A Tentative Analysis of the Stability of A Competitive Economy with Externalities

Masaaki Homma*
Osaka University

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1. Introduction

It has long been recognised that a consumer's preference or a firm's production possibility is itself affected by the allocation of resources among other consumers and firms. The presence of such dependent effects, usually referred to as "externalities" shows that not all the economic behaviour is mediated through the market price system. While the problem of externalities has received increasing attention, analysis of its effects on the stability of a competitive economy appears to have suffered comparative neglected.[1/]

In recent years, however, two important features inherent to models with externalities have been observed and examined by several authors. These special features suggest the importance of considering the problem of what happens to time paths of prices generated by a competitive economy with externalities. One of the features is related to the "anomalous" qualitative properties investigated by Diamond and Mirrlees [9] and others.[2/] A price anomaly indicates the possibility that even in the absence of income effects the aggregate quantity of an externality causing commodity increases as its price increases. As is well known, the only source of destabilising a competitive economy without externalities is the presence of asymmetrical income effects among consumers. However, the suspicion stems from the anomalous behaviour due to externalities that the Walrasian pricing rule may not lead the economy with externalities to a stable competitive equilibrium even in the absence of the such asymmetrical income effects.

Another feature is concerned with the game theoretical situations discussed by Sandmo [21] and others.[3/] When externalities prevail in the economy, an individual agent must choose his consumption or production taking the decisions of each of others as parametrically given. This implies that the demand functions or the supply functions are functions not only of prices but also of consumption allocation or production allocation of others. It follows from the direct interdependent effects through externalities that a similar difficulty to the Cournot-Nash type model of oligopoly is brought into models with externalities even when prices are kept constant. The quantity adjustment with this aspect may form an another obstacle for a competitive economy with externalities to attain its equilibrium prices.

The purpose of this paper is to provide a tentative approach to the stability analysis which investigates the qualitative behaviour of the dynamics of a competitive economy with externalities. We will concentrate our attention on an economy with reciprocal externalities between consumers and firms. Within the framework, we will show that the economy with externalities can only attain its equilibrium prices under fairly stringent conditions even if we limit our concern to the case where the stability of a competitive economy without externalities is guaranteed. We will further clarify the relationship between the stability and the separability of utility (or production) functions discussed by Davis and Whinston [7] and others.

In Section 2, we introduce notation, preliminary assumptions and then define a competitive equilibrium with externalities. In Section 3, we investigate the qualitative structure of the response of demand and supply functions to a change in prices and in externalities. In Section

4, we describe the dynamics of adjustment process with externalities and then examine the stability of the dynamic system. In the last section, concluding remarks are briefly mentioned.

2. The Model

Our concern is limited to an economy consisting of a representative consumer and a representative firm, though it is easy to extend our analysis to the economy with many consumers and many firms. There are $(n + 1)$ commodities. The numeraire is labelled 0, and may be considered as minus labour. Its demand and supply quantities are denoted by x_0 and y_0, respectively, and its price set at unity without loss of generality. The other commodities are labelled $1, 2, . . . n$. The demand vector and supply vector of their commodities are written as x and y, respectively, and the vector of their prices as q. For the sake of simplicity, we assume that the consumer has no initial endowments of commodities except the numeraire $\bar{x}_0$. This assumption can be easily removed without any difficulties.

We concentrate our attention on the case where reciprocal externalities take place between the consumer and the firm through other commodities except the numeraire. For the analysis that follows, it is convenient to distinguish the actual level of externalities from the expected or perceived one received by the consumer (or the firm). Let $c = (c_1 . . . c_n)$ be the expected level of externalities received by the consumer and $e = (e_1, . . ., e_n)$ be that of externalities received by the firm. We will hereafter refer to c (or e) as the consumption (or production) environment facing the consumer (or the firm). Under such a situation, the utility function of the consumer is represented by

$$U = U(x_0, x; c) \tag{1}$$

and the transformation function of the firm by

$$F(y_0, y; e) = 0 \tag{2}$$

We impose the following rather stringent assumptions on the utility function and the transformation function : (i) U is twice continuously differentiable on $(x_0, x; c)$, (ii) U is strictly concave and increasing with respect to (x_0, x), (iii) F is twice continuously differentiable on $(y_0, y; e)$ and (iv) F is strictly convex and increasing with respect to (y_0, y) . It should be emphasised that we assume concavity or convexity only with respect to its own decision variables (not including the consumption or production environment).[4/] This implies that we deal with the Marshallian type of externalities.

It remains in this section to define a competitive equilibrium with externalities.

Definition : The set of a consumption allocation $(\hat{x}_0, \hat{x})$, a production allocation $(\hat{y}_0, \hat{y})$, a consumption environment facing the consumer $(\hat{c})$, a production environment facing the firm $(\hat{e})$ and a price vector $(\hat{q})$ are a competitive equilibrium with externalities if

(i) $(\hat{x} - \hat{y}) \leqslant 0$, $\hat{q} \geqslant 0$;

(ii) $(\hat{y}_0, \hat{y})$ maximises $\Pi = y_0 + qy$
subject to $F(y_0, y; e) = 0$ given the production
environment $e = \hat{e}$;

(iii) $(\hat{x}_0, \hat{x})$ maximises $U = U(x_0, x; c)$
subject to $x_0 + \hat{q}x \leq \bar{x}_0 + \Pi$ given the consumption
environment $c = \hat{c}$:

(iv) $\hat{e} = \hat{x}$, $\hat{c} = \hat{y}$.

A few comments will help. Part (i) is self-evident. The part (ii) and (iii) require that the firm (or the consumer) forms his own expectation on the production (or consumption) environment and maximises his profit (or utility) subject to the transformation function (or the budget) taking it as parametrically given. The part (iv) means that the (expected) environment facing the firm (or the consumer) is realized in the sense that it coincides with the actual level of externalities.

There may be problems putting expected externality levels into an objective technological relationship. We will, however, be able to escape from the difficulty by limiting our stability analysis to the Walrasian tâtonnement process. The reason why such is a case is that the tâtonnement process is the price adjustment one in which no actual production takes place until equilibrium is reached.

3. Qualitative Structure of the Model with Externalities

We begin with the investigation of qualitative structure of the model with externalities. Let us first consider the supply side of the economy. We denote $\Pi(q; e)$ the level of profit attained at the maximisation of the problem in part (ii) of the definition, that is,

$$\Pi(q; e) = \max_{y_0, y} \{y_0 + qy \mid F(y_0, y; e) = 0\}^{5/} \quad (3)$$

The partial derivatives of the profit-function Π with respect to prices give the supply functions for the corresponding commodities. Using the subscript notation, we can write the supplies as

$$y(q; e) = \Pi_q(q; e) \quad (4)$$

Taking total differentials of (4), we find

$$dy = \Pi_{qq}\, dq + \Pi_{qe}\, d\, e\,, \quad (5)$$

where Π_{qq} is the matrix of second-order partial derivatives of Π with respect to the prices of all commodities except the numeraire and Π_{qe} is the matrix of partial derivatives of Π_q with respect to the externalities.

Let us denote the marginal rate of transformation of y_j (or e_j) for the numeraire, i.e., F_j/F_0 (or F_{n+j}/F_0) by ψ_j (or ψ_{n+j}) and the partial derivative of ψ_i with respect to y_j (or e_j) by ψ_{ij} (or $\psi_{i\ n+j}$). Then we obtain from (5) the matrix of partial derivatives of the supply functions with respect to the prices, denoted by y_q , and that of supply functions with respect to the externalities, denoted by y_e , as follows :

$$y_q = \Pi_{qq} = [\psi_{yy} - \psi_{y0}\,\psi_y]^{-1} \tag{6}$$

and

$$y_e = \Pi_{qe} = -[\psi_{yy} - \psi_{y0}\,\psi_y]^{-1}[\psi_{ye} - \psi_{y0}\,\psi_e] \tag{7}$$

where $[\psi_{yy} - \psi_{y0}\psi_y]$ is the matrix with the (i, j) elements $(\psi_{ij} - \psi_{i0}\,\psi_j)$ and $[\psi_{ye} - \psi_{y0}\psi_e]$ is the matrix with the (i, j) elements $(\psi_{i\ n+j} - \psi_{io}\psi_{n+j})$ for all i, j . The matrix $[\psi_{yy} - \psi_{y0}\psi_y]$ is referred to as the Antonelli matrix for the firm, which is symmetric and positive definite.[6/] The inverse matrix $[\psi_{yy} - \psi_{y0}\psi_y]^{-1}$ has the same property and so does y_q from (6).

We consider the dual to the maximisation problem in part (iii) of the definition and obtain the demand functions from an expenditure function. Write the expenditure function as E(q; c; u) , which relates the minimum level of expenditure to achieve a given utility level u to the vector of prices and that of externalities.[7/] More formally,

$$E(q;\ c;\ u) = \min_{x_0,x} \{x_0 + qy \mid U(x_0,\ x;\ c) \geq u\} \tag{8}$$

The partial derivatives of the expenditure function with respect to the prices are precisely the compensated demand functions for the corresponding commodities. We thus have

$$x = E_q(q;\ c;\ u)\ , \tag{9}$$

from which we find by differentiation

$$dx = E_{qq}\,dq + E_{qc}\,dc + E_{qu}\,du\,. \tag{10}$$

Let $\phi = (\phi_x; \phi_c) = (\phi_1, \ldots, \phi_n; \phi_{n+1}, \ldots, \phi_{2n})$ be the vector of the marginal rates of substitution of x and of c for the numeraire. Total differentiation of the utility function (1) with the budget equation and arrangement leads to

$$du = U_0\left[(\Pi_q - x)dq + \phi_c\,dc + \Pi_e\,de\right]$$

$$= U_0\left[(y - x)\,dq + \phi_c\,dc + \Pi_e\,de\right]. \tag{11}$$

Substituting (11) into (10), we obtain

$$dx = \left[E_{qq} + E_{qu}U_0(y - x)\right]dq + \left[E_{qc} + E_{qu}U_0\,\phi_c\right]dc$$

$$+ E_{qu}U_0\,\Pi_e\,de\,, \tag{12}$$

which yields

$$x_q = E_{qq} + E_{qu}U_0(y - x) = (\phi_{xx} - \phi_{x0}\phi_x)^{-1}(I - \phi_{x0}(y - x)), \tag{13}$$

$$x_c = E_{qc} + E_{qu}U_0\,\phi_c = -(\phi_{xx} - \phi_{x0}\phi_x)^{-1}((\phi_{xc} - \phi_{x0}\phi_c) + \phi_{x0}\phi_c) \tag{14}$$

and

$$x_e = E_{qu}U_0\,\Pi_e = -(\phi_{xx} - \phi_{x0}\phi_x)^{-1}\,\phi_{x0}\,\Pi_e\,. \tag{15}$$

The expression in (13) corresponds to a familiar result that the matrix of partial derivatives of the demand functions with respect to prices is partitioned into the substitution term matrix and the income term matrix. We also know from (14) that the response of the demand functions to a change in externalities can be divided into the corresponding terms. It should be noticed from (15) that the demand functions depend on the production environment facing the firm by way of the profit which is distributed as the divident to the consumer. The matrix $(\phi_{xx} - \phi_{x0}\phi_x)$ is the Antonelli one for the consumer and hence the substitution matrix E_{qq} is symmetric and negative definite from (13).

4. The Dynamic System and Its Stability

We are now concerned with the main subject of this paper, that is, the stability of a Competitive economy with externalities. From now on, all variables are assumed to be functions of time. However, the time argument will be omitted unless confusion would arise. In addition, a dot over a variable denotes its time-derivative.

We consider first the pricing rule of a competitive economy with externalities. It is natural to follow the traditional formulation of the Walrasian tâtonnement pricing rule. We will work with the stability analysis of a situation in which the prices are called in terms of numéraire. The process of pricing is then represented by the following differential equation system:

$$\text{(W.P)} \qquad \dot{q} = [\alpha]\,[x(q;\, c;\, e) - y(q;\, e)] \,,$$

where $[\alpha]$ is the diagonal matrix with positive elements α . [8/]

We consider next the expectation formation hypothesis on the level of externalities. Assuming that the expected level of externalities received by each agent is an exponentially weighted average of past externality levels, we introduce the following adaptive expectation rule into the model:[9/]

$$\dot{c} = [\beta]\,[y\,(q;\,e) - c]$$

(A.E)

$$\dot{e} = [\gamma]\,[x\,(q;\,c;\,e) - e]$$

where $[\beta]$ (or $[\gamma]$) is the diagonal matrix with positive elements β_i (or γ_i).

We are now in a position to prove the stability of the simultaneous price and quantity adjustment process composed of (W.P) and (A.E). We will accomplish this by imposing the following regularity condition on the demand and supply functions:

(i) <u>The Column Diagonal Dominance Condition</u>; The "own" price demand (or supply) elasticity dominates in the sense that it is greater than or equal to the sum of the absolute values of cross price elasticities and those with respect to externalities.

(ii) <u>The Column Norm Condition</u>; The sum of the absolute values of all the elasticities is less than or equal to unity. The above conditions can be integrated and expressed for the demand functions as

$$\left(\sum_{j\neq i}^{n} |\varepsilon^{q}_{ij}| + \sum_{j=1}^{n} |\varepsilon^{c}_{ij}| + \sum_{j=1}^{n} |\varepsilon^{e}_{ij}|\right) \leq |\varepsilon^{q}_{ii}| \leq 1 - \left(\sum_{j\neq 1}^{n} |\varepsilon^{q}_{ij}| + \sum_{j=1}^{n} |\varepsilon^{c}_{ij}| + \sum_{j=1}^{n} |\varepsilon^{e}_{ij}|\right)$$

and for the supply functions as

$$\left(\sum_{j\neq i}^{n}|\eta^{q}_{ij}|+\sum_{j=1}^{n}|\eta^{e}_{ij}|\right) \leq |\eta^{q}_{ii}| \leq 1-\left(\sum_{j\neq i}^{n}|\eta^{q}_{ij}|+\sum_{j=1}^{n}|\eta^{e}_{ij}|\right) ,$$

where

$$\varepsilon^{q}_{ij} = \frac{\partial \log x_i}{\partial \log q_j} , \quad \varepsilon^{c}_{ij} = \frac{\partial \log x_i}{\partial \log c_j} , \quad \varepsilon^{e}_{ij} = \frac{\partial \log x_i}{\partial \log e_j} , \quad \eta^{q}_{ij} = \frac{\partial \log y_i}{\partial \log q_j}$$

and $\eta^{e}_{ij} = \dfrac{\partial \log y_i}{\partial \log e_j}$.

Then we establish:

Theorem 1. If the economy with externalities satisfies the regularity condition, then the dynamic system consisting of simultaneous price and quantity adjustment processes is locally asymptotically stable around the equilibrium values $(\hat{q}, \hat{c}, \hat{e})$.

Proof. Taking a linearisation of the entire system, (W.P) and (A.E), we find

$$\dot{\delta} = [D]\,[H]\,\delta , \tag{16}$$

where $\delta = [\delta; c; e]'$, $\dot{\delta} = [\dot{q}; \dot{c}; \dot{e}]'$,

$$[D] = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \quad \text{and} \quad [H] = \begin{bmatrix} x_q - y_q & x_c & x_e - y_e \\ y_q & -I & y_e \\ x_q & x_c & x_e - I \end{bmatrix} .$$

From the diagonal dominance condition (i), we obtain with the use of the definition of each elasticity for all i

$$|x^q_{ii}|\ |\hat{q}_i| \geq \sum_{j\neq i}^{n} |x^q_{ij}|\ |\hat{q}_j| + \sum_{j=1}^{n} |x^c_{ij}|\ |\hat{c}_j| + \sum_{j=1}^{n} |x^e_{ij}|\ |\hat{e}_j| \qquad (17)$$

and

$$|y^q_{ii}|\ |\hat{q}_i| \geq \sum_{j\neq i}^{n} |y^q_{ij}|\ |\hat{q}_j| + \sum_{j=1}^{n} |y^e_{ij}|\ |\hat{e}_j|\ , \qquad (18)$$

in which all partial derivatives are evaluated at the equilibrium values $(\hat{q}, \hat{c}, \hat{e})$. All the income effects are evacuated in the equilibrium $\hat{x} = \hat{y}$ in view of (13), and therefore $\hat{x}_q$ is reduced to E_{qq} only. It is well known that E_{qq} is symmetric and negative definite and then x^q_{ii} is shown to be negative. Moreover, y^q_{ii} is positive from the positive definiteness of $y_q = \Pi_{qq}$. Thus, summing (17) and (18) leads to

$$|x^q_{ii} - y^q_{ii}|\ |\hat{q}_i| = (-x^q_{ii} + y^q_{ii})|\hat{q}_i| \geq \sum_{j\neq i}^{n} (|x^q_{ij}|+|y^q_{ij}|)|\hat{q}_j|$$

$$+ \sum_{j=1}^{n} |x^c_{ij}|\ |\hat{c}_j| + \sum_{j=1}^{n} (|x^e_{ij}|+|y^e_{ij}|)|\hat{e}_j| \geq \sum_{j\neq i} |x^q_{ij} - y^q_{ij}|\ |\hat{q}_j| \qquad (19)$$

$$+ \sum_{j=1}^{n} |x^c_{ij}|\ |\hat{c}_j| + \sum_{j=1}^{n} |x^e_{ij} - y^e_{ij}|\ |\hat{e}_j| \text{ for } i = 1, \ldots, n.$$

It follows from the norm condition (ii) for the demands that

$$\sum_{j=1}^{n} |x^q_{ij}|\ |\hat{q}_j| + \sum_{j=1}^{n} |x^c_{ij}|\ |\hat{c}_j| + \sum_{i\neq j} |x^e_{ij}|\ |\hat{e}_j| \leq (1 - |x^e_{ii}|)\,|\hat{e}_i|$$

$$\leq (1 - x^e_{ii})|\hat{e}_i| = |x^e_{ii} - 1|\ |\hat{e}_i|\ . \qquad (20)$$

Directly from the norm condition (ii) for the supplies, we have

$$\sum_{j=1}^{n} |y^q_{ij}| \; |\hat{q}_j| + \sum_{j=1}^{n} |y^e_{ij}| \; |\hat{e}_j| \leq |-1| \; |\hat{c}_i| \; . \tag{21}$$

A careful inspection of (19), (20) and (21) reveals that

$$[D] \; [H^*] \; \delta^* \leq 0 \; ,$$

where $[H^*]$ is the matrix which replaces all the off-diagonal elements of H by its absolute values and δ^* is the vector which replaces all the components of δ by its absolute values. This inequality system shows that the matrix $[D] \; [H]$ has a negative dominant diagonal and hence characteristic roots with only negative real parts owing to the well-known theorem by Mckenzie [16]. This completes the proof.

If we limit our attention to a partial analysis in which prices are kept constant, we can easily show stability of the Cournot-Nash equilibrium of the economy with externalities relative to a fixed set of prices.

Corollary 1. The quantity (or expectation) adjustment (A.E) is locally asymptotically stable around the equilibrium values $(\hat{c}(q); \hat{e}(q))$ relative to q if

$$1 \geq \sum_{j=1}^{n} |\varepsilon^c_{ij}| + \sum_{j=1}^{n} |\varepsilon^e_{ij}| \quad \text{and} \quad 1 \geq \sum_{j=1}^{n} |\eta^e_{ij}| \quad \text{for all } i \; .$$

To our knowledge, only two other papers discuss stability of quantity adjustment within the class of models considered here. Sandmo [21] and Cornes [6] restrict their attention to a simple two-commodity (n = 1) two-consumer pure exchange economy holding prices constant.[10/]

The analysis developed so far suggests that the instability results from the strong effects of externalities on each agent's decision. It is Davis and Whinston [7] who first pointed out its possibility and considered the functional condition under which each agent's decision with respect to commodities is independent of his externality environment. They provided a special form of additive separability (in terms of cost functions) which is sufficient for it, although they made no explicit analysis on the stability. Dusansky and Kalman [11] developed further research on this problem and extended the Davis and Whinston argument to a wider class of additive separability.

On the other hand, as is already mentioned, another source of difficulties comes from the presence of consumers' income effects. This is the reason why Diamond and Mirrlees [9] and their followers use a special form of utility functions in analysing anomalous behaviour in models with externalities.11/ Since our analysis is limited to the case in which there is only one consumer, all the income effects with respect to the prices are invalidated in (13) at the equilibrium values. This does not, however, imply that the income effects with respect to the externalities are also vanished in the consumer's choice.

Motivated by the present stage of discussions, we are interested in elucidating the exact relationship between the stability of a competitive economy with externalities and the functional form of transformation (or utility) functions. It is required for the transformation function that the firm's decision with respect to commodities is independent of his production environment, i.e., $y_e = \Pi_{qe} = 0$. A careful observation of (7) reveals that this is satisfied if $\psi_{ye} = \psi_{y0} = 0$. Similarly, for the utility function the consumer's decision with respect to commodities does not depend on the externality environment if $\phi_{xc} = \phi_{x0} = 0$ in view of (14) and (15). Together with these observations, we can summarise that the externality environment is neutral to the firm's or consumer's decision with respect to commodities except the numéraire if the following two conditions are fulfilled:[12/] (i) A change in externalities does not affect the marginal rates of transformation or substitution of other commodities for the numéraire (ii) A change in the quantity of the numéraire has no effect on the marginal rates of transformation or substitution of other commodities for the numéraire.

It is well known that the condition (i) is satisfied if and only if the production or consumption environment facing the firm or consumer is "weakly separable" from his own decision with respect to commodities. We can thus state with the aid of the theorem by Goldman and Uzawa [12] :

Lemma 1. The production or consumption environment facing the firm or consumer is "weakly separable" from his own decision with respect to commodities if and only if the transformation or utility function is written as

$$F(f^{I}(y_0, y), f^{II}(e)) = 0 \tag{22}$$

or as

$$U = U(u^{I}(y_0, y), u^{II}(c)) . \tag{23}$$

Our task is to add a new concept of separability satisfying the condition (ii) to the weakly separable function (22) or (23). We are ready to prove :

Lemma 2. A change in the quantity of the numeraire has no effects on the marginal rates of transformation or substitution of any other commodities for the numeraire if and only if the "weakly separable" transformation or utility function is represented by

$$F(f^{I}(y_0 + \sigma(y); f^{II}(e)) = 0 \tag{24}$$

or

$$U = U(u^{I}(x_0 + \theta(x); u^{II}(c)). \tag{25}$$

Proof. We only give the proof to the case of transformation function because the proof is similar to the case of utility function. The necessity is obvious. Consider the sufficiency for the "weak separable" transformation function. The requirement stated in lemma leads to write

$$\psi_{i0} = \frac{\partial}{\partial y_0}\left(\frac{f^{I}_{i}}{f^{I}_{0}}\right) = \frac{f^{I}_{i0}\, f^{I}_{0} - f^{I}_{i}\, f^{I}_{00}}{(f^{I}_{0})^{2}} = 0$$

for all i , from which we obtain

$$\frac{f^{I}_{00}}{f^{I}_{0}} = \ldots = \frac{f^{I}_{i0}}{f^{I}_{i}} = \ldots = \frac{f^{I}_{n0}}{f^{I}_{n}} . \tag{26}$$

By applying Lemma 1 of Goldman and Uzawa [12] to (26), there exists a function B such that

$$f_0^I(y_0, y) = B(f^I(y)) \ . \tag{27}$$

We find from (27)

$$A(f^I) \equiv \int^{f^I} \frac{1}{B(f^I)} \, df^I = y_0 + \delta(y) \ , \tag{28}$$

where σ is an arbitrary function of integration. Then we have

$$f^I(y_0, y) = A^{-1}(y_0 + \delta(y)) = f^I(y_0 + \delta(y))$$

which yields the desired result, i.e., $F(f^I(y_0 + \delta(y)); f^{II}(e)) = 0$.

The following theorem shows the fundamental relationship between the stability of a competitive economy with externalities and the functional form of transformation or utility functions:

Theorem 2. If the economy is characterised by the transformation function (24) and by the utility function (25), the dynamic system with (W.P) and (A.E) is locally asymptotically stable around the equilibrium values $(\hat{q}, \hat{c}, \hat{e})$.

Proof. In view of Lemma 1 and Lemma 2, we find that $x_c = x_e = y_e = 0$ under the assumptions made in theorem. The linearised system of (W.P) and (A.E) is now simplified as

$$\dot{\delta} = [D]\,[H^{**}]\,\delta ,$$

where

$$[H^{**}] = \begin{bmatrix} E_{qq} - \Pi_{qq} & 0 & 0 \\ \Pi_{qq} & -I & 0 \\ E_{qq} & 0 & -I \end{bmatrix} .$$

Consider the characteristic equation of $[D]\,[H^{**}]$ and then obtain

$$0 = \left| \lambda I - [D]\,[H^{**}] \right| = \prod_{i=1}^{n} (\lambda + \beta_i) \cdot \prod_{i=1}^{n} (\lambda + \gamma_i) \left| \lambda I - [\alpha]\,[E_{qq} - \Pi_{qq}] \right| .$$

In order the dynamic system to be stable, it suffices to show that the matrix $[\alpha]\,[E_{qq} - \Pi_{qq}]$ has characteristic roots with only negative real parts since $\beta \geqslant 0$ and $\gamma \geqslant 0$. As is already noted , E_{qq} (or Π_{qq}) is negative (or positive) definite and thus $E_{qq} - \Pi_{qq}$ is negative definite. Then, owing to the theorem by Arrow and McManus [2], $E_{qq} - \Pi_{qq}$ is the so-called D-stable matrix and hence the desired result.

Theorem 2 yields a well known result on the stability of a competitive equilibrium:

<u>Corollary 2.</u> <u>If there exist no asymmetrical income effects, the Walrasian price adjustment rule leads the economy without externalities to be a stable competitive equilibrium.</u>

5. Concluding Remarks

Although the theory of externalities has been manifested in many broad areas, no attempt has been made to explore its effects on the stability of a competitive economy with externalities. This paper has described the dynamic behaviour consisting the Walrasian pricing and the adaptive expectation formation and has made an analysis of stability of its dynamic system.

The analysis has suggested that even in the situation where price stability of a competitive economy is guaranteed, the strong effects of externalities violate its stability of the entire dynamic adjustment process. The paper has also devoted to elucidating the fundamental relationship between the stability of a competitive economy with externalities and the functional form of transformation and utility functions.

The analysis of stability with externalities is in itself quite interesting and important from a theoretical point of views. Beyond this, however, emphasis should be placed on the fact that stability considerations play a crucial role in judging economic impacts of piecemeal policy recommendations (for example the Pigovian tax policy) in models with externalities. In this connection, further stability analysis will be necessary for obtaining fruitful qualitative comparative static results in this area.

Notes:

1/ The existence proof of a competitive economy with externalities may be found in Arrow and Hahn [1] and Osana [19].

2/ See Buchanan and Kafoglis [4] and Sheshinski [22].

3/ A related topic to the aspect has been appeared in Williams [24], Olson and Zeckhausen [18] and Connolly [5].

4/ The possibility of non-convex production set arising from externalities has been discussed by Starrett [23], Inada and Kuga [13] and Osana [20].

5/ The properties of the profit function are given, for example, by Lau [15].

6/ See Mosishima [17] and Katzner [14].

7/ See, for instance, Diewert [10].

8/ It is not necessary to consider explicitly the pricing process for the numeraire by taking Walras' Law into consideration.

9/ One can introduce alternative expectation rule, for example the stationary or extrapolative one, into the model.

10/ If we limit our concern to such a two-commodity economy in which the consumers' income effects are neglected as they assume, we can easily prove that the Cournot-Nash equilibrium is locally stable if $1 > \varepsilon_{ii}^{c} \cdot \eta_{ii}^{e}$.

11/ The utility functions are assumed to be linear in the numeraire commodity.

12/ Diamond [8] directly assumed these two conditions in his analysis. See also Balcer [3].

References:

1. Arrow, K.J. and F.H. Hahn, *General Competitive Analysis* (Holden-Day), 1971.

2. Arrow, K.J. and M. MacManus, A Note on Dynamic Stability, *Econometrica* 26 (1958), 448-454.

3. Balcer, L., Taxation of Externalities: Direct versus Indirect, *Journal of Public Economics* 13(1980), 121-129.

4. Buchanan, M. and M. Kafoglis, A Note on Public Good Supply, *American Economic Review* 53 (1963) 403-414.

5. Connolly, M., Public Goods, Externalities and International Relations, *Journal of Political Economy* 78 (1970), 279-290.

6. Cornes, R., External Effects: An Alternative Formulation, Warwick Economic Research Papers, no. 159.

7. Davis, D.A. and A.B. Whinston, Externalities, Welfare and the Theory of Games, *Journal of Political Economy* 70 (1962), 241-262.

8. Diamond, P., Consumption Externalities and Imperfect Corrective Pricing, *Bell Journal of Economics and Management* 4 (1973), 526-538.

9. Diamond, P. and J. Mirrlees, Aggregate Production with Consumption Externalities, *Quarterly Journal of Economics* 87 (1973), 1-24.

10. Diewert, W.E., Applications of Duality Theory, in Intriligator, M.D. and D.A. Kendrick (eds), *Frontiers of Quantitative Economics*, Vol.III (North-Holland), (1974), 107-171.

11. Dusansky, R. and P.J. Kalman, Externalities, Welfare and the Feasibility of Corrective Taxes, *Journal of Political Economy* 80 (1972), 1045-1051.

12. Goldman, S.M. and H. Uzawa, A Note of Separability in Demand Analysis, *Econometrica*, 32 (1964), 387-398.

13. Inada, K. and K. Kuga, Limitations of the 'Coase Theorem' on Liability Rules, *Journal of Economic Theory*, 6 (1973) 606-613.

14. Katzner, D.W., *Static Demand Theory* (Macmillan), 1970.

15. Lau, L.J., A Characterization of the Normalized Restricted Profit-functions, *Journal of Economic Theory* 12 (1976), 131-163.

16. Mckenzie, L.W.,Matrices with Dominant Diagonals and Economic Theory, in Arrow, K.J., S. Karlin and P. Suppes (eds), *Mathematical Methods in the Social Sciences*, (Stanford University Press), (1960), 47-62.

17. Morishima, M., A Note on Definition of Related Goods, Review of Economic Studies 26 (1955-56), 131-134.

18. Olson, M. and R. Zeckhauser, An Economic Theory of Alliances, Review of Economics and Statistics 43 (1966), 266-279.

19. Osana, H., On the Boundedness of an Economy with Externalities, Review of Economic Studies 40 (1973), 321-331.

20. Osana, H., Optimal Tax-Subsidy System for an Economy with Marshallian Externalities, Econometrica 45 (1977), 329-340.

21. Sandmo, A., Anomaly and Stability in the Theory of Externalities, forthcoming in Quarterly Journal of Economics.

22. Sheshinski, E., Note on the Relation between Quantity and Price Anomalies under Externalities, Economics Letters 1 (1978), 111-115.

23. Starrett, D.A., Fundamental Nonconvexity in the Theory of Externalities, Journal of Economic Theory 4 (1972), 180-189.

24. Williams, A., The Optimal Provision of Public Goods in a System of Local Government, Journal of Political Economy 74 (1966), 18-33.