Testing Recursiveness in a Triangular Simultaneous Equation Model

by

Alberto Holly*

NUMBER 154

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

UNIVERSITY OF WARWICK
COVENTRY

WARWICK ECONOMIC RESEARCH PAPERS NUMBER 154

Testing Recursiveness in a Triangular Simultaneous Equation Model

by

Alberto Holly*

NUMBER 154

July, 1979.

* I am very grateful to Pascal Mazodier for many helpful suggestions and encouragement during the preparation of this paper. Of course, he is absolved from blame for any of the errors that remain. The present version was written during the 1979 Warwick Summer Workshop, with support from the Social Science Research Council.

Author's addresses:

Université de Lausanne,
Ecole des H.E.C., 1015-Lausanne,
Switzerland.

and

Laboratoire d'Econométrie, Ecole Polytechnique,
17 rue Descartes, 75005 Paris, France.

I. INTRODUCTION

In applied work in macroeconomics using simultaneous equation systems relationships between variables are sometimes described by means of a triangular model. However, in a simultaneous equation spirit the *a priori* assumption of full recursivity is typically not made. The purpose of this paper is to suggest a recursiveness test for models which are already written in a triangular form. It is a score test (Rao (1948)) applied to the concentrated likelihood function, which is equivalent to Neyman's $C(\alpha)$ test (1959). As indicated in Holly (1978) this type of procedure is quite general and can be applied to a large variety of tests of model specification. It is based on estimators of the model under the null hypothesis, which are, in the particular case of the recursity test the O.L.S. estimators of each structural equation.

In Section 2 of this paper we define some notations and state, without proof, the statistical properties of this procedure when applied to the recursiveness test. Moreover, as these properties depend on the non-singularity of the information matrix under the null hypothesis, it is shown in Section 3 that this recursiveness test is valid if and only if each of the structural equations is identified by means of zero restrictions. This result is not *a priori* obvious since under the null hypothesis the model is always identified.

The explicit form of the test statistic is given in Section 4. It is a weighted sum of the estimators of the correlation coefficients between the disturbances of each pair of structural equations. In the case of a two equation model, it is shown that the weight is related to the ratio of the variance of the disturbance of the first equation to what is known in the literature as the concentration coefficient of the first equation.

2. NOTATIONS AND PRELIMINARY RESULTS

We consider the following simultaneous equation model

$$YA' + XB' = U \tag{2.1}$$

where Y is a $T \times G$ matrix, X a $T \times K$ matrix, A a $G \times G$ matrix, B a $G \times K$ matrix and U a $T \times G$ matrix. The reduced form may be written as

$$Y = XC' + V \tag{2.2}$$

By a triangular model we mean a model in which the elements $a_{gg'}$ of A satisfy the conditions $a_{gg} = 1$ for $g = 1, \ldots, G$ and $a_{gg'} = 0$ for $g' > g$. Additional zero restrictions can be also imposed on the elements of A .

The T observations on the g-th structural equation may be written as

$$y_g = Y_g \alpha_g + X_g \beta_g + u_g \tag{2.3}$$

where

$$Y_g = Y S_{ag}$$

and S_{ag} is a $G \times N_g$ selection matrix. Also,

$$X_g = X S_{bg}$$

where S_{bg} is a $K \times K_g$ selection matrix.

Equation (2.3) may be written also as

$$y_g = Z_g \delta_g + u_g$$

where

$$Z_g = (Y, X) S_g$$

and

$$S_g = \begin{pmatrix} S_{ag} & 0 \\ 0 & S_{bg} \end{pmatrix}$$

The complete model may be written as

$$y = Z\delta + u$$

where

$$Z = (I_G \otimes (Y, X)) S$$

and

$$S = (\text{diag } S_g)$$

We also have

$$E(uu') = \Sigma \otimes I_T$$

with

$$\Sigma = (\sigma_{gg'})$$

Let us define

$$\sigma = \text{vec } \Sigma$$

and θ_1 as the $G(G - 1)/2 \times 1$ vector containing the off-diagonal elements of Σ, each entered only once, that is,

$$\theta_1' = (\sigma_{12}, \ldots, \sigma_{1G}, \sigma_{23}, \ldots, \sigma_{2G}, \ldots, \sigma_{G-1,G})$$

and let θ_2 be the $(G + \Sigma_g N_g + \Sigma_g K_g) \times 1$ vector such that

$$\theta_2' = (\sigma_{11}^2, \ldots, \sigma_{gg}^2, \ldots, \sigma_{GG}^2, \delta_1', \ldots, \delta_g', \ldots \delta_G')$$

Also define $\theta = (\theta_1, \theta_2)$.

Let $L(\theta_1, \theta_2)$ be the likelihood function and $\hat{\theta}_2(\theta_1)$ be the (unique) value of θ_2 at which $L(\theta_1, \theta_2)$ is maximised conditionally upon θ_1. The concentrated likelihood function is defined as

$$L^*(\theta_1) = L(\theta_1, \hat{\theta}_2(\theta_1))$$

The recursiveness test which is suggested in this paper for testing the null hypothesis $\theta_1 = 0$ against the alternative hypothesis $\theta_1 \neq 0$ is a score test (or "Rao's efficient score test", Rao (1948)) applied to $L^*(\theta_1)$.

If $\hat{\theta}_2(0)$ satisfies the first order condition

$$\frac{\partial L}{\partial \theta_2}(0, \hat{\theta}_2(0)) = 0$$

then, as proved in Koopmans, Rubin and Leipnik (1950) and Barnett (1976),

$$\frac{\partial L^*}{\partial \theta_1}(0) = \frac{\partial L}{\partial \theta_1}(0, \hat{\theta}_2(0)) \tag{2.4}$$

and

$$\frac{\partial^2 L^*(0)}{\partial \theta_1 \partial \theta_1'} = \left(\frac{\partial^2 L}{\partial \theta_1 \partial \theta_1'} - \frac{\partial^2 L}{\partial \theta_1 \partial \theta_2'} \left(\frac{\partial^2 L}{\partial \theta_2 \partial \theta_2'} \right)^{-1} \frac{\partial^2 L}{\partial \theta_2 \partial \theta_1'} \right)_{(0, \hat{\theta}_2(0))} \tag{2.5}$$

Under general conditions, the following statistic

$$R_T = \left(T^{-\frac{1}{2}} \frac{\partial L^*}{\partial \theta_1}(0) \right)' \left[\text{plim} \left(- T^{-1} \frac{\partial^2 L^*(0)}{\partial \theta_1 \partial \theta_1'} \right) \right]^{-1} \left(T^{-\frac{1}{2}} \frac{\partial L^*}{\partial \theta_1}(0) \right) \tag{2.6}$$

is asymptotically distributed as a central chi-square with $G(G - 1)/2$ degrees of freedom. Moreover, under a sequence of local alternatives $\{\bar{\theta}_{1T}\}$ such that

$$\bar{\theta}_{1T} = T^{-\frac{1}{2}} d_T$$

with

$$\lim_{T\to\infty} d_T = d$$

R_T is asymptotically distributed as a non-central chi-square with $G(G - 1)/2$ degrees of freedom and non-centrality parameter equal to

$$d' \left(\text{plim} \; - T^{-1} \frac{\partial^2 L^*(0)}{\partial\theta_1 \partial\theta_1'} \right)^{-1} d$$

3. VALIDITY OF THE TEST STATISTIC AND IDENTIFIABILITY CONDITIONS

In this section, we show that the test statistic R_T is valid if and only if each structural equation is identified by zero restrictions. This result may be proved by showing that

$$\text{plim}\left[\left(-T^{-1}\frac{\partial^2 L}{\partial\theta\partial\theta'}\right)_{\theta} = (0,\ \hat{\theta}_2(0))\right] \tag{3.1}$$

is non-singular if and only if each equation is identified by zero restrictions.

The Log-likelihood function of the model is[1/]

$$L = C^{te} - \frac{T}{2}\log\det\Sigma - \frac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}U'U$$

Its first differential is

$$dL = -\frac{T}{2}\,\mathrm{tr}\,(\Sigma^{-1} - \frac{1}{T}\Sigma^{-1}U'U\;\Sigma^{-1})d\Sigma - \frac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}d(U'U) \tag{3.2}$$

and its second differential is

$$d^2L = -\frac{T}{2}\,\mathrm{tr}\,[\Sigma^{-1}d\,\Sigma(2\Sigma^{-1}\frac{U'U}{T} - I)\,\Sigma^{-1}d\Sigma]$$

$$+ 2\,\mathrm{tr}\,[\Sigma^{-1}d\,\Sigma\,\Sigma^{-1}(dU)'U] - \mathrm{tr}\,\Sigma^{-1}(dU)'(dU)$$

Since we have

1/ It must be noted that the results of this paper do not depend crucially on the normality of the disturbances. They are valid for the quasi-maximum likelihood procedure.

$$\text{plim}\ \frac{1}{T}\ U'U \ = \ \Sigma$$

we may write,

$$\text{plim}(-\frac{1}{T}\ d^2L) = \frac{1}{2}\ \text{tr}(\Sigma^{-1}d\Sigma\Sigma^{-1}d\Sigma) - 2\ \text{tr}[\Sigma^{-1}d\Sigma\Sigma^{-1}\ \text{plim}\ \frac{1}{T}\ (dU)'U]$$

$$+\ \text{tr}\ [\Sigma^{-1}\ \text{plim}\ \frac{1}{T}\ (dU)'\ (dU)]$$

which may be written in our notation as,

$$\text{(3.3)} \qquad \text{plim}(-\frac{1}{T}d^2L) = \frac{1}{2}(d\sigma)'(\Sigma^{-1} \otimes \Sigma^{-1})d\sigma - 2(d\sigma)'[\Sigma^{-1} \otimes \Sigma^{-1}\text{plim}\ \frac{1}{T}\ U'(Y,X)]Sd\delta$$

$$+\ (d\delta)'\ [\text{plim}\ \frac{1}{T}\ Z'(\Sigma^{-1} \otimes I)Z]\ d\delta .$$

Now, we need to take proper account of the symmetry of Σ when evaluating the first and second order partial derivatives of the likelihood function. This problem has already been stressed by Richard (1975) and Balestra (1976). A more extensive treatment has been given recently by Magnus and Neudecker (1978) whose results cannot be applied without modifications to our problem. For this reason we develop here a procedure, which is parallel to the one adopted by Magnus and Neudecker (1978), in order to obtain (3.1) in a convenient matrix form.

Let M be a square matrix of order G and define $sd(M)$ the vector obtained by retaining the under diagonal elements of each successive columns of M . For example, if

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix}$$

then we have

$$sd(M) = \begin{pmatrix} m_{12} \\ m_{13} \\ m_{23} \end{pmatrix}$$

Now consider the identity matrix of order $G(G-1)/2$ and its partition into columns

$$I_{\frac{1}{2}G(G-1)} = (e_{11} \; e_{21} \; \cdots \; e_{G-1,1} \; e_{22} \; \cdots \qquad e_{G-1,G-1})$$

Taking into account the symmetry of M, we may write

(3.4) $$sd(M) = \sum_{g=1}^{G-1} \sum_{g'>g} m_{gg'} \, e_{g'-1,g}$$

Note that

(3.5) $$\sum_{g=1}^{G-1} \sum_{g'>g} e_{g'-1,g} \; e'_{g'-1,g} = I_{\frac{1}{2}G(G-1)}$$

and

(3.6) $$e'_{g'-1,g}\ sd(M) = m_{gg'}$$

Let us now consider the relationship between vec(M) and sd(M) . The vector $e_g (g = 1,\dots, G)$ is defined as a $G \times 1$ vector whose elements are equal to zero except its g-th element which is equal to 1.

Taking into account the symmetry of M , it can be verified that

$$\text{vec } M = \sum_{g=1}^{G-1} \sum_{g'>g} m_{gg'} \text{ vec}(e_g e'_{g'} + e_{g'} e'_g) + \sum_{g=1}^{G} m_{gg} \text{ vec}(e_g e'_g)$$

Since for any vector x and y of any order we have

(3.7) $$\text{vec } y x' = x \otimes y$$

we may write

$$\text{vec } M = \sum_{g=1}^{G-1} \sum_{g'>g} m_{gg'}(e_{g'} \otimes e_g + e_g \otimes e_{g'}) + \sum_{g=1}^{G} m_{gg}(e_g \otimes e_g)$$

Taking into account (3.6) we have

(3.8) $$\text{vec } M = \left[\sum_{g=1}^{G-1} \sum_{g'>g} (e_{g'} \otimes e_g + e_g \otimes e_{g'}) e'_{g'-1,g}\right] sd(M) + \sum_{g=1}^{G} m_{gg}(e_g \otimes e_g)$$

We define the $G^2 \times G(G - 1)/2$ matrix D as

(3.9) $$D = \sum_{g=1}^{G-1} \sum_{g'>g} (e_{g'} \otimes e_g + e_g \otimes e_{g'})\ e'_{g'-1,g}$$

If we apply (3.8) to $d\Sigma$, we obtain

(3.10) $$d\sigma = Dd\theta_1 + \sum_{g=1}^{G} d\sigma_g^2 (e_g \otimes e_g)$$

In order to evaluate (3.3) we need to evaluate expressions of the form

(3.11) $$(e_i' \otimes e_j') (\Sigma^{-1} \otimes \Sigma^{-1} \operatorname{plim} \frac{1}{T} U' (Y, X)) Sd\delta$$

for different values of i and j . We can observe that, under the null,

$$e_i' \Sigma^{-1} = \frac{1}{\sigma_i^2} e_i'$$

and hence (3.11) may be written

(3.12) $$\frac{1}{\sigma_i^2\sigma_j^2} (e_i' \otimes e_j') (I \otimes \operatorname{plim} \frac{1}{T} U'(Y,X))Sd\delta$$

We can easily verify that (3.12) is of the following form

$$\frac{1}{\sigma_i^2\sigma_j^2} (0 \quad 0 \; \ldots . \; e_j'(\operatorname{plim} \frac{1}{T} U'(Y, X)) \; S_i dS_i \; 0 \; \ldots . \; 0)$$

and the term $e_j'(\operatorname{plim} \frac{1}{T} U'(Y, X))S_i$ occupies the i-th block

Now, since $\operatorname{plim} \frac{1}{T} U'X = 0$, we may write

$$\operatorname{plim} \frac{1}{T} U'(Y, X) \; S_i = (\Sigma \; A'^{-1} \; S_{ai}, \; 0)$$

and hence, under the null hypothesis

(3.13) $$e_j' \left(\text{plim} \frac{1}{T} U'(Y, X)\right) S_i = (\sigma_j^2 \, e_j' \, A'^{-1} \, S_{ai}, 0)$$

It is now crucial to observe that, because the model is already in a triangular form, $A'^{-1}S_{ai}$ has the following feature

$$A'^{-1}S = \begin{pmatrix} \overbrace{\qquad}^{N_i} \\ \cdots\cdots \\ 0 \end{pmatrix} \begin{matrix} \} \; i-1 \\ \\ \} \; G-(i-1) \end{matrix}$$

and hence,

(3.14) $$e_j' \, A'^{-1} \, S_{ai} = 0 \quad \text{for} \quad j \geq i$$

As a consequence, it follows from (3.10) and (3.9) that

$$(d\sigma)' \left(\Sigma^{-1} \otimes \Sigma^{-1} \, \text{plim} \frac{1}{T} U'(Y,X)\right) S d\delta =$$

$$(d\theta_1)' \left(\sum_{g=1}^{G-1} \sum_{g'>g} e_{g'-1,\, g} \quad (e_{g'}' \otimes e_g') \right) (\Sigma^{-1} \otimes (A'^{-1}, 0)) S d\delta$$

We define F as

(3.15) $$F = \sum_{g=1}^{G-1} \sum_{g'>g} (e_{g'} \otimes e_g) \, e_{g'-1,g}'$$

We can see that the information matrix is of the following form

$$
\begin{array}{c|c|c|c}
 & \theta_1' & \sigma_1^2 \; \cdots\cdots \; \sigma_G^2 & \delta' \\
\hline
\theta_1 & \begin{matrix} x & & & & 0 \\ & x & & & \\ & & x & & \\ & & & x & \\ 0 & & & & x \end{matrix} & 0 & F'(\Sigma^{-1} \otimes (A'^{-1}, 0))S \\
\hline
\begin{matrix} \sigma_1^2 \\ \vdots \\ \sigma_G^2 \end{matrix} & 0 & \begin{matrix} x & & & & 0 \\ & x & & & \\ & & \ddots & & \\ & & & x & \\ 0 & & & & x \end{matrix} & 0 \\
\hline
\delta & S'(\Sigma^{-1} \otimes \begin{pmatrix} \bar{A}^{-1} \\ 0 \end{pmatrix})F & 0 & \text{plim}\, \frac{1}{T} Z'(\Sigma^{-1} \otimes I)Z
\end{array}
$$

Because of this particular form of the information matrix we only need to consider the regularity of the matrix obtained by deleting the column and line blocks corresponding to the diagonal elements of Σ .

It is also important to verify that we may write, under the null hypothesis

$$
\frac{1}{2} (d\sigma)'(\Sigma^{-1} \otimes \Sigma^{-1})d\sigma = (d\theta_1)' \, F' \, (\Sigma^{-1} \otimes \Sigma^{-1})Fd\theta_1
$$

$$
+ \frac{1}{2} \sum_g \, (d\sigma_g^2)^2 (e_g' \otimes e_g')(\Sigma^{-1} \otimes \Sigma^{-1})(e_g \otimes e_g)
$$

Consequently, the matrix in which we are interested may be written as

$$
(3.16) \quad \begin{pmatrix} F' & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma^{-1} \otimes \Sigma^{-1} & (\Sigma^{-1} \otimes (A'^{-1}, 0))S \\ S'(\Sigma^{-1} \otimes (A^{-1}, 0)) & \text{plim}\, \frac{1}{T} Z'(\Sigma^{-1} \otimes I)Z \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix}
$$

It should be noted that F has rank $G(G-1)/2$. In effect, we can easily verify according to (3.5) that

$$
F' \, F = I_{G(G-1)/2}
$$

and hence, rank (F) = rank (F'F) = G(G - 1)/2. Consequently and according to (3.16) the information matrix is non-singular under the null hypothesis if and only if

$$(3.17)\qquad \begin{pmatrix} \Sigma^{-1} \otimes \Sigma^{-1} & (\Sigma^{-1} \otimes (A'^{-1},0))S \\ S'(\Sigma^{-1} \otimes (A^{-1},0)) & \text{plim}\, \frac{1}{T} Z'(\Sigma^{-1} \otimes I)Z \end{pmatrix}$$

is non-singular.

It can be easily verified that

$$(3.18)\qquad \text{plim}\, \frac{1}{T} Z'(\Sigma^{-1} \otimes I)Z = S'\left(\Sigma^{-1} \otimes \begin{pmatrix} CQC' + A^{-1}\Sigma A'^{-1} & CQ \\ QC' & Q \end{pmatrix}\right) S$$

where

$$(3.19)\qquad Q = \lim_{T\to\infty} \frac{X'X}{T}$$

and we assume that Q is positive definite.

Since $\Sigma^{-1} \otimes \Sigma^{-1}$ is a non-singular matrix, (3.17) is non-singular if and only if the following matrix

$$(3.20)\qquad S'\left[\Sigma^{-1} \otimes \begin{pmatrix} CQC' + A^{-1}\Sigma A'^{-1} & CQ \\ QC' & Q \end{pmatrix} - \left(\Sigma^{-1} \otimes \begin{pmatrix} A^{-1} \\ 0 \end{pmatrix}\right)\left(\Sigma \otimes \Sigma\right)\left(\Sigma^{-1} \otimes \left(A'^{-1}, 0\right)\right)\right] S$$

is non-singular. But it is easy to verify that (3.20) is equal to

$$S' \left(\Sigma^{-1} \otimes \begin{pmatrix} CQC' & CQ \\ QC' & Q \end{pmatrix} \right) S \tag{3.21}$$

which, under the null hypothesis is block diagonal, and its g-th block multiplied by σ_g^2 is equal to

$$S'_g \binom{C}{I} Q \, (C' \; I) \, S_g \tag{3.22}$$

which is a non-singular if and only if

$$\text{rank} \left[(C' \; I) \, S_g \right] = N_g + K_g$$

which is the well known identifiability condition under zero restrictions of the g-th equation.

To summarize the discussion of this section, we have proved that the recursiveness test that we would like to suggest is valid if and only if each of the structural equation is identified by zero restrictions, although the test statistic depends on the estimators under the null hypothesis.

4. THE RECURSIVENESS TEST STATISTIC

Rather than considering the inversion of the information matrix in the form given by (3.16) it is simpler to observe that the first block diagonal element of the inverse of (3.16) is itself a diagonal matrix. The element which has the (g, g') position is the first block diagonal element of

$$(4.1)\qquad \begin{pmatrix} \frac{1}{\sigma_g^2\,\sigma_{g'}^2} & \frac{1}{\sigma_{g'}^2}\,(e_g'\,A'^{-1}\,S_{ag}\;,\;0) \\ \frac{1}{\sigma_{g'}^2}\begin{pmatrix} S'_{ag'}\;A^{-1}e_g \\ 0 \end{pmatrix} & \frac{1}{\sigma_{g'}^2}\left[S'_{g'}\begin{pmatrix} CQC' + A^{-1}\,\Sigma\,A'^{-1} & CQ \\ QC' & Q \end{pmatrix} S_g \right] \end{pmatrix}^{-1}$$

By applying standard results on the inversion of a matrix in term of submatrices, we may find that the first block diagonal element of (4.1) is equal, after simplifications, to

$$(4.2)\qquad \sigma_g^2\,\sigma_{g'}^2 \left[1 + \sigma_g^2\,(e_g'\;A'^{-1}\;S_{ag'},\;0)\;\;H_{g'}^{-1}\begin{pmatrix} S'_{ag'}\;A^{-1}e_g \\ 0 \end{pmatrix} \right]$$

with

$$(4.3)\qquad H_{g'} \;=\; S'_{g'}\begin{pmatrix} CQC' + A^{-1}(\Sigma - \sigma_g^2\;e_g\;e_g')\;A'^{-1} & CQ \\ QC' & Q \end{pmatrix} S_{g'}$$

According to (3.2) we have

$$(4.4)\qquad \frac{\partial L^*(0)}{\partial \sigma_{gg'}} \;=\; \frac{\hat{u}_g'\;\hat{u}_{g'}}{\hat{\sigma}_g^2\;\hat{\sigma}_{g'}^2}$$

Consequently, application of (2.6) to the recursiveness test leads to

$$(4.5) \qquad R_T = T^{-1} \sum_{g=1}^{G-1} \sum_{g'>g} \frac{(\hat{u}_g' \hat{u}_{g'})^2}{\hat{\sigma}_g^2 \hat{\sigma}_{g'}^2} (1 + \hat{\mu}_{gg'})$$

with

$$(4.6) \qquad \mu_{gg'} = \sigma_g^2 (e_g' A'^{-1} S_{ag'}, 0) \; H_{g'}^{-1} \begin{pmatrix} S_{ag'}' & A^{-1} e_g \\ & 0 \end{pmatrix}$$

and $\tilde{\mu}_{gg'}$ is obtained by replacing the unknown elements in $\mu_{gg'}$ by their O.L.S. estimators and Q by $X'X/T$.

As mentioned in section 2, R_T is asymptotically distributed as a central chi-square with $G(G - 1)/2$ degrees of freedom under the null hypothesis. Moreover, under a sequence of local alternatives $\{\sigma_{gg', T}\}$ such that

$$\sigma_{gg', T} = T^{-\frac{1}{2}} d_T$$

with

$$\lim_{T\to\infty} d_T = d$$

R_T is asymptotically distributed as a non-central chi-square with $G(G - 1)/2$ degrees of freedom and non-centrality parameter

$$(4.7) \qquad \sum_{g=1}^{G-1} \sum_{g'>g} d_{gg'}^2 \, \sigma_g^2 \, \sigma_{g'}^2 \, (1 + \mu_{gg'})$$

In order to get a better insight into the interpretation of R_T let us consider the following two equation model

$$y_1 = X_1 \beta_1 + u_1$$

$$y_2 = ay_1 + X_2 \beta_2 + u_2$$

We assume that the second equation is identified by zero restrictions

We have

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

$$S_{a2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A'^{-1} S_{a2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Sigma - \sigma_1^2 e_1 e_1' = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

and

$$S_{a2}' A^{-1} (\Sigma - \sigma_1^2 e_1 e_1') A'^{-1} S_{a2} = 0$$

Consequently, (4.3) simplifies as

(4.8)
$$H_2 = \begin{pmatrix} S'_{a2}\ C\ Q\ C'\ S_{a2} & S'_{a2}\ C\ Q\ S_{b2} \\ S'_{b2}\ \ Q\ C'\ Sa_2 & S'_{b2}\ \ Q\ Sb_2 \end{pmatrix}$$

since

$$X_2 = X\ S_{b2}$$

we may write (4.8) as

(4.9)
$$H_2 = \lim_{T\to\infty} \begin{pmatrix} S'_{a2}\ C\ \frac{X'X}{T}\ C'\ Sa_2 & S'_{a2}\ C\ \frac{X'X_g}{T} \\ \frac{X'_g X\ C'\ S_{a2}}{T} & \frac{X'_g X_g}{T} \end{pmatrix}$$

The first block diagonal element of the inverse of H_2 is the inverse of

(4.10)
$$S'_{a2}\ C\ \left[\lim_{T\to\infty}\left(\frac{X'X}{T} - \frac{X'X_2}{T}\left(\frac{X_2'X_2}{T}\right)^{-1}\frac{X'_2X}{T}\right)\right]\ C'S_{a2}$$

and μ_{12} is equal to

(4.11)
$$\mu_{12} = \frac{\sigma_1^2}{S'_{a2}\ C\ \left[\lim_{T\to\infty}\left(\frac{X'X}{T} - \frac{X'X_2}{T}\left(\frac{X_2'\ X_2}{T}\right)^{-1}\frac{X_2'\ X}{T}\right)\right]\ C'\ S_{a2}}$$

The expression which appears in (4.11) may be recognised as the numerator of what is known in the literature as the concentration coefficient of the first structural equation. We may also interpret μ_{12} as being the ratio of the structural disturbance in the first equation to the variability of the deterministic term in the reduced form of y_1

projected on the subspace generated by the columns of X which is orthogonal to the subspace generated by the columns of X_2. In other words, a large μ_{12} means that y_1 is "relatively" more random than deterministic which induces a large value of the test statistic. On the contrary, a small μ_{12} means that y_1 is "relatively" more deterministic than random.

More generally, it may be observed that the test statistic R_T measures the product of two effects. The first one is the correlation coefficient between the disturbances, and the second is a measure of the "relative randomness" of the endogenous variables which appear in each structural equation.

References

BALESTRA, P. (1976) La dérivation matricielle Collection de l'IME no.12, Sirey, Paris.

BARNETT, W.A. (1976) "Maximum likelihood and iterated Aitken estimation of non-linear systems of equations" J.A.S.A., 71, 354-360.

HOLLY, A. (1978) "A note on tests of serial correlation based upon instrumental variables estimators", Cahier No A 186 0578, Laboratoire d'Econometrie de l'Ecole Polytechnique, Paris.

KOOPMANS, T.C., H. RUBIN and R.B. LEIPNIK (1950), "Measuring the equation systems of dynamic economics" in Statistical Inference in Dynamic Economic Models, T.C. KOOPMANS, (ed.) J. Wiley & Sons, Inc., New York.

MAGNUS, J.R. and H. NEUDECKER (1978) "The elimination matrix: some theorems and applications". Report: AE 2/78, Instituut voor Actuariat en Econometrie, Universiteit van Amsterdam.

NEYMAN, J. (1959) "Optimal asymptotic tests of composite statistical hypotheses" in Probability and Statistics, U. Grenander (ed.) J. Wiley, 213-234.

RAO, C.R. (1948) "Large sample tests of statistical hypothesis concerning several parameters with applications to problems of estimation" Proc. Cam. Phil. Soc., 44, 50-57.

RICHARD, J.F. (1975) "A note on the information matrix of the multivariate normal distribution" Journal of Econometrics, 3, 57-60.