ON HARTWICK'S RULE FOR CONSTANT UTILITY AND REGULAR MAXIMIN PATHS OF CAPITAL ACCUMULATION AND RESOURCE DEPLETION

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ABSTRACT

Hartwick's rule of investing resource rents in an economy with producible capital and exhaustible resources becomes, in a general model of heterogeneous stocks, a rule whereby the total value of net investment (resource depletion counting negative) is equal to zero. It is shown that holding the discounted present value of net investment constant is necessary and sufficient for a competitive path to give constant utility. Conditions for the general rule to give a maximin path are also discussed.

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1. Introduction

In a series of recent papers, Hartwick (1977 a,b,c, 1978 b,c) has shown in a number of special models that keeping investment equal to the rents (really profits from the flow of depletion) from exhaustible resources under competitive pricing yields a path of constant consumption. Our purpose is to examine this striking rule in a general context. We shall allow many types of consumption goods and endogenous labour supplies. We shall also allow heterogeneous capital goods, and treat exhaustible or renewable resources as special capital goods: exhaustible resources can be depleted but not produced, renewable ones can also be produced. In this general framework the Hartwick rule becomes "keep the total value of net investment under competitive pricing equal to zero". This is then shown to be sufficient to give a constant utility path. The desirability of such a simple unified treatment should be evident. It even proves possible to generalize the rule to "keep the present discounted value of total net investment under competitive pricing constant over time"; indeed, the generalized Hartwick rule is necessary and sufficient for constant utility.

More importantly, while these rules give intergenerational equality, it remains to be seen whether they yield the best paths of this kind, i.e. Rawlsian paths. We therefore examine when the generalized Hartwick rule is sufficient to give a constant utility maximin path, or a little more precisely, a "regular" maximin path as defined in Burmeister and Hammond (1977).

2. A General Model and Constant Utility Paths

Let x denote a current net output vector, whose positive components represent outputs of consumption goods and whose negative components represent supplies of different types of labour. We shall assume that each generation's tastes are described by a utility function $u(x)$, without explicit time-dependence. This is only sensible when population is constant.

Let k denote a vector of stocks. Some components of k will refer to capital stocks, in which case the corresponding component of $\dot{k}$ can be positive and is the rate of net investment in that capital good. Other components of k will refer to exhaustible resources, when the corresponding component of $\dot{k}$ must be non-positive, the corresponding component of $-\dot{k}$ being the rate of depletion of that resource. Still other components of k can be renewable resources, when the corresponding component of $\dot{k}$ will be the difference between the rates of augmentation and depletion.

We shall assume that the economy has stationary and convex production possibilities described by a set Y consisting of feasible triples $(x,k,\dot{k})$. It can be seen that all the stationary models of Hartwick are special cases of this one. Exogenous population growth and technical progress considered in Hartwick (1977b) are not allowed here, but as they require a fortuitous coincidence of different exogenous rates if the rule is to remain valid, we do not think it worthwhile to attempt that generalization.

To describe a competitive path, we shall use present value prices, with a price vector p for current net outputs, and q for the current net additions to stocks. Being the supporting prices for the integrated feasible set Y, these reflect marginal rates of transformation. For an exhaustible resource, the component of q gives the marginal social profit, i.e. the market price minus the marginal extraction cost. If extraction costs are constant, this is just the unit profit or rent, and this part of the net investment equals the profit or rent on the flow being depleted.

A feasible path $(x^*(t), k^*(t), \dot{k}^*(t))_{t=0}^{\infty}$ will be called a competitive path at prices $(p(t), q(t))_{t=0}^{\infty}$ and utility-discount factors $\lambda(t) > 0$ if and only if

(i) for each t, $(x^*(t), k^*(t), \dot{k}^*(t))$ maximises instantaneous profit $p(t)x + q(t)\dot{k} + \dot{q}(t)\ k$ subject to $(x, k, \dot{k}) \in Y$, the juxtaposition of two vectors indicating their inner product,

(ii) for each t, $x^*(t)$ maximizes $\lambda(t)\ u(x) - p(t)x$ over all x.

A brief explanation will clarify this definition and put it in terms that may be more familiar to some readers. We could write the instantaneous profit as $p(t)x + q(t)\dot{k} - r(t)\ k$, where $r(t)$ is the vector of imputed rents to the assets, expressed in present values. The rents are connected to the asset prices $q(t)$ by the arbitrage conditions

$$r(t) = -\dot{q}(t),$$

which include the familiar user cost expression of capital good rents, and the Hotelling rule for resources. To write this in an even more common way, let us recall that if we choose utils of date 0 as the common numeraire for all these prices, we will be using a normalization $\lambda(0) = 1$. If instead we were to use current value prices, with utils of date t as the numeraire for prices of date t, and write these prices using the corresponding upper case letters, they will be rated to the present value prices by

$$P(t) = p(t)/\lambda(t), \quad Q(t) = q(t)/\lambda(t) \quad , \quad R(t) = r(t)/\lambda(t).$$

The interest rate in terms of utils will be $i(t) = - \dot{\lambda}(t)/\lambda(t)$. Then it is easy to verify that the arbitrage equation will be

$$R(t) = i(t)\ Q(t) - \dot{Q}(t),$$

a very well known form.

The conditions for the consumer follow from the intertemporal maximization problem

$$\text{maximize} \quad \int_0^\infty \lambda(t)\ u(x(t))\ dt \quad \text{subject to} \quad \int_0^\infty p(t)\ x(t) = \text{constant},$$

and setting the Lagrange multiplier on the budget constraint equal to 1 is simply a consequence of our choice of normalization.

Using present values with the above normalization, the value of net investment at time t is

$$I(t) = q(t)\ \dot{k}(t) \quad .$$

In current prices, it would be $I(t)/\lambda(t)$. Hartwick's rule is stated in terms of current prices. So long as the rule is to equate the value of net investment to zero, it makes no difference which price system we use.

We shall also consider a <u>generalized Hartwick Rule</u> where $I(t)$ = constant, not necessarily zero. For this it is crucial that present prices rather than current prices are used when valuing stock changes. If the constant is positive, in terms of current prices the rule will be to keep net investment growing at the rate of discount.

Hartwick proves that when there is a single consumption good, consumption will be constant if his rule is followed. The obvious extension and generalization in our model is that utility is constant if and only if a generalized Hartwick rule is followed. In fact we have:

<u>Theorem 1</u>: Suppose the economy follows a path $(x^*(t), k^*(t), \dot{k}^*(t))$ which is competitive at prices $(p(t), q(t))$. Suppose the production set Y has a smooth frontier and that the utility function is differentiable. Then $u(x^*(t))$ is constant if and only if $q(t)\,\dot{k}^*(t)$ is constant.

<u>Proof</u>: Consider times t and t+δt. Since $(x^*, k^*, \dot{k}^*)$ at both times are feasible with the same technology, profit-maximisation implies

$$p(t)\ x^*(t+\delta t) + \dot{q}(t)\ k^*(t+\delta t) + q(t)\ \dot{k}^*(t+\delta t)$$

$$\leq\ p(t)\ x^*(t) + \dot{q}(t)\ k^*(t) + q(t)\ \dot{k}^*(t)$$

Collect terms, divide by δt, and take limits as δt goes to zero both from the right and from the left. Given smoothness $p(t)\ \dot{x}^*(t) + \dot{q}(t)\dot{k}^*(t) + q(t)\ddot{k}^*(t)$ exists; one limit ensures it is non-negative, the other that it is non-positive. Thus we have

$$p(t)\ \dot{x}^*(t) + \dot{q}(t)\ \dot{k}^*(t) + q(t)\ \ddot{k}^*(t) \ = \ 0$$

Using the other part of the definition of a competitive path, we similarly write

$$\lambda(t)\ u(x^*(t+\delta t)) - p(t)\ x^*(t+\delta t) \leq \lambda(t)\ u(x^*(t)) - p(t)\ x^*(t),$$

and given differentiability, we have

$$\lambda(t)\ du(x^*(t))/dt \ = \ p(t)\ \dot{x}^*(t)$$

Putting these two results together,

$$\lambda(t)\ du(x^*(t))/dt \ = \ -\dot{q}(t)\ \dot{k}^*(t) - q(t)\ \ddot{k}^*(t)$$

$$= \ -d(q(t)\dot{k}^*(t))/dt$$

which completes the proof.

3. Regular Maximin Paths

Following Burmeister and Hammond (1977), we shall call a path $(x^*(t), k^*(t), \dot{k}^*(t))_{t=0}^{\infty}$ a regular maximin path if it is competitive at prices $(p(t), q(t))_{t=0}^{\infty}$ and positive discount factors $\lambda(t)$, and also

(a) $u(x^*(t)) = u^*$, constant, for all t

(b) $\int_0^{\infty} \lambda(t)\ dt$ is positive and finite

(c) $q(t)\ k^*(t) \to 0$ as $t \to \infty$.

A familiar argument establishes

Theorem 2: A regular maximin path maximizes $\inf_t u(x(t))$ over the collection of feasible paths.

Proof: (cf. Burmeister and Hammond (1977, p.860)) It suffices to show that for all feasible paths $(x(t), k(t), \dot{k}(t))$,

$$\liminf_{T \to \infty} \int_0^T \lambda(t) \left[u(x(t)) - u^* \right] dt \ \leq \ 0$$

Now $\int_0^T \lambda(t) \left[u(x(t))-u^* \right] dt$

$$= \int_0^T \lambda(t) \left[u(x(t))-u(x^*(t)) \right] dt$$

$$\leq \int_0^T p(t) \left[x(t) - x^*(t) \right] dt \quad \text{by (ii) for a competitive path}$$

$$\leq \int_0^T \left[q(t)(\dot{k}^*(t)-\dot{k}(t)) + \dot{q}(t)(k^*(t)-k(t)) \right] dt$$

by (i) for a competitive path

$$= \int_0^T d \left[q(t) \ (k^*(t)-k(t)) \right] /dt \quad dt$$

$$= q(T) \left[k^*(T)-k(T) \right] - q(0) \left[k^*(0)-k(0) \right]$$

$$\leq q(T) \ k^*(T) \quad \text{since } k(0) \leq k^*(0), \ q(0) \geq 0, \ q(T) \geq 0, \ k(T) \geq 0$$

Since $q(T) \ k^*(T) \to 0$ as $T \to \infty$, the result follows.

Regularity is not an inevitable feature of a maximin path, however. In particular, as Calvo (1978) has shown, if there is one capital good whose initial supply exceeds the golden rule capital stock in a capital accumulation model, then there is no regular maximin path, because any constant utility path is inefficient. Moreover, any golden rule stationary path violates the Malinvaud transversality condition $q(t) \ k^*(t) \to 0$. It does seem possible, however, that if no golden rule stationary path exists (as will be the case with essential non-renewable resources), or if the initial stocks are not sufficient for the economy to be able to sustain the golden rule utility level, then, provided a maximin path does exist, it must be regular. We offer this as a conjecture.

Returning to the Hartwick rule, it is an immediate corollary of Theorem 1 that any regular maximin path must satisfy a generalized Hartwick rule. It has not been shown, however, that the rule is the original one. The possibility that a regular maximin path may have the present value of net investment at a constant but non-zero level remains to be investigated. For many models with heterogeneous capital but without exhaustible resources, we often have a regular maximin path with $k^*(t)$ converging to a steady state value $\bar{k}$ as $t \to \infty$. Then the limit of $\dot{k}^*(t)$, if it exists, must be zero, and so it must be true that $I = 0$. It is precisely when there are exhaustible resources that the original Hartwick rule becomes more problematic as a criterion for achieving a regular maximin path. For, with capital goods and exhaustible resources, it is often the case that a constant utility path involves resource stocks tending to zero and capital stocks to infinity. Even so, for resources the present value prices will typically be constant, so the resource part of $q\dot{k}$ will converge to zero. For capital goods whose stocks are becoming infinite, prices must tend to zero (and do so sufficiently fast) if the Malinvaud condition is to be satisfied. Then provided the rate of accumulation $\dot{k}$ of such stocks is bounded, it must be true that on a regular maximin path their contribution to $q\dot{k}$ goes to zero. Under such conditions, the original Hartwick rule $I = 0$ is valid. It can be verified that in the Cobb-Douglas case where the capital/output ratio grows linearly, this is the case. But a general result appears to be a loose end.

4. Concluding Remarks

We leave it to interested readers to obtain the various special cases considered by Hartwick from our general formulation above. What is needed is to understand the appropriate significance of the prices q and $\dot{q}$ in each case. For example, it was stated that for a resource, the component of q gives its market price minus its marginal extraction cost. With constant average extraction costs, this element of $q\dot{k}$ is simply the market profit calculated on the flow depletion of the resource. Exponential depreciation of capital is easy to interpret, but more general depreciation, and also the case of stock-dependent resource extraction costs, present problems in relating the general rule to market prices.

Our analysis has not dealt with production sets Y whose frontiers have corners. This is an important omission when some resources have good substitutes, for the stocks of such resources may well get run down to zero. For each stock which is run down to zero, we have two regimes, one when the stock is still being depleted, and another when it has all been used up. Within each separate regime, the generalized Hartwick rule applies. But it is possible for the present value of net investment to jump at the moment when the resource in question just becomes exhausted. However, we expect utility to remain continuous across such a junction in a strictly convex Ramsey problem given some minimal intertemporal transformation possibility both ways. Therefore we do not expect this feature to pose any special problems for regular maximin paths either. In Hartwick's (1978b) case of a homogeneous output being produced using capital and different grades of a given physical resource, for example, at a junction point the market price of the resource is continuous, but the extraction cost jumps up. The value of net investment jumps down by the same amount. Total output is continuous, and consumption being net of both investment and extraction costs, is also continuous. Hartwick (1978a) states in great detail the determination and characterization of such junction points.

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