Duality, Quantity Constraints
and Consumer Behaviour

by

Richard Cornes[*]
Australian National University
Canberra

No.149

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

UNIVERSITY OF WARWICK
COVENTRY

Duality, Quantity Constraints
and Consumer Behaviour

by

Richard Cornes[*]
Australian National University
Canberra

No.149

June 1979

Until the recent revival of interest among macroeconomists whose work is surveyed by Malinvaud (1977), the systematic analysis of quantity-constrained behaviour in a multimarket setting has attracted surprisingly little attention since the classic survey by Tobin (1952). The few exceptions, in addition to the works cited by Malinvaud, are the discussion by Gould and Henry (1967) of price control, the systematic analysis by Pollak (1969) of conditional demand functions, and a recent attempt by Howard (1977), not entirely successful, to extend the scope of the earlier treatment of consumer choice under quantity rationing by Tobin and Houthakker (1951).[1/]

This paper exploits the minimum expenditure function approach to simplify the analysis of quantity-constrained consumer choice.[2/] Section I introduces the "restricted" minimum expenditure function and "restricted" compensated demand functions, which provide the basis of our approach. While our treatment of the formal rationing problem is similar to that of Neary and Roberts, we are also concerned to stress alternative applications of these functions, particularly to situations in which the quantity constraints are interpreted as externalities or public goods. Section II discusses applications of the analysis to a generalisation of the Tobin-Howthakker analysis and to the price control problem raised by Gould and Henry. Section III comments on the "virtual price system" used by Neary and Roberts, and draws attention to the formal similarity between the rationing and externality problems. Finally, section IV takes up the problem of price control in a general equilibrium context.

I.

We denote by $\underline{x}$ the consumption vector of a utility-maximising individual. $\underline{x}$ is partitioned into $\underline{x}_F$ and $\underline{x}_R$, reflecting the classification of commodities into two mutually exclusive and exhaustive categories. $\underline{x}_F$ is a vector of commodities whose quantities may be freely chosen by the individual subject only to an overall budget constraint, while $\underline{x}_R$ is the vector of the remaining preassigned, or rationed, commodities. These the individual consumes in quantities which are not wholly under his control.

The notion of preassigned commodities has a variety of specific interpretations. Such commodities may be marketed goods or services, whose prices are sticky and fail to clear the market, so that the individual faces nonprice rationing. Alternatively, they may be collectively provided commodities, such as "free" school milk or health services, distribution of which is deliberately effected through channels other than the price mechanism. In particular, governments may compel the individual to consume either more or less of a good than he would prefer, or may induce such behaviour, for example by offering a subsidy which is conditional upon the consumption of a certain minimum quantity, or quality, which itself may exceed that which the individual would freely choose at the subsidized price. Finally, they may be interpreted either as public goods in Samuelson's sense, or as externalities. If a public good (or bad) is non-optional, in the sense that each individual is compelled to consume the whole of the quantity produced, or if the total supply of a public good falls short of the quantity most preferred at zero price, then from the individual's point of view it has the attributes of a preassigned commodity. Externalities may be modelled in the same way, as commodities made available to, or forced

upon, the consumer at zero price but in quantities which are beyond his control.

A general treatment of many of these problems would state the quantity constraints as inequalities. In order to keep the treatment simple, the present analysis formulates them as equality constraints.

Whatever the precise interpretation of $\underline{x}$ the consumer's problem allows one to define a "restricted" expenditure function,

$$e^*(\underline{p}_F \ \underline{q}_R \ u) \equiv \text{Min} \ (\underline{p}_F \ \underline{x}_F \ | \ u(\underline{x}) \overset{\geq}{=} u, \ \underline{x}_R = \underline{q}_R) \tag{1}$$

where $\underline{q}_R$ is the vector of exogenous quantity constraints, and $\underline{p}_F$ the price vector of freely chosen commodities. The complete price vector is written as $\underline{p} \equiv (\underline{p}_F \ \underline{p}_R)$. Typically the utility level is itself endogenously determined once the level of money income, m, is known, since

$$e^*(\underline{p}_F \ \underline{q}_R \ u) = m - \underline{p}_R \ \underline{q}_R. \tag{2}$$

(a)   *Properties of the Restricted Minimum Expenditure Function*

The function $e^*(\underline{p}_F \ \underline{q}_R \ u)$ has properties similar to those of the standard minimum expenditure function as summarised by Diamond and McFadden (1974). In the absence of quantity constraints the individual is optimising in n-dimensional space, where n is the total number of commodities. The properties of the resulting functional relationships follow from the standard properties of the utility function - for example, those of continuity, strict quasi-concavity and local non-satiation. The imposition of quantity constraints in the form of equalities reduces the dimension of the space in which the individual is optimising. But if the form of the utility function in this subspace retains the same qualitative properties, so too will the expenditure function.

Partial differentiation of $e^*(\ )$ with respect to $p_i (i \epsilon F)$ yields a restricted compensated demand function for the ith commodity. To illustrate this, and the discussion in the previous paragraph, we consider the case of three commodities. In the absence of quantity constraints one can imagine a set of indifference surfaces and a budget plane in three dimensions. If the further constraint, $x_3 = q_3$, is imposed, the individual is thereby restricted to a 2-dimensional cross-section, as shown in Figs. 1(a) and 1(b).

Suppose that at the price vector $\underline{p}_F = (p_1 \ p_2)$ the budget constraint is BB and the optimal consumption point is $\underline{x}^*_F$. Then

$$e^*(\underline{p}_F \ q_3 \ u) \ = \ p_1 \ x_1 \ + \ p_2 \ x_2$$

Now consider any positive price vector $\underline{p}'_F$ with associated optimal consumption

$\underline{x}'_F$. Clearly, from figure 1(b),

$$e^*(\underline{p}'_F \; q_3 \; \overline{u}) \;=\; \underline{p}'_F \; \underline{x}'_F \;\leq\; \underline{p}'_F \; \underline{x}^*_F$$

with the inequality becoming an equality where $\underline{p}'_F = \underline{p}_F$. That is, at the point $\underline{p}_F$ the expression $\phi(\underline{p}'_F) \;=\; \underline{p}'_F \; x^*_F \;-\; e^*(\underline{p}'_F \; q_3 \; \overline{u})$ attains a minimum. The first order conditions yield the result that, evaluated at $\underline{p}_F$,

$$\partial\phi/\partial p'_i \;=\; \partial e^*(\underline{p}_F \; q_3 \; \overline{u})/\partial p'_i \;-\; x^*_i \;=\; 0.$$

We can therefore generate the restricted compensated demand functions, which we denote by $x^*_i \;=\; c^*_i \; (\underline{p}_F \; q_{\underline{R}} \; u)$ from now on.[3]

The partial derivatives $\partial e^*(\;)/\partial q_i$ have a simple interpretation. $\partial e^*(\;)/\partial q_K$ is a measure of the individual's valuation of a marginal unit of the Kth preassigned, commodity. It is the amount by which expenditure on freely chosen commodities just change in order to compensate the individual for a change in q_K.

Suppose that x_K is a marketed commodity with a price p_K. Then $(-\partial e^*/\partial q_K)$ will exceed, equal, or be less than p_K according to whether q_K is less than, equal to or greater than the quantity which the individual would freely choose. The term $(p_K + \partial e^*/\partial q_K)$ is important in generating the real income consequences of a change in q_K. If all other quantity constraints, together with all prices and money income, remain constant, then differentiation of (2) yields

$$du/dq_K = -(p_K - e_K^*)/e_u^*$$ (3)

where $e_K^* \equiv \partial e^*(\ )/\partial q_K$ and $e_u^* \equiv \partial e^*(\ )/\partial u$. If commodity K is interpreted as an externality, then $p_K = 0$. Then the sign of du/dq_K depends on the sign of e_K^*, which is an indicator of whether an increase in the externality is beneficial or detrimental to the recipient.

(b)   *Properties of Restricted Compensated Demand Functions*

Consider now the function $\underline{c}_F^*(\underline{p}_F \; \underline{q}_R \; u)$. The partial derivatives $\partial c_i^*/\partial p_j$ $(i,j \in F)$ are "restricted" Slutsky terms. Their precise interpretation differs from the usual one, since the individual is explicitly constrained to consume $\underline{q}_R$. For a given $\underline{q}_R$, however, they obey the usual sign patterns and symmetry conditions. The square matrix of such terms will be denoted by $\underline{C}_{FF}^*$.

The partial derivatives $\partial c_i/\partial q_K$ $(i \in F, \; k \in R)$ have a straightforward interpretation, being the change in optimal consumption of x_i in response to a change in q_K while all other preassigned quantities, all prices and utility are held constant. By definition

$$e^*(\underline{p}_F \; \underline{q}_R \; u) = \underline{p}_F \cdot \underline{c}_F^*(\underline{p}_F \; \underline{q}_R \; u)$$

$$\therefore \quad \partial e^*/\partial q_K = \sum_{i \in f} p_i \cdot \partial c_i^*/\partial q_K$$

Whence the price-weighted sum of the responses $\partial c_i^*/\partial q_K$, where the summation is over all freely chosen commodities, will be negative, zero or positive

according to whether an increase in q_K benefits the individual, leaves the welfare unchanged, or harms him. Section III takes up the relationship between the individual $\partial c_i^* / \partial q_K$ terms and the conventional parameters that reflect the substitute/complement relationships between commodities.

II

The results of Section I permit a brief and simple analysis of consumer behaviour under parametric quantity rationing. In this section we look at two specific problems. First, we discuss a generalisation of the Tobin-Houthakker analysis of rationing. Also, we consider the price control problem discussed by Gould and Henry (1967).

(a)   *Straight Rationing: the General Case*

We are interested in the response of demand for a freely chosen commodity, x_i, to changes in prices, rations and income in a quantity-rationed regime. First consider a change in the ration q_K when all prices, all other quantity constraints, and income are held constant. The 3-commodity example provides an illustration. Suppose the consumer is at Q in Fig. 2. To facilitate comparison with other treatments, consider a reduction in q_3. This induces a shift in the indifference map which has two components. First, the curve ii will move outwards away from the origin, reflecting the need for greater expenditure on unrationed goods in order to compensate for the tightening of the ration. Conceivably, the new compensated equilibrium at unchanged prices may be at S, where x_1 and x_2 have changed by the same proportion. Generally, however, there will be some degree of bias in the shift, so that at unchanged prices the compensated equilibrium is a point such as V, which is off the ray OQ. It seems reasonable to call goods close substitutes for K for which the elasticity $(\partial c_i^*/\partial q_K) \cdot (q_K/c_i^*)$ has a relatively high negative value. The results of Section III enable us to investigate this matter more rigorously.

Tobin and Houthakker assume that the initial value of q_K is precisely the quantity which would be freely chosen. In other words, the ration is only just binding before the change. As argued above, this implies that $(p_K + e_K^*)$ is zero. The real income effect of dq_K is then zero, since the amount of expenditure 'released' by the tightening of the ration is precisely equal to the change in expenditure on unrationed goods required to effect exact compensation. Consequently, the new compensated equilibrium is the actual equilibrium. If, however, q_K is initially below the level which the individual would have chosen, the outward shift in the budget line in Figure 2 brought about by the tightening of the ration is insufficient to maintain an unchanged utility level. This has already been shown in (3). The consequence for x_i is easily shown algebraically:

$$x_i = c_i^*(\underline{p}_F \ \underline{q}_R \ u)$$

$$dx_i/dq_K = c_{iK}^* + c_{iu}^*(du/dq_K)$$

$$= c_{iK}^* - c_{iu}^*(p_K + e_K^*)/e_u^*$$

or

$$dx_i/dq_K = c_{iK}^* - \mu_i^*(p_K + e_K^*) \tag{4}$$

where μ_i^* is simply an income response, dx_i/dm.[4] As mentioned above, in the Tobin-Houthakker analysis the term $(p_K + e_K^*)$ becomes zero.

Further results which may be obtained easily by using the restricted compensated demand function together with equation (2) are, first, the response

of x_i to a change in the price of an unrationed commodity, p_j :

$$dx_i/dp_j \;=\; c_{ij}^{*} + c_{iu}^{*}(du/dp_j)$$

$$=\; c_{ij}^{*} - \mu_i^{*} c_j^{*}, \qquad\qquad i, j \; \epsilon \; F$$

and finally, the response of x_i to a change in the price of a rationed commodity, p_K:

$$dx_i/dp_K \;=\; c_{iu}^{*} \; du/dp_K$$

$$=\; \mu_i^{*} q_K \qquad\qquad (i \; \epsilon \; F, \;\; k \; \epsilon \; R)$$

Note that a change in p_K works in the same way as a change in money income. The response is weighted by the importance of the Kth commodity in the consumer's bundle. Note, too, the straightforward decomposition of dx_i/dp_j into a pure substitution effect and an income effect.

In Section III, we discuss the relationships between the restricted parameters which appear in these results and the parameters of unrestricted choice.

(b)   *Price Control*

Quantity rationing is often accompanied by, and indeed is a natural consequence of, discretionary price control. Gould and Henry consider a situation in which the price of, say, commodity K is reduced – by decree, perhaps, of a prices justification tribunal or rent control body – while the resulting excess demand for that commodity is handled by the allocation of a fixed ration to each purchaser such that the total supply of the controlled commodity is just exhausted.

We now consider the response of demand for an unrationed commodity by a representative consumer to such a scheme. For simplicity, we assume 3 commodities, of which x_3 is the controlled commodity whose price is depressed while other prices and money income are held constant. Consider the demand for commodity 1. The equations of the system are

$$x_1 = c_1^*(p_1\ p_2\ q_3\ u)$$

$$e^*(p_1\ p_2\ q_3\ u) = m - p_3\ q_3$$

$$q_3 = q_3(p_3)$$

Of these, only the third requires explanation. It is the exogeneous rationing rule imposed on the individual. We assume it to be differentiable, and also that dq_3/dp_3 is positive. With a conventional upward-sloping supply curve, the typical consumer is **constrained** to consume less at the lower controlled price.

Differentiating, we obtain

$$dx_1/dp_3 = c_{13}^*(dq_3/dp_3) - \mu_1^*(p_3 + e_3^*)(dq_3/dp_3) - \mu_1^* q_3$$

which we may write as

$$dx_1/dp_3 = A - \mu_1^* B - \mu_1^* C \qquad (5)$$

Figure 3 illustrates the decomposition provided by (5). The initial equilibrium is I. The term A in (5) is reflected in the shift of the indifference curve ii brought about by the reduction in the ration, dq_3. It indicates by how much x_1 would change if dq_3 were accompanied by a compensating income change. The point J represents such a compensated equilibrium, at which the individual is no better or worse off than at I. As drawn, the compensated change has increased x_1 proportionally more than x_2. Both are Hicksian substitutes for x_3, but x_1 may be considered the closer substitute.

The term $\mu_1^* B$ captures the real income effect of the reduction in the ration. As already seen, the reduction in q_3 releases expenditure available for the unrationed commodities. If initially the ration is just binding, such additional expenditure exactly compensates for the loss of units of x_3. However, if the ration is effectively binding at all times, the individual will become worse off. In short, the term $\mu_1^* B$ will in general tend to push the budget constraint below and to the left of J — say, to DD, with consumption at K. Finally, the term $\mu_1^* C$ captures the beneficial real income effect of the cheapening of x_3. This produces an outward shift of the budget line, which may be sufficient to make the individual better off than at I.

This decomposition does, I believe, clarify the discussion by Gould and Henry (1967, p.45) of the response of demand for close substitutes to a change in the price of a controlled commodity. For an individual, the term A certainly works to make dx_1/dp_3 negative if c_{13}^* is negative. Also, μ_1^*C will strengthen this effect if $\mu_1^* > 0$. However, the middle term, μ_1^*B, will tend to pull in the other direction. Figure 4 shows some possibilities. In each case, I is the initial equilibrium, allowing for the real income effects μ_1^*B and μ_1^*C. Also, in each case, commodity 2 is a relatively close substitute for commodity 3 in the sense that the absolute value of the elasticity $(q_3c_{13}^*/x_1)$ exceeds that of $(q_3c_{23}^*/x_2)$. Geometrically, J lies below the ray OI. Figure 4(b) demonstrates the influence of an adverse real income effect in reducing the absolute value of dx_1/dp_3, while in 4(c) the effect is so strong as to make dx_1/dp_3 positive, even though both goods have a normal income response, as emphasised by the slope of the Engel curve, EE.

This last example appears to conflict with the Gould-Henry analysis. From their analysis they conclude that there cannot be a fall in demand for a substitute in response to the reduction of p_3 when both unrationed goods are normal. Figure 4 (c) is quite consistent with commodities 1 and 2 being normal, and yet the equilibrium moves from I to F, resulting in a reduction in x_1.

While the assumptions of the Gould-Henry analysis are not entirely clear, the source of the conflict appears to be the following. Their appeal to the Tobin-Houthakker results means that in their paper the initial situation is an unrationed equilibrium. Hence, for differential changes, $(p_3 + e_3^*) = 0$, so that the favourable real income effect of the price

change is dominant. The perverse result is then associated with a negative value for μ_1^*, the "restricted" income response. In the more general case, however, $(p_3 + e_3^*)$ may well be dominant. More important, in view of Gould and Henry's emphasis on interdependence and concern with the general equilibrium implications, is the fact that in a general equilibrium context the net real income effect associated with a move away from an unrationed equilibrium must be zero or negative. If we interpret the model as a closed general equilibrium system, the equilibrium F cannot be on a higher indifference curve than J, and will generally lie on a lower indifference curve.

III.

In their analysis of rationing, Neary and Roberts follow Rothbarth (1941) in defining a virtual price system. The idea is very simple. Suppose a consumer faces a set of prices $\underline{p}_F$ and of quantity constraints q_R, and chooses $\underline{c}_F^*$. Under fairly general assumptions set out by Neary and Roberts one may define a hypothetical, or virtual, price vector $\underline{\tilde{p}}_R$ such that when faced with $(\underline{p}_F \ \underline{\tilde{p}}_R \ m)$ the consumer chooses $(\underline{c}_F^* \ \underline{q}_R)$. The procedure simply involves being able to define a vector of inverse demand functions, $\underline{\tilde{p}}_R = \underline{\tilde{p}}_R(\underline{c}_F^* \ \underline{q}_R)$.

The notion of virtual prices is interesting for two reasons. First, they enable us to express the parameters of choice under quantity consttaints in terms of the more conventional parameters associated with choice subject to the budget constraint alone. By virtue of the definition of $\underline{\tilde{p}}_F$, it must be true that

$$\underline{c}_F^*(\underline{p}_F \ \underline{q}_R \ u) \equiv \underline{c}_F(\underline{p}_F \ \underline{\tilde{p}}_R \ u)$$

and
$$\underline{q}_R \equiv \underline{c}_R(\underline{p}_F \ \underline{\tilde{p}}_R \ u)$$

Hence, $\underline{C}_{FF}^* \ d\underline{p}_F + \underline{C}_{FR}^* \ dq_R + C_{Fu}^* \ du = \underline{C}_{FF} \ d\underline{p}_F + \underline{C}_{FR} d\underline{\tilde{p}}_R + \underline{c}_{Fu} \ du$ \hfill (6a)

$$d\underline{q}_R = \underline{C}_{RF} d\underline{p}_F + \underline{C}_{RR} d\underline{\tilde{p}}_R + \underline{c}_{Ru} \ du \hfill (6b)$$

where $\underline{C}_{FF}^* \equiv \{\partial c_i^*/\partial_{pj}\}$, $\underline{C}_{FR}^* \equiv \{\partial c_i^*/\partial q_K\}$, $c_{FU} \equiv \{\partial c_i^*/\partial u\}$,

for $i, j \in F$ and $k \in R$. The definitions of the unasterisked matrices

and vectors on the right-hand side are self-evident.

The vector $d\underset{\sim}{\underline{p}}_R$, having served its purpose, may now be eliminated, yielding

$$\underline{c}^*_{FF} \, d\underline{p}_F + \underline{c}^*_{FR} \, d\underline{q}_R + \underline{c}^*_{Fu} du = \underline{C}_{FF} \, d\underline{p}_F + \underline{C}_{FR}(\underline{C}_{RR})^{-1}) d\underline{q}_R - \underline{C}_{RF} - c_{Ru} du).$$

Putting appropriate differentials equal to zero yields the results that

$$\underline{C}^*_{FF} = \underline{C}_{FF} - \underline{C}_{FR}(\underline{C}_{RR})^{-1} \underline{C}_{RF}$$

$$\underline{C}^*_{FR} = \underline{C}_{FR}(\underline{C}_{RR})^{-1}$$

and

$$\underline{c}^*_{Fu} = \underline{c}_{Fu} - \underline{C}_{FR}(\underline{C}_{RR})^{-1} \underline{c}_{Ru}$$

If there is a single quantity constraint of the form $x_k = q_k$, then we have, for i and $j \in F$,

$$c^*_{ij} = c_{ij} = (c_{iK} \, c_{Kj}/c_{KK})$$

$$c^*_{iK} = c_{iK}/c_{KK}$$

$$c^*_{iu} = c_{iu} - (c_{iK} \, c_{Ku}/c_{KK}).$$

These results are similar to those found in Tobin and Houthakker (1951), but for reasons already given they represent generalisations of that analysis.

The second use of the virtual price system is in the evaluation
of welfare changes in a quantity-constrained regime. If virtual prices
can be estimated, quantity-constrained choice may be modelled using
conventional demand functionsin which the quantity constraints are replaced
by prices. Evaluation of the move from $(\underline{p}_F{}^a \ \underline{q}_R{}^a \ m^a)$ to $(\underline{p}_F{}^b \ \underline{q}_R{}^b \ m^b)$,
where m denotes lump sum income, then becomes a comparison between
$(\underline{p}_F{}^a \ \underline{\tilde{p}}_R{}^a \ m^a)$ and $(\underline{p}_F{}^b \ \underline{\tilde{p}}_R{}^b \ m^b)$. This is a standard problem which has
received a clear treatment from Dixit and Weller (1979).

It should be stressed that the use of virtual prices is essential
if the welfare evaluation of quantity-constrained choice is to have as firm
a logical foundation as the conventional analysis of unconstrained choice.
At the same time, their estimation may pose serious problems. Both $\underline{\tilde{p}}_R$
and the unconstrained parameters appearing in (6) relate to the point
$(\underline{c}_F{}^* \ \underline{q}_R)$. Suppose that $\underline{q}_R$ represents a vector of rations which
significantly affect the individual's choice. Then it is quite likely
that observations used to provide our estimates of unrestrained parameters
do not relate to the neighbourhood of $(\underline{c}_F{}^* \ \underline{q}_R)$.

Since the functional forms chosen by econometricians, and the
resulting parameter estimates are, strictly speaking, only local approx-
imations, care must be exercised in attempting to extrapolate their values
elsewhere in the consumption space.

If reliable direct estimates of virtual prices are unavailable,
alternative less direct methods may be explored. The various problems
which face attempts to estimate virtual prices indirectly have been
extensively discussed in the literature on externalities, and a broad
survey may be found in Pearce (1978). In situations where $\underline{q}_R$ is a

vector of formal rations, accurate preference revelation may be thwarted
by the free rider problem. If a respondent to a questionnaire believes
that his stated valuation of marginal units may influence the size of his
future ration, he may have a strong incentive to misstate his true preferences.
Hence the welfare evaluation of regimes involving formal or informal quantity
rationing of marketed commodities faces the same problems as the public
goods literature. The formal similarity between quantity rations and
externalities is most clearly seen by noting that what Neary and Roberts
call virtual prices are simply the marginal valuations of externality
recipients which welfare theorists attempt to measure in their analyses of
problems such as environmental damage.

IV

This section considers the response of prices of "uncontrolled" commodities to price control in one market. This was the question posed by Gould and Henry, but not explicitly answered by them. The present analysis demonstrates clearly the relevance of the concept of effective demand to the problem, and provides a simple example of a destabilising spillover in the presence of a quantity constraint.

We consider a simple 3-commodity example. Commodity 0 is the numeraire, and price controllers fix p_2, rationing any resulting excess demand or supply for commodity 2. Given the rationing constraint, p_1 is determined by equating the effective demands and supplies in the two remaining markets.

A complete analysis must allow for the possibility that it is producers who experience quantity constraints. We assume that in the absence of constraints on the supply side, aggregate supply functions may be defined which are of the form $s(p_0\ p_1\ p_2)$ with the usual properties {Pearce (1970) ch.16}. If producers are constrained to produce q_2, then restricted aggregate supply functions are defined which take the form $s_i^*(p_0\ p_1\ q_2)$ for $i = 0, 1$. This is discussed in more detail in Cornes (1979). On the demand side, we assume that behaviour can be described as if there were a single individual.

Restricting attention to points in the neighbourhood of the Walrasian equilibrium, so that real income effects may be ignored, we wish

to plot values of p_1/p_0 and p_2/p_0 consistent with equality of effective demands and supplies in the uncontrolled markets. To do this, first consider the conditions which characterise such an equilibrium when consumers of commodity 2 are rationed. They are

$$c_1^* (p_1 \; q_2) = s_k (p_1 \; p_2)$$

and

$$q_2 = s_2 (p_1 \; p_2)$$

whence $dp_1/dp_2 = (s_{12} - c_{12}^* s_{22})/(c_{11}^* - s_{11} + c_{12}^* s_{21}).$

Using the results obtained in Section III, this expression for the slope of the constrained equilibrium price locus may be written using parameters of unrestricted choice

$$dp_1/dp_2 = (-\{c_{12} - s_{12}\} + \{c_{22} - s_{22}\}\{c_{12}/c_{22}\})/(\{c_{11} - s_{11}\} - \{c_{21} - s_{21}\}\{c_{12}/c_{22}\}). \quad (7)$$

In the alternative situation, in which producers are subject to quantity rationing, the equilibrium conditions are

$$c_1 (p_1 \; p_2) = s_1^* (p_1 \; q_2)$$

$$c_2 (p_1 \; p_2) = q_2$$

The slope of the resulting locus is the same as that given in (7) except that the term $\{c_{12}/c_{22}\}$ is replaced in both the numerator and the denominator by $\{s_{12}/s_{22}\}$. We assume that the quantity rationing is always

imposed on the long side of the market. This determines which portions
of the two loci are the relevant ones.

Before depicting possible outcomes graphically, we must explore
the stability issue. Even in neighbourhood of a Walrasian equilibrium
in single-household economy, the spillover effects associated with quantity
constraints can make for instability. For example, suppose the consumer
is constrained. Varying P_1/P_0 while holding P_2/P_0 constant, and
remembering that the level of q_2 is endogenous, we have

$$z_1^* \equiv c_1^*(P_1 \; q_2) \; - \; s_1(P_1 \; P_2)$$

$$dz_1^*/dp_1 \; \bullet \; c_{11}^* + c_{12}^* \; s_{21} - s_{11}$$

$$\bullet \; (c_{11} - s_{11}) - (c_{21} - s_{21})/(c_{12}/c_{22})$$

or $\qquad dz_1/dp_1 \; = \; (c_{11} c_{22} - c_{12} c_{21})/(c_{22} - s_{11} + s_{21} c_{12}/c_{22}.$

We assume that commodities 1 and 2 are substitutes on both the demand and
the supply sides. It is then the term $(s_{21} c_{12}/c_{22})$ which can make for
instability under an adjustment rule which makes dp_1/dt an increasing function
of the excess effective demand for commodity 1. Briefly, this mechanism
runs as follows. An increase in P_1 will, <u>inter alia</u>, reduce output of
commodity 2 if s_{21} is negative. <u>Ceteris paribus</u>, this diverts demand
away from 2 towards the uncontrolled commodities. To the extent to which
this "spillover" is diverted to x_1, the rise in P_1 will, through this
mechanism, stimulate demand for x_1. This informal argument suggests that
instability of quantity constrained equilibria is associated with high

numerical values for s_{21} and c_{12}, suggesting a high degree of substitutability between commodities 1 and 2 on both the demand and supply side. One can go further, and show that for instability to occur it is necessary that either s_{10} be positive or c_{20} be negative, or both.

Limitations of space preclude a systematic analysis of the various possibilities, but a single numerical example serves to illustrate one such possible outcome. Let all prices be normalised at unity at the Walrasian equilibrium of the 3-commodity economy. Suppose that the matrices of substitution terms are as follows:

$$\{c_{ij}\} = \begin{pmatrix} -1 & 3 & -2 \\ 3 & -10 & 7 \\ -2 & 7 & -5 \end{pmatrix} \qquad \{s_{ij}\} = \begin{pmatrix} 30 & 15 & -45 \\ 15 & 10 & -25 \\ -45 & -25 & 70 \end{pmatrix}$$

Then it may be confirmed that the locus of values of P_1/P_0 and P_2/P_0 which equate effective demands and supplies given that consumers are rationed in the market for commodity 2 in the line K2 in Figure 5. The line P2 is the resulting locus given rationing of producers. The continuous portions of the two loci are the relevant portions when rationing is imposed on the long side of the market. This is most clearly seen by drawing the locus 22 in the neighbourhood of the Walrasian equilibrium along which the notional excess demand for commodity 2 is zero. At points below and to the right of 22, there is excess notional supply of commodity 2, so that producers are rationed, while above and to the left the converse is true.

Also shown in Figure 5 are arrows indicating the direction of change of p_1 implied by the adjustment mechanism already mentioned. Starting from the Walrasian equilibrium E, a small reduction in p_2/p_0 coupled with quantity rationing of commodity 2 is inconsistent with a quantity constrained equilibrium in the neighbourhood of E. A global analysis may reveal a "bending back" of K2 and an associated stable equilibrium in which consumers are constrained. The behaviour of p_1/p_0 exhibits a discontinuity, and the response is not amenable to the differential calculus.

Of equal interest, perhaps, is the possibility of instability when p_2/p_0 is frozen at its Walrasian level. Here, a small upward movement of p_1/p_0 will generate further increase, precisely for the reasons given in our discussion of stability. The presence of a stable equilibrium at a point such as S in Figure 5 depends upon the growing importance of real income effects and/or changes in the values of the c_{ij} parameters as p_1/p_0 diverges further from its Walrasian level.

V

The use of dual formulations of both producer and consumer behaviour has already made substantial contributions to our understanding of these problems. This paper has tried to demonstrate the usefulness of the expenditures function in approaching demand problems with quantity constraints. Rather than summarise results we indicate some of the problems not dealt with by the present analysis.

In the first place, many quantity constraints are not completely fixed, but are to some extent affected by the individual's action, without being completely controllable by him. War time petrol rations have usually taken account of the recipient's occupation, and have been known to be allocated to vehicles rather than to owners. Choice of residential location, too, many affect ones consumption of smog or noise. Second, the distributional aspects both of rationing schemes and of externalities problems are often of prime importance for both positive and normative analysis. Third, the assumption of certainty should be relaxed for some applications, particularly those in which intertemporal aspects are important. Finally, where quantity constraints are determined partly by others, and partly by the individual himself, strategic considerations may complicate the analysis of general equilibrium systems with quantity constraints.
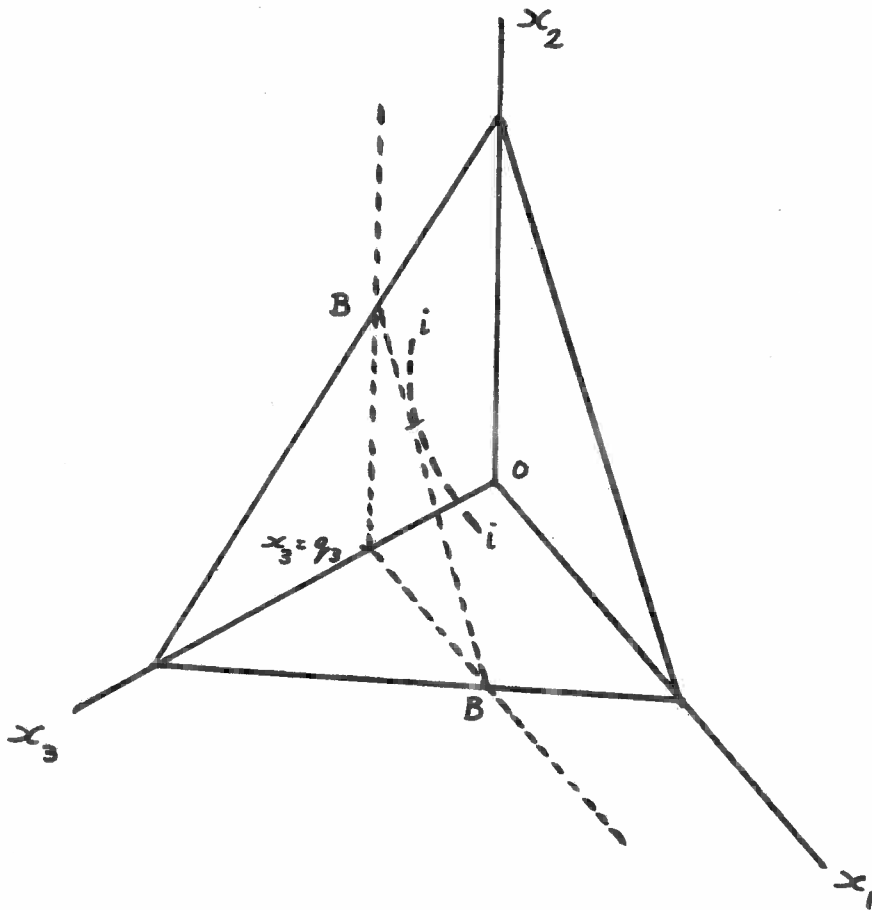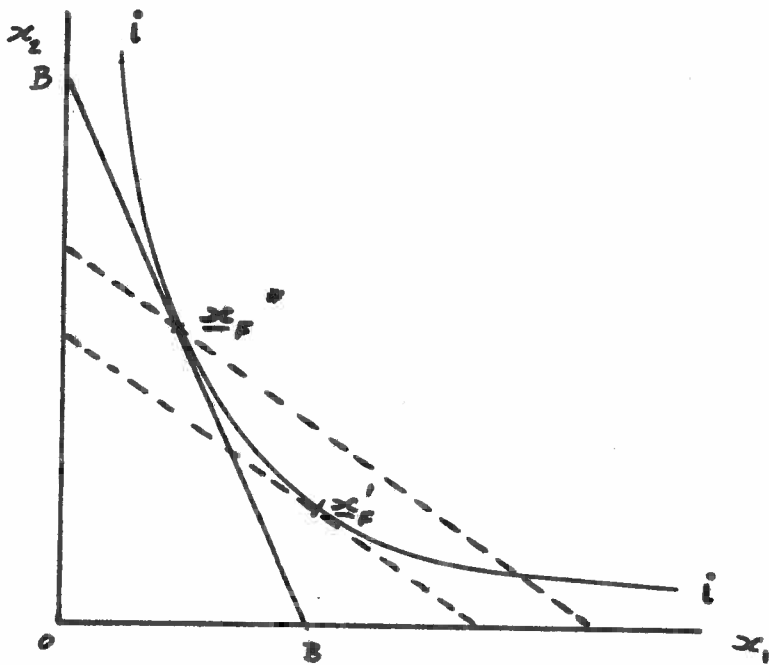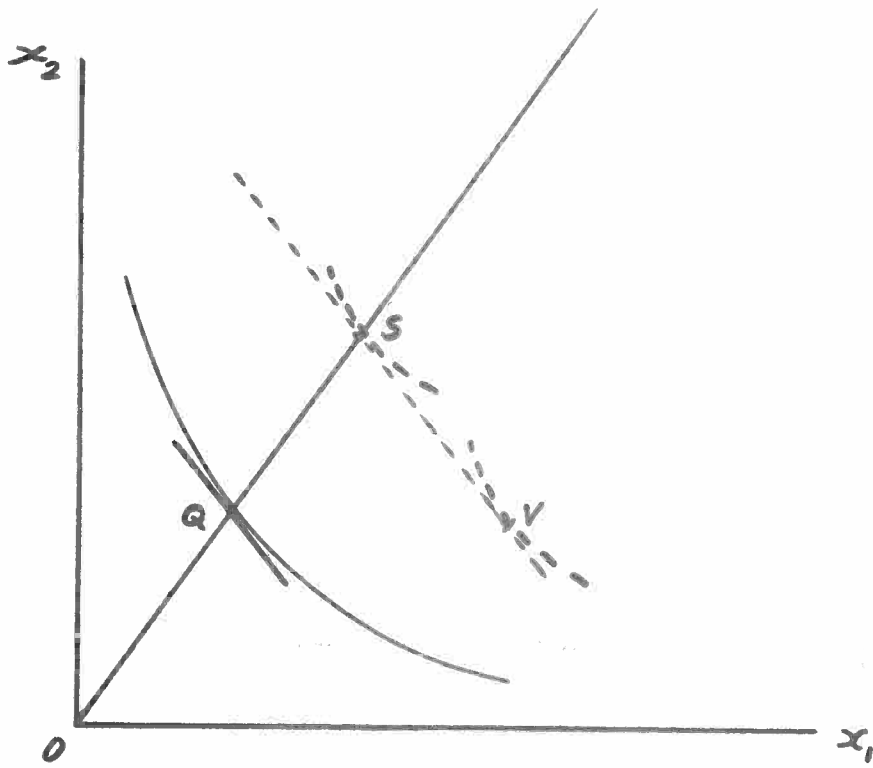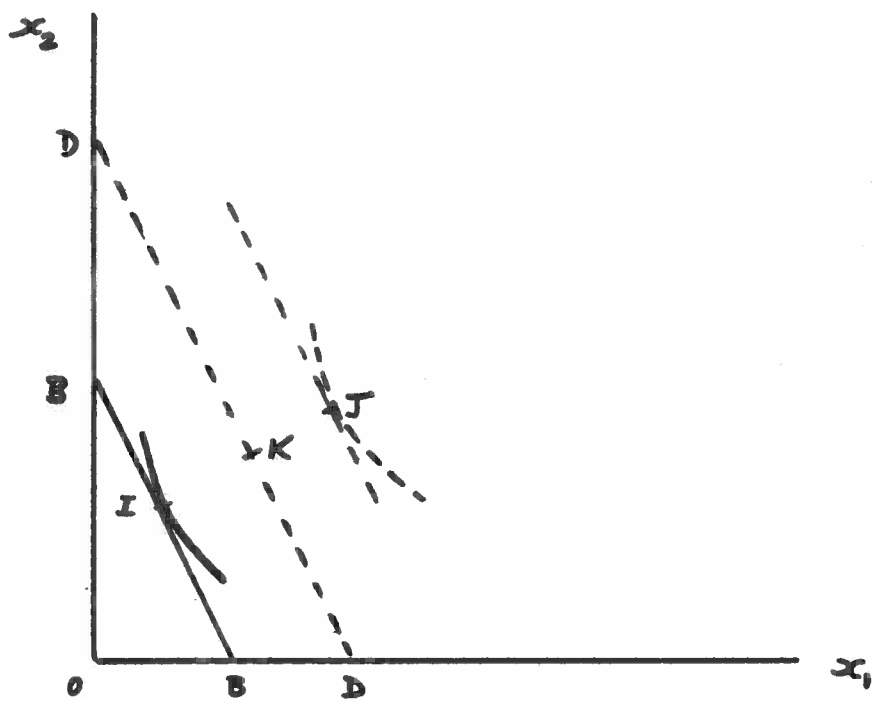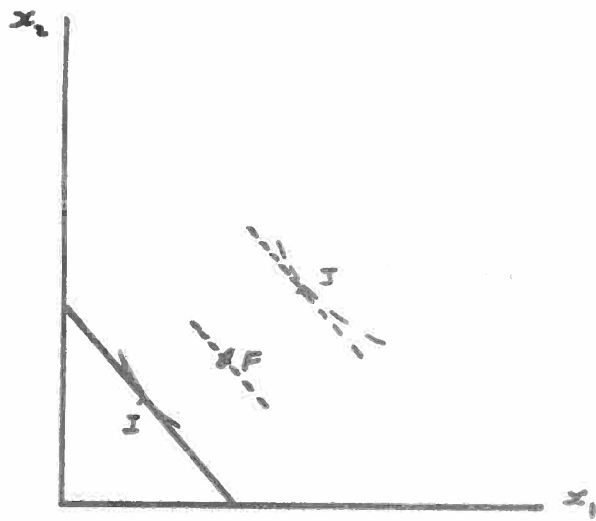
FIG. 1(a)
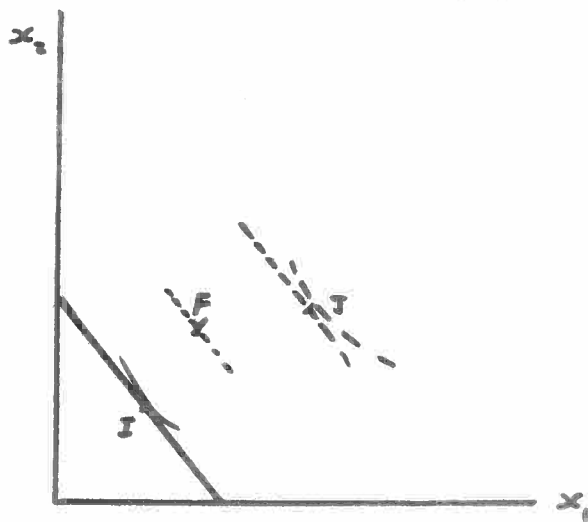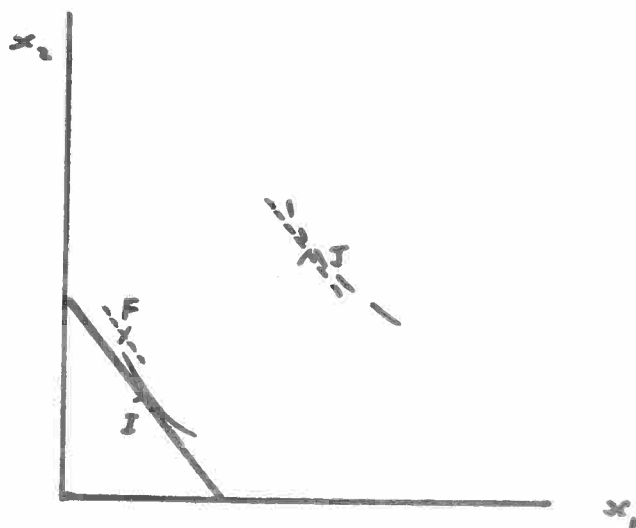

FIG. 1(b)

FIG. 2

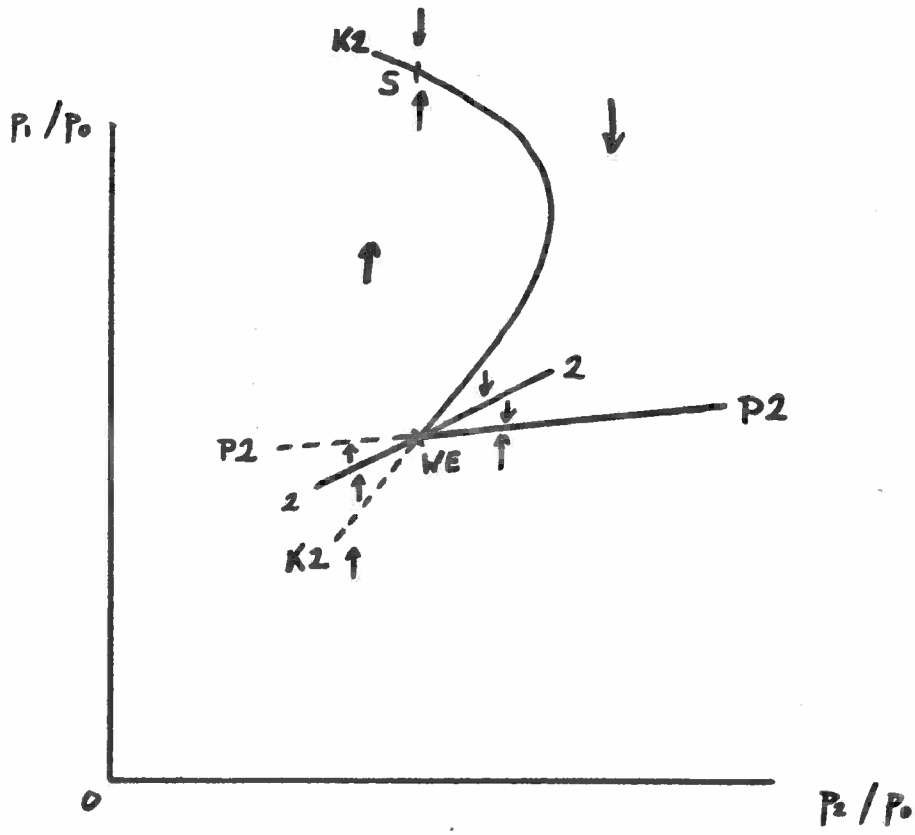FIG. 3

(a)

(b)

(c)

FIG. 4

FIG. 5

FOOTNOTES

1/ A further paper, which draws on some of the earlier rationing literature, merits some discussion. Diamond and Yaari (1972) use the indirect utility function to analyse points rationing. Their concern is not so much to develop that analysis, but to exploit its mathematical equivalence to that of choice under uncertainty. Their consumer maximises $U(\underline{x})$, where $\underline{x} \equiv (x_1, \ldots, x_n)$ subject to m budget constraints,

$\underline{p}_i\underline{x} = E_i$, $i = 1, 2, \ldots, m$, where $\underline{p}_i$ is an $n \times 1$ vector of prices associated with the ith budget constraint. The indirect utility function is then

$$U = V(\underline{p}_1, \ldots, \underline{p}_m, E_1, \ldots, E_m)$$

Since there are many budget constraints, the choice of one as the image of a minimum expenditure function is somewhat arbitrary, but such a function can be defined - for example

$$E_1 = E_1(\underline{p}_1, \ldots, \underline{p}_m, E_2, \ldots, E_m, U).$$

In the present paper, in which attention is confined to straight rationing, the indirect utility function is written as

$$U = V(\underline{p}_F \ \underline{q}_R \ E)$$

from which the minimum expenditure is obtained as

$$E = E(\underline{p}_F \ \underline{q}_R \ U).$$

In either case, the minimum expenditure function is a helpful formulation, especially for the analysis of compensated changes.

2/ Gorman (1976) briefly indicates how such a formulation may facilitate the analysis of quantity-constrained choice, but does not pursue the matter in any detail.

3/ This argument is formally identical with that of Diamond and McFadden. The geometric illustration in figure 1(b) demonstrates the simplicity of the approach.

4. $$x_i = c_i^*(\underline{p}_F \, \underline{q}_R \, U)$$

$$dx_i/dm = c_{iu}^*(du/dm)$$

But $$e^*(\underline{p}_F \, \underline{q}_R \, u) = m - \underline{p}_R \, \underline{q}_R$$

$\therefore$ $$e_u^* \, du = dm$$

$\therefore$ $$du/dm = 1/e_u^*$$

$\therefore$ $$dx_i/dm = c_{iu}^*/e_u^* \equiv \mu_i^*$$

References

Cornes, R.   (1979)   'The Comparative Static Analysis of Quantity-
          Constrained Regimes', Unpublished Manuscript.

Diamond, P.A. and D.L.McFadden (1974) "Some Uses of the Expenditure
          Function in Public Finance", *Journal of Public Economics,*
          3, 3-21.

Diamond, P.A. and M. Yaari, "Implications of the Theory of Rationing
          for Consumer Choice under Uncertainty", *American Economic
          Review,* 62, 333-43.

Dixit, A.K. and P.Weller (1979) "The Three Consumer's Surplus",
          Economica, forthcoming.

Gorman, W.M.   (1976) "Tricks with Utility Functions" in *Essays in
          Economic Analysis,* ed. by Artis, M.J. and Nobay, A.R.,
          Cambridge University Press.

Gould, J.R. and S.G.B.Henry (1967) "The Effects of Price Control on
          A Related Market", *Economica,* 34, 42-9.

Howard, D., (1977) "Rationing, Quantity Constraints and Consumption
          Theory", *Econometrica,* 45, 299-412.

Malinvaud, E., (1977) *The Theory of Unemployment Reconsidered*
          (Blackwell, Oxford).

Neary, J.P., and K.W.S.Roberts (1979) "The Theory of Household
          Behaviour under Rationing", *European Economic Review,*
          forthcoming.

Pearce, D.W. (1978) *"The Valuation of Social Cost,* George Allen and
          Unwin.

Pearce, I.F., (1970) *International Trade,* (Macmillan, London).

Pollak, R., (1969) "Conditional Demand Functions and Consumption
          Theory", *Quarterly Journal of Economics,* 83, pp.60-78.

Rothbarth, E., (1940-41) "The Measurement of Changes in Real Income
          under Conditions of Rationing", *Review of Economic Studies,*
          8, 87-104.

Tobin, J., (1952) "A Survey of the theory of Rationing", *Econometrica,*
          20, 521-53.

——   and H.S.Houthakker (1951) "The Effects of Rationing on Demand
          Elasticities", *Review of Economic Studies,* 18, 140-53.