Limit Theorems on the Core of a Many Good Economy with Individual Risks

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Introduction

The relationship between the core of an exchange economy and competitive equilibrium is well-known from the work of Debreu and Scarf [1963]. If we define a group of individuals of the same type to be a group with the same preferences and endowments, Debreu and Scarf have shown that if there are a fixed number of types in an economy, and if the economy is expanded by increasing equal numbers of each type, then the set of core allocations converges to the set of competitive allocations.

There are a number of different ways in which we can introduce uncertainty into an exchange economy. The most straightforward is to assume that there is a given set S of possible states of the world. We introduce state-contingent markets for each good and simply reinterpret the Debreu-Scarf theorem. Individuals of the same type now have the same preferences for certain outcomes, the same probability distribution over states of the world, and the same distribution of endowments. In addition they are assumed to maximise expected utility. A coalition is said to block an allocation if a redistribution of endowments within the coalition leaves at least one member of the coalition with higher expected utility, and none with lower. The set of core allocations again converges to the set of competitive allocations.

However, there is a problem that is immediately apparent in this kind of approach. The set of states of the world is assumed to be given, and must therefore be unaffected by the process of replication necessary to generate the convergence result. This will only be true if the characteristics of any risk state can be specified independently of the circumstances or characteristics of any individual agent. In other words, we must confine ourselves to considering only communal risks. If we wish to consider individual risks, then we must allow for the possibility that the set of states of the world will expand as the number of agents increases. A natural way of introducing individual risks is to suppose that agents have their initial endowments determined by personal risk states.

Caspi [1978] has examined the problem of core convergence in a simple exchange economy with individual risks where agents have the same tastes and differ only in their initial endowment distributions, and where there is only one good. We shall first consider the case of an economy with many goods, and will show that in the core of a large economy risk-spreading will occur, and almost all individuals will receive an allocation close to the expected initial endowment which is the competitive (no trade) equilibrium in the mean value economy, where everybody receives the expected value of his endowment. The particular method of proof used allows us then to consider the introduction of a simple form of production. We show that in the same sense as indicated above, the core will converge to the competitive equilibrium allocation in the mean value economy, where the random variables in the production functions are replaced by their expected values. In the case where there exist non-convexities in production, the core converges to the competitive equilibrium in the convexified mean value economy.

The Model

Consider an exchange economy E^r in which there are r traders who have identical continuous, strictly concave utility functions. There are n goods in the economy, and a set of states of the world $\{s^r\}$ S^r. The endowment vector of any trader is determined by his personal state $s_i \in S_i$. The subjective and objective probability of s_i is $\Pi(s_i)$. The set of states of the world S^r we define as

$$S^r = \underset{i=1}{\overset{r}{X}} S_i^{\ r} \qquad (1)$$

We use $\Pi(s^r)$ to denote the probability of any state of the world $s^r = (s_1, s_2, \ldots s_r)$. The endowment of individual i in state s^r, $x_i(s^r)$, is determined only by his personal state, so that $x_i(s^r) = x_i(s'^r)$ if $s_i = s_i'$. We assume that

$$E(x_j \mid x_1, x_2, \ldots x_{j-1}) = Ex \qquad j = 1,2,\ldots r \qquad (2)$$

where x_j is an n-vector random variable.
Note that this implies neither independence nor identical distribution of random endowments, as was assumed by Caspi. [1978]

Traders are assumed to maximise expected utility, and an allocation $[y_1^r(\cdot), \ldots y_r^r(\cdot)]$ in E^r is feasible if

$$\sum_{i=1}^{r} y_i^r(s^r) \leq \sum_{i=1}^{r} x_i(s^r) \quad \text{for all } s \in S^r$$

Results

We wish to show a result which has been demonstrated for the one good economy by Caspi [1978] under the more restrictive assumptions mentioned above. The result states that, as the economy is increased in size, so the proportion of traders who receive in the core an allocation which does not converge in probability to Ex, the common expected value of initial endowments, becomes arbitrarily small. The result on convergence in probability, although stated in Caspi [1978], is not proved correctly.

Lemma 1. Let $[y_1^r(\cdot), \ldots y_r^r(\cdot)]$ be an allocation in the core of E^r. Then for every $\varepsilon > o$ and α, $o < \alpha < 1$, there exists R such that for $r \geq R$

$$\overset{r}{\text{Prop}} \; \{ \, |Eu(y_i^r) - u(Ex)| < \varepsilon \, \} > 1 - \alpha \qquad (3)$$

$\overset{r}{\text{Prop}} \{p\}$ denotes the proportion of individuals in E^r who satisfy proposition p.

Proof: As Theorem 1 of Caspi [1978] noting that (2) in conjunction with the condition

$$\sum_{j=1}^{\infty} \frac{E(|x_j|^r)}{j^r} < \infty \quad \text{for some } r \text{ in } [1,2], \text{ clearly}$$

satisfied here, is sufficient to establish that

$$\frac{1}{r} \sum_{i=1}^{r} x_i \xrightarrow{\text{a.c.}} Ex \qquad (4)$$

(see Theorem 9.5.1. of Whittle [1970]).

Lemma 2. Let $[y_1^r(\cdot), \ldots y_r^r(\cdot)]$ be an allocation in the core of E^r. Then for every $\varepsilon > o$ and α, $o < \alpha < 1$, there exists R such that for $r \geq R$

$$\overset{r}{\text{Prop}} \; \{ u(Ey_i^r) \geq u(Ex) - \varepsilon \} > 1 - \alpha \qquad (5)$$

Proof : This follows straightforwardly from lemma 1, once one observes that $u(Ey_i^r) > Eu(y_i^r)$ by the strict concavity of u.

Lemma 3. Let X, Y be finite dimensional Euclidean spaces, and let $\phi : X \to Y$ be an upper hemi-continuous (u.h.c.) compact-valued correspondence. Define $\psi(x) = \text{con}\, \phi(x)$ (i.e. the convex hull of $\phi(x)$). Then $\psi(x)$ is also a u.h.c. correspondence.

Proof : See Hildenbrand and Kirman {1976}, A.III.4.

Now we state the first result

Theorem 1. Let $[y_1^r(\cdot), \ldots\ y_r^r(\cdot)]$ be an allocation in the core of E^r. Then for every $\varepsilon > o$, and α, $o < \alpha < 1$, there exists R such that for $r \geq R$

$$\overset{r}{\text{Prop}} \{ \| Ey_i^r - Ex \| < \varepsilon \} > \alpha \tag{6}$$

Suppose the theorem is false. Then there must exist a sequence $\{r_k\}$ and numbers $\bar{\varepsilon} > o$, $o < \bar{\alpha} < 1$, such that

$$\overset{r_k}{\text{Prop}} \{ \|Ey_i^{r_k} - Ex\| < \bar{\varepsilon} \} < 1 - \bar{\alpha} \qquad k = 1,2,\ldots \tag{7}$$

Define the set

$$A(\varepsilon) = \{Ey \mid u(Ey) \geq u(Ex) - \varepsilon \;; \quad \|Ey - Ex\| \geq \bar{\varepsilon} \} \tag{8}$$

Then it follows from lemma 2 and (7) that there exists K such that for $k \geq K$

$$\overset{r_k}{\text{Prop}} \{Ey_i^{r_k} \in A(\varepsilon)\} > o \tag{9}$$

Now define

$$B(\varepsilon) = \{Ey \mid u(Ey) \geq u(Ex) - \varepsilon \; ; \; ||Ey - Ex|| < \bar{\varepsilon}\}$$

$$C(\varepsilon, \lambda^{r_k}) = \{Ez \mid \lambda^{r_k} Ey + (1 - \lambda^{r_k}) Ez \leq Ex \; ; \; Ey \in \text{con } A(\varepsilon)\}$$

where $$\lambda^{r_k} = \frac{\text{Prop}^{r_k} \{ Ey_i^{r_k} \in A(\varepsilon)\}}{\text{Prop}^{r_k} \{Ey_i^{r_k} \in A(\varepsilon)\} + \text{Prop}^{r_k} \{Ey_i^{r_k} \in B(\varepsilon)\}}$$

and con $A(\varepsilon)$ is the convex hull of $A(\varepsilon)$.

Consider the tangent hyperplane $H = \{Ey \mid p\, Ey = p\, Ex\}$ to the indifference surface at Ex. By construction $A(o)$ does not contain any point in H. Also $A(o)$ is a subset of $D = \{Ey \mid u(Ey) \geq u(Ex)\}$, the upper contour set at Ex. Since D is strictly convex, con $A(o)$ is also a subset of D. But Ex is an extreme point of D, by the strict concavity of u. Since $A(o)$ does not contain Ex, neither does con $A(o)$. But $H \cap D = \{Ex\}$, so that con $A(0) \cap H = \phi$.

Now it is clear that $A(\varepsilon)$ is a u.h.c. correspondence. From lemma 3 we infer that con $A(\varepsilon)$ is also u.h.c. So there must exist $\hat{\varepsilon} > o$ such that

$$\text{con } A(\varepsilon) \cap H = \phi \qquad \text{for all} \quad \varepsilon, \; o \leq \varepsilon \leq \hat{\varepsilon} . \tag{10}$$

The above argument can be illustrated as in Figs. 1 and 2.

Figure 1:

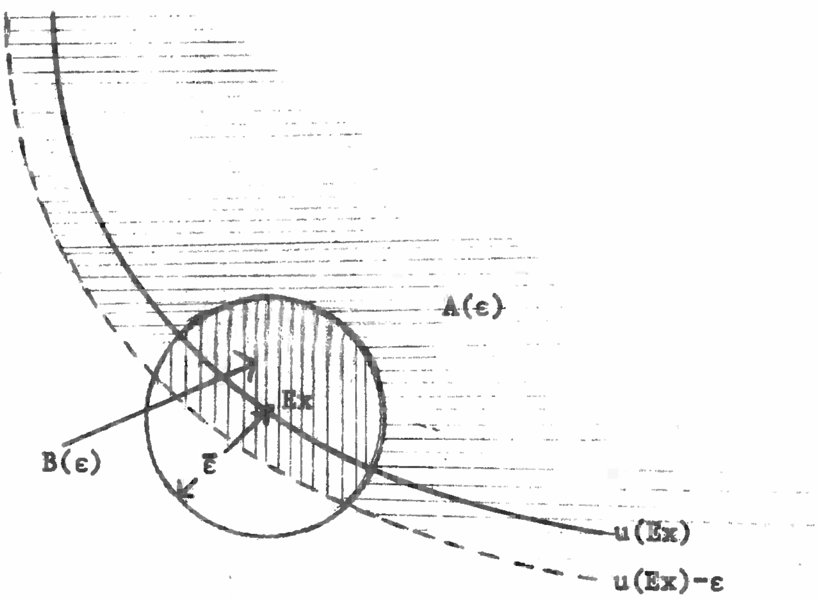
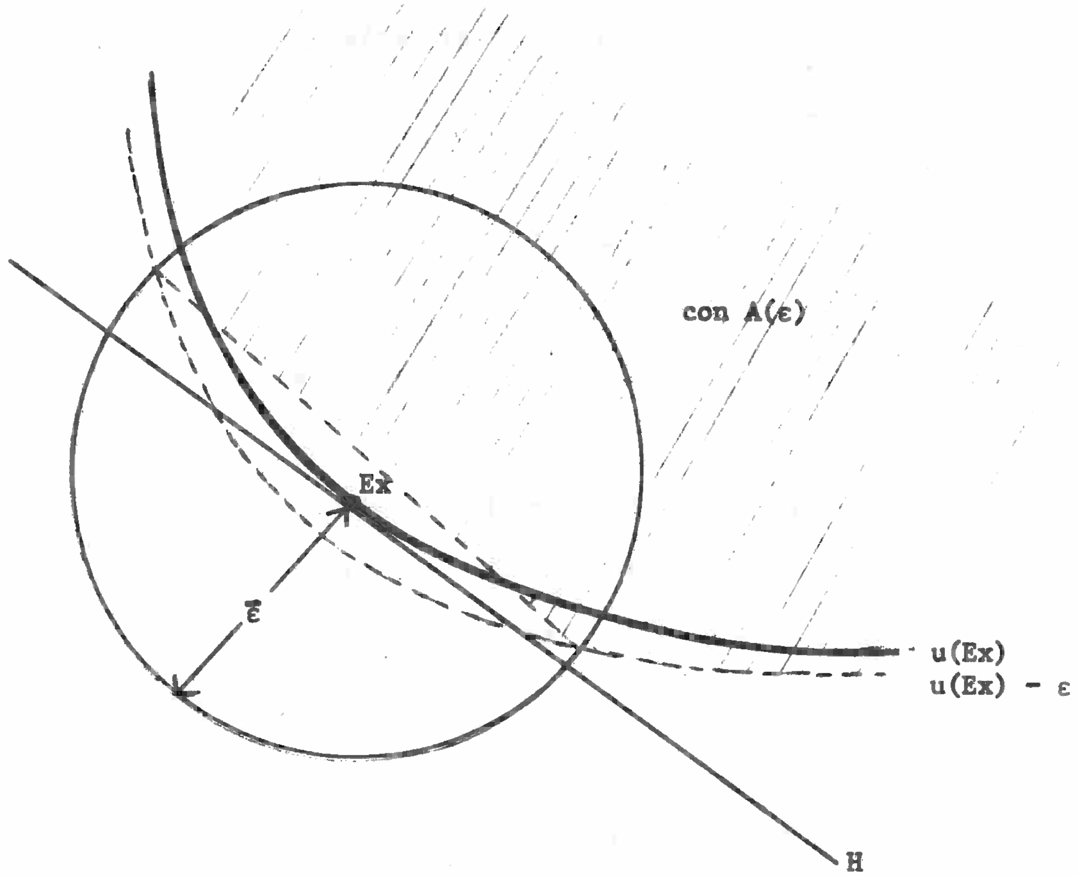


Figure 2:



If $Ez \in C(\varepsilon, \lambda^{r_k})$, then $\lambda^{r_k} Ey + (1 - \lambda^{r_k}) Ez \leq Ex$ for some $Ey \in \text{con } A(\varepsilon)$. Therefore $\lambda^{r_k} p\, Ey + (1 - \lambda^{r_k}) p\, Ez \leq p\, Ex$. We also know that

$$\min_{Ey \in \text{con } A(\varepsilon)} [p\, Ey] > p\, Ex \quad \text{for } o \leq \varepsilon \leq \hat{\varepsilon} \quad \text{by the theorem}$$

of the separating hyperplane. From Lemma 2 and (7) we know that $\liminf_{k \to \infty} \lambda^{r_k} \geq \bar{\alpha} > o$.

If we define

$$\eta(\varepsilon) = \limsup_{k \to \infty} \left[\sup_{Ez \in C(\varepsilon, \lambda^{r_k})} p\, Ez \right] \tag{11}$$

then $\eta(\varepsilon) < p\, Ex$ for $o \leq \varepsilon \leq \hat{\varepsilon}$.

Since $\inf_{B(o)} [p\, Ey] = p\, Ex$, there exists $\bar{\varepsilon}$, satisfying $o < \bar{\varepsilon} \leq \hat{\varepsilon}$, such that

$$\inf_{B(\varepsilon)} [p\, Ey] > \eta(\varepsilon) \quad \text{for } o \leq \varepsilon \leq \bar{\varepsilon} \tag{12}$$

We see this from the fact that the closure of $B(\varepsilon)$ is u.h.c.

Using the theorem of the separating hyperplane, (11) and (12) imply that for all ε, $o \leq \varepsilon \leq \bar{\varepsilon}$, there exists $\hat{K}$ such that for $k \geq \hat{K}$, $C(\varepsilon, \lambda^{r_k})$ and the closure of $B(\varepsilon)$ are disjoint.

We conclude that, for any $Ey \in \text{con } A(\bar{\varepsilon})$, $Ez \in B(\bar{\varepsilon})$, there exists $\theta > o$ such that for $k \geq \max(K, \hat{K})$

$$e_j(\lambda^{r_k} Ey + (1 - \lambda^{r_k}) Ez - Ex) > \theta \quad \text{for some } j \tag{13}$$

where e_j is the j^{th} unit vector.

Suppose we write $\lambda^{r_k} = \dfrac{a(r_k)}{a(r_k) + b(r_k)}$ where

$$a(r_k) = r_k \operatorname{Prop}^{r_k} \{Ey_i^{r_k} \in A(\bar{\varepsilon})\}, \quad b(r_k) = r_k \operatorname{Prop}^{r_k} \{Ey_i^{r_k} \in B(\bar{\varepsilon})\}$$

and that $I^{r_k}(C)$ is the set of agents in E^{r_k} whose allocations lie in C.

Then (13) tells us that

$$e_j \left[\frac{a(r_k)}{a(r_k)+b(r_k)} \left(\frac{1}{a(r_k)} \sum_{i \in I^{r_k}(A(\bar{\varepsilon}))} Ey_i^{r_k} \right) + \frac{b(r_k)}{a(r_k)+b(r_k)} \left(\frac{1}{b(r_k)} \sum_{i \in I^{r_k}(B(\bar{\varepsilon}))} Ey_i^{r_k} \right) - Ex \right] > \theta \qquad (14)$$

for some j.

This follows from the fact that $\dfrac{1}{a(r_k)} \sum_{i \in I^{r_k}(A(\bar{\varepsilon}))} Ey_i^{r_k}$ is a convex combination of points in $A(\bar{\varepsilon})$ and must therefore be in con $A(\bar{\varepsilon})$. Similarly, since $B(\bar{\varepsilon})$ is convex, a convex combination of points in $B(\bar{\varepsilon})$ must lie in $B(\bar{\varepsilon})$.

It follows that

$$e_j \left(\sum_{i \in I^{r_k}(A(\bar{\varepsilon}))} Ey_i^{r_k} + \sum_{i \in I^{r_k}(B(\bar{\varepsilon}))} Ey_i^{r_k} \right) > (a(r_k) + b(r_k))(e_j Ex + \theta) \quad (15)$$

for some j.

Since

$$\frac{a(r_k)+b(r_k)}{r_k} = \overset{r_k}{\text{Prop}} \{ u(Ey_i^{r_k}) \geq u(Ex) - \varepsilon \}$$

we know from lemma 2 that there exists $\bar{K} \geq \max(K, \hat{K})$ such that for $k \geq \bar{K}$

$$\frac{a(r_k) + b(r_k)}{r_k} > \frac{e_j\ Ex}{e_j\ Ex+\theta} \tag{16}$$

Thus for $k \geq \bar{K}$

$$e_j \sum_{i \in I^{r_k}(A(\bar{\varepsilon})) \cup B(\bar{\varepsilon}))} Ey_i^{r_k} > r_k\, e_j\, Ex \quad \text{for some } j. \tag{17}$$

Feasibility requires that

$$\sum_{i=1}^{r} x_i(s) \geq \sum_{i=1}^{r} y_i^r(s)$$

for all $s \in S$. This implies that

$$\sum_{i=1}^{r} Ey_i^r \leq r\ Ex \qquad \text{for all } r \tag{18}$$

But (18) is inconsistent with (17). It is not possible that $[y_1^{r_k}(\cdot), \ldots y_{r_k}^{r_k}(\cdot)]$ are core allocations as assumed. Therefore, contrary to hypothesis, (6) must hold.

Theorem 1 only establishes that most individuals will receive similar allocations in <u>expected value</u> in the core. It is not correct then directly to infer that allocations converge in probability to Ex, as Caspi does. We need to use the results contained in Lemma 1 and Theorem 1. We proceed as follows.[1/]

Define a function

$$\phi(y) = u(\bar{y}) + (y - \bar{y})\, u_y(\bar{y}) - u(y) \tag{19}$$

Since u is strictly concave, ϕ is strictly convex and possesses the properties $\phi(\bar{y}) = 0$, $\phi(y) > 0$ for $y \neq \bar{y}$. Fix any $\varepsilon > 0$ and let $\inf \{\phi(y) \mid \; ||y - \bar{y}|| = \varepsilon\} = K(\varepsilon, \bar{y}) > 0$

Lemma 4. If $||y - \bar{y}|| > \varepsilon$ then $\phi(y) > K(\varepsilon, \bar{y})$

Proof: Suppose not. Let z be such that

$$z = \lambda y + (1 - \lambda)\bar{y}\,; \quad ||z - \bar{y}|| = \varepsilon \tag{20}$$

By strict convexity

$$\phi(z) < \lambda\phi(y) + (1 - \lambda)\,\phi(\bar{y}) \tag{21}$$

$$< \lambda K(\varepsilon,\bar{y}) + (1 - \lambda)\cdot 0 \tag{22}$$

$$< K(\varepsilon,\bar{y}) \tag{23}$$

which contradicts the definition of K.

Lemma 5. (i) $\lim_{r\to\infty} Eu(y_i^r) = u(Ex)$ (24)

(ii) $\lim_{r\to\infty} Ey_i^r = Ex$ (25)

1/ I am indebted to Avinash Dixit for supplying the proof given here.

then, for all $\varepsilon > 0$

$$\lim_{r\to\infty} \text{Prob}\ \{||y_i^r - Ex|| > \varepsilon\} = 0 \qquad (26)$$

Proof: Suppose not. Then there exists a sequence $\{r_k\}$ and numbers $\bar{\varepsilon}$, $\bar{\delta} > 0$ such that

$$\text{Prob}\ \{||y_i^{r_k} - Ex|| > \bar{\varepsilon}\} > \bar{\delta} \qquad k = 1, 2, \ldots \qquad (27)$$

Then, using Lemma 4, and letting $\bar{y}_i^{r_k} = E_{y_i}^{r_k}$

$$E\phi(y_i^{r_k}) \geq \text{Prob}\ \{||y_i^{r_k} - Ey_i^{r_k}|| > \bar{\varepsilon}\}.\inf\ \{\phi(y_i^{r_k})\ |\ ||y_i^{r_k} - Ey_i^{r_k}|| > \bar{\varepsilon}\} \quad (28)$$

Therefore

$$\liminf_{k\to\infty} E\phi(y_i^{r_k}) \geq \bar{\delta}\ .\ K(\bar{\varepsilon}, Ex) > 0 \qquad (29)$$

But $E\phi(y_i^{r_k}) = u(Ey_i^{r_k}) - Eu(y_i^{r_k})$ (30)

and (29) contradicts (24).

We now state

Theorem 2. Let $\{y_1^r(\cdot), \ldots y_r^r(\cdot)\}$ be an allocation in the core of E^r. Then for every $\varepsilon > 0$, and α, β, $0 < \alpha < 1$, $0 < \beta < 1$ there exist R such that for $r \geq R$

$$\operatorname{Prop}^{r}\left\{\operatorname{Prob}\left(\|y_i^r - Ex\| < \varepsilon\right) > \alpha\right\} > \beta \tag{31}$$

Proof: This follows directly from Lemmas 1 and 5 and Theorem 1.

A model with production

Let us assume that each individual i has a stochastic production function

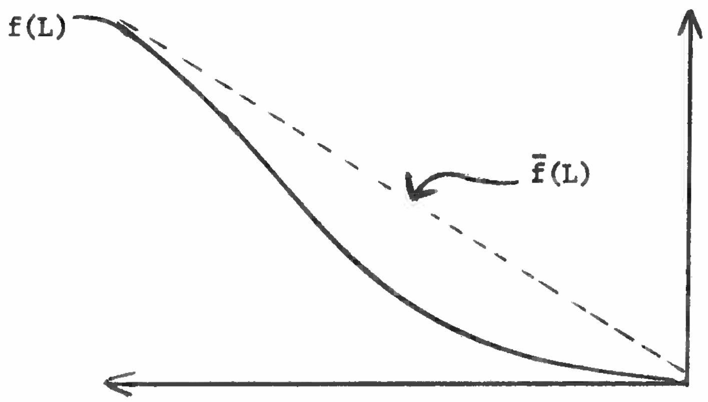
$$Y^i(s_i) = f^i(L_i, s_i) \tag{32}$$

where Y^i is the output of a single good, L_i is the labour input of individual i, and s_i is the personal risk state of individual i. We make the assumption that

$$E(f^i(L_i) \mid f^{i-1}(L_{i-1}), \; f^{i-2}(L_{i-2}) \; \ldots.) = f(L_i) \qquad i = 1,2,\ldots \tag{33}$$

which is a generalisation of the assumption made above on endowment distributions. We do not assume that $f(L)$ is concave, but write $\bar{f}(L)$ for the concavified version of $f(L)$. This is illustrated in fig. 3.

Figure 3:



What we are doing is to take the convex hull of the production set associated with $f(L)$, thus generating a new production function $\bar{f}(L)$.

Then we may state the following result :

Theorem 3. If $\bar{L}$ maximises $u(\bar{f}(L),L)$, and $[(y_1^r(\cdot), L_1^r), \ldots. (y_r^r(\cdot), L_r^r)]$ is a core allocation in E^r, then for all $\varepsilon > o$ and α, $o < \alpha < 1$, there exists R such that for $r \geq R$

$$\overset{r}{\text{Prop}} \{ \mid Eu(y_i^r, L_i^r) - u(\bar{f}(\bar{L}),\bar{L}) \mid \geq \varepsilon \} < \alpha \tag{34}$$

Proof: Suppose that (34) is not true. Then there exists a sequence $\{r_k\}$, and numbers $\bar{\varepsilon} > o$, α, $o < \bar{\alpha} < 1$ such that

$$\overset{r_k}{\text{Prop}} \{ \mid Eu(y_i^{r_k}, L_i^{r_k}) - u(\bar{f}(\bar{L}),\bar{L}) \mid \geq \bar{\varepsilon} \} \geq \bar{\alpha} \qquad k = 1,2,\ldots \tag{35}$$

If (35) holds, it must be true that there exists $\bar{\beta} > o$ such that either

$$\overset{r_k}{\text{Prop}} \{ Eu(y_i^{r_k}, L_i^{r_k}) \geq u(\bar{f}(\bar{L}),\bar{L}) + \bar{\varepsilon} \} \geq \bar{\beta} \qquad k = 1,2,\ldots \tag{36}$$

or

$$\overset{r_k}{\text{Prop}} \{ Eu(y_i^{r_k}, L_i^{r_k}) \leq u(\bar{f}(\bar{L}),\bar{L}) - \bar{\varepsilon} \} \geq \bar{\beta} \qquad k = 1,2,\ldots \tag{37}$$

We show first that if (36) holds, then there must exist $\bar{\gamma} > o$ and $\bar{\delta}$, $o < \bar{\delta} < 1$ such that

$$\overset{r_k}{\text{Prop}} \{ Eu(y_i^{r_k}, L_i^{r_k}) \leq u(\bar{f}(\bar{L}),\bar{L}) - \bar{\gamma} \} \geq \bar{\delta} \qquad k = 1,2,\ldots \tag{38}$$

Define

$$A = \{(Ey,L) \mid u(Ey,L) \geq u(\bar{f}(\bar{L}),\bar{L}) + \bar{\varepsilon}\}$$

$$B(\gamma) = \{(Ey,L) \mid u(\bar{f}(\bar{L}),\bar{L}) + \bar{\varepsilon} > u(Ey,L) > u(\bar{f}(\bar{L}),\bar{L}) - \gamma\}$$

$$C(\gamma) = \{(Ey,L) \mid u(Ey,L) \leq u(\bar{f}(\bar{L}),\bar{L}) - \gamma\} .$$

We note first that if $(Ey,L) \in A$, then $Ey > \bar{f}(L)$, since $\bar{L}$ is assumed to maximise $u(\bar{f}(L),L)$. So

$$\inf_{(Ey,L) \in A} [Ey - \bar{f}(L)] = \Theta > 0 \tag{39}$$

Let $\inf_{(Ey,L) \in B(\gamma)} [Ey - \bar{f}(L)] = \eta(\gamma)$. It is clear that

$\eta(\gamma) \leq 0$ and that $\lim_{\gamma \to 0} \eta(\gamma) = 0$. Further, if the maximum amount of labour an individual can supply is L^*, then

$$\inf_{(Ey,L) \in C(\gamma)} [Ey - \bar{f}(L)] = - \bar{f}(L^*) \tag{40}$$

(assuming that a negative allocation of the output is ruled out).

Putting this together, we find that

$$\sum_{i=1}^{r_k} (Ey_i^{r_k} - \bar{f}(L_i^{r_k})) \geq (\Theta \bar{\beta} + \eta(\gamma) - \bar{f}(L^*) \delta(r_k)) r_k \tag{41}$$

where $\delta(r_k) = \text{Prop}^{r_k} [(Ey_i^{r_k}, L_i^{r_k}) \in C(\gamma)]$.

The feasibility condition

$$\sum_{i=1}^{r_k} f^i (L_i^{r_k}, s) \geq \sum_{i=1}^{r_k} y_i^{r_k} (s) \quad \text{for all} \quad s \in S ,$$

implies that

$$\sum_{i=1}^{r_k} f(L_i^{r_k}) \geq \sum_{i=1}^{r_k} Ey_i^{r_k} \tag{42}$$

But since $\bar{f}(L) \geq f(L)$, we see from (41) that feasibility will be violated if

$$\theta\,\bar{\beta} + \eta(\gamma) - \bar{f}(L^*)\,\delta(r_k) > o. \tag{43}$$

We can always choose γ sufficiently small that $\theta\,\bar{\beta} + \eta(\gamma) > o$. So if the allocation is feasible, it must be true that, for some $\bar{\gamma} > o$,

$$\lim_{k\to\infty} \inf \delta(r_k) = \bar{\delta} > o \tag{44}$$

So if (35) holds, (38) must always hold, since (37) obviously implies (38).

Now we need to consider the feasibility of the allocation $(\bar{f}(\bar{L}),\bar{L})$. It will always be possible to write

$$\bar{f}(\bar{L}) = \alpha\, f(L_1) + (1-\alpha)\, f(L_2) \qquad \text{for some} \quad \alpha,\ o \leq \alpha \leq 1, \tag{45}$$

$$\bar{L} = \alpha\, L_1 + (1-\alpha) L_2.$$

Then if we denote by $[a]$ the largest integer less than a

$$\lim_{n\to\infty} \frac{1}{n} \{ [\alpha n]\, f(L_1) + (n - [\alpha n])\, f(L_2) \} = \bar{f}(\bar{L}) \tag{46}$$

What (46) tells us is that if each member of an n-member coalition supplies labour $\bar{L}$, then an equal division of output will produce individual shares which approach progressively closer to $\bar{f}(\bar{L})$, in the mean-value economy.

But our assumption on the random variables f^i enable us to use the strong law of large numbers to obtain

$$\lim_{r\to\infty} \quad Eu\left(\frac{1}{r}\sum_{i=1}^{r} f^i(\bar{L},s),\ \bar{L}\right) = u\left(\bar{f}(\bar{L}),\bar{L}\right) \tag{47}$$

Now consider the coalition, call it V^{r_k}, whose members have expected utility no greater than $u(\bar{f}(\bar{L}),\bar{L}) - \bar{\gamma}$. We have established from (44) that their number, n_k, is at least $\bar{\delta} r_k$. From (46) and (47) there must exist some K such that for $k \geq K$

$$Eu\left(\frac{1}{n_k}\sum_{i\in V^{r_k}} f^i\left(\frac{1}{n_k}\left([\alpha n_k]\ f(L_1) + (n - [\alpha n_k])\ f(L_2)\right)\right),\ \bar{L}\right) >$$

$$u(\bar{f}(\bar{L}),\bar{L}) - \bar{\gamma} \tag{48}$$

which means that V^{r_k} will be a blocking coalition. So (34) is true and the theorem holds.

Before we proceed to establish the analogue to Theorem 2 in our economy with production, we observe first that the following general result is true

Theorem 4. Let $[y_1^r(\cdot),\ldots y_r^r(\cdot)]$ be a core allocation in E^r.

Then if

i) $\lim_{r\to\infty} \frac{1}{r} \sum_i Ey_i^r = \overline{y}$

ii) for all $\varepsilon > 0$, and α, $0 < \alpha < 1$ there exists R such that for $r \geq R$

$$\overset{r}{\text{Prop}} \{ |Eu(y_i^r) - u(\overline{y})| \leq \varepsilon \} \geq \alpha \tag{49}$$

then for all $\varepsilon > 0$ and α, β, $0 < \alpha < 1$, $0 < \beta < 1$ there exists R such that for $r \geq R$

$$\overset{r}{\text{Prop}} \{ \text{Prob} (||y_i^r - \overline{y}|| \leq \varepsilon) \geq \beta \} \geq \alpha \tag{50}$$

Proof: As for Theorem 2.

We now state our final result.

Theorem 5. If $\overline{L}$ maximises $u(\overline{f}(L),L)$ and $\{(y_1^r(\cdot),L_1^r),\dots(y_r^r(\cdot),L_r^r)\}$ is a core allocation in E^r, then for all $\varepsilon > 0$ and α, β, $0 < \alpha < \beta 1$ $0 < \beta < 1$ there exists R such that for $r \geq R$

$$\overset{r}{\text{Prop}} \{ \text{Prob} (|| (y_i^r, L_i^r) - (\overline{f}(\overline{L}), \overline{L}) || \leq \varepsilon) \geq \beta\} \geq \alpha \tag{51}$$

Proof: In view of Theorems 3 and 4, we need only show that

$$\lim_{r\to\infty} \left(\frac{1}{r} \sum_i Ey_i^r , \frac{1}{r} \sum_i L_i^r\right) = (\overline{f}(\overline{L}), \overline{L}). \tag{52}$$

From Lemma 2 and Theorem 3 we see that for every $\varepsilon > 0$ and α, $0 < \alpha < 1$, there exists R such that for $r \geq R$

$$\overset{r}{\text{Prop}} \{ u(Ey_i^r , L_i^r) \geq u(\overline{f}(\overline{L}), \overline{L}) - \varepsilon \} \geq \alpha \tag{53}$$

If p is the vector of prices which produces equilibrium in the mean-value economy at $(\overline{f}(\overline{L}) , \overline{L})$, then we must show that

$$\liminf_{r\to\infty} \; p.(\frac{1}{r_i} \sum_i Ey_i^r , \frac{1}{r} \sum_i L_i^r) \geq p.(\overline{f}(\overline{L}), \overline{L}) \tag{54}$$

If (54) is not true

$$\liminf_{r\to\infty} \; p.(\frac{1}{r} \sum_i Ey_i^r , \frac{1}{r} \sum_i L_i^r) < p.(\overline{f}(\overline{L}), \overline{L}) \tag{55}$$

Then

$$\liminf_{r\to\infty} \; u(\frac{1}{r} \sum_i Ey_i^r , \frac{1}{r} \sum_i L_i^r) < u(\overline{f}(\overline{L}) , \overline{L}) \tag{56}$$

But strict concavity of u guarantees that

$$\frac{1}{r} \sum_i u(Ey_i^r , L_i^r) < u(\frac{1}{r} \sum_i Ey_i^r , \frac{1}{r} \sum_i L_i^r) \tag{57}$$

and so

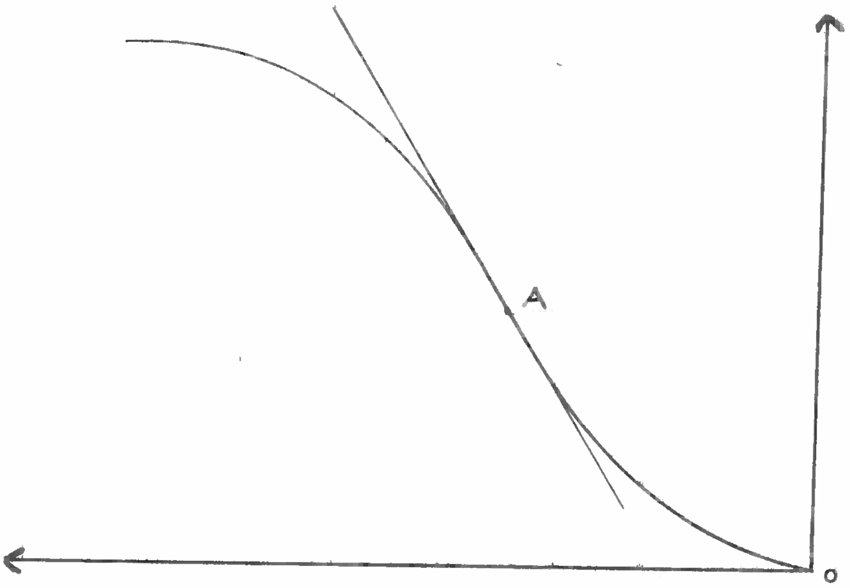
$$\liminf_{r\to\infty} \frac{1}{r} \sum_i u(Ey_i^r , L_i^r) < u(\overline{f}(\overline{L}) , \overline{L}) \tag{58}$$

which is clearly inconsistent with (53).

If (54) holds, then so must (52), in order to preserve feasibility.

It is worth remarking upon the absence of any concavity assumption for the function f. It is well-known that the presence of non-convexities in production may mean that a competitive equilibrium does not exist. The reason for this is that there may only exist market-clearing prices at which some firms are required to make losses. This is illustrated in figure 4, where A is the market-clearing production plan for the firm. But the firm

Figure 4



would prefer to shut down at these prices. So, if there exists no competitive equilibrium in the mean-value economy, to what is the core converging ?

It is known from the work of Farrell [1959], that the conditions under which core convergence is examined are precisely those which tell us that we need not worry about non-convexities. When an economy is sufficiently large, we can average out non-convexities and consider equilibrium in a convexified economy. This is what we do here. We allow individuals to provide labour as an input to somebody else's production function. If we suppose, for example, that we are dealing with an economy of peasant farmers, whose production functions relate to levels of output on their own plots of land, then we are assuming that individuals can offer to work on somebody else's plot in return for a share of the output.

Conclusion

We have shown that in a many-good exchange economy in which traders face individual risks which affect their endowments, as the economy becomes large, so the core allocations of an arbitrarily large proportion of traders are likely to be close to the common expected endowment vector. Another way of characterising this result is to say that individual risks are spread in the core. We examined also the nature of core allocations in a simple production economy. Without non-convexities in production, core allocations in a large economy are now likely to be close to the competitive equilibrium in the mean-value economy. In the presence of non-convexity, not only is risk spread but the economy is convexified, and core allocations are likely to be close to the competitive equilibrium in the convexified mean-value economy.

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