CONSUMER'S SURPLUS AND LINEARITY OF ENGEL'S CURVES

by

JESUS SEADE*

NUMBER 95

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

UNIVERSITY OF WARWICK
COVENTRY

CONSUMER'S SURPLUS AND LINEARITY OF ENGEL'S CURVES

by

JESUS SEADE*

NUMBER 95

October, 1976.

* University of Warwick, Coventry, England. I am indebted to Professors P. Diamond, A. Dixit and J. Mirrlees for helpful comments on a previous draft and to J. Flemming, N. Stern and P. Weller for useful discussions. Views expressed and errors remaining are of course mine. The paper was read at the 1976 European Meeting of the Econometric Society (Helsinki). Financial support from the Ford Foundation is gratefully acknowledged.

This paper is circulated for discussion purposes only and its contents should be considered preliminary.

Consumer's Surplus and Linearity of Engel's Curves

Abstract

The Marshallian measure of surplus (M) is well known to require extremely restrictive assumptions on the behaviour of demands. Harberger's influential contribution to this topic, though, originated the now not uncommon belief that M is a correct measure of surplus also when the marginal utility of income (λ) changes, which indirectly removes the underlying restrictions. This is discussed and shown to be incorrect. However, by merely assuming linearity of Engel's curves in the relevant price-income region, one can calculate rigorously the Hicksian measures of Consumer's Surplus in terms of actual demands, with λ determined as an explicit endogenous variable. An extra bonus of the linearity-assumption is that, as is well known, it is the main ingredient for one to be able to define community preferences, thus rendering the standard one-consumer's surplus analysis more credible.

1. Introduction

The notion of Consumer's Surplus - the money measure of a consumer's utility gains and losses when some prices he faces or quantities he consumes change - can be expressed rigorously in terms of compensated demands, the demand curves that would obtain were income adjusted so as to leave the consumer on a constant utility level.[1] Actual (uncompensated) demands do not naturally lend themselves to rigorous application to this problem, for the reason that in the course of the price or quantity change, the marginal utility of income changes, upsetting the direct equivalence between a change in "utils" and the compensating change in money-incomes. However, it usually is rather difficult to calculate the compensated demands needed, for they require that our data on price- and income-derivatives of demands be integrated twice, which in turn requires not only integrability but, besides, a large amount of information not commonly available, whose normal content of errors will necessarily be increased manyfold by these complicated manipulations. It is for this reason that practitioners of applied welfare economics, rightly less concerned with rigour than with obtaining policy prescriptions otherwise unavailable, have by and large resorted to the use of the Marshallian measure of surplus - the area under the curve of (actual) demand.

The assumptions needed to justify the Marshallian measure, though, are unduly restrictive: as emphasised by Hicks (e.g. in [7, p. 105]), one needs all income effects to be 'small' - namely zero, which implies zero

income-derivatives or zero expenditure shares of the relevant commodities. The requirement is in fact even stronger than that: one specifically needs the income-derivatives of all price-changing commodities to be zero. This is shown in passing in section 4, which also contains a discussion of Harberger's bold and influential attempt to accommodate a varying marginal utility of income.

However, this prohibitively strong requirement need not be imposed. In this essay we introduce a mild stylization of the economy, one that has received considerable attention in other corners of consumption theory, namely, that within the region of prices and incomes relevant to a given problem, all Engel's curves (at least those of price-changing commodities) be linear, and we show that for this case one obtains a workable, explicit formula for Consumer's Surplus in terms of actual demands (demand curves and income elasticities). No further restrictions need to be placed on the marginal utility of income, which is treated as an endogenous variable. In the special case when our linear Engel's curves point towards the origin (equivalent to unitary income elasticities), the formula for surplus takes a particularly nice form, as a simple function of the Marshallian measure.(2)

The requirement that Engel's (income-consumption) curves be linear is equivalent to that of linear expansion paths. The class of preference structures giving rise to this behaviour of demands has been studied by Gorman [4, 5], Afriat [1] and others, largely in the contexts of aggregation across commodities (price indices) and across consumers. In

particular, when preferences are of this type (and, basically, for a given distribution of relative incomes), community preferences can be defined (see Gorman [4, 5]), for different people with different incomes, a convenient feature that lends more credibility to the familiar but crude assumption of a one consumer community which we shall make.

2. Defining Consumer's Surplus

Consider a one-consumer community. Once we have introduced the assumption of linear Engel's curves, below, we shall be able to interpret this "consumer" as in fact consisting of a number of consumers (typically a town, a certain group in a city, etc.), so long as the project or policy measure to be introduced (whose effect on "surplus" we are interested in) affects them only collectively via prices or, if it also changes their unearned incomes, it does so in a proportional way. Since this consumer's or group's ranking of alternatives is not altered by which particular cardinalization is chosen to reflect his (their) preferences, we shall be free to do the choice ourselves.

Let us state some basic notation : $\underline{x}$ is a consumption vector whose i^{th} element is x_i; $\underline{p}$ is a price vector with i^{th} element p_i; y is the consumer's income and $u(\underline{x})$ his well behaved direct utility function. Denote by $v(\underline{p},y)$ the consumer's indirect utility function :

$$v(\underline{p},y) \equiv \max_{\underline{x}} \{u(\underline{x}) \mid \underline{p}.\underline{x} \leq y\}. \qquad (1)$$

I shall consider Consumer's Surplus to be given by either of the two main Hicksian variants : the compensating variation C and the

equivalent variation E. The former is the amount of income the consumer can give up (or requires) if he is to stay on his initial level of indifference when prices change from $\underline{p}^o$ to $\underline{p}^1$, whereas the latter is his required change in income to be applied instead of the price change being contemplated and which would produce the same utility change as the latter. That is, C is the consumer's gain (loss) given that the change is effected, while E is his loss (gain) given that the change is not effected. We can define these two variables, implicitly, by means of the indirect utility function :

$$v(\underline{p}^o, y^o) = v(\underline{p}^1, y^1 - C) , \qquad (2)$$

$$v(\underline{p}^1, y^1) = v(\underline{p}^o, y^o + E) . \qquad (3)$$

Notice that in both cases the income associated with $\underline{p}^1$ (i.e. if the "project" is implemented) is y^1, so as to allow for the possibility that the project-package may change the (unearned) income of our consumer or group, alongside changing the net prices they face. The problem facing the cost-benefit analyst is, then, to find whether C (or E) is greater than F, the cost of the project, typically borne by a different (or larger) group from those who benefit from the project, often by the government, i.e., the tax payers at large. The important point is that any fixed cost cannot <u>a priori</u> be assumed to be paid out of the unearned income y of the group in question. Changes in local income-or excise-tax rates, for example, only act through $\underline{p}$, and are usually not relied on very heavily. Neither are lump-sum or poll-taxes, which would affect y. It is important to keep the distinction clear as to who pays for the project, for both utility levels and behaviour of "our" consumer depend on this, and the whole outcome of a given cost-benefit analysis may be very

sensitive to the formulation adopted. This point was overlooked by Diamond and McFadden [2], who assume the same consumer to be both benefitting from the project and paying its full cost, in such a way that no resource cost exogenous to him can be conceived.(3) This of course needs to be the case if one takes the one consumer model to its extreme form (one country), but not in the typical, more central problem of the "one consumer" community (or group therein) immersed in a larger background country.

I shall as from now assume that we have agreed to identify surplus with C rather than E (a point I touch upon below, pp. 16-17) and only reproduce at the end the results for E, alongside those for C. To derive a more convenient expression for the latter, we write (2) as

$$v(\underline{p}^o, y^o) - v(\underline{p}^1, y^o) = v(\underline{p}^1, y^1 - C) - v(\underline{p}^1, y^o) \qquad (4)$$

whose two sides give the same total change in utility brought about by the price changes and by the compensated change in income.

Denote by $\lambda \equiv \lambda(\underline{p}, y)$ the marginal utility of income :

$$\lambda(\underline{p}, y) \equiv \partial v(\underline{p}, y)/\partial y. \qquad (5)$$

Then, from Roy's Theorem

$$\partial v/\partial p_i = - \lambda x_i, \qquad (6)$$

we can express the left-hand side of (4) as :

$$v(\underline{p}^o,y^o) - v(\underline{p}^1,y^o) = \int_{\underline{p}^1}^{\underline{p}^o} [\partial v(\underline{p},y^o)/\partial \underline{p}] \cdot d\underline{p}$$

$$= -\int_{\underline{p}^1}^{\underline{p}^o} \lambda(\underline{p},y^o)\underline{x}(\underline{p},y^o).d\underline{p} \qquad (7)$$

denoting line integrals with a dot, as inner products.

Similarly, the right-hand side of (4) is

$$v(\underline{p}^1,y^1 - C) - v(\underline{p}^1,y^o) = \int_{y^o}^{y^1 - C} [\partial v(\underline{p}^1,y)/\partial y]\, dy$$

$$= \int_{y^o}^{y^1 - C} \lambda(\underline{p}^1,y)\, dy, \qquad (8)$$

which with (4) and (7) yields :

$$\int_{y^o}^{y^1 - C} \lambda(\underline{p}^1,y)\, dy = \int_{\underline{p}^1}^{\underline{p}^o} \lambda(\underline{p},y^o)\underline{x}(\underline{p},y^o).d\underline{p} \qquad (9)$$

We therefore have C implicitly defined by this equation. To actually find it from here we need to know more about the function $\lambda(\underline{p},y)$.

3. Changes in the Marginal Utility of Income

Consider a change in the price of one commodity, p_i. The corresponding change of $\lambda(\underline{p},y) \equiv \partial v(\underline{p},y)/\partial y$ is given by

$$\begin{aligned} \partial\lambda/\partial p_i &= \partial(\partial v/\partial y)/\partial p_i \\ &= \partial(\partial v/\partial p_i)/\partial y \\ &= -\partial(\lambda x_i)/\partial y \qquad \text{(by (6))} \\ &= -x_i\partial\lambda/\partial y - \lambda\partial x_i/\partial y, \qquad (10) \end{aligned}$$

which can also be expressed in terms of elasticities:[4]

$$\epsilon_{\lambda i} = w_i(\eta_\lambda - \eta_i), \qquad (11)$$

where $\epsilon_{\lambda i}$ = i^{th} price elasticity of λ;

w_i = expenditure share of good i;

η_λ = income elasticity of λ (with sign changed);

η_i = income elasticity of demand for good i.

Since η_i and w_i are empirically observed, this equation leaves (only) one degree of freedom for λ. That is, if we could impose a suitable arbitrary condition on the function $\lambda(\underline{p},y)$, equation (11) would then determine λ as an endogenous variable. We shall presently return to this, but let us first use what we already have to analyse the Marshallian measure of Consumer's Surplus.

4. The Marshallian measure of surplus

If we were to obtain from (7) a measure of the utility change in terms of areas to the left of demand curves, we would require λ to be independent of all the prices that actually change :

$$-\Delta\Psi/\lambda\ (\underline{p},y^o) = \int_{\underline{p}^1}^{\underline{p}^o} \underline{x}.d\underline{p} \equiv M, \qquad (12)$$

where $\partial\lambda/\partial p_i = 0$ for each i with $p_i^1 \neq p_i^o$. (13)

On the other hand, when solving equation (8) for Hicksian surplus, we get the same value of $\Delta\Psi/\lambda$ provided λ does not change with income, in which case (8) solves to :

$$-\Delta\Psi/\lambda(\underline{p}^1,y) = C - \Delta y, \qquad (14)$$

provided $\partial\lambda/\partial y = 0$, (15)

where $(C - \Delta y)$ is the conventional notion of surplus (at constant income, prices varying), to which the direct income increase (or fall) to the consumer, Δy, is to be added to obtain the full measure of his gain. Hence for surplus to be correctly measured by the Marshallian measure $(C = M + \Delta y)$, generally, conditions (13) and (15) must hold, i.e. the correct notion of "constancy of λ" required is its independence both of income and of every price that changes between $\underline{p}^o$ and $\underline{p}^1$. What this requires is then obvious from (10) : that $\partial x_i/\partial y = 0$ for all the relevant i's. The

requirement is too unrealistic in most cases, no matter how permissive we may be - all the more so when we define our commodities as relatively broad aggregates, as is often the case.

Although with varying degrees of precision as to what precisely one needs so as to have $C = M + \Delta y$, it is common knowledge that some form of "constancy" of λ must be assumed. However, in recent years many writers on Consumer's Surplus have come to believe that a variable λ does not alter the validity of M as a measure of surplus. The argument, originated by Harberger [6], is based on a simple and seemingly innocuous procedure: from Roy's Theorem, we can write

$$dv/\lambda = - x_i \, dp_i, \tag{16}$$

whose integral gives M (for a one-price change) on the right-hand side and, it is claimed, a money measure of the utility change on the left-side, because we are "transforming utility into money continuously through the integration process always at the $[\lambda]$ prevailing at that point." [6, p. 788n.].(5)

There should be little doubt that this argument must in fact be wrong, if only for its implications. First, because we know that there is not just one way of converting utility- into income-changes, but at least two (generally different, moreover, to M itself), and second, because each of the familiar rigorous measures is well defined, path-independent, unlike M which, when applied to many prices changing (as the proponents of this approach freely do) is a line integral generally dependent on the path of integration, on the way we conceive $\underline{p}^o$

to move on towards $\underline{p}^1$.

But we would still want to know why $\int dv/\lambda$ is not a proper money measure of the utility change. To understand the reason we must recall the definition of consumer's surplus as an amount of money that, if given to the consumer, would bring about a certain utility change. We imagine that we give the consumer penny after penny, transform these into utils using the prevailing value of λ and stop the process at the required level of $u(\cdot)$. But notice that the relevant "prevailing" values of λ are determined by the way λ depends on <u>income</u>, i.e. by the function $\lambda(\bar{\underline{p}},y)$ as y changes. This has nothing to do with the values of λ prevailing when <u>prices</u> change, an altogether different path of values of $(\underline{p},y)$, with $\lambda \equiv \lambda(\underline{p},\bar{y})$ a function of $\underline{p}$. It is these latter values that appear in the left-hand side of (16), and they are of no use to the consumer willing to transform utils into pounds. The dimensions of an expression such as $\int dv/\lambda$ are money units for whichever λ's we may be using, and we may easily be misled into making a theorem out of what really is only a dimensionality check.

5. The Main Discussion

I shall retain one of the restrictions placed on λ by the Marshallian measure of surplus, that it be independent of income, and leave unrestricted its dependence on prices. The reason why we may be able to impose income-independence is that, by using our freedom to select a specific utility function for the consumer, his preferences given, we may try to construct it so as to achieve $\partial\lambda/\partial y = 0$ at each point. It will presently be clear that this cannot generally be done for all price and income vectors unless some conditions are met.

Let $v(\underline{p},y)$ be an arbitrary indirect utility function for the consumer. We want to find a strictly increasing transformation of it,

$$v^* = F(v(\underline{p},y)), \quad \text{with} \quad F' > 0 \quad \text{for all} \quad v,$$

such that $\partial\lambda^*/\partial y = 0$, where $\lambda^* \equiv \partial v^*/\partial y$.

That is, we want

$$\begin{aligned} \partial\lambda^*/\partial y &= \partial(\partial v^*/\partial y)/\partial y \\ &= \partial(F'v_y)/\partial y \\ &= F'' v_y^2 + F' v_{yy} \\ &= 0 , \end{aligned}$$

which says that the function F should be chosen so as to have

$$F''(v)/F'(v) \quad = \quad - v_{yy}/v_y^2 \ , \qquad (17)$$

with non-satiation ensuring that $v_y > 0$.

Equation (17) clearly must be restrictive, for since its left-hand side depends only on v, so must its right-hand side depend only on v too, that is, on income and prices appearing in such a way that they can be reduced to an expression in terms of v only (by means of $v(\underline{p},y)$). To bring out the implicit requirement, notice that the right-hand side of (17) is equal to $\partial(1/v_y)/\partial y$, and since the inverse of the marginal utility of income is the marginal cost of utility m_u, where $m \equiv m(\underline{p},u)$ is the expenditure function, (17) can be written as[6]

$$(1/m_u)\partial m_u/\partial u \quad = \quad F''(u)/F'(u), \qquad (17')$$

which can be integrated twice to obtain

$$m = A(\underline{p})\ F(u) + B(\underline{p}) \qquad (18)$$

where $A(\underline{p})$ and $B(\underline{p})$ are 'constants' of integration. Hence the expenditure function being of the form (18) is a necessary condition for us to be able to choose an index of utilities $u^* = F(u)$ $(F' > 0)$ such that $\partial\lambda^*/\partial y = 0$ holds in the relevant region of prices and income. That (18) is also sufficient can be shown solving for $v^* \equiv F(v)$ $(= F(u))$ from (18) and writing y for m:

$$\psi^*(\underline{p},y) = \frac{y}{A(\underline{p})} - \frac{B(\underline{p})}{A(\underline{p})} \quad , \tag{19}$$

which clearly yields $\partial\lambda^*/\partial y = \psi^*_{yy} = 0$, so that (18) is iff for $(\partial\lambda^*/\partial y) = 0$[7].

The class of preference structures giving rise to the expenditure function (18) has been studied by Gorman [4,5], Afriat [1] and others. Their empirical implications are best seen by deriving the Engel's (and demand) curves and the expansion paths associated with them. From (18), using $m_{\underline{p}} = \underline{x}$, we get

$$\underline{x} = A_{\underline{p}} F(u) + B_{\underline{p}} \tag{20}$$

$$= y \frac{A_{\underline{p}}}{A} + (B_{\underline{p}} - B \frac{A_{\underline{p}}}{A}), \tag{20'}$$

using (19). Hence Engel's curves (i.e. $\underline{x}(y)$ for given $\underline{p}$) are straight lines. However, their intercepts and slopes can show wide differences from good to good, which allows for there to be all the range of commodities from very income-elastic to inferior goods.[8] Similarly, no obvious strong restrictions seem to be imposed on demand curves. Alternatively, we can look at the expansion paths, in $\underline{x}$-space. From (20), for any two different i,j, we have

$$x_i = A_i u + B_i,$$

$$x_j = A_j u + B_j, \qquad \forall\, i,j$$

which yield, constructing $x_i - (A_i/A_j)x_j$, the following relations :

$$x_i = x_j C_{ij} + D_{ij}, \qquad \forall\, i,j \tag{21}$$

where $C_{ij} \equiv A_i/A_j$, $D_{ij} \equiv B_i - C_{ij} B_j$.

Preferences that give rise to any of the above conditions, ((18),(19),(20') and (21)) give rise to the other three, which are equivalent descriptions of the same class of utility functions (Gorman [5]). As mentioned in the introduction, this is the main ingredient of Gorman's aggregation theorem stating that in this case one can define aggregate preferences on aggregate $\underline{x}$ for a given community, provided relative incomes are kept fixed. One can even allow for differences in tastes to exist, provided the linear expansion paths are parallel lines. Such preferences are admittedly restrictive if assumed to hold in a large region on $(\underline{p},y)$, as one probably requires in certain applications. But if all the consumers in our aggregated group or economy do not differ too much in y (wealth plus Becker's "full income") and if $\underline{p}^0$ and $\underline{p}^1$ are not excessively different, then the mere assumption of linearity does not seem too bad. Indeed, for small enough changes ("near" the uninteresting infinitesimal end) the requirement is only that of differentiability of, say, expansion paths, otherwise allowing for full generality.

Hence, on the assumption that demands do resemble, say, (21), we are free to impose

$$\partial\lambda/\partial y = 0 , \qquad (22)$$

by suitable choice of an (imaginary) utility index. This reduces (10) to

$$\partial\lambda/\partial p_i = - \lambda \partial x_i/\partial y,$$

which can be integrated to yield the following formula, determining the marginal utility of income as an explicit endogenous variable :

$$\lambda(\underline{p}) = \lambda(\underline{p}^o) \exp \{ - \int_{\underline{p}^o}^{\underline{p}} \underline{x}_y . d\underline{p} \}, \qquad (23)$$

where I have reverted to vector notation. Since by (22) λ is now independent of y, I have written it as $\lambda(\underline{p}) \equiv \lambda(\underline{p},y)$. The fact that the constant of integration $\lambda(\underline{p}^o)$ appears multiplicatively will allow us to get rid of it below.

Equation (23) contains a line integral, but one which is independent of the path of integration. A necessary and sufficient condition for path-independence is symmetry of the matrix of cross derivatives, in this case that

$$\partial^2 x_i/\partial y \, \partial p_j = \partial^2 x_j/\partial y \, \partial p_i,$$

but from (20')

$$\partial x_i/\partial y = A_i/A,$$

which upon differentiation yields

$$\partial^2 x_i/\partial y \partial p_j = A_{ij}/A - A_i A_j/A^2 = \partial^2 x_j/\partial y \, \partial p_i.$$

Let us now recall equation (9), our implicit definition of surplus. Using $\partial\lambda/\partial y = 0$ and integrating its left-hand side, this equation

becomes (writing $\Delta y \equiv y^1 - y^0$),

$$C - \Delta y = \int_{\underline{p}^1}^{\underline{p}^0} \lambda(\underline{p})/\lambda(\underline{p}^1)\ \underline{x}\ (p,y^0)\cdot d\underline{p},$$

which, from $V_{\underline{p}} = -\lambda \underline{x}$, obviously is independent of the path of integration. We now use (23) to finally write

$$C = \Delta y + \int_{\underline{p}^1}^{\underline{p}^0} \exp\{-\int_{\underline{p}^1}^{\underline{p}} \underline{x}_y.d\underline{p}\}\ \underline{x}.d\underline{p}\ , \qquad (24)$$

with demands $\underline{x}$ evaluated at base income y^0 (whereas $\underline{x}_y$, by (20'), is independent of y).

We can quote Harberger [6, p.788]: "The first term on the right-hand side of [(24)] measures the first-order change in utility, and can be identified with the change in national income (or, more properly, net national product)[9] expressed in constant prices. The second term measures the second-order change in utility, and can be identified with the change in consumer surplus." In our terminology, following Diamond and McFadden [2], all of (24) is (the change in) surplus. Since C is to be regarded as the gain when the move from $(\underline{p}^0,y^0)$ to $(\underline{p}^1,y^1)$ *is* effected, a final *approval* of the project is to be based on it, i.e. on having $C > F$, where F is the part of the cost paid by the government or, more generally, the part not internalized in the accounts and decisions of *this* consumer/(or group) whose surplus-gain we are trying to measure. Similarly, since the equivalent variation E is to be regarded as the loss when the change from $(\underline{p}^0,y^0)$ to $(\underline{p}^1,y^1)$ is *not* effected, a final

rejection of the project would be based on it, i.e. on having $E < F$. Since we normally expect $E > C$, this rule would not lead to inconsistencies, but it might yield the familiar ambiguous outcome, when $C < F < E$. Since "not going" from $(\underline{p}^o, y^o)$ to $(\underline{p}^1, y^1)$ is technically the same thing as going from $(\underline{p}^1, y^1)$ to $(\underline{p}^o, y^o)$, and since $-E$ for one problem is just C for the reverse problem, the above rule simply says our criterion would always take the form of positive-direction comparisons ("going", rather than "not going"), always using the corresponding value of C. Perhaps this accounts for the slightly greater popularity C seems to have over E in the literature. The formula for E, derived in a similar fashion as (24), is :

$$E = \Delta y + \int_{\underline{p}^1}^{\underline{p}^o} \exp\left\{ - \int_{\underline{p}^o}^{\underline{p}} \underline{x}_y . d\underline{p} \right\} \underline{x} . d\underline{p} , \qquad (25)$$

with demands $\underline{x}$ evaluated at final income y_1.

We have thus arrived at a pair of relatively simple and manageable formulas for the rigorous Hicksian values of surplus, for the rather general case where Engel's curves are approximately linear in a region around the relevant pairs of points $(\underline{p}, y)$ used in the definitions of C and E. The integrals involved, as such, are not computationally that more complicated than the Marshallian $\int \underline{x} . d\underline{p}$ itself. The requirements of information, on the other hand, although admittedly larger, do not seem prohibitive, and one could hope to get reasonable approximations with commonly available data on price and income elasticities of the goods whose prices change as a result of the implementation of the project.

If in a given case the assumption of local homotheticity seems justified, these expressions can be integrated explicitly (using $\underline{x}_y = \underline{x}/y$, with the change of variables $\xi = \int \underline{x}.d\underline{p}$), to yield the following[10]:

$$\left.\begin{aligned} C/y^o &= y^1/y^o - e^{-M(y^o)/y^o} \ , \\ \text{and} \qquad E/y^1 &= e^{M(y^1)/y^1} - y^o/y^1 \ , \end{aligned}\right\} \qquad (26)$$

which for the sake of symmetry have been expressed as fractions of initial and final income, respectively. Notice that the Marshallian measure $M \equiv \int_{\underline{p}^1}^{\underline{p}^o} \underline{x}.d\underline{p}$ is to be evaluated at different values of income in the two expressions.[11]

A final approximative pair of formulas can be given. If only one price changes, the integrals inside the exponentials in (24) and (25) take the form $-(1/y) \int \eta \, x \, dp$, where $\eta \equiv (\underline{p},y)$ is the income-elasticity of the commodity whose price changes, evaluated at y^o for C and at y^1 for E. We can then invoke the Mean Value Theorem and write $-(1/y) \int \eta \, x \, dp = -(\eta^{*o}/y) \int x \, dp$ for some constant η^{*o} lying somewhere in between the two extreme values of η in the expression for C. Similarly, with another η^{*1}, for E, which yield :

$$\left.\begin{aligned} C &\approx \Delta y + (y^o/\eta^{*o})(1 - e^{-\eta^{*o} M^o/y^o}), \\ \text{and} \qquad E &\approx \Delta y + (y^1/\eta^{*1})(e^{\eta^{*1} M^1/y^1} - 1) \ , \end{aligned}\right\} \qquad (27)$$

with (η^{*o},M^o) valued at y^o, and (η^{*1},M^1) valued at y^1. With

any luck, if the corresponding extreme values of η in the relevant ranges of $(\underline{p},y)$ do not fall too far apart, we can use these expressions with reasonable confidence. In fact we may have to make use of approximations of this kind, if the information needed in (24) and (25), basically income elasticities at <u>different</u> prices, is too unreliable or unavailable. Approximations similar to (27) can be justified also if more than one prices change, provided the corresponding income elasticities are numerically equal at each point.(12)

6. Concluding Remarks

It is clear that what enabled us to derive the explicit formulae for C and E (eqs. (24) and (25)), our main result here, is the assumption of linearity of Engel's curves or, equivalently, linearity of expansion paths. The assumption no doubt is restrictive, particularly when interpreted in a global sense, but it does however seem to provide an excellent approximation to the way demands for most commodities behave in a majority of cases, at any rate if the "project" under scrutiny would not alter the whole structure of prices too radically if implemented. In particular, the assumption allows of rather different commodities to be present in the model, with different and largely unrelated demand behaviour across them, only constrained by the requirement of linearity within the relevant region.

It may be objected that, with the information required by equation (24), one could just as well calculate compensated demands and hence surplus, without imposing any conditions. For the one-price-change, for example, one needs first an econometric fit of the right-hand side of the Slutsky equation :

$$\partial x_i^c/\partial p_i = \partial x_i/\partial p_i + x_i\, \partial x_i/\partial y$$

$$= \psi\,(p_i, y)\ ,$$

and then, since

$$\partial m/\partial p_i = x_i^c\ ,$$

where m is the expenditure function, one can write

$$\partial^2 m/\partial p_i^2 = \psi\,(p_i, m)$$

(using $m = y$), which upon double integration yields $m(p_i, u)$, u appearing in constants of integration. Only then we would be able to find the compensated demand curve, as the differential of the expenditure function thus found. The method is indeed complicated, and it necessarily introduces a considerable amount of extra errors into the estimates. Of course, upon reflection, what equations (24) and (25) do is precisely to calculate the required areas to the left of compensated demand curves, directly from observed data rather than indirectly as outlined above, with the consequent economy in data-requirements and in errors. On the other hand, for it to be true that in the approach outlined above one is not imposing any conditions, one must in fact have, literally, a <u>one</u> consumer economy (with arbitrary preferences) or else assume conditions that ensure consistent aggregation, which is in fact what we have done.

Footnotes:

(1) See, e.g, Hicks' final version in [7]. For an excellent exposition using duality methods, also working with compensated demands, see Diamond and McFadden [2].

(2) The formula for the unitary-elasticity case (equations(26') below) appears also in Willig [10] (although in need of a sign-change), together with other very interesting results for other more general cases. I only knew about his contribution after the first draft of this paper had been written. However, both the approaches taken and the directions of extension from the common special case mentioned above are different.

(3) It then follows, as the cited authors show [2 ,footnote (12)], that the change in Consumer's Surplus (net of the change in y) has the same sign whether we measure it by C or E, giving an unambiguous answer. Dixit and Weller [3] extended this result, showing that whenever the line-integral defining Marshallian surplus M is path-independent, it also takes the sign of C and E. But what we normally want to know is whether C or E exceed the given cost F or not, and the sign of (C-F) or (E-F) does depend on the actual numbers involved.

(4) This expression appears in Samuelson [8, equation (13)].

(5) There is no essential difference between the argument in this form, presented by Harberger in a footnote, and his main text's approximative formula, using the mid-value of λ to 'transform' Δv. We are here concerned with the principles of the method, whether in its 'exact' form or otherwise. One's first instinct at criticising Harberger's formula is for his invalid use of the Mean Value Theorem (for a line integral !), implicit in his use of mid-λ and almost explicit in his footnote 2. But in fact this is not the essence of the difficulty, which lies elsewhere.

(6) The expenditure function is defined as $m(\underline{p},u) \equiv \min_{\underline{x}} \{\underline{p}.\underline{x} \mid u(\underline{x}) \geq u\}$. For its properties see, e.g., Shephard [9]. By non-satiation and since $\underline{x}$ includes all relevant commodities, we must have $m(\underline{p},v(\underline{p},y)) = y$, which by differentiation yields $m_u v_y = 1$, as stated in the text. Finally, (17') is obtained by calculating $\partial m_u/\partial y$, y being regarded as the inverse-function of $v(p,y)$.

(7) An alternative proof of the necessity of (18) for $\partial\lambda/\partial y = 0$ is to integrate twice $\partial\lambda/\partial y = 0$ to obtain (19), which then implies (18). However, as an aid to intuition, I prefer the less mechanistic derivation given above, deriving directly the implied form of the expenditure function.

(8) However, for preference sets to be convex, we must restrict ourselves to points, in the space of prices, lying in a region bounded below by an indifference surface that does not meet the axes (except, only, for the homothetic case, $B \equiv 0$, which admits all of R^n_+). This region must be further restricted if inferior goods are to be allowed and we insist on interpreting our one "consumer" as an aggregate of different individuals. See Gorman [4,pp.66,77].

(9) Or, I would add, for a smaller unit, simply change in fixed income.

(10) These equations hold in the slightly more general case of unitary income elasticities of all price-changing goods, instead of local homotheticity over <u>all</u> commodities.

(11) If, moreover, (unearned) income is not directly affected by the project, both M's are the same and these expressions further simplify to

$$\left.\begin{aligned} C/y &= 1 - e^{-M/y}, \\ \text{and} \qquad E/y &= e^{M/y} - 1. \end{aligned}\right\} \qquad (26')$$

These equations appear in Willig [10].

(12) Dixit and Weller [3] show that a necessary and sufficient condition for M to be path-independent (at any given y) is that all price-changing commodities should have equal income elasticities of demand, without constraining the functional dependence of that common value on prices and income. Hence all the values of M in (26), (26') and (27) are path-independent.

References

[1] Afriat, S.F., "The theory of international comparisons of real income and prices", in D.J. Daly, ed., International Comparisons of Prices and Output (National Bureau of Economic Research, New York, 1972), 13-69.

[2] Diamond, P.A., and D.L. McFadden, 1974, "Some uses of the expenditure function in public finance", Journal of Public Economics 3, 1-19.

[3] Dixit, A.K., and P.A. Weller, 1976, "The three consumers' surpluses", University of Warwick, April, mimeo.

[4] Gorman, W.M., 1953, "Community preference fields", Econometrica 21, 63-80.

[5] Gorman, W.M., 1961, "On a class of preference fields", Metroeconomica 13, 53-56.

[6] Harberger, A.C., 1971, "Three basic postulates for applied welfare economics : an interpretive essay", Journal of Economic Literature, 785-797.

[7] Hicks, J.R., A Revision of Demand Theory (Oxford University Press, Oxford, 1956.).

[8] Samuelson, P.A., 1942, "Constancy of the marginal utility of income", in J.E. Stiglitz, ed., The Collected Scientific Papers of Paul A. Samuelson (The M.I.T. Press, Cambridge, Massachusetts, 1966), pp. 37-53 .

[9] Shephard, R.W., Theory of Cost and Production Functions, (Princeton University Press, Princeton, New Jersey, 1970), pp. 93ff.

[10] Willig, R., 1973, "Consumer's surplus: a rigorous cookbook", Stanford University, Institute for Mathematical Studies in the Social Sciences, Technical Report No. 98.