either inclusivity or common interest is violated.

Case 1: $\Phi(S_3(\omega_3)) < N_3 + 0.5(N_2 - N_3)$. We show that in this case any coalition $S_1 = (-\infty, z_1]$ either does not affect the outcome or flips the election away from candidate 2. This implies that conditional on case 1 occurring, all voters in $(\theta_{12}, z_1]$ are strictly worse off: Their instrumental payoff cannot increase and their expressive payoff is strictly decreased, violating common interest.

To see this, let $\Delta = \Phi(S_3(\omega_3)) - N_3$ be the number of expressive candidate 2 supporters who vote for candidate 3. Absent strategic voting for candidate 1, candidate 2 would receive more votes than candidate 3 if $N_2 - \Delta > N_3 + \Delta$, or equivalently if $\Phi(S_3(\omega_3)) < N_3 + 0.5(N_2 - N_3)$. Thus, if a non-trivial coalition S_1 forms, then the outcome can only flip away from candidate 2.

Case 2: $\Phi(S_3(\omega_3)) \geq N_3 + 0.5(N_2 - N_3)$ and $N_3 + 0.5(N_2 - N_3) < N_1$. I If $N_3 + 0.5(N_2 - N_3) < N_1$, then absent strategic voting candidate 1 wins. Thus, voting strategically for candidate 1 only incurs expressive costs without changing the outcome. Thus, common interest is violated for any coalition $S_1 = (-\infty, z_1]$ conditional on case 2 occurring.

Case 3: $\Phi(S_3(\omega_3)) \geq N_3 + 0.5(N_2 - N_3)$ and $N_3 + 0.5(N_2 - N_3) = N_1$. Because $\underline{z}_1 = \theta_{12}$ there exist ω_1 in the support of λ_1 where $z_1(\omega_1)$ is marginally larger than θ_{12}. Then voter $z_1(\omega_1)$ is strictly better off if the coalition forms because the winning probability of candidate 3 is decreased by a discrete amount at an arbitrarily small expressive cost. This violates common interest and inclusivity, because $z_1(\omega_1)$ must be indifferent between the coalition forming and not forming, a contradiction.

Case 4: $\Phi(S_3(\omega_3)) \geq N_3 + 0.5(N_2 - N_3)$ and $N_3 + 0.5(N_2 - N_3) > N_1$. Let $0 < \varepsilon < N_3 + 0.5(N_2 - N_3) - N_1$. Because $\underline{z}_1 = \theta_{12}$ by assumption, there exist ω_1 in the support of λ_1 with $\theta_{12} < z_1(\omega_1) < \theta_{12} + \varepsilon$. Then regardless of whether or not coalition $S_1(\omega_1)$ forms, candidate 3 wins: In expectation strategic voters for candidate 1 are worse off, violating common interest.

Similarly, let $\underline{z}_3 = \sup_{\omega_3 \in \Omega_3} z_3(\omega_3)$ s.t. $z_3(\omega_3) < \theta_{23}$. Then an analogous argument yields that $\underline{z}_3 < \theta_{23}$. Because the sets $S_1(\omega_1)$ and $S_3(\omega_3)$ are closed, it follows that $\underline{z}_i$, $i = 1, 3$ are the minimum and maximum respectively, i.e., there exist smallest strategic coalitions that are not equal to E_1 and E_3, respectively.

Next, we show that $\lambda_i(\{\omega_i | z_i(\omega_i) = \underline{z}_i\}) > 0$ cannot hold for both $i = 1, 3$. If both choose the smallest coalition (with bounds $\underline{z}_i$), there must be a tie between candidates 1 and 3. But then by marginally increasing the size of the coalition the winning probability can be increased by a discrete amount, violating inclusivity.

Now, suppose that $\lambda_3(\{\omega_3 | z_3(\omega_3) = \underline{z}_3\}) = 0$. Because there must be a tie between candidates 1 and 3 when the smallest strategic coalitions form (else we get an immediate contradiction), candidate 1 loses with probability 1 when coalition $S_1 = (-\infty, \underline{z}_1]$ forms. But then common interest is violated for all voters in $[\underline{z}_1, \theta_{23})$. A similar contradiction obtains when $\lambda_1(\{\omega_1 | z_1(\omega_1) = \underline{z}_1\}) = 0$. ■

Proof of Proposition 2. An equilibrium with only expressive voting exists if and only if it is not optimal for any group of citizens to vote strategically and make themselves better off in the process. If $N_2 \geq N_1$, then if candidate 2 supporters vote expressively so would all expressive supporters of other parties—common interest would be violated if expressive supporters of the other parties voted strategically for another party, as either candidate 2 still wins or they vote in sufficient numbers for their least favored party, that it wins.

Now suppose $N_2 > N_1$ and another pure coalitional voting equilibrium exists. In such a pure coalitional equilibrium there would have to be exactly two coalitions S_1 and S_3, and there has to be a tie between them—else it is immediate that common interest is violated. However, a marginal increase of say coalition S_1 would result in candidate 1 winning with probability 1, making the marginal voter strictly better off, thus violating inclusivity. ■

Proof of Proposition 3. Suppose that $N_1 \geq \max\{N_2 + M_{32}, N_3 + M_{23}\}$. Then it is immediate that candidate 1 wins: Even if the largest coalitions form for either candidates 2 or 3, then this coalitions are at most large enough to generate a tie. However, if only a tie can be achieved then common interest is violated because the marginal voters y_{23} or y_{32} are only indifferent when strategic voting results in a win with probability 1.

Conversely, suppose that we have an equilibrium in which candidate 1 always wins, but the condition is violated. First suppose that $N_2 + M_{32} > N_1$. Then there exists $\delta > 0$ such that coalition $S_2 = [\theta_{12}, y_{32} - \delta]$ still wins. All members of S_2 are strictly better off by at least some $\epsilon > 0$ if the coalitions is formed, which violates inclusivity, a contradiction. The argument for $N_3 + M_{23} > N_1$ is analogous.

Finally, when the condition holds then any strategic voting would violate common interest, because the electoral outcome would not change. Hence, everyone votes expressively. ■

Proof of Proposition 4. *Proof of statement 1:* Let $N_2 + M_{32} > N_1$. Because $N_2 < N_1$ candidate 2 loses unless there is strategic voting. Using the same argument as in the proof of Proposition 3, it follows that a strategic coalition would form for candidate 2 to win by inclusivity. Similarly, candidate 2 cannot win with just the votes of expressive candidate 3 supporters unless $N_2 + M_{32} > N_1$. Next, note that by common interest, candidate 1 supporters will not strategically vote for candidate 2 in a pure coalitional equilibrium, because candidate 1 already wins without strategic voting. Conversely, suppose that $N_2 + M_{32} \leq N_1$. Then as in the proof of Proposition 3 we can conclude that candidate 2 cannot win.

Proof of statement 2: If $N_2 + M_{32} > N_1$ holds then statement 1 shows that an equilibrium exists in which candidate 2 always wins. If $N_3 + M_{23} \leq N_1$ then there are not enough potential strategic voters to flip the election to candidate 3 in either a pure or mixed coalitional equilibrium.

Now suppose that either $N_2 + M_{32} \leq N_1$ or $N_3 + M_{23} > N_1$. The proof of statement 1 already shows that candidate 2 cannot win if $N_2 + M_{32} \leq N_1$. Thus, suppose that $N_3 + M_{23} > N_1$. Then we show in Propositions 5 and 6 that either an equilibrium exists in which candidate 3 always wins, or there exists a mixed coalitional equilibrium in which both candidates 1 and 3 win with strictly positive probability. ■

Proof of Proposition 5. First, note that $M_{21} > 0$ because Φ has a strictly positive density, and $\beta < 1$, i.e., some citizens marginally to the right of θ_{12} would be willing to vote strategically for candidate 1, if candidate 1 wins as a result. Thus, $N_3 + M_{23} > N_1$. This implies that there exists a sufficiently number of strategic voters such that candidate 3 could win. This would be a coalitional equilibrium unless it is the case that strategic voting by leftist candidate 2 supporters would flip the election back to candidate 1. However, such a strategic coalition would only form if and only if $M_{21} + N_1 > N_3 + M_{23}$. ■

Proof of Proposition 6. We first establish necessary conditions that must hold in any mixed coalitional voting equilibrium. The proof of Lemma 2 shows that coalitions are closed intervals, i.e., $S_1(\omega_1) =$

$(-\infty, \theta_{12}, \theta_{12} + y_1(\omega_1)]$ and $S_3(\omega_3) = [\theta_{23} - y_3(\omega_3), \infty)$. We can thus describe coalitions by the distributions of $y_i(\omega_i)$. Let $q_i = P(\{y_i(\omega_i) = 0\})$. Thus, q_i is the probability that nobody votes strategically for candidate i.

Claim 1: In any mixed coalitional voting equilibrium, the supports of the distributions F_i of $y_i(\omega_i)$ conditional on $y_i(\omega_i) > 0$ are intervals.

Proof: Suppose by way of contradiction that the support D_i of one of the distributions, say F_i over $y_i > 0$, does not have an interval support. Suppose that $i = 1$. Then there exists an open interval $V = (y_{1,L}, y_{1,H}) \cap D_1 = \emptyset$ with $y_{1,L}, y_{1,H} \in D_1$, and $0 < F_1(y_{1,L}) = F_1(y_{1,H}) < 1$. Let $y_{3,L}$ and $y_{3,H}$ be defined such that the vote ends in a tie when the endpoints of the intervals are $y_{1,L}$, $y_{3,H}$, respectively. There cannot be mass points at $y_{1,L}$ and $y_{1,H}$, else a marginal increase of the opposing coalitions via increases of $y_{3,L}$ or $y_{3,H}$ would make all coalition members strictly better off as the winning probability would be strictly increased. This implies that the winning probabilities of coalitions $S_{1,L} = (-\infty, y_{1,L}]$ and $S_{1,H} = (-\infty, y_{1,H}]$ are equal. Following the argument in Lemma 2 common interest and inclusivity imply that a voter at $y_{1,L}$ must be indifferent between being in the coalition or having no coalition form. But then voter $y_{1,H}$ must be strictly worse off if coalition $S_{1,H}$ forms, because the winning probability is the same, but $y_{1,H}$ has a strictly larger cost of voting against expressive preferences. Thus, common interest is violated for $S_{1,H}$, a contradiction. □

Claim 2: The vote shares must be equal if the largest coalitions are chosen.

Proof: Because all realized coalitions are intervals, the largest realized coalitions are $\bar{S}_1$ and $\bar{S}_3$ defined in Section 4.2. Suppose by way of contradiction that candidate 1 wins with a strict majority of votes if these coalitions form. Let $\bar{y}_1$ be the upper end point of the interval $\bar{S}_1$. Recall that indifference must hold at all $y_1(\omega_1)$ for all ω_1. However, in a neighborhood of $\bar{y}_1$ the winning probability of candidate 1 remains 1. Let y_1 be marginally smaller than $\bar{y}_1$. If coalitions $(-\infty, y_1]$ forms then y_1 must be indifferent between having the coalition form or not. However, because the winning probability remains 1 if y_1 is increased to $\bar{y}_1$, and $\bar{y}_1$ has a higher expressive cost of voting strategically for candidate 1, this implies that coalition $\bar{S}_1$ violates common interest. □

Claim 3: There is no point mass at the upper end of the distributions.

Proof: Suppose without loss of generality there is a point mass at $\bar{y}_1$. Claim 2 established that there must be a tie when the coalitions are maximal. But then coalition $\bar{S}_3$ could be marginally increased, resulting in a discrete increase in candidate 3's winning probability, violating inclusivity for coalition $\bar{S}_3$. □

Recall that $\theta_{ij} = 0.5(x_i + x_j)$. Let Z be the minimum amount of strategic voting for candidate 3 to have a chance of winning, i.e., Z solves

$$N_1 = N_3 + \Phi(\theta_{23}) - \Phi(\theta_{23} - Z). \tag{13}$$

If y_1 and y_3 are the cutoffs for strategic voters, then define $z_1 = y_1 - \theta_{12}$ and $z_3 = \theta_{23} - y_3 - Z$. Then $z_1, z_3 \geq 0$ for all coalitions that involve strategic voting, i.e., where $y_1 > \theta_{12}$ and $y_3 < \theta_{23}$. Define the total (expressive plus strategic) vote shares for candidates 1 and 3 by

$$H_1(z_1) \equiv N_1 + \Phi(\theta_{12} + z_1) - \Phi(\theta_{12}) \text{ and } H_3(z_3) \equiv N_3 + \Phi(\theta_{23}) - \Phi(\theta_{23} - Z - z_3). \tag{14}$$

Then candidate 1 wins if $H_1(z_1) > H_3(z_3)$. Because Φ is strictly increasing, this is equivalent to $z_1 > H_1^{-1}(H_3(z_3))$. Candidate 3 wins if the inequality is reversed.

Let $F_i(z_i)$ be the cdf that describes the distribution of the z_i. Recall that q_i as defined at the beginning of the proof is the probability that candidate i only receives expressive support.

The indifference condition for each realized marginal type y_1 is

$$\begin{aligned}
&-\beta(y_1 - x_2)^2 - (1-\beta)\left(q_3(y_1 - x_1)^2 + (1-q_3)(y_1 - x_3)^2\right) \\
&\quad = -\beta(y_1 - x_1)^2 - (1-\beta)\Big(\big(q_3 + (1-q_3)F_3(H_3^{-1}(H_1(z_1)))\big)(y_1 - x_1)^2 \\
&\qquad\qquad + \big(1 - q_3 - (1-q_3)F_3(H_3^{-1}(H_1(z_1)))\big)(y_1 - x_3)^2\Big).
\end{aligned} \tag{15}$$

The left-hand side is the expected payoff if all members of the realized coalition $S_1 = \{\theta : \theta \leq y_1\}$ who expressively prefer candidate 2 vote for 2, which leads to 1 winning if and only if $y_3 = \theta_{23}$, which happens with probability q_3, because $V_1^E > V_3^E$. The right-hand side is the expected payoff if all members of the realized coalition S_1 vote for candidate 1. Now, candidate 1 wins either when $y_3 \leq \theta_{23} - Z$, which happens with probability $1 - q_3$; or when the positive measure of the realized coalition S_3 is less than S_1, which happens when $z_3 \leq H_3^{-1}(H_1(z_1))$. Thus, candidate 1 wins with probability $F_3(H_3^{-1}(H_1(z_1)))$.

The analogous indifference condition to (15) for each realized marginal type y_3 is

$$\begin{aligned}
&-\beta(y_3 - x_2)^2 - (1-\beta)(y_3 - x_1)^2 \\
&\quad = -\beta(y_3 - x_3)^2 - (1-\beta)\Big(\big(q_1 + (1-q_1)F_1(H_1^{-1}(H_3(z_3)))\big)(y_3 - x_3)^2 \\
&\qquad\qquad + \big(1 - q_1 - (1-q_1)F_1(H_1^{-1}(H_3(z_3)))\big)(y_3 - x_1)^2\Big).
\end{aligned} \tag{16}$$

A citizen at $\theta_{13} = 0.5(x_1 + x_3)$ is indifferent between either extreme candidate winning. Thus, θ_{13} must be strictly outside any strategic voter coalition as long as voters place weight $\beta > 0$ on expressive preferences.

Note that

$$x_2 - x_1 = 2(\theta_{23} - \theta_{13}),\ x_3 - x_1 = 2(\theta_{23} - \theta_{12}), \text{ and } x_3 - x_2 = 2(\theta_{13} - \theta_{12}). \tag{17}$$

Solving equation (15) for F_3 using $y_1 = \theta_{12} + z_1$ and (17) yields

$$F_3(H_3^{-1}(H_1(z_1))) = \frac{\beta}{1-\beta}\frac{1}{1-q_3}\frac{(\theta_{23} - \theta_{13})\, z_1}{(\theta_{23} - \theta_{12})\,(\theta_{13} - \theta_{12} - z_1)}. \tag{18}$$

Similarly, solving (16) for F_1, using $y_3 = \theta_{23} - Z - z_3$ and (17) yields

$$F_1(H_1^{-1}(H_3(z_3))) = \frac{\beta}{1-\beta}\frac{1}{1-q_1}\frac{(\theta_{13} - \theta_{12})\,(Z + z_3)}{(\theta_{23} - \theta_{12})\,(\theta_{23} - \theta_{13} - (Z + z_3))} - \frac{q_1}{1-q_1}. \tag{19}$$

Thus,

$$F_1(z) = \frac{\beta}{1-\beta}\frac{1}{1-q_1}\frac{(\theta_{13}-\theta_{12})\left(Z+H_3^{-1}(H_1(z))\right)}{(\theta_{23}-\theta_{12})\left(\theta_{23}-\theta_{13}-(Z+H_3^{-1}(H_1(z)))\right)} - \frac{q_1}{1-q_1}; \tag{20}$$

$$F_3(z) = \frac{\beta}{1-\beta}\frac{1}{1-q_3}\frac{(\theta_{23}-\theta_{13})\,H_1^{-1}(H_3(z))}{(\theta_{23}-\theta_{12})\left(\theta_{13}-\theta_{12}-H_1^{-1}(H_3(z))\right)}. \tag{21}$$

Note that $H_3^{-1}(H_1(z))$ and $H_1^{-1}(H_3(z))$ are strictly monotone in z and therefore F_1 and F_3 are strictly increasing on their supports. Further, (13) implies $H_3^{-1}(H_1(0)) = 0$ and $H_1^{-1}(H_3(0)) = 0$. Hence, $F_3(0) = 0$.

From the definition of q_1, $F_1(0) = 0$. Substituting $F_1(0) = 0$ and $H_3^{-1}(H_1(0)) = 0$ in (20) we solve for:

$$q_1 = \frac{\beta}{1-\beta}\frac{(\theta_{13}-\theta_{12})Z}{(\theta_{23}-\theta_{12})(\theta_{23}-\theta_{13}-Z)}. \tag{22}$$

Note that $q_1 \geq 0$ because the interval of strategic voting $[\theta_{23}-Z-z_3, \theta_{23}]$ of candidate 2 supporters who vote for candidate 3 must be strictly to the right of the voter θ_{13} who is indifferent between candidates 1 and 3. If the solution has $q_1 \geq 1$ then there is no mixed coalitional voting equilibrium. Either candidate 1's base support advantage is sufficiently large to win (Proposition 3), or we get the pure coalitional voting equilibrium in which enough expressive candidate 3 supporters vote strategically for candidate 2 that 2 wins (Proposition 5).

Next, let $[0, \bar{z}_i]$ be the support of F_i. Then claim 3 implies $H_1(\bar{z}_1) = H_3(\bar{z}_3)$. Further, claim 4 implies that there cannot be a mass point at the upper end of either distribution. Thus, in this equilibrium we set $H_1(\bar{z}_1) = H_3(\bar{z}_3)$, which pins down q_3.

Setting the right-hand sides of (20) to 1 and solving for $\bar{z}_i$, $i = 1, 3$ yields

$$H_3^{-1}(H_1(\bar{z}_1)) = \frac{(1-\beta)(\theta_{23}-\theta_{12})(\theta_{23}-\theta_{13}) - (\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13}))Z}{\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13})}. \tag{23}$$

Similarly we get

$$H_1^{-1}(H_3(\bar{z}_3)) = \frac{(1-\beta)(1-q_3)(\theta_{13}-\theta_{12})(\theta_{23}-\theta_{12})}{(1-\beta)(1-q_3)(\theta_{23}-\theta_{12})+\beta(\theta_{23}-\theta_{13})}. \tag{24}$$

Assuming that $H_1(\bar{z}_1) = H_3(\bar{z}_3)$ we get

$$\begin{aligned} &H_3\left(\frac{(1-\beta)(\theta_{23}-\theta_{12})(\theta_{23}-\theta_{13}) - (\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13}))Z}{\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13})}\right) \\ &\qquad = H_1\left(\frac{(1-\beta)(1-q_3)(\theta_{13}-\theta_{12})(\theta_{23}-\theta_{12})}{(1-\beta)(1-q_3)(\theta_{23}-\theta_{12})+\beta(\theta_{23}-\theta_{13})}\right). \end{aligned}$$

Thus, we can solve:

$$q_3 = \frac{(\theta_{13}-\theta_{12})((1-\beta)(\theta_{23}-\theta_{12})+\beta C) - (\theta_{23}-\theta_{12})C}{(1-\beta)(\theta_{23}-\theta_{12})(\theta_{13}-\theta_{12}-C)}, \tag{25}$$

where

$$C = H_1^{-1}\left(H_3\left(\frac{(1-\beta)(\theta_{23}-\theta_{12})(\theta_{23}-\theta_{13}) - (\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13}))Z}{\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13})}\right)\right). \tag{26}$$

It remains to prove that $0 \leq q_3 \leq 1$. If the largest coalition $\bar{S}_1 = (-\infty, \bar{y}_1]$ forms, then candidate 1 must win with probability 1 by claims 2 and 3. Substituting $F_3(\cdot) = 1$ into (15) implies

$$-\beta(\bar{y}_1 - x_2)^2 - (1-\beta)\left(q_3(\bar{y}_1 - x_1)^2 + (1-q_3)(\bar{y}_1 - x_3)^2\right) = -(\bar{y}_1 - x_1)^2 \tag{27}$$

This equation implicitly defines $\bar{y}_1$ as a function of q_3: It is immediate that $\bar{y}_1(q_3)$ is continuous and that $\bar{y}_1(0) = y_{21}$, and $\bar{y}_1(1) = \theta_{12}$ (where y_{ij} is defined in (6)). Further, $\bar{y}_1(q_3)$ is strictly decreasing. If both maximal coalitions form, then by claim 2, the election must end in a tie, i.e.,

$$\Phi((-\infty, \bar{y}_1(q_3)] + \rho_1 = \Phi([y_{23}, \infty)] + \rho_3. \tag{28}$$

The right-hand side of (28) is $M_{23} + N_3$, where M_{ij} defined in Section 5 is the maximum number of expressive i supporters who would be willing to vote for j, if j wins with probability 1 as a result. The left-hand side of (28) is $M_{21} + N_1$ when $q_3 = 0$, and it is N_1 when $q_3 = 1$. In order for a solution to (28) exist for $0 \leq q_3 < 1$ it must be the case that the left-hand side is greater than or equal to $M_{23} + N_3$ for $q_3 = 0$ and it must be less than $M_{23} + N_3$ for $q_3 = 1$. Thus, $M_{21} + N_1 \geq M_{23} + N_3$ and $N_1 < M_{23} + N_3$, which are the necessary and sufficient conditions given in the Proposition.

We have already shown that $q_1 \geq 0$. It remains to prove that $q_1 < 1$. Let $\tilde{F}_1(z_1) = q_1 + (1-q_1)F_1(z_1)$. Then (20) immediately implies that $\tilde{F}_1(z_1)$ is strictly increasing, and $\tilde{F}_1(\bar{z}_1) = F_1(\bar{z}) = 1$. Thus, $\tilde{F}_1$ is a well defined cdf, which describes the distribution of z_1 including $z_1 = 0$. Hence $q_1 = \tilde{F}_1(0) < \tilde{F}_1(\bar{z}_1) = 1$. ■

Proof of Proposition 7. Suppose that $N_1 + \Phi([\theta_{12}, \theta_{13}]) > N_3 + \Phi([\theta_{13}, \theta_{23}])$. Suppose the probability of victory for candidate 1 is bounded from above by $q < 1$ for arbitrarily small β. But then, for such small β, a type marginally to the left of θ_{13} would be willing to vote strategically for candidate 1 to increase the probability that candidate 1 wins from q to 1. Further all individuals to the left of this marginal type would similarly be strictly better off. By assumption, if these voters join an existing voting coalition they would all be uniformly strictly better off. This violates inclusivity, a contradiction. The argument when the inequality is reversed is analogous. ■

Proof of Proposition 8. In the mixed coalitional voting equilibrium, the marginal realized coalition member is indifferent (by common interest and inclusivity) between having all members vote expressively or all voting strategically. Further, regardless of the level of candidate 1's base support advantage, if members of a realized coalition for candidate 3 vote expressively, candidate 1 is sure to win. It follows that the indifference condition for each possible marginal realized coalition member $\theta = y_3(\omega_3)$ who votes for candidate 3 but expressively prefers candidate 2 given ρ_1' takes the form:

$$-\beta(\theta - x_2)^2 - (1-\beta)(\theta - x_1)^2 = -\beta(\theta - x_3)^2 - (1-\beta)(\Pr(1 \text{ wins})(\theta - x_1)^2 + \Pr(3 \text{ wins})(\theta - x_3)^2). \tag{29}$$

The left-hand side of (29) does not depend on the level of candidate 1's ex ante base support advantage and is therefore not affected by changing ρ_1 to ρ_1'. It follows that the probability that candidate 3 wins when all voters to the right of θ vote for candidate 3 given ρ_1' is the same as that given ρ_1. Since this holds for each such θ, it follows that the distribution over the sizes of strategic coalitions supporting candidate 1 is shifted to the left by $\rho_1' - \rho_1$. Further, setting Pr(3 wins) to one in (29) implies that the size of the maximal possible coalition supporting candidate 3 is independent of ρ_1.

Next, observe that the smallest non-trivial coalition for candidate 3 increases due to candidate 1's increased base support advantage. Because the marginal member of this coalition has the same probability of winning given ρ_1' as ρ_1, it follows that $q_1' > q_1$.

Further, the requirement that there be a tie in vote share when both coalitions are maximal together with the size of the maximal coalition supporting candidate 3 being unchanged, it follows that the size of the maximal coalition supporting candidate 1 is reduced by $\rho_1' - \rho_1$. The indifference condition for the marginal maximal coalition member takes the form:

$$-\beta(\theta - x_2)^2 - (1-\beta)(q_3'(\theta - x_1)^2 + (1-q_3')(\theta - x_3)^2 = -\beta(\theta - x_1)^2 - (1-\beta)(\theta - x_1)^2,$$

since candidate 1 wins with probability 1 when the maximal coalition forms. Because the increase to ρ_1' shifts θ to the left, the right-hand side of this indifference condition increases. To preserve equality, the left-hand side must increase, i.e., $q_3' > q_3$. For smaller coalitions, the marginal coalition member's indifference condition takes the form

$$-\beta(\theta-x_2)^2-(1-\beta)(q_3'(\theta-x_1)^2+(1-q_3')(\theta-x_3)^2 = -\beta(\theta-x_1)^2-(1-\beta)(\text{Pr(1 wins)}(\theta-x_1)^2+\text{Pr(3 wins)}(\theta-x_3)^2).$$

Because $q_3' > q_3$, the left-hand side is larger given ρ_1' than ρ_1, implying that the right-hand side must also be larger, i.e., it must be that candidate 1 is more likely to win. ■

9 Online Appendix

Proof of Proposition 9. The first-order conditions for the candidates' optimization problems are given by:

$$g(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right)-\phi(y_{32})\frac{\partial y_{32}}{\partial x_1}\right)\left(\frac{x_1+x_2}{2}-\theta_1\right)(x_2-x_1)-(x_1-\theta_1)G(\bar{\rho})=0 \tag{30}$$

and

$$-g(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right)-\phi(y_{32})\frac{\partial y_{32}}{\partial x_2}\right)\left(\theta_2-\frac{x_1+x_2}{2}\right)(x_2-x_1)+(\theta_2-x_2)(1-G(\bar{\rho}))=0. \tag{31}$$

If $\beta \leq \bar{\beta}$, then $y_{32}=\infty$, so $\bar{\rho}=2\Phi((x_1+x_2)/2)-1$. Hence, the terms $\phi(y_{32})\frac{\partial y_{32}}{\partial x_1}$ in the first-order conditions (30) and (31) disappear, which yields

$$g(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right)\right)\left(\frac{x_1+x_2}{2}-\theta_1\right)(x_2-x_1)-(x_1-\theta_1)G(\bar{\rho})=0$$

and

$$-g(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right)\right)\left(\theta_2-\frac{x_1+x_2}{2}\right)(x_2-x_1)+(\theta_2-x_2)(1-G(\bar{\rho}))=0.$$

Substitute $\theta_1=-\theta_2$ and impose symmetry to solve the first-order conditions for the equilibrium platforms:

$$x_2=-x_1=\frac{\theta_2}{1+4\theta_2 g(0)\phi(0)}, \tag{32}$$

which also implies $\bar{\rho}=0$.

We now show that we have enough structure to apply the implicit function theorem to characterize the equilibrium candidate location responses to slight increases in β above $\bar{\beta}$. Define

$$f(\beta)=\phi(y_{32})\frac{\partial y_{32}}{\partial x_1}.$$

Clearly, f is continuously differentiable for $\beta \neq \bar{\beta}$. We next show that f is also continuously differentiable at $\bar{\beta}$. Because the left-derivative of f at $\bar{\beta}$ is trivially zero, it is sufficient to show that $\lim_{\beta\downarrow\bar{\beta}} f'(\beta)=0$.

For $\beta > \bar{\beta}$

$$\frac{\partial y_{32}}{\partial \beta}=-\frac{1}{2}\frac{(x_2-x_1)(x_3-x_1)(x_3-x_2)}{((1-\beta)x_1+\beta x_3-x_2)^2}. \tag{33}$$

and

$$\frac{\partial^2 y_{32}}{\partial\beta\partial x_1}=\frac{1}{2}\frac{(x_3-x_2)^2\,(x_2+\beta x_3-(1+\beta)x_1)}{((1-\beta)x_1+\beta x_3-x_2)^3}. \tag{34}$$

Because $\frac{\partial y_{32}}{\partial \beta}$ and $\frac{\partial y_{32}}{\partial x_1}$ both go to infinity at rate $\frac{1}{(\beta-\bar{\beta})^2}$ as $\beta\downarrow\bar{\beta}$, for $\beta>\bar{\beta}$ there exists $K_1, K_2>0$ such that

$$\left|f'(\beta)\right|=\left|\phi'(y_{32})\frac{\partial y_{32}}{\partial\beta}\frac{\partial y_{32}}{\partial x_1}+\phi(y_{32})\frac{\partial^2 y_{32}}{\partial\beta\partial x_1}\right|\leq\left|\phi'(y_{32})\right|\frac{K_1}{(\beta-\bar{\beta})^4}+\phi(y_{32})\frac{K_2}{(\beta-\bar{\beta})^3}. \tag{35}$$

Further, y_{32} goes to infinity at the rate $1/(\beta-\bar{\beta})$ as $\beta \downarrow \bar{\beta}$. Because the fourth moment of Φ is finite, it follows that $\lim_{x\to\infty} x^4\phi(x) = 0$. Integration by parts yields that $\int_0^\infty x^4\phi'(x)\,dx = x^4\phi(x)|_0^\infty - 4\int_0^\infty x^3\phi(x)\,dx$. Hence, $\int_0^\infty x^4\phi'(x)\,dx$ is finite, which implies that $\lim_{x\to\infty} x^4\phi'(x) = 0$. This and (35) yield $\lim_{\beta\downarrow\bar{\beta}} f'(\beta) = 0$.

To show differentiability at $\bar{\beta}$, it is sufficient to prove that

$$\lim_{\beta\downarrow\bar{\beta}} \frac{f(\beta)-f(\bar{\beta})}{\beta-\bar{\beta}} = 0. \tag{36}$$

The argument is similar to above. Note that $f(\beta) < \hat{K}/(\beta-\bar{\beta})^2$ for β marginally larger than $\bar{\beta}$ and some $\hat{K} > 0$. Further, we have shown that $\phi(y_{32}) < \varepsilon(\beta-\bar{\beta})^4$, for β near $\bar{\beta}$ because $\lim_{x\to\infty} x^4\phi(x) = 0$. Thus, the limit in (36) exists and is zero. Hence, $f(\beta)$ is continuously differentiable, and $f'(\bar{\beta}) = 0$.

An analogous argument shows that $\phi(y_{32})\frac{\partial y_{32}}{\partial x_2}$ is continuously differentiable, and that the first derivative with respect to β at $\bar{\beta}$ is zero.

Next differentiate candidate 1's first-order condition (30) with respect to β to obtain:

$$\begin{aligned}\frac{\partial \mathrm{FOC}_1}{\partial\beta} =&\phi(y_{32})\frac{\partial y_{32}}{\partial\beta}\left(-g'(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right)-\phi(y_{32})\frac{\partial y_{32}}{\partial x_1}\right)\left(\frac{x_1+x_2}{2}-\theta_1\right)(x_2-x_1)\right)\\ &-\phi(y_{32})\frac{\partial y_{32}}{\partial\beta}g(\bar{\rho})\left(\left(\frac{\phi'(y_{32})}{\phi(y_{32})}\frac{\partial y_{32}}{\partial x_1}+\frac{\frac{\partial^2 y_{32}}{\partial x_1\partial\beta}}{\frac{\partial y_{32}}{\partial\beta}}\right)\left(\frac{x_1+x_2}{2}-\theta_1\right)(x_2-x_1)+(x_1-\theta_1)\right).\end{aligned} \tag{37}$$

Similarly, differentiating candidate 2's first-order condition (31) with respect to β yields

$$\begin{aligned}\frac{\partial \mathrm{FOC}_2}{\partial\beta} =&\phi(y_{32})\frac{\partial y_{32}}{\partial\beta}\left(g'(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right)-\phi(y_{32})\frac{\partial y_{32}}{\partial x_2}\right)\left(\theta_2-\frac{x_1+x_2}{2}\right)(x_2-x_1)\right)\\ &+\phi(y_{32})\frac{\partial y_{32}}{\partial\beta}g(\bar{\rho})\left(\left(\frac{\phi'(y_{32})}{\phi(y_{32})}\frac{\partial y_{32}}{\partial x_2}+\frac{\frac{\partial^2 y_{32}}{\partial x_2\partial\beta}}{\frac{\partial y_{32}}{\partial\beta}}\right)\left(\theta_2-\frac{x_1+x_2}{2}\right)(x_2-x_1)+(\theta_2-x_2)\right).\end{aligned} \tag{38}$$

Note that the term in the large parentheses on the first lines of (37) and (38) are zero, respectively at $\beta=\bar{\beta}$. In contrast, the terms inside the large parentheses on the second lines of (37) and (38) go to infinity. Thus,

$$\lim_{\beta\downarrow\bar{\beta}} \frac{\frac{\partial \mathrm{FOC}_1}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} = -\lim_{\beta\downarrow\bar{\beta}} \frac{\frac{\phi'(y_{32})}{\phi(y_{32})}\frac{\partial y_{32}}{\partial x_1}+\frac{\frac{\partial^2 y_{32}}{\partial x_1\partial\beta}}{\frac{\partial y_{32}}{\partial\beta}}}{\frac{\phi'(y_{32})}{\phi(y_{32})}\frac{\partial y_{32}}{\partial x_2}+\frac{\frac{\partial^2 y_{32}}{\partial x_2\partial\beta}}{\frac{\partial y_{32}}{\partial\beta}}} = \frac{x_3-x_2}{x_3-x_1}. \tag{39}$$

Next, differentiate the first-order conditions with respect to x_1 and x_2 at $\beta=\bar{\beta}$. Note that $\bar{\rho}=0$ at $\bar{\beta}$, and hence $g'(\bar{\rho})=0$. Similarly, because $x_1=-x_2$ in equilibrium at $\bar{\beta}$, $\phi'((x_1+x_2)/2)=0$. Further, $\phi(y_{32})\frac{\partial y_{32}}{\partial x_1}=0$ and $\frac{\partial G(\bar{\rho})}{\partial x_i}=g(0)\phi(0)$. Therefore,

$$\left.\frac{\partial \mathrm{FOC}_1}{\partial x_1}\right|_{\beta=\bar{\beta}} = -2g(0)\phi(0)(x_1-\theta_1)-\frac{1}{2}<0, \quad \left.\frac{\partial \mathrm{FOC}_1}{\partial x_2}\right|_{\beta=\bar{\beta}} = g(0)\phi(0)(x_2-x_1)>0, \tag{40}$$

and

$$\left.\frac{\partial \mathrm{FOC}_2}{\partial x_1}\right|_{\beta=\bar{\beta}} = g(0)\phi(0)(x_2 - x_1) > 0, \quad \left.\frac{\partial \mathrm{FOC}_2}{\partial x_2}\right|_{\beta=\bar{\beta}} = -2g(0)\phi(0)(\theta_2 - x_2) - \frac{1}{2} < 0. \tag{41}$$

Let $x_1(\beta)$ and $x_2(\beta)$ be the optimal policy of candidates 1 and 2 given β. Further, let $\tilde{x}_1(\beta, x_2)$ be the solution to the first-order conditions (30) with respect to x_1. Similarly, let $\tilde{x}_2(\beta x_1)$ be the solution to (31) with respect to x_2. Then $x_1(\beta) = \tilde{x}_1(\beta, x_2(\beta))$ and $x_2(\beta) = \tilde{x}_2(\beta, x_1(\beta))$. Differentiating with respect to β yields

$$x_1'(\beta) = \frac{\partial \tilde{x}_1(\beta, x_2)}{\partial \beta} + \frac{\partial \tilde{x}_1(\beta, x_2)}{\partial x_2} x_2'(\beta); \tag{42}$$

$$x_2'(\beta) = \frac{\partial \tilde{x}_2(\beta, x_1)}{\partial \beta} + \frac{\partial \tilde{x}_2(\beta, x_1)}{\partial x_1} x_1'(\beta). \tag{43}$$

Solving these equations for $x_1'(\beta)$ and $x_2'(\beta)$ yields

$$x_1'(\beta) = \frac{\frac{\partial \tilde{x}_1(\beta,x_2)}{\partial x_2}\frac{\partial \tilde{x}_2(\beta,x_1)}{\partial \beta} + \frac{\partial \tilde{x}_1(\beta,x_2)}{\partial \beta}}{1 - \frac{\partial \tilde{x}_1(\beta,x_2)}{\partial x_2}\frac{\partial \tilde{x}_2(\beta,x_1)}{\partial x_1}}; \tag{44}$$

$$x_2'(\beta) = \frac{\frac{\partial \tilde{x}_2(\beta,x_1)}{\partial x_1}\frac{\partial \tilde{x}_1(\beta,x_2)}{\partial \beta} + \frac{\partial \tilde{x}_2(\beta,x_1)}{\partial \beta}}{1 - \frac{\partial \tilde{x}_1(\beta,x_2)}{\partial x_2}\frac{\partial \tilde{x}_2(\beta,x_1)}{\partial x_1}}. \tag{45}$$

The implicit function theorem implies that for $i \neq j$,

$$\frac{\partial \tilde{x}_i(\beta, x_j)}{\partial \beta} = -\frac{\frac{\partial \mathrm{FOC}_i}{\partial \beta}}{\frac{\partial \mathrm{FOC}_i}{\partial x_i}}, \text{ and } \frac{\partial \tilde{x}_i(\beta, x_j)}{\partial x_j} = -\frac{\frac{\partial \mathrm{FOC}_i}{\partial x_j}}{\frac{\partial \mathrm{FOC}_i}{\partial x_i}} \tag{46}$$

Let D be the denominator term in equations (44) and (45). Substituting (40), (41), and (46) into D yields.

$$\lim_{\beta\to\bar{\beta}} D = 1 - \frac{\frac{\partial \mathrm{FOC}_1}{\partial x_2}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}}\frac{\frac{\partial \mathrm{FOC}_2}{\partial x_1}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}} = 1 - \frac{4g(0)^2\phi(0)^2(x_2 - x_1)^2}{(1 + 4g(0)\phi(0)(\theta_2 - x_2))\,(1 + 4g(0)\phi(0)(x_1 - \theta_1))}. \tag{47}$$

Using symmetry, i.e., $\theta_2 = -\theta_1$ and x_1 and x_2 from the symmetric solution (32) implies

$$\lim_{\beta\to\bar{\beta}} D = \frac{(1 + 4g(0)\phi(0)\theta_2)^2\left(1 + 16g(0)^2\phi(0)^2\theta_2^2\right)}{\left(1 + 4g(0)\phi(0)\theta_2 + 16g(0)^2\phi(0)^2\theta_2^2\right)^2} > 0. \tag{48}$$

Equations (44) and (46) imply

$$D\frac{\partial x_1(\beta)}{\partial \beta} = \frac{\frac{\partial \mathrm{FOC}_1}{\partial x_2}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}}\frac{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}} - \frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}} = \frac{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}}\left(\frac{\frac{\partial \mathrm{FOC}_1}{\partial x_2}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}} - \frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}\right) \tag{49}$$

$$D\frac{\partial x_2(\beta)}{\partial \beta} = \frac{\frac{\partial \mathrm{FOC}_2}{\partial x_1}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}}\frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}} - \frac{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}} = \frac{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}}\left(\frac{\frac{\partial \mathrm{FOC}_2}{\partial x_1}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}}\frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} - 1\right) \tag{50}$$

Substituting the above derivatives we get

$$\lim_{\beta\downarrow\bar{\beta}} D\frac{\frac{\partial x_1(\beta)}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} = \frac{2(x_3 - x_2) + 8g(0)\phi(0)(\theta_2(x_3 - x_2) + 2x_2^2)}{(x_2 + x_3)\,(1 + 4g(0)\phi(0)(\theta_2 - x_2))^2}. \tag{51}$$

Thus, $\frac{\partial x_1(\beta)}{\partial\beta} > 0$ for β that are marginally larger than $\bar{\beta}$, because $D > 0$. Similarly,

$$\lim_{\beta\downarrow\bar{\beta}} D\frac{\frac{\partial x_2(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} = \frac{2(x_2+x_3)+8g(0)\phi(0)(\theta_2(x_2+x_3)-2x_2^2)}{(x_2+x_3)\,(1+4g(0)\phi(0)(\theta_2-x_2))^2}, \tag{52}$$

which implies $\frac{\partial x_2(\beta)}{\partial\beta} > 0$ for β marginally larger than $\bar{\beta}$, because $\theta_2, x_3 > x_2$.

Next, we prove that candidate 2 moves her policy by more to the right than candidate 1 moves her policy:

$$\lim_{\beta\downarrow\bar{\beta}} D\frac{\frac{\partial x_2(\beta)}{\partial\beta}-\frac{\partial x_1(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} = \frac{4x_2\,(1+4g(0)\phi(0)\theta_2-8g(0)\phi(0)x_2)}{(x_2+x_3)\,(1+4g(0)\phi(0)(\theta_2-x_2))^2}, \tag{53}$$

which is strictly positive because (12) implies

$$1+4g(0)\phi(0)\theta_2-8g(0)\phi(0)x_2 = \frac{1+16g(0)^2\phi(0)^2\theta_2^2}{1+4g(0)\phi(0)\theta_2} > 0. \tag{54}$$

Finally, we prove that the spoiler is better off for β marginally larger than $\bar{\beta}$. The spoiler's utility is

$$U_3(\beta) = -G(\bar{\rho})(x_1(\beta)-x_3)^2-(1-G(\bar{\rho}))(x_2(\beta)-x_3)^2. \tag{55}$$

Differentiating with respect to β yields

$$\begin{aligned} U_3'(\beta) = &- g(\bar{\rho})\left(\phi\left(\frac{x_1(\beta)+x_2(\beta)}{2}\right)\frac{x_1'(\beta)+x_2'(\beta)}{2}-\phi(y_{32})\left(\frac{\partial y_{32}}{\partial\beta}+\frac{\partial y_{32}}{\partial x_1}x_1'(\beta)+\frac{\partial y_{32}}{\partial x_2}x_2'(\beta)\right)\right)\\ &\cdot\left((x_1(\beta)-x_3)^2-(x_2(\beta)-x_3)^2\right)\\ &+2G(\bar{\rho})(x_3-x_1(\beta))x_1'(\beta)+2(1-G(\bar{\rho}))(x_3-x_2(\beta))x_2'(\beta). \end{aligned} \tag{56}$$

Note that

$$\lim_{\beta\downarrow\bar{\beta}}\left|\frac{\frac{\partial y_{32}}{\partial\beta}\phi(y_{32})}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}}\right| = \lim_{\beta\downarrow\bar{\beta}}\left|\frac{\phi(y_{32})}{\phi'(y_{32})g(\bar{\rho})\frac{\partial y_{32}}{\partial\beta}\theta_2(x_2-x_1)}\right| \le \lim_{\beta\downarrow\bar{\beta}} C(\beta-\bar{\beta})^{2-b} = 0, \tag{57}$$

where $C > 0$ is some constant. The last inequality follows because $|\phi'(x)/\phi(x)| > \varepsilon/x^b$ for $0 < b < 1$ and $\frac{\partial y_{32}}{\partial\beta}$ goes to infinity at the rate $1/(\beta-\bar{\beta})^2$. Further,

$$\lim_{\beta\downarrow\bar{\beta}}\frac{\frac{\partial y_{32}}{\partial x_i}x_i'(\beta)}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} = 0. \tag{58}$$

because $\frac{\partial y_{32}}{\partial x_i}$ goes to zero as $\beta\downarrow\bar{\beta}$. Thus,

$$\lim_{\beta\downarrow\bar{\beta}}\frac{U_3'(\beta)}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} = \lim_{\beta\downarrow\bar{\beta}} -2g(0)\phi(0)\left(\frac{\frac{\partial x_1(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}}+\frac{\frac{\partial x_2(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}}\right)x_2(\bar{\beta})x_3+(x_3+x_2(\bar{\beta}))\frac{\frac{\partial x_1(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}}+(x_3-x_2(\bar{\beta}))\frac{\frac{\partial x_2(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}}. \tag{59}$$

Substituting (51) and (52) and writing x_2 for $x_2(\bar{\beta})$ yields

$$\lim_{\beta \downarrow \bar{\beta}} D \frac{U_3'(\beta)}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} = \frac{4\,(1+4g(0)\phi(0)\theta_2)\,(x_3^2 - x_2^2) + 8g(0)\phi(0)x_2\left(4x_2^2 - x_3^2\,(1+4g(0)\phi(0)\theta_2)\right)}{(x_2+x_3)(1+4g(0)\phi(0)(\theta_2 - x_2))^2}. \tag{60}$$

The denominator of (60) is strictly positive.

To verify that the numerator is strictly positive for $x_3 \geq \theta_2$ substitute the solutions for x_1 and x_2 from (32) into the numerator and evaluate at $x_3 = \theta_2$ to get

$$8\theta_2^3\phi(0)g(0)\frac{80\theta_2^2\phi(0)^2g(0)^2 + 64\theta_2^3\phi(0)^3g(0)^3 + 28\theta_2\phi(0)g(0) + 7}{(4\theta_2\phi(0)g(0)+1)^3} > 0. \tag{61}$$

Differentiating the numerator of (60) with respect to x_3 and again using (32) yields

$$-8x_3\,(4\theta_2\phi(0)g(0)+1)\,(2\phi_0 g(0)x_2 - 1) = 8x_3\,(2\theta_2\phi(0)g(0)+1) > 0. \tag{62}$$

Thus, (60) is strictly positive for $x_3 \geq \theta_3$. Hence, candidate 3's utility from entry is increasing in β for β slightly larger than $\bar{\beta}$. ■