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Peter J. Law

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This paper is circulated for discussion purposes only and its contents should be considered preliminary.

ABSTRACT

It is argued that where static bicriteria models of the firm are specified with a profit-input ratio and sales (or output) in the managerial utility function, the normal restrictions on the managerial utility function, and on revenue and production functions, are insufficient in general to satisfy the relevant second order conditions for a utility maximum at the point where first derivatives are zero. The problem is at its most severe in the case of a linear objective function where it is shown that, for the C.E.S.production function, the second order conditions are never satisfied.

Sales and Profit-Input Ratios in the Firm's Objective Function*

Profit-input ratios have for some time attracted considerable attention within the theory of the firm. The implications of a constraint on permitted profits per unit of capital have been explored in work on the regulated firm, some of which is surveyed by Baumol and Klevorick {1}. The theory of the firm which maximises profits per worker has also been developed and a recent discussion of this analysis is provided by Meade {6}.

The purpose of the present paper is to indicate a problem which may arise in static models where the firm's objective function contains the level of sales (or output) as well as a profit-input ratio and, in particular, where the objective function is linear in these two variables.

Such a bicriteria objective function has recently been advanced by Bonin {2} in a discussion of the behavior of the Soviet firm after the economic reforms of 1966. According to Bonin the Soviet manager may in some circumstances attempt to maximise the following objective function:

$$V(K,L) = a\left[\frac{q(K,L)}{K} - \frac{wL}{K} - r\right] + b'\left[\frac{q(K,L)-\bar{Q}}{\bar{Q}}\right] \qquad (1)$$

In (1), K and L are homogeneous inputs of capital and labor which are combined to produce a single, homogeneous output, Q. The production function, q(K,L), is assumed by Bonin to be homogeneous, concave and differentiable and to exhibit constant returns to scale up to a threshold output Q' and diminishing returns to scale for higher levels of output. Profits, Π, are defined as follows:

$$\Pi = q(K,L) - wL - rK \qquad (2)$$

where w and r are the constant unit rates at which inputs of labor and capital may be hired and it is assumed that the firm sells its output at a constant price which, without loss of generality, is set equal to unity. In (1), a and b' are constant coefficients set by the planners and $\overline{Q}$ is the firm's initial level of output.

It is sometimes assumed in the economic analysis of the labor-managed firm that such a firm will attempt to maximise income per worker. This assumption has been criticised on a number of grounds which need not be reviewed here. However, for present purposes it is relevant to note that one modification of the assumption might be to have both the level of sales and income per worker in the objective function. Since income per worker is revenue per worker less capital costs per worker or $w + \Pi/L$ the expanded objective function like (1) above would contain a profit-input ratio. Such a maximand might be relevant where worker cooperatives have managers (appointed perhaps by the state or by members of the cooperatives) and conditions are such that the managers enjoy enough discretionary power to pursue their own goals. Income per worker might remain important but the level of sales may also be a pertinent managerial goal. Thus just as it has been suggested by Brown and Revankar {3} and Landsberger and Subotnik {4} that, where a capitalist firm has market power and managerial discretion exist, the managerial utility function will contain revenue and profits, so in the case of the worker cooperative in similar circumstances managerial utility may depend on both revenue and profits per worker.

In the next section the case of an objective function which is linear in sales (output) and a profit-input ratio is considered.

1. The Linear Case

This case is exemplified by (1) above and since $\overline{Q}$ is a constant maximising V is equivalent to maximising

$$Z = a\frac{\Pi}{K} + bQ \qquad (1)'$$

where $b = b'/\overline{Q}$. Bonin recognises that, when the production function and profits are homogeneous of degree one, profitability (Π/K) is homogeneous of degree zero. An implication of this is that, if capital and labor inputs can be varied, the firm will never be in equilibrium at a level of output in the constant returns range because it is always possible to increase sales, Q, without reducing profitability by a proportionate increase in both inputs. Consequently the following discussion will make the assumption that the production function is twice differentiable and strictly concave.

Setting the partial derivatives of Z with respect to L and K equal to zero yields the following conditions:

$$Z_K = \left(\frac{a}{K} + b\right)q_K - \frac{a}{K}\left(r + \frac{\Pi}{K}\right) = 0 \qquad (3)$$

$$Z_L = \left(\frac{a}{K} + b\right)q_L - \frac{a}{K} w = 0 \qquad (4)$$

Bonin discusses the choice of technique by the firm by examining the ratio q_L/q_K derived from (3) and (4). However (3) and (4) will only characterise the solution for maximum Z if the relevant second order conditions are satisfied. Further differentiation and substitution from (3) and (4) yields:

$$Z_{KK} = \left(\frac{a}{K} + b\right)q_{KK} + \frac{2b}{K} q_K \qquad (5)$$

$$Z_{LL} = \left(\frac{a}{K} + b\right)q_{LL} \qquad (6)$$

$$Z_{KL} = \left(\frac{a}{K} + b\right) q_{KL} + \frac{b}{K} q_L \qquad (7)$$

It is clear that although Z_{LL} is negative by the assumption that the marginal product of labor is diminishing, the sign of Z_{KK} remains ambiguous. Moreover, let H denote $Z_{KK}Z_{LL} - Z_{KL}^2$ then:

$$H = \left(\frac{a}{K} + b\right)^2 \left(q_{KK}\, q_{LL} - q_{KL}^2\right)$$

$$+ \frac{2b}{K}\left(\frac{a}{K} + b\right)\left(q_{LL}\, q_K - q_{KL}\, q_L\right)$$

$$- \frac{b^2}{K^2}\, q_L{}^2 \qquad (8)$$

The sign of H is ambiguous. The product of the first two brackets is positive by the assumption of strict concavity of the production function and the last term (ignoring the minus sign) is, of course, positive. The sign of the remaining term will be the same as the sign of $\left(q_{LL}\, q_K - q_{KL}\, q_L\right)$ which is in general ambiguous. However, Baumol and Klevorick {1, 178} have shown in another context that this latter expression will be negative whenever capital and labor are complements in production and more generally it will be negative "so long as labor is a better substitute for labor than it is for capital". Landsberger and Subotnik {4, 595} show that for cost-minimising firms negativity of this bracket is a necessary condition for inputs being non-inferior.

Equations (3) and (4) characterise a maximum value of Z if $Z_{KK} < 0$, $Z_{LL} < 0$ and $H > 0$. Clearly given the ambiguity of the sign of Z_{KK} and H there exists a possibility that these (sufficient) conditions are not met. More specifically, if $Z_{KK} > 0$ and/or $H < 0$, Z will not be at a maximum when $Z_K = Z_L = 0$ and consequently (3) and (4) will be an incorrect description of the optimum. This problem may be investigated further by examining the case of a particular production function. Let the general production function $q(K,L)$ be replaced by the C.E.S. production function,

$$Q = \gamma\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-\nu/\rho} \qquad (9)$$

where $\nu < 1$ by the assumption of strict concavity and $\rho > -1$.

Equation (9) may be appropriately differentiated and the obvious substitutions made in (3) and (4) which can now be solved simultaneously to yield that value of K, say K^*, which satisfies them both[1],

$$K^* = \frac{a(1-\nu)}{b\nu} \qquad (10)$$

Note that K^* is independent of both rental rates r and w. Utilising (10) and the appropriate second order partial derivatives of (9) the sign of Z_{KK} and H may be investigated for the case of the CES production function. Thus, the following expressions can be derived,

$$Z_{KK} = \frac{b\nu}{(1-\nu)}\ (1+\rho)\ \delta\gamma\left(\frac{Q}{\gamma}\right)^{1+\rho/\nu} K^{-\rho-2}\left[\frac{(\nu+\rho)}{(1+\rho)}\ \delta\left(\frac{Q}{\gamma}\right)^{\rho/\nu} K^{-\rho}-1 + \frac{2(1-\nu)}{(1+\rho)}\right] \qquad (5)'$$

$$H = \frac{-b^2\nu^2}{(1-\nu)}\ (1+\rho)\ \delta\ (1-\delta)\gamma^2\left(\frac{Q}{\gamma}\right)^{2+2\rho/\nu} K^{-\rho-2}\ L^{-\rho-2}\quad -\frac{b^2}{K^2}\ q_L{}^2 \qquad (8)'$$

The values of K and L in (5)' and (8)' will of course be those which satisfy (3) and (4) for the C.E.S. production function and q_L in$_2$ (8)' is the marginal product of labour for the same production function.

From (8)' it is easily seen that $H < 0$ for the C.E.S. production function and so, in this case, (3) and (4) cannot characterise the solution for maximum Z. Moreover from (5)' Z_{KK} is ambiguous in sign.

It is interesting to examine the Cobb-Douglas production function $Q = \gamma K^{\alpha} L^{\beta}$ as a special case of (9), when $\rho = 0$, $\alpha = \delta\nu$, $\beta = (1-\delta)\nu$ and $\alpha + \beta < 1$. In this case of an elasticity of substitution of unity, by a similar procedure to the C.E.S. case above, it is possible to show

$$Z_{KK} = \frac{b\alpha(1-\alpha-2\beta)\gamma\, K^{\alpha-2}L^{\beta}}{(1-\alpha-\beta)} \qquad (5)''$$

$$H = \frac{-b^2\beta\{\alpha+\beta(1-\alpha-\beta)\}\gamma^2 K^{2\alpha-2}L^{2\beta-2}}{(1-\alpha-\beta)} \qquad (8)''$$

The values of K and L in (5)'' and (8)'' must of course satisfy (3) and (4) for the Cobb-Douglas case. From equations (5)'' and (8)'' it is easily seen that $H < 0$ and $Z_{KK} \lesseqgtr 0$ as $(\alpha + 2\beta) \gtreqless 1$. When $(\alpha + 2\beta) < 1$ the solution $Z_K = Z_L = 0$ represents a local minimum for Z with respect to K and a local maximum with respect to L - in other words the position described is that of a conventional saddle point with the ridge line of the saddle lying parallel with the K axis in K, L-space. When $(\alpha+2\beta) > 1$ then $Z_{KK} < 0$, $Z_{LL} < 0$ and $H < 0$ where $Z_K = Z_L = 0$. In this case a change in K alone or in L alone will reduce the value of the objective function, however there exists a set of combined input variations which will increase Z above its value where $Z_K = Z_L = 0$. Finally it may be noted that in the case of a fixed coefficient, zero elasticity of substitution, production function it can be shown that $Z_{QQ} > 0$ along the expansion path.

Thus these examples show that there may well be circumstances where the optimum solution, contrary to Bonin's suggestion, will not be characterised by (3) and (4)[3].

2. The Non-Linear Case

The argument may be extended to the non-linear case. As an example consider a managerial utility function which is quasi-concave in sales, R, and income per worker, y. Thus

$$U = U(y, R) \tag{11}$$

where $y = w + \Pi/L = (R - rK)/L$ and $R = P(Q)Q$ with $P = P(Q)$, the selling price of the firm's output and $P_Q < 0$. The following expression can be derived for U_{LL} when $U_K = U_L = 0$.

$$U_{LL} = U_y y_{LL} + U_R R_{LL} + \frac{y_L^2 D}{U_R^2} \tag{12}$$

where $D = U_R^2 U_{yy} - 2U_y U_R U_{yR} + U_y^2 U_{RR}$ which is negative by the assumption of quasiconcavity of U and R_{LL} is negative because $R(K,L)$ is assumed strictly concave[4]. However the sign of U_{LL} is ambiguous, because y_{LL} (like the second partial derivative of Π/K with respect to K in the example of the last section) may be positive or negative. Note that if the algebraic sum of the first two terms is positive this may still be offset to make U_{LL} negative if the indifference curves have suitable convexity. The sign of U_{KK} is, of course, negative.

The expression for H, the value of the Hessian determinant of second partial derivatives associated with (11) can be written as follows when $U_K = U_L = 0$,

$$\bar{H} = \left(\frac{U_y}{L} + U_R\right)^2 \left(R_{KK}R_{LL} - R_{KL}^2\right) + \frac{2U_R}{L}\left(\frac{U_y}{L} + U_R\right)\left(R_{KK}R_L - R_{KL}R_K\right)$$

$$- \frac{U_R^2}{L^2} R_K^2 + \frac{D}{U_y^2}\left(\frac{U_y}{L} + U_R\right)\left(R_{LL}R_K^2 - 2R_K R_L R_{KL} + R_{KK}R_L^2\right) \qquad (13)$$

The product of the first two brackets is positive by concavity of R as is the last complex product, $D(\cdot)(\cdot)/U_y^2$. The expression $(R_{KK}R_L - R_{KL}R_K) = R_Q^2(X_{KK}X_L - X_{KL}X_K)$ and so may be either positive or negative for reasons discussed in the previous section; $U_R^2 R_K^2/L^2$ is of course positive. Thus although the non-linearity in the form of convex indifference curves moves H in the positive direction by virtue of the positive term $D(\cdot)(\cdot)/U_y^2$, which is of course absent in the linear case, the ambiguity concerning the sign of H is not removed. In short, concavity of U in R and y and of R in K and L are not, in general, sufficient to ensure that $U_K = U_L = 0$ defines a maximum.

Finally it may be of some interest to note that if L is the only variable input, the firm's level of capital input is fixed and $U_{LL} < 0$, the direction of response of the firm's optimal labor input (and optimal output) to changes in demand and in fixed costs is, in general ambiguous. This contrasts with the well-known results for the firm which maximises y alone.

3. Concluding Comment

It has been shown that where static bicriteria models of the firm are specified with a profit-input ratio and sales (or output) in the objective function the normal restrictions on managerial utility functions and on revenue and production functions are insufficient in general to satisfy the relevant second order conditions for a utility maximum at a position where first derivatives are zero. The problem is at its most severe in the case of a linear objective function where it has been shown that, for the C.E.S. production function, the second order conditions are never satisfied.

FOOTNOTES

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1. Equations (3) and (4) may be written, in the C.E.S. case as

$$\left(\frac{a}{K} + b\right) \nu\delta\gamma \left(\frac{Q}{\gamma}\right)^{1+\rho/\nu} K^{-\rho} = \frac{a}{K}\, Q - \frac{a}{K}\, wL$$

$$\left(\frac{a}{K} + b\right) \nu(1-\delta)\, \gamma \left(\frac{Q}{\gamma}\right)^{1+\rho/\nu} L^{-\rho} = \frac{a}{K}\, wL$$

$$\therefore \left(\frac{a}{K} + b\right) \nu Q = \frac{a}{K}\, Q.$$

The last expression can be solved for K^*.

2. Equation (8)' can be written:

$$H = \frac{-b^2(1+\rho)q_K q_L}{(1-\nu)KL} - \frac{b^2}{K^2}\, q_L^2$$

3. Profit and sales targets and constraints on input availability are important in pracrice in Soviet industry. The argument here has abstracted from these considerations. Bonin {2} devotes some discussion to quantity targets and related issues.

4. For a recent discussion of this assumption see Landsberger and Subotnik {5}.

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