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ABSTRACT

This paper simplifies and extends the theory of household behaviour under rationing, using duality and the concept of 'virtual' prices. Slutsky-type equations, decomposing the derivatives of the rationed demand functions into income and substitution effects, are derived, and these derivatives are related to the corresponding derivatives of the unrationed demand functions for finite as well as infinitesimal ration levels. The results imply that the Keynesian demand multiplier and the Barro-Grossman supply multiplier are more likely to have their expected signs the further the household is from its unconstrained equilibrium.

1. Introduction

Following its relative neglect during most of the 1950's and 1960's, there has recently been a revival of interest in the implications of rationing, or, more generally, of quantity constraints, for a number of different branches of economic theory.[1] However, the basic theory of household behaviour in the presence of rationing remains in an unsatisfactory state. The principal results in this area, derived by Tobin and Houthakker (1950-51) and restated by Pollak (1969), apply only for the case where the ration "just" bites, in the sense that the ration levels coincide with the quantities which would have been chosen by an unrationed household facing the same prices and income. Since, to a first order, such a ration has no income effect, it is not surprising that these results express the derivatives of the rationed demand functions in terms of the utility-compensated derivatives of the unconstrained demand functions. Empirical applications of these results, for example, to measuring the true cost of unemployment, have therefore resorted to Taylor Series approximations around the unrationed equilibrium point. (See, for example, Ashenfelter (1977)).

The results of Tobin and Houthakker have been extended to the more interesting and realistic case where the ration constraint strictly bites by Howard (1977). However, he interpreted his results in terms of whether the rationed and unrationed commodites were "substitutes or complements in disequilibrium" (Howard, p. 405). This criterion suffers from the disadvantages that it is not related to the usual definition of substitutes and complements, and that it cannot obviously be determined from a knowledge of the properties of unrationed demand functions.

The object of the present paper is to present a concise and complete analysis of the properties of household demand and supply functions under rationing which avoids both of these difficulties. Thus we wish to provide a complete characterisation of rationed demand and supply functions which relates their properties to the properties of the unrationed demand and supply functions. However it should be clear that it is fruitless to attempt such an exercise if the latter are evaluated at the consumption bundle which the household would choose in the absence of rationing, since in general the consumption bundles involved, and hence the local properties of the utility function, will be very different. Instead we make use of the concept of "virtual" prices, first suggested by Rothbarth (1940-41), i.e., those prices which would induce an unrationed household to behave in the same manner as when faced with a given vector of ration constraints. We show that all the properties of the rationed demand and supply functions may be expressed in terms of the properties of the unrationed functions, provided the latter are evaluated at the virtual prices. Of course, these virtual prices are themselves functions of the ration levels, as well as of the household's income and of the prices of the unrationed commodities. However this does not mean that they cannot be calculated from a knowledge of the unconstrained demand and supply functions, as we illustrate for the case of the Linear Expenditure System in Section 6.

Before making use of the virtual prices, it is necessary to show that they exist and to characterise their form. This is the purpose of Section 2. Sections 3 to 5 then characterise the household's behaviour under rationing in terms of its unconstrained behaviour when faced with the virtual prices. Adopting a duality approach, it is shown that the results of Tobin and Houthakker generalise to the case of a strictly binding ration, provided the consumer is compensated for the utility loss due to the imposition of the ration. When this compensation is not carried out, there is an additional income effect which must be taken into account, and it is shown that all the derivatives of the rationed demand and supply functions may be decomposed into income and substitution effects in a manner identical to the familiar Slutsky equations for unrationed demand and supply functions. Section 6 then notes some applications of the results, in particular their implications for the signs of the Keynesian demand multiplier and the supply multiplier of Barro and Grossman (1974).

2. Preliminaries

Our interest is in the behaviour of a household which is free to purchase goods in some markets but is forced to purchase certain levels of goods in other markets. This is formally equivalent to a situation where the household faces rationing constraints, the constraints binding in some markets but not in others. Since we wish to express the properties of rationed demand and supply functions in terms of the properties of unrationed functions, it is necessary to consider how choices made by the household subject to rationing could arise in a no rationing situation. This leads to an investigation of virtual prices which support particular bundles in commodity space.

Let there be n goods in the economy, labour being treated as minus leisure, and assume that the household has a well-defined preference ordering R over commodity bundles. The collection of goods may be partitioned into two subsets, the partitioning being determined by whether or not the household is forced to consume a particular amount of any good. If x is the vector of goods freely chosen and y is the vector of goods imposed, then a consumption bundle is a pair (x,y). If p and q are the vectors of prices associated with x and y, respectively, and b is the lump-sum income of the household, then x must be chosen subject to the budget constraint

$$p.x + q.y \leq b \tag{1}$$

(for simplicity, it will be assumed that a feasible x always exists).

The household also faces a constraint of the form $y = \bar{y}$ (by definition). If $\bar{x}$ is demanded then $(\bar{x},\bar{y})$ is the consumption bundle of the household. To be able to say how the rationed demand $\bar{x}$ changes with parameter changes, and to express this change in terms of unrationed demand changes, it is necessary to show that $(\bar{x},\bar{y})$ would be demanded by the household at some prices and income when faced with no rationing constraints. The following three conditions are sufficient to ensure that any allocation can be supported with suitable prices:

A.1. R is convex, i.e. $\{z : zRz'\}$ is convex for all commodity bundles z'.

A.2. R is continuous, i.e. $\{z: zRz'\}$ and $\{z: z'Rz\}$ are closed for all commodity bundles z'.

A.3. R is strictly monotonic, i.e. for all commodity bundles, z,z' such that $z > z'$, zRz' but not $z'Rz$.[2]

The role of convexity is clear. However, convexity is not sufficient. For consider the well-known lexicographic preference ordering associated with Debreu (1959), (z_1, z_2) being strictly preferred to (z_1',z_2') if $z_1 > z_1'$ or $z_1 = z_1'$ and $z_2 > z_2'$. It is clear that no allocation in the interior of the consumption set can be supported. To overcome such problems, continuity is imposed. Finally, strict monotonicity is imposed so that support prices will be strictly positive. This assumption also allows a slightly stronger characterization of virtual prices to be made.

With conditions A.1 - A.3 imposed, there is no loss in generality in assuming that there exists a continuous quasi-concave utility function $v(x,y)$ which represents the household's preference ordering (see Debreu (1954)).

It will be shown that $(\bar{x},\bar{y})$ can be supported by prices p and some $\bar{q}$, i.e. the price vector for x which supports $(\bar{x},\bar{y})$ is the price vector faced by the household when $\bar{x}$ is chosen subject to the condition that y is fixed at $\bar{y}$. Define the partial indirect utility function V as follows:

$$V(r,y,p) = \max_{x} \{v(x,y): \; p.x = r\}. \tag{2}$$

As $\bar{x}$ is the optimal value of x under the budget constraint (1) and the ration constraint $y = \bar{y}$:

$$V(p.\bar{x},\bar{y},p) = v(\bar{x},\bar{y}). \tag{3}$$

For fixed p, V is quasi-concave in r and y. To see this consider any r,r',y,y',p such that

$$V(r,y,p) \geq V(r',y',p) \tag{4}$$

and let

$$V(r,y,p) = v(x,y), \qquad p.x = r, \tag{5}$$

$$V(r',y',p) = v(x',y'), \qquad p.x' = r'. \tag{6}$$

As v is quasi-concave:

$$v(\lambda x+(1-\lambda)x',\lambda y+(1-\lambda)y') \geq v(x',y') \tag{7}$$

for all $0 \leq \lambda \leq 1$, and

$$p.(\lambda x+(1-\lambda)x') = \lambda r+(1-\lambda)r'. \tag{8}$$

Thus, applying (2) and (6):

$$V(\lambda r+(1-\lambda)r', \lambda y+(1-\lambda)y', p) \geq V(r',y',p) \tag{9}$$

which proves the quasi-concavity of V in r and y.

Applying a supporting hyperplane theorem, the preferred region to a point in (r,y) space may be supported by a hyperplane through that point, i.e. there exist μ, $\overline{q}$ such that

$$V(r,y,p) > V(p.\overline{x},\overline{y},p) \rightarrow \mu r + \overline{q}.y > \mu p.\overline{x} + \overline{q}.\overline{y}. \tag{10}$$

As preferences are strictly monotonic, $\mu > 0$ and $\overline{q} >> 0$. Thus μ may be normalized at unity.[3] It is now easy to show that $(p,\overline{q})$ supports $(\overline{x},\overline{y})$. For assume not. There then exists an (x^*,y^*) such that

$$v(x^*,y^*) > v(\overline{x},\overline{y}) \tag{11}$$

and

$$p.x^* + \overline{q}.y^* \leq p.\overline{x} + \overline{q}.\overline{y}. \tag{12}$$

Applying (11) and (2) gives

$$V(p.x^*,y^*,p) > V(p.\overline{x},\overline{y},p). \tag{13}$$

Finally, combining (10) and (13) gives

$$p.x^* + \overline{q}.y^* > p.\overline{x} + \overline{q}.\overline{y} \tag{14}$$

which contradicts (12). Thus $(p,\overline{q})$ supports the bundle $(\overline{x},\overline{y})$.

$(p,\overline{q})$ is the virtual price system associated with the bundle $(\overline{x},\overline{y})$. Notice that the virtual prices for unrationed goods coincide with actual prices. The term virtual prices may therefore be retained exclusively for the vector $\overline{q}$.

As we shall be looking at how the demand for unrationed goods changes with various parameter changes, it will be convenient to assume that demand functions are appropriately differentiable. Thus, in the sequel, it will be implicitly assumed that preferences are suitably smooth and possess strict Gaussian curvature.

3. Notional and Constrained Expenditure Functions

Having shown that virtual prices $\bar{q}$ must exist we now proceed to introduce the expenditure (or "cost-of utility") functions with and without rationing.

When the consumer is unconstrained the expenditure function takes the familiar form:

$$m(p,q,u) = \text{Min}\ [p.x + q.y: \ \ v(x,y) \geq u] \tag{15}$$

The properties of this function are well known: m is increasing and concave in p and q, and linear homogeneous in p and q together. Moreover, by Shephard's Lemma, the partial derivatives of m with respect to p and q are the Hicksian demand functions for x and y respectively.

When the consumer is forced to consume an amount $\bar{y}$ of y a constrained expenditure function may be defined in an analogous manner:

$$\tilde{m}(\bar{y},p,q,u) = \text{Min}[p.x + q.\bar{y}: \ \ v(x,\bar{y}) \geq u]. \tag{16}$$

It is easily seen that this is increasing and concave in p and q, and that its partial derivatives with respect to p and q are the compensated constrained demand function for x and the ration level $\bar{y}$ respectively:

$$\tilde{m}_p\ (\bar{y},p,q,u) = \tilde{x}^c\ (\bar{y},p,q,u) \tag{17}$$

$$\tilde{m}_q\ (\bar{y},p,q,u) = \bar{y}. \tag{18}$$

Note that (18) implies that the partial derivatives of $\tilde{m}_q$ with respect to p, q, and u are zero:

$$\tilde{m}_{q\alpha} = \tilde{m}_{\alpha q} = 0 \qquad\qquad \alpha = p,q,u. \tag{19}$$

To establish the partial derivative of $\tilde{m}$ with respect to the ration level $\bar{y}$, we note that $\tilde{m}$ may be related to the unconstrained expenditure function (15), where the latter is evaluated at the virtual prices $\bar{q}$:

$$\begin{aligned}\tilde{m}\,(\bar{y},\, p,q,u) &= p.\tilde{x}^c(\bar{y},p,q,u) + q.\bar{y} \\ &= p.x^c(p,\bar{q},u) + q.y^c(p,\bar{q},u) \\ &= m(p,\bar{q},u) + (q-\bar{q}).\; y^c(p,\bar{q},u).\end{aligned} \tag{20}$$

The virtual prices are defined as an implicit function of $\bar{y}$, p and u by the restriction that they are those prices which would induce an unconstrained household to purchase the ration level $\bar{y}$:

$$\bar{y} = y^c(p,\bar{q},u). \tag{21}$$

Differentiating (20) with respect to $\bar{y}$ yields:

$$\begin{aligned}\tilde{m}_y &= (m_q - \bar{y})\,\frac{\partial \bar{q}}{\partial \bar{y}} + (q-\bar{q}) \\ &= q-\bar{q}\;.\end{aligned} \tag{22}$$

Equation (22) gives a precise measure of the benefit or loss to the household of a change in $\bar{y}$: a small increase in the amount of y which the household is forced to consume reduces the expenditure required to attain the same utility level u by the difference between the virtual and the actual price of y.

It may be noted that there is nothing in the preceding derivations which requires that $\bar{y}$ lies below the desired consumption level of y. The implications of this may be seen more clearly by considering an alternative measure of the cost to the household of being constrained to purchase $\bar{y}$, namely, the marginal utility of y at the constrained consumption point. It is clear that this must equal $\lambda\bar{q}$, where λ is the marginal utility of income in the constrained situation. Comparing this with the first-order conditions for a utility maximum obtained from an analysis using the primal approach, it follows that:

$$\tilde{q} = q + \frac{\phi}{\lambda} \tag{23}$$

where ϕ is the vector of shadow prices of the y constraints, each component of which, ϕ_j, is positive or negative according as the household would prefer to consume more or less than the corresponding constrained level of consumption, $\bar{y}_j$, given the levels of the other constraints $\bar{y}_k$, $k \neq j$.[4] Equation (23) therefore shows that $\bar{q}_j$ exceeds q_j when the household is (locally) constrained to consume less of commodity y_j that it wishes, and vice versa. In subsequent sections we shall give intuitive explanations of our results in terms of the familiar case of goods rationing ($\bar{q}>q$), but it should be kept in mind that the results apply with equal validity to the alternative case of forced consumption or a constraint on the amount supplied of a commodity (e.g., labour).[5]

4. Comparative Statics: Changes in the Ration Levels

In this section we make use of the functions introduced above to derive the effects of changes in the ration levels $\bar{y}$ on the demand for the unrationed goods. Given that virtual prices exist, we know that at the rationed consumption point:

$$\tilde{x}^c(\bar{y},p,q,u) = x^c(p,\bar{q},u) \tag{24}$$

where $\bar{q}$ is defined by (21). Differentiating (24) with respect to $\bar{y}$ yields:

$$\begin{aligned} \tilde{x}^c_y &= x^c_q \frac{\partial \bar{q}}{\partial \bar{y}} \\ &= x^c_q (y^c_q)^{-1} . \end{aligned} \tag{25}$$

This resembles equation 3.8 of Tobin and Houthakker but in fact it is a considerable generalisation of theirs, since (25) holds not only at the point where the constraint just bites but at all other points, provided the matrices of derivatives x^c_q and y^c_q are evaluated at the virtual prices $\bar{q}$.[6]

Equation (25) also shows that the Tobin-Houthakker result gives only the utility-compensated effect of a change in the ration level. When any ration constraint strictly bites a change in a ration level will have an income effect. To establish what this will be, we adopt the duality technique for deriving the Slutsky equation (used for example by Cook (1972):

$$\tilde{x}^c(\bar{y},p,q,u) = \tilde{x}[\bar{y},p,q,\tilde{m}(\bar{y},p,q,u)]. \tag{26}$$

Equation (26) states that the Hicksian and Marshallian rationed demand functions coincide when the household's total expenditure equals the minimum expenditure needed to reach utility level u in the face of prices p and q and the ration levels $\bar{y}$. Differentiating with respect to $\bar{y}$:

$$\tilde{x}^c_y = \tilde{x}_y + \tilde{x}_b\,\tilde{m}_y.$$

Therefore, invoking (22):

$$\tilde{x}_y = \tilde{x}^c_y + \tilde{x}_b(\bar{q}-q). \tag{27}$$

Equation (27) is an exact analogue of the usual Slutsky equation for the effect on demand of a price change. It shows that a relaxation of a ration constraint has a substitution effect, tending (from (25)) to reduce demand for substitutes and to increase demand for complements of the rationed good. In addition, the change has an income effect: just as in the usual Slutsky equation, e.g., (31) below, an increase in the price of good y reduces real income to a first order approximation by the amount of consumption of that good, so here an increase in the ration of y raises income by an amount which equals the saving in expenditures required to reach the same utility level, which in turn, from (22), equals the difference between the virtual and the actual price of the rationed good. This must be multiplied by the income derivative of the rationed demand function to determine the magnitude of its effect on the demand of x. Of course, when the ration just bites, the virtual and actual prices coincide, a small change in the ration has no income effect, and so the Tobin-Houthakker result (25) gives the total (and not just the utility-compensated) effect of the change in the ration on the demand for the unrationed goods.

5. Comparative Statics: Changes In Income and Prices

As they stand, (27) and (25) together do not completely express $\tilde{x}_y$ in terms of the derivatives of the unrationed demand functions, since the income derivative $\tilde{x}_b$ has not yet been related to the corresponding derivative of the unrationed demand function. To do this, we note that the rationed and unrationed Marshallian demand functions for x may be equated, provided the latter are evaluated at the virtual prices $\overline{q}$ and at the expenditure necessary to attain utility level u when the household faces prices p and $\overline{q}$, namely, $m(p,\overline{q},u)$. From (20), this equals actual expenditure under rationing $\tilde{m}$, i.e., b, plus an additional amount $(\overline{q} - q).\overline{y}$ needed to compensate the household for the imposition of the ration. Hence:

$$\tilde{x}(\overline{y},p,q,b) = x[p,\overline{q},b + (\overline{q}-q).\overline{y}]. \tag{28}$$

Since utility is no longer being held constant, the virtual prices must be such as to equate the Marshallian demands for the rationed commodities to the ration levels:

$$\overline{y} = y[p,\overline{q},b + (\overline{q}-q).\overline{y}]. \tag{29}$$

Differentiating (28) with respect to b and making use of (29) yields the expression sought for the income derivative of the rationed demand function which (when combined with (25)) expresses it in terms of derivatives of the unrationed demand functions only:

$$\tilde{x}_b = x_b - \tilde{x}^c_y \, y_b \qquad (30)$$

Equation (30), which is a generalisation of equation 6.1 of Tobin and Houthakker, shows that an increase in the budget affects the demand for an unrationed good in two distinct ways: firstly, it directly affects demand in the same way as it would in the absense of rationing (though evaluated at the virtual prices of course); and secondly, assuming the rationed commodities are normal, the increase in the budget amounts to a tightening of the ration constraints, and so, from (25), it increases demand for an unrationed commodity if and only if it is a net substitute for the rationed one.

Our final task is to derive the price derivatives of the rationed demand functions. We may note first that the familiar Slutsky equations hold for the unrationed demand functions. Thus differentiating (26) with respect to p and q yields:

$$\tilde{x}^c_p = \tilde{x}_p + \tilde{x}_b \tilde{x} \qquad (31)$$

and:

$$\tilde{x}^c_q = \tilde{x}_q + \tilde{x}_b \, \bar{y}. \qquad (32)$$

Since, from (19), a change in the price of the rationed good does not affect demand when utility is held constant, we may rewrite (32) immediately as:

$$\tilde{x}_q = -\tilde{x}_b \, \bar{y} \qquad (33)$$

Thus an increase in the price of the rationed good has an income effect only, forcing the household to pay more for the rationed consumption levels $\bar{y}$, and so reducing the demand for normal unrationed goods.

Finally, to relate the own-price derivatives of demand for the rationed goods to the corresponding derivatives of the unrationed demand functions, we first differentiate (24), making use of (21), to obtain:

$$\tilde{x}_p^c = x_p^c - \tilde{x}_y^c \, y_p^c . \tag{34}$$

Using (25) and the symmetry of the Slutsky substitution matrix we may rewrite (34) in a form which generalises another of Tobin and Houthakker's results:

$$\tilde{x}_p^c - x_p^c = - y_p^c \, (y_q^c)^{-1} \, y_p^c \tag{35}$$

Hence, irrespective of the extent of rationing, the difference between the matrices of rationed and unrationed compensated price derivatives is a positive definite matrix. This general Le Chatelier-type result implies that holding utility constant, rationing reduces the responsiveness of any Hicksian composite commodity formed from the set of unrationed commodities and, a fortiori, of any individual unrationed commodity to changes in its own price.

A result identical to (20) for the uncompensated own-price derivative may be obtained by differentiating (28), making use of (29):

$$\tilde{x}_p = x_p - \tilde{x}_y^c \, y_p \quad . \tag{36}$$

The rationale for this result, which generalises equation 6.2 of Tobin and Houthakker, is the same as for (30): a change in the price of an unrationed good has both a direct price effect, and, to the extent that it changes the "notional" demand for the rationed goods, an indirect effect identical to that of an equivalent tightening or relaxation of the ration constraint.

6. Some Applications

In this section we consider briefly some implications of our results for macroeconomics, for the measurement of consumer surplus, and for the empirical prediction of behaviour under rationing.

A. Macroeconomics

Recent work on macroeconomics by Barro and Grossman (1971), Malinvaud (1977) and others has argued that an appropriate short-run macro model consistent with microeconomic theory should be one where prices do not change to clear all markets, and where, as a result, agents on the long side of a market are faced with ration constraints, which in turn affect their behaviour in other markets. The signs of these feedback effects are crucial for the predictions of these models, but to date these signs have not been rigorously analysed. Indeed, presumably in the light of the results of Tobin and Houthakker, the impression has been given by Barro and Grossman (1971), p. 91) that the feedback effects of interest have their expected signs only if all commodities are substitutes in demand, and Howard (1977) claims to prove that this is the case. It is easy to show using our results however that, especially for large displacements from an unrationed equilibrium, the presumption that these feedback effects will have their expected signs is much stronger.

To illustrate this, we consider the household side of the simplest model of this kind, where utility is a function of leisure (i.e., total endowment of time, H, less hours worked, ℓ), the quantity consumed of a composite consumption good, x, and the amount of money balances carried over to the next period, m. When the household is unconstrained, its decision problem is therefore:

$$\max_{\ell,c,m} u(H-\ell,x,m) \text{ subject to: } px + m \leqslant w\ell + I \tag{37}$$

where I is the level of lump-sum income (including initial money balances), w is the wage rate and p the price of consumption goods. The solution to this problem yields, in the terminology of Clower (1965), notional demand functions for consumption and money balances, and a notional supply function for labour:

$$x(p,w,I),\ m(p,w,I) \text{ and } \ell(p,w,I). \tag{38}$$

Suppose instead however, that the household is prevented from realising its notional supply of labour by a constraint $\ell < \bar{\ell}$. The outcome of the problem (37) subject to this additional constraint is a set of effective demand functions for consumption and money balances, which depend explicitly on the employment constraint $\bar{\ell}$:

$$\tilde{x}(\bar{\ell},p,w,I) \text{ and } \tilde{m}(\bar{\ell},p,w,I). \tag{39}$$

It is now a simple matter of translating the results of earlier sections to relate the derivatives of the effective demand functions (39) to the corresponding derivatives of the notional functions in (37). The principal derivatives of interest are the marginal propensities to consume different types of income. Considering first the marginal propensity to consume lump-sum income, we find, making use of (30), that it equals:

$$\begin{aligned} MPC_I &= p\,\tilde{x}_I \\ &= p[x_I - \tilde{x}^c_{\bar{\ell}}\,\ell_I] \end{aligned} \tag{40}$$

where the compensated demand derivative with respect to the labour ration is, from (25):

$$\tilde{x}^c_{\bar{\ell}} = x^c_w\,(\ell^c_w)^{-1} \tag{41}$$

Equations (40) and (41) show that an increase in lump-sum income affects demand both directly, as shown by the term px_I, and indirectly, by changing the household's desired labour supply and so the effective level of the labour constraint. (Note that all the derivatives of the unconstrained functions (38) must be evaluated at the virtual wage $\bar{w}$, which is less than the actual wage w, and the corresponding virtual lump-sum income, $I + (w-\bar{w})\bar{\ell})$. If consumption and leisure are normal ($x_I > 0$ and $\ell_I < 0$) and net substitutes ($x_w^c > 0$), then both effects work in the same direction, leaving a positive marginal propensity to consume out of lump-sum income.

Turning to the effects of a change in the wage rate, since we know from Section 5 that it has an income effect only, we may define a marginal propensity to consume out of a wage rate increase in an analogous manner:

$$MPC_w = \frac{p}{\bar{\ell}} \tilde{x}_w . \tag{42}$$

Making use of (33), this is easily shown to be identical to (40):

$$MPC_w = p \tilde{x}_I = MPC_I . \tag{43}$$

Hence an unemployed household's marginal propensity to consume out of lump-sum income and out of a wage rate increase are identical. This provides the rationale for aggregating wage and non-wage income in the Keynesian consumption function.

However, the marginal propensity to consume following an increase in employment, i.e., a relaxation of the labour constraint, will generally be different:

$$\begin{aligned} MPC_\ell &= \frac{p}{w} \tilde{x}_{\bar{\ell}} \\ &= \frac{p}{w} [\tilde{x}^c_{\bar{\ell}} + (w-\bar{w}) \tilde{x}_I] \end{aligned} \tag{44}$$

where we have invoked (27). From (25), the first term on the right-hand side of (44), representing the substitution effect of the change in $\bar{\ell}$, is positive if and only if consumption and leisure are net substitutes. However, from the discussion of (40), the second term is likely to be positive, and to dominate the first for sufficiently large displacements from the unrationed equilibrium, i.e. sufficiently low values of $\bar{w}$. Using (40), (44) may be written in an alternative form as:

$$\frac{w}{p} \cdot MPC_{\ell} = [1 - (w-\bar{w})\ell_I]\ \tilde{x}^c_{\ell} + (w-\bar{w})x_I. \qquad (45)$$

Even if consumption and leisure are net complements, (45) is likely to be positive for reasonable parameter values, whenever the degree of disequilibrium, as measured by $(w-\bar{w})$, is substantial.

Comparing the marginal propensities to consume following changes in employment and lump-sum income, (40) and (44) yield, after some manipulations:

$$MPC_{\ell} - MPC_I = \frac{p}{w}\,(\ell^c_w)^{-1}\ [x^c_w\, m_I - m^c_w\, x_I] \qquad (46)$$

A sufficient condition for the final bracketed term to be zero is that consumption and money balances be weakly separable from leisure in the household's utility function. In this case, the only influence of labour supply upon consumption is through the aggregate income effect upon consumption.

Finally, an upper bound on the marginal propensity to consume may be examined by considering the marginal propensity to save. The foregoing analysis is almost directly applicable with x being replaced by m. For instance, the analogue of (45) gives

$$w(1-MPC_{\ell}) = [1 - (w-\bar{w})\ell_I]\ \tilde{m}^c_{\ell} + (w-\bar{w})\ m_I. \qquad (47)$$

Thus, when the degree of disequilibrium is small, i.e. $\bar{w}$ is close to w, $0 < MPC_{\ell} < 1$ requires the assumption that both consumption and money balances are net substitutes to leisure; but when the degree of disequilibrium is large, $0 < MPC_{\ell} < 1$ follows from the assumption that consumption and money balances are normal goods.

Similar arguments may be used to derive the marginal propensities to consume leisure following a relaxation of a goods market constraint, $x \leq \bar{x}$, in a situation of repressed inflation. The solution of the problem (37) subject to this additional constraint yields effective supply of labour and demand for money functions:

$$\tilde{\ell}\ (\bar{x},p,w,I) \text{ and } \tilde{m}\ (\bar{x},p,w,I) \tag{48}$$

As in the unemployment case therefore, we may use the results of previous sections to calculate the marginal propensities to supply labour (or marginal propensities to work, MPW) following a change in any one of the components of effective income, $I - p\bar{x}$:

$$MPW_I = MPW_p = w\tilde{\ell}_I = w[\ell_I - \tilde{\ell}^c_{\bar{x}}\ x_I] \tag{49}$$

and:

$$MPW_{\bar{x}} = \frac{w}{p}\,\tilde{\ell}_{\bar{x}} = \frac{w}{p}[\tilde{\ell}^c_{\bar{x}} + (\bar{p}-p)\ \tilde{\ell}_I] \tag{50}$$

where the derivatives of the unconstrained demand and supply functions are evaluated at $\bar{p}$, the virtual price of consumption which would induce an unconstrained household to behave in the same way as a rationed one. The interpretation of these equations is the same as that of those in the case of unemployment: an increase in lump-sum income or a reduction in the price of the rationed consumption good have the same direct and indirect effects on labour supply, both of which tend to increase it if consumption and leisure are normal and net substitutes; while a relaxation of the goods ration has a different effect on labour supply, which (provided only that leisure is normal) is more likely to be positive the further the household is from its unconstrained equilibrium.

In conclusion, it should be noted that these results imply that both the simple Keynesian demand multiplier and the supply multiplier of Barro and Grossman (1974) are more likely to have their expected signs (implying that a reduction in unemployment will increase consumption and an increase in the extent of excess demand for goods will reduce labour supply) the greater the extent of the initial disequilibrium from the point of view of households.[7] This both reinforces the conventional wisdom concerning the signs of these multipliers, and calls into question existing estimates of the marginal propensity to consume, which take no account of the fact that it may be expected to vary with the extent of unemployment.

B. Consumers' Surplus

The amount by which a household must be compensated for a small change in one of the prices p is given by the partial derivative of the constrained expenditure function with respect to the price in question, i.e., from (17), by the compensated demand for the commodity whose price has changed. Hence for finite changes in one or more prices, integrating over $\tilde{x}_p^c$ gives a measure of the money cost to the household of reaching the same utility level following the price changes, and thus a true measure of consumers' surplus. Of course, for large price changes, the usual problems of path dependence of the integral and the choice of reference utility level must be faced. However, apart from computational difficulties (which may be substantial, since for each p between the initial and final levels a new set of virtual prices must be calculated), the presence of rationing poses no new difficulties for the calculation of valid measures of consumers' surplus.

Finally, as noted above, equation (22) shows that the difference between the actual and virtual price of rationed commodity measures by how much a household must be compensated for a change in the ration level. Hence, for example, if labour supply is constrained, integrating over (22) provides an exact measure of "true" unemployment compensation, by constrast with the quadratic approximation at the unconstrained point which previous researchers such as Ashenfelter (1977) have adopted.

Of course, in order to be able to operationalise these measures of consumers' surplus, it is necessary to estimate the virtual prices. We turn therefore in the next sub-section to demonstrate how this can be done, at least for one simple and widely-used demand system.

C. Estimation and Prediction of Behaviour under Rationing

One of the important implications of the results presented above is that they enable a household's behaviour under rationing to be fully predicted from a knowledge of its unconstrained demand functions, even when no rationing has ever taken place. This makes possible some stringent tests of maintained hypotheses in empirical demand analysis, and it also suggests that the existing practice in such studies of omitting those years in which rationing was severe discards valuable information.

To illustrate how our results might be empirically implemented, we work through the case of the Linear Expenditure System, where the unconstrained demand functions (returning to the notation of previous sections, and considering the case of a single rationed good only) are:

$$p_i\ x_i = p_i\ \gamma_i + \beta_i(b-\Sigma p_i\gamma_i - q\gamma_0) \qquad i = 1, .,N \tag{51}$$

$$q\ y = q\ \gamma_0 + \beta_0(b-\Sigma p_i\gamma_i - q\gamma_0). \tag{52}$$

When commodity y is subject to a ration, the constrained demand functions for the unrationed commodities x are (making use of 28)):

$$p_i\ \tilde{x}_i = p_i\ \gamma_i + \beta_i[b+(\bar{q}-q)y - \Sigma p_i\gamma_i - \bar{q}\gamma_0] \tag{53}$$

$$i = 1,..,N.$$

The virtual price of the rationed good may be computed by solving the corresponding equation for y, which yields:

$$\bar{q} = \frac{\beta_o}{(1-\beta_o)(\bar{y}-\gamma_o)}\ [b - \Sigma p_i\gamma_i - q\ \bar{y}]. \tag{54}$$

This virtual price may be used for the welfare purposes discussed in the previous sub-section. Substituting from (54) into (53) yields the rationed demand functions for x, which depend linearly on observable exogenous variables only:

$$p_i \tilde{x}_i = p_i \gamma_i + \frac{\beta_i}{1 - \beta_0} (b - \Sigma p_i \gamma_i - q \bar{y}) \quad i = 1, \ldots N. \qquad (55)$$

Of course (55) could be calculated directly from the constrained primal problem, as was done by Ashenfelter (1977), who also discusses the estimation of (55) from aggregate data when some but not all households are constrained. (If all individuals possess identical preferences then the average demand will be identical to the demand of an individual with average income who faces an average ration). It is worth noting that if labour supply is rationed then (55) shows that aggregate consumption will be a linear function of aggregate income (wage and non-wage income). Thus the Linear Expenditure System provides a rationale for the simple linear Keynesian consumption function.

7. Summary and Conclusion

In this paper we have shown how duality theory and the concept of virtual prices may be used to simplify and extend the theory of household behaviour under rationing. We have shown that all the derivatives of the rationed demand functions may be decomposed into income and substitution effects which may in turn be related to the derivatives of the corresponding unconstrained demand functions. Among the implications of our results are the fact that behaviour under rationing may be fully predicted from a knowledge of unconstrained demand functions, and that the Keynesian demand multiplier and the Barro-Grossman supply multiplier are more likely to have their expected signs the further the household is from its unconstrained equilibrium.

FOOTNOTES

1. Areas in which the implications of quantity constraints have recently been studied include macroeconomics (Barro and Grossman (1971) and Malinvaud (1977)), portfolio choice (Diamond and Yaari (1972)), public economics (Roberts (1978)), and labour supply (Ashenfelter (1977)).

2. We adopt the standard vector inequality notation:

$\alpha \geqslant \beta$ iff $\alpha_i \geqslant \beta_i \ \forall i$

$\alpha > \beta$ iff $\alpha_i \geqslant \beta_i \ \forall i \ \& \ \exists j : \alpha_j > \beta_j$

$\alpha >> \beta$ iff $\alpha_i > \beta_i \ \forall i$

3. If the household's preferences are not strictly monotonic then it is possible that $\mu = 0$; in this case, $(\bar{x}, \bar{y})$ will be supported by $(0, \bar{q})$ which cannot be expressed in the form $(p, \bar{\bar{q}})$ for any well-defined $\bar{\bar{q}}$.

4. If more than one commodity is subject to a constraint, then a household might wish to consume more of a particular constrained commodity y_j (implying that ϕ_j is positive) at given levels of the other constraints, even though at the fully unconstrained equilibrium it would wish to consume less than the constraint level $\bar{y}_j$.

5. Of course if $\bar{y}_j$ is actually greater than y_j for a commodity demanded or less than y_j for a commodity supplied the household cannot be trading voluntarily. This interpretation avoids the asymetry in the Tobin-Houthakker results noted by Pollak (1969), p. 73).

6. We may note that $\tilde{m}_{py}$ equals $\tilde{m}_{yp}$ by Young's Theorem, which implies (from (8)) that an increase in a ration constraint $\bar{y}_j$ raises the compensated demand for an unrationed good x_i by the same amount

as an increase in p_i reduces the virtual price of the ration, q_j. We are grateful to Angus Deaton for pointing out that this was noted (if not in so many words) by Gorman (1976), p. 215.

7. These statements must be qualified by the fact that the further are firms from their notional equilibrium, the lower is the Keynesian multiplier, for a given value of the marginal propensity to consume. However in the case of a repressed inflation regime the two effects reinforce each other, so that we may say unambiguously that the Barro-Grossman supply multiplier is likely to be more negative the greater the extent of disequilibrium. See Neary (1978).

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