THE THREE CONSUMER'S SURPLUSES

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I. INTRODUCTION

Recent years have seen a revival of interest in the concept of consumer's surplus. Among the new ideas and methods that have emerged, the use of the expenditure function is particularly important. Diamond and McFadden (1974) show how it simplifies the discussion of Hicksian notions of the surplus based on compensated demand functions, viz. the compensating and equivalent variations. Seade (1976) and Willig (1976) show that in certain special cases, a simple functional relation exists between these compensated surpluses and the conventional 'Marshallian' surplus defined using uncompensated or market demand functions. Willig introduces the ingenious idea of forgetting about the problem of the path-dependence of the Marshallian surplus by making it a convention to choose a particular path by a specified rule, and examines how well this surplus can approximate one of the Hicksian surpluses. The purpose of this paper is complementary; we show that the very conditions which make Marshallian surplus path-independent go a long way towards yielding bounds for it in terms of the Hicksian surpluses.

This turns out to be useful in answering a common question of cost-benefit analysis. The problem is to find which of two equilibria of an economy has a higher level of welfare, using only tests based on demand functions. We begin with a one-consumer economy, and establish two results that have been recognized only imperfectly in existing work.

These concern tests of the form: welfare is higher in equilibrium b than in equilibrium a if and only if the consumer's surplus gained in going from a to b exceeds the loss of transfer income in the move. We show that such a test is valid whether the surplus is interpreted as the compensating variation or as the equivalent variation. We then turn to the Marshallian surplus and show that when it is path-independent, there are several important cases when it lies between the two Hicksian surpluses. It follows as a corollary that the above test is also valid when the surplus is interpreted in the Marshallian sense. This is an exact result, and does not depend on choosing a particular path and taking approximations as in Willig (1976), or upon sleight of hand as in Harberger (1971). Incidentally, we see that the conditions required for the Marshallian surplus to be path-independent, while restrictive, are far less so than is often believed.

We then turn to the case of several consumers with different lump sum incomes. The conditions for the Marshallian surplus to be well-defined again take us a long way in enabling aggregation over consumers. However, the use of a simple test like that above remains problematic owing to the differences between market aggregation and welfare evaluation.

II. THE ONE-CONSUMER CASE

Consider an economy in which convex social indifference curves exist. These can be described by means of an expenditure function $E(p,u)$, where p is a vector of prices and u is the level of utility or social welfare, which need only have ordinal significance. Let m denote lump sum or transfer income; this is to be distinguished from income earned by the sale of labour services, which is handled by letting labour services be another commodity and incorporating the wage rate in the vector p.

Now consider two equilibria (p^a, m^a, u^a) and (p^b, m^b, u^b), differing in some aspects of production, organization, or government policy in ways that need not concern us here. We wish to find out whether u^b exceeds u^a or vice versa, using criteria based on demand functions. The two equilibrium conditions are simply

$$m^a = E(p^a, u^a) \quad , \qquad m^b = E(p^b, u^b) \quad . \tag{1}$$

The compensated demand functions are the partial derivatives of the expenditure function with respect to the prices for each commodity. Denoting function labels by superscripts and partial derivatives by subscripts, we have for each i :

$$C^i(p,u) = E_i(p,u) \tag{2}$$

as the compensated demand function for commodity i . The Hicksian surpluses in going from a to b are now easily defined. The compensating variation is

$$\begin{aligned} CV(a,b) &= E(p^a,u^a) - E(p^b,u^a) \\ &= - \int_{p^a}^{p^b} C(p,u^a) \, . \, dp \end{aligned} \tag{3}$$

and the equivalent variation is

$$\begin{aligned} EV(a,b) &= E(p^a,u^b) - E(p^b,u^b) \\ &= - \int_{p^a}^{p^b} C(p,u^b) \, . \, dp \quad , \end{aligned} \tag{4}$$

The expressions being integrated are the inner products of the appropriate compensated demand vector and the differential price change dp. These are exact differentials by Slutsky symmetry ($C^i_j = C^j_i$ for all i and j); thus integrability conditions are satisfied and the integrals are well-defined, i.e. path-independent.

Utility comparisons are easy in terms of these surpluses. The basic result is that since E is an increasing function of u for any fixed p,

$$u^b > u^a \quad \text{if and only if} \quad E(p,u^b) > E(p,u^a) \quad \text{for any fixed} \quad p \,.$$

Using this for $p = p^a$, the criterion for $u^b > u^a$ becomes

$$E(p^a,u^b) > E(p^a,u^a).$$

Subtracting $E(p^b,u^b)$ from both sides and using (1) and (4), this becomes

$$EV(a,b) > m^a - m^b \,. \tag{5}$$

On the other hand, if we use $p = p^b$, the criterion becomes

$$E(p^b,u^b) > E(p^b,u^a),$$

and subtracting each side from $E(p^a,u^a)$ and using (1) and (3), we have

$$CV(a,b) > m^a - m^b \,. \tag{6}$$

This establishes our basic test for utility increase in going from a to b: the gain in the consumer's surplus should exceed any loss in lump sum income. The interesting conclusion is that it does not matter which of the two Hicksian surpluses we use. The actual values of CV(a,b) and EV(a,b) will of course differ in general, but they must lie on the same side of $(m^a - m^b)$. This result is implicit in the treatment of Diamond and McFadden (1974), but is not explicitly pointed out.

Note that the result is independent of the cause of the shift of the equilibrium from a to b: any comparative static change which leaves consumer tastes unaltered fits into the framework. Also, no commodities need be given any special role. If prices of some commodities do not change, we need to evaluate the integrals in (3) and (4) only with respect to the ones whose prices do change, and we can in fact aggregate the commodities with fixed prices if we so wish. A significant case of this kind is where one commodity is chosen as the numeraire. A careful consideration of the production side of the economy is of course necessary in order to evaluate the transfer income correctly in an economy with pure profits.

For practical reasons, it is often claimed to be preferable to work with the market or uncompensated demand functions $D^i(p,m)$. Let us use these to define the Marshallian surplus for fixed m as

$$MS(a,b,m) = -\int_{p^a}^{p^b} D(p,m) \cdot dp \quad . \tag{7}$$

We will normally want to use this for $m = m^a$ or m^b . If we do not know these values in advance and are seeking a general theorem, we should allow a general fixed value of m at the outset. The surplus is well-defined iff the integral is independent of the path of integration. This will be so if and only if there is a scalar function $F(p,m)$ such that the function D^i is the i^{th} partial derivative of F, viz. F_i, and then

$$MS(a,b,m) = F(p^a,m) - F(p^b,m) . \tag{8}$$

The condition for such integrability is that the cross-partial derivatives of the component functions be equal : $D^i_j = D^j_i$ for all i and j. This can be cast in a simpler form using the Slutsky equation. If m and u are consistently chosen, i.e. $m = E(p,u)$, then we have

$C^i(p,u) = D^i(p,m)$, and differentiation with respect to p_j yields

$$C^i_j(p,u) = D^i_j(p,m) + D^j(p,m) \quad D^i_m(p,m) \quad .$$

Using Slutsky symmetry, the integrability condition becomes

$$D^j(p,m) \quad D^i_m(p,m) = D^i(p,m) \quad D^j_m(p,m) \quad ,$$

or

$$m \, D^i_m(p,m) \;/\; D^i(p,m) = \alpha(p,m) \, , \tag{9}$$

say, for all i. This requires the income elasticities of demand for all the commodities in question to be equal, although the common value α is allowed to be an arbitrary function of (p,m).

There is a further restriction if the symmetry is required for literally all the commodities in the economy. We use the budget constraint $p.D(p,m) = m$ (where the dot is again the inner product operator). On differentiating it with respect to m, and using (9), we find

$$m \, \alpha(p,m) = m \, ,$$

which implies either $m = 0$ or $\alpha(p,m) = 1$. If there is non-zero lump sum income, therefore, the condition reduces to that of unitary income elasticities of demand. Further, the case of $m = 0$ does not get us far. For example, in the case where the first commodity is <u>minus</u> the labour supply, on writing x_1 as the demand for it, we would want $x_1 < 0$ and $\partial x_1/\partial m > 0$. This requires $\alpha < 0$, which is incompatible with the other commodities being normal and having positive demands.

However, the usual setting of cost-benefit analysis does not require unitary income elasticities for all commodities. This is because there is often at least one commodity which has the same price in the two equilibria.

So long as we confine the paths of integration to the subspace of those commodity prices that do change, we need to impose symmetry only for that subset of demands. Then (9) is required only for this subset, and the function α can be arbitrary. The traditional case where one commodity is the numeraire, and all the others have zero income effects, is an obvious and very special case of this. In the conventional diagrammatic case where only one price is changing, (9) is automatically satisfied. Thus it appears that several cases of practical interest allow a meaningful definition of the Marshallian surplus under much more general conditions than is commonly realized.[1]

We therefore proceed on the assumption that there is at least one unchanging price in the two equilibria, and partition the commodities into two subsets, P being the set of changing prices and Q that of fixed prices. Typical vectors in the respective subsets will be denoted by p and q. Now (9) can be rewritten as

$$m\, D^i_m(p,q,m) \;/\; D^i(p,q,m) \;=\; \alpha(p,q,m) \;,$$

for $i \in P$. We integrate this with respect to m between the limits m' and m''. Writing $\beta(p,q,m) = \alpha(p,q,m)/m$ for brevity, we have

$$D^i(p,q,m'') \;=\; D^i(p,q,m') \;\; \exp\left\{ \int_{m'}^{m''} \beta(p,q,m)\, dm \right\} \qquad (10)$$

We will write the second term on the right hand side as $\phi(p,q,m',m'')$. It should be emphasised that (10) is a direct consequence of the condition for the Marshallian surplus to be path-independent, and most of what follows is a consequence of simple manipulations of (10).

The first line of argument takes particular limits of integration,

$m' = m^a$ and $m'' = E(p,q,u)$ for a general point (p,q,u). Now the demands on the left hand side equal the corresponding compensated demands, so (10) becomes

$$C^i(p,q,u) = D^i(p,q,m^a) \quad \phi(p,q,m^a,E(p,q,u)) .$$

Since the compensated demands are the price partial derivatives of the expenditure function, and since the integrability conditions for the set P yield as in (8) a function $F(p,q,m)$ such that the uncompensated demands in this subset are the corresponding partial derivatives of F, we have

$$E_i(p,q,u) = F_i(p,q,m^a) \quad \phi(p,q,m^a,E(p,q,u)) , \tag{11}$$

for $i \in P$. Consider these relations for fixed q and u as functions of p alone. We see that the gradients of the functions E and F are parallel at each point (although their relative magnitudes can change from one point to another). Therefore the indifference surfaces of the two functions in P-space must coincide, and E must be a scalar transform of F. Of course the functional form of this transform can depend on the values of q and u chosen. Thus there exists a scalar function Π such that

$$E(p,q,u) = \Pi(F(p,q,m^a),q,u) . \tag{12}$$

This is a slight generalization of a result of Goldman and Uzawa (1964); a formal statement and proof is given in the Appendix.

Differentiating with respect to p_i and comparing the outcome with (11), we find that

$$\phi(p,q,m^a,E(p,q,u)) = \Pi_F(F(p,q,m^a),q,u) , \tag{13}$$

where of course E and F are understood to be related by (12). We see that although we started without any restrictions concerning the dependence of β or ϕ on p, in fact the dependence is somewhat circumscribed since it must be channelled through F.

Evaluating (13) at the a equilibrium, we have

$$\Pi_F(F(p^a,q,m^a),q,u^a) = \phi(p^a,q,m^a,m^a) = \exp(0) = 1 .$$

where q is the common value $q^a = q^b$. Differentiating again,

$$\Pi_{FF}(F,q,u) = \{ \phi(p,q,m^a,\Pi(F)) \}^2 \ \beta(p,q,\Pi(F)) .$$

This has the sign of β. Therefore, if the commodities in the subset P are all normal, Π is a convex function of F. We proceed for a while assuming this to be the case. We have

$$\begin{aligned} - CV(a,b) &= E(p^b,q,u^a) - E(p^a,q,u^a) \\ &= \Pi(F(p^b,q,m^a),q,u^a) - \Pi(F(p^a,q,m^a),q,u^a) \\ &\geq \Pi_F(F(p^a,q,m^a),q,u^a) \ \{ F(p^b,q,m^a) - F(p^a,q,m^a) \} \quad \text{using convexity} \\ &= 1 \ . \ \{ - MS(a,b,m^a) \} \quad \text{by definition and above result.} \end{aligned}$$

In other words, we have

$$CV(a,b) \leq MS(a,b,m^a) . \tag{14}$$

Noting that $EV(a,b) = - CV(b,a)$ and $MS(a,b,m) = - MS(b,a,m)$, we have a similar result for the equivalent variation:

$$EV(a,b) \geq MS(a,b,m^b) . \tag{15}$$

If all commodities in the subset P were inferior, both inequalities would be reversed, while if they could change from being normal to being inferior in the course of the move from a to b, we would not be able to have a clear result. Our results are easily modified for inferior commodities, while the case of a change we regard as unlikely. Therefore we will henceforth deal only with the normal case.

Equations (14) and (15) embody several useful results, and it would help to enumerate them.

1. First we have two one-tailed tests, respectively for the acceptance and the rejection of the move from a to b:

if $\quad m^a - m^b > MS(a,b,m^a)$, then $u^b < u^a$,

if $\quad MS(a,b,m^b) > m^a - m^b$, then $u^b > u^a$.

These follow at once when we combine (14) and (15) with (5) and (6).

2. If $m^a = m^b = m$, say, we have

$$CV(a,b) \leq MS(a,b,m) \leq EV(a,b) .$$

This is a multi-price-change generalization of a result that is well known for the case of one price change, and is illustrated in Figure 1.

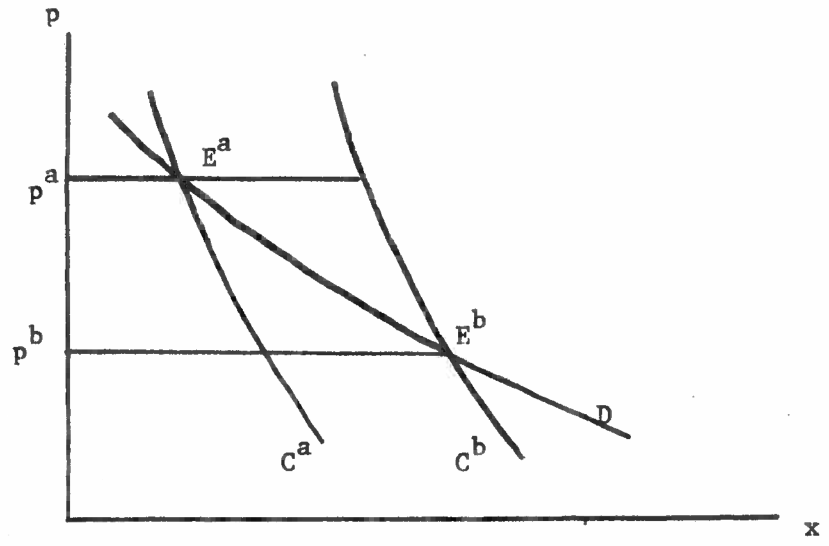

Figure 1

E^a and E^b are the two equilibria, C^a and C^b the two compensated demand curves, and D the uncompensated demand curve.[2] Each consumer's surplus is the area to the left of the appropriate demand curve, bounded by the lines at prices p^a and p^b.

Note that our result follows almost directly from the integrability conditions; the only added assumption is that commodities do not change from being normal to being inferior.

3. If we could say that for $m^a \leq m^b$, we had $MS(a,b,m^a) \leq MS(a,b,m^b)$, then a complete chain of inequalities would follow:

$$CV(a,b) \leq MS(a,b,m^a) \leq MS(a,b,m^b) \leq EV(a,b) .$$

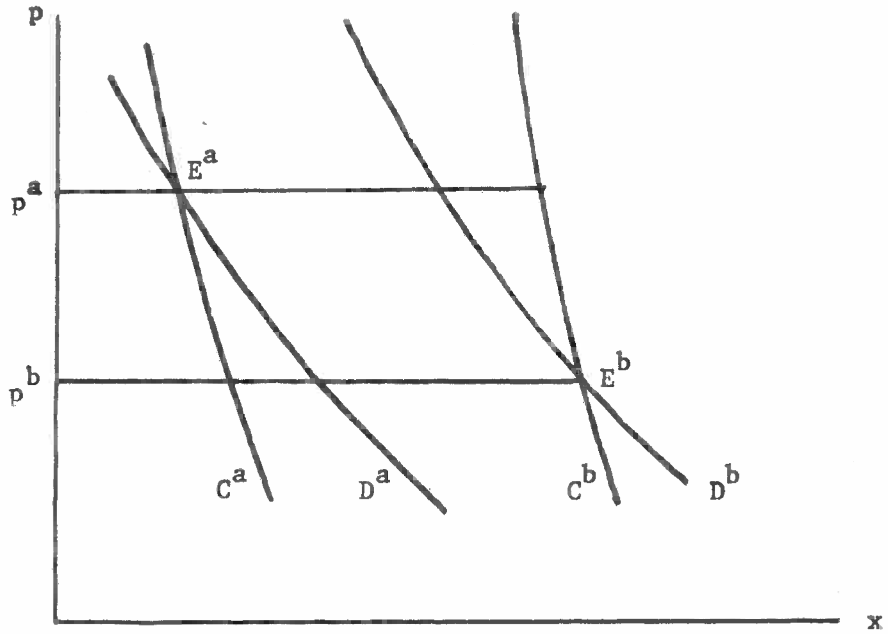

Figure 2

This would be an obvious multi-price-change generalization of the obvious generalization of the one-price-change of Figure 1, which is depicted in Figure 2. This would be very desirable, because we can always label the equilibria so that $m^a \leq m^b$, and then the necessary and sufficient condition for $u^b > u^a$ given by

$$MS(a,b,m) > m^a - m^b , \tag{16}$$

where m is either of m^a and m^b, or indeed any intermediate value, would become a valid test.

Unfortunately, integrability alone does not secure this. The problem is seen on writing (10) as

$$F_i(p,q,m^b) = F_i(p,q,m^a) \quad \phi(p,q,m^a,m^b) .$$

If we integrate the left hand side with respect to p from p^a to p^b to get $MS(a,b,m^b)$, on the right hand side we would have to integrate by parts, yielding an unwanted term which is the integral of ϕ with respect to p. This spoils any simple conclusion. However, if β is independent of p, then so is ϕ, and integration gives

$$MS(a,b,m^b) = MS(a,b,m^a) \quad \phi(q,m^a,m^b) \tag{17}$$

It is easy to see that $\phi \gtrless 1$ according as $m^b \gtrless m^a$, and the desired inequality follows.

It is not to be thought that the independence of income elasticities from the changing prices is a realistic assumption. However, it is worth knowing stumbling blocks in attempts at generalizing one-price-change results as well as knowing successful generalizations. Further, it must be granted that quite a lot can be done without this assumption.

4. In special cases, e.g. unitary income elasticities, or income elasticities independent of (p,q,m), it is possible to find the functional form of Π exactly, and thus establish exact formulae linking the compensated and uncompensated surpluses. This is done by Seade (1976) and Willig (1976). Our equation (10) provides an alternative approach that is sometimes simpler, but we leave the development of it to the reader.

III. THE MANY-CONSUMER CASE

We now consider an economy consisting of H consumers with identical tastes but differing transfer incomes m_h for $h = 1,2, .. H$. We assume that the integrability condition (10) is satisfied for each consumer. This turns out to allow aggregation over consumers. This could be done using general theorems following Gorman (1959), but it is simpler to give a more direct demonstration. What we want is a function

$$\bar{m} = M(p,q,m_1,m_2,\dots m_H)$$

which can be regarded as yielding the representative income in the sense that

$$D^i(p,q,\bar{m}) = H^{-1} \sum_{h=1}^{H} D^i(p,q,m_h) \tag{18}$$

for all $i \in P$. Using (10) for each h with $m' = \bar{m}$ and $m'' = m_h$, this amounts to requiring

$$\sum_{h=1}^{H} \phi(p,q,\bar{m},m_h) = H . \tag{19}$$

Since ϕ is decreasing in $\bar{m}$ in each term, and since each term exceeds 1 when $\bar{m} < \min_h m_h$ and is less than 1 when $\bar{m} > \max_h m_h$, we know that (19) is satisfied by a unique $\bar{m}$ for each given $(p,q,m_1,m_2,\dots m_H)$. This is the implicit definition of the function we are looking for.

Note the importance of assuming that at least one price does not change. If the set Q were empty, the budget constraint would hold over P alone. Multiplying (18) by p_i and adding over i, we would find

$$\bar{m} = H^{-1} \sum_{h=1}^{H} m_h ,$$

and only the arithmetic mean would be a permissible representative income.

With at least one unchanging price, there is no rigid budget constraint applying to the subset P alone, and we have much more freedom. For example, if the commodities in P have a common and constant income elasticity of demand α , it is easy to verify that

$$\bar{m} = \{ H^{-1} \sum_{h=1}^{H} m_h^{\alpha} \}^{1/\alpha} .$$

Now define $\bar{u}$ by the relation

$$\bar{m} = E(p,q,\bar{u}) . \qquad (20)$$

All the tests of Section II can be expressed in the form: $\bar{u}^b$ exceeds $\bar{u}^a$ if and only if the gain in the 'representative consumer's surplus' exceeds the loss in 'representative transfer income'. Now $\bar{u}$ is a function of p, q, and $\bar{m}$, while $\bar{m}$ is a function of p, q, and the m_h's. But each m_h itself equals $E(p,q,u_h)$. Thus $\bar{u}$ becomes a function of the u_h's. If this happens to be our social welfare function, the representative consumer's surplus criterion gives correct answers as to which of the two equilibria has a higher level of social welfare. We see that there has to be a coincidental consistency between market aggregation of demands and the social aggregation of utilities if the simple procedure is to work.

But there is a problem more serious than that of the need for a coincidence. The social welfare function derived above will in general depend on prices. The general form (12) of the expenditure function entailed by the integrability condition leaves this possibility open. It is only in further special cases that a price-independent social welfare function results. The strongest result we have been able to discover is for the class of functions given by the expenditure function

$$E(p,q,u) = A(p,q)\, u + B(A(p,q),q) , \qquad (21)$$

where A and B are required to satisfy the conditions that will make E a legitimate expenditure function. This is clearly a special case of (12). It implies linear Engel's curves, but not necessarily through the origin, and with possibly differing intercepts and slopes for the different commodities. For this class, it is easy to verify that

$$\bar{m} = H^{-1} \sum_{h=1}^{H} m_h \quad \text{and} \quad \bar{u} = H^{-1} \sum_{h=1}^{H} u_h ,$$

so the arithmetic mean income and a Benthamite social welfare function are consistent with the use of the representative consumer's surplus.

This is not meant to be a result in support of the use of such a surplus concept for many-consumer problems. On the contrary, it is meant to illustrate the strong restrictions that must be placed on the social welfare function as well as the demand functions before the so-called distributional neutrality of consumer's surplus can be given a welfare interpretation. In particular, it seems unlikely that any significant generalization beyond linear Engel's curves and strict utilitarianism will be possible.

APPENDIX

Lemma: Let $f(x,y)$ and $g(x,y)$ be two scalar-valued continuously differentiable functions of vector variables $x \in R^n$ and $y \in R^m$. Suppose that for each given y, each indifference surface of g in x-space is arc-connected, and there is a scalar-valued function $h(x,y)$ such that

$$f_{x_i}(x,y) = h(x,y) \quad g_{x_i}(x,y) \tag{i}$$

for all (x,y) and $i = 1,2,.. n$. Then there exists a funotion $H(t,y)$ of a scalar argument t and the vector y such that, for all (x,y)

$$f(x,y) = H(g(x,y),y) . \tag{ii}$$

Further, such a function $h(x,v)$ satisfying (i) must have the form

$$h(x,y) = J(g(x,y),y), \tag{iii}$$

for a function $J(t,y)$ of a scalar argument t and the vector y.

Proof: This follows Goldman and Uzawa (1964), but the process is carried out for each y separately, and therefore y enters all the resulting functions as a parametric argument. The idea is to show that f depends on x only through g, i.e. if $g(x',y) = g(x'',y)$, then $f(x',y) = f(x'',y)$. We construct an arc $x = x(\theta)$, going from x' to x'' and lying entirely in an indifference surface of g. Along it,

$$dg/d\theta = \sum_i g_{x_i}(x,y)\, dx_i/d\theta = 0 .$$

Then, using (i),

$$\begin{aligned} df/d\theta &= \sum_i f_{x_i}(x,y)\, dx_i/d\theta \\ &= h(x,y) \sum_i g_{x_i}(x,y)\, dx_i/d\theta = 0 . \end{aligned}$$

It is then easy to see that $J(t,y) = H_t(t,y)$ in obvious notation.

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FOOTNOTES

1. Note that we are committing ourselves to a choice of numeraire in the subset of commodities with fixed prices. If we were to require invariance of the results when one of the remaining commodities is made the numeraire, we would need (9) to hold for all commodities , and thus unit income elasticities for all of them.

2. Robert Willig of Bell Laboratories has shown us an alternative proof for this case. The move from a to b is to be split into parts. From a to c, utility stays constant, all but one of the changing prices move to their b values, and the last price accommodates to keep utility constant. From c to b, only this price changes to its b value, thus reaching b utility. Now CV = MS from a to c, and the one-price-change argument applies from c to b. Adding, we have the result.