A Note on the Speed of Convergence
of Prices in Random Exchange Economies

by

Paul A. Weller

Number 179

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

UNIVERSITY OF WARWICK
COVENTRY

A Note on the Speed of Convergence of Prices in Random Exchange Economies

by

Paul A. Weller

Number 179

October 1980

I am grateful to Mukul Majumdar for suggesting the problem and to him and R.Bhattacharya for useful suggestions. The research reported here was supported in part by NSF Grant SOC 79-06540 to Cornell University during the author's stay there as a visiting Assistant Professor in 1979-80.

I. Introduction

This paper is a continuation of work by Hildenbrand (1971) and Bhattacharya and Majumdar (1973) (henceforth B - M). They consider pure exchange economies in which both preferences and endowments are random. Hildenbrand examines the convergence behaviour of price vectors for which total expected excess demand is zero. He shows that as an economy increases in size, if agents are stochastically independent, then the limit of such a sequence of price vectors is an equilibrium price vector in a suitably defined limit economy. B - M consider the case where prices guarantee equilibrium in almost all states of the world. In other words, market-clearing prices are treated as random vectors. In particular, B - M show that under suitable assumptions there will exist a sequence of such random price vectors displaying almost sure convergence to any equilibrium price vector in a deterministic limit economy. As in the case of any convergence result, the speed of convergence is a natural question to investigate and we will be concerned in this note with establishing a result on the speed of convergence of the random price vectors as the economy increases in size. In order to provide a characterization of the result in a simple case, let us suppose that the deterministic limit economy has a unique equilibrium p_o. Assume also that the random economy consists of individuals with the same independently distributed random preferences and endowments. Then our result states that the probability that the random equilibrium price vector is further from p_o than some distance which converges to zero slower than $N^{-1/2}$ (where N is the number of agents in the random economy) is less than a term which converges to zero faster than $N^{-1/2}$. In the more general case we consider both distance and probability will depend upon the proportions of different types of agent in the economy. In addition, the distance will also depend upon the rate at which the proportions of agents of different types approach their limiting values.

I. The Model.

Let the commodity space be R^ℓ, and let P be the set of all continuous preference orderings on R^ℓ. Then we may describe the characteristics of an agent in a deterministic pure exchange economy by a point in $P \times R^\ell$, thus specifying the agent's preferences and initial endowment of commodities.

If we restrict our attention to the set $Q \subset P$ of orderings defined on the positive orthant R^ℓ_+ which are monotonic and strictly convex, we can introduce the concept of an agent's demand function. Let the set Δ be defined as

$\Delta = \{p = (p_1, \ldots p_{\ell-1}) : p \gg 0, \sum_{i-1}^{\ell-1} p_i < 1\}$[1] and R_{++} the set of strictly positive reals. Then the demand function for an agent is a continuous function

$f : \Delta \times R_{++} \to R^\ell_+$.

If we denote

$A = Q \times T$

where T is a closed and bounded subset of the strictly positive orthant, R^ℓ_{++} then we may write the demand function of an agent of type $a \in A$ as $\phi(a,p)$. The <u>excess demand function</u> is then $\tilde{\zeta}(a,p) = \phi(a,p) - e$. It gives the desired net trades at prices p for an agent of type a . Since, if the agent satisfies his budget constraint, the value of net trades must sum to zero, we may equivalently restrict our attention to the $(\ell-1)$-vector valued function $\zeta(a,p)$.

---

[1] We will write $x \geq 0$ to indicate that the vector x is non-negative i.e. if each element $x^i \geq 0$: $x \gg 0$ will indicate that $x^i > 0$ for each i .

A random agent is represented as a random variable defined on the space of agents' characteristics i.e. a measurable mapping $\alpha(\ )$ of a probability space $(\Omega,\mathcal{B},\mathcal{P})$ into A. Ω is the set of all possible states of the world. Hence for every Borel subset G of A the set $\{\omega \in \Omega \mid \alpha(\omega) \in G\} = \alpha^{-1}(G)$ belongs to $\mathcal{B}$. We can summarize all the information we need about a random agent by considering the distribution of the random variable.

A random pure exchange economy $\mathcal{E} = \{\alpha_i\}_{i \in I}$ is a finite collection of random agents. Consider a sequence $(\mathcal{E}_n)_n$ of such economies. Let $(\mu_1,\mu_2,\ldots)$ be a countable set of measures on A representing the possible distributions of the random agents, and define C_n^k/N_n as the fraction of agents in $\mathcal{E}_n$ with distribution μ_k. Two agents are said to be of the same type if they are identically distributed.

The sequence $(\mathcal{E}_n)_n$ is called a regular increasing sequence if

(i) all agents in $\mathcal{E}_n$ are stochastically independent,

(ii) the number of agents in $\mathcal{E}_n$, N_n is strictly increasing in n,

(iii) $\mathcal{E}_n \subset \mathcal{E}_{n+1}$,

(iv) the fraction C_n^k/N_n of agents with distribution μ_k converges to c_k as $n\to\infty$, and $\sum_{k=1}^{\infty} c_k = 1$.

We will call the measure μ defined on A by

$$\mu(B) = \sum_{k=1}^{\infty} c_k\mu_k(B) \tag{1}$$

the asymptotic distribution of the sequence $(\mathcal{E}_n)_n$.

The distribution μ on the space of agents' characteristics can be interpreted as determining a non-random pure exchange economy with many agents. The set of equilibrium price vectors for such an economy is

$$W(\mu) = \{p : p \in \Delta, \int_A \zeta(a,p) d\mu = 0\} . \quad (2)$$

We now introduce the concept of a random price equilibrium as defined in B - M. Define for each $\omega \in \Omega$

$$W(\mathscr{E}_n^{\omega}) = \{p : p \in \Delta, \sum_{i=1}^{N_n} \zeta(\alpha_i(\omega),p) = 0\} \quad (3)$$

$W(\mathscr{E}_n^{\omega})$ is the set of all equilibrium price vectors for economy $\mathscr{E}_n$ in state ω. Then a random price equilibrium for $\mathscr{E}_n$ is a measurable mapping $p_n(\ )$ on $(\Omega,\mathscr{B},\mathscr{P})$ into Δ such that for almost every state ω, $p_n(\omega) \in W(\mathscr{E}_n^{\omega})$.

The excess demand of individual j of type k in state ω will be written as $\zeta_j^k(\omega,p)$. We will denote the matrix of first partial derivatives of the excess demand function $\frac{\partial \zeta_j^k}{\partial p}(\omega,p)$, which is a square $(\ell-1)$ - dimensional matrix.

We will need the following assumption:

(A) The matrix $\sum_{(j,k) \in I(\mathscr{E}_n)} \frac{\partial \zeta_j^k}{\partial p}(\omega,p)$ is nonsingular a.e. for all $p \in \Delta$, and for all n.

Remark. For a motivation behind this assumption, see B - M (Section IV.3). We than have the following

Theorem Let $(\mathscr{E}_n)$ be a regular increasing sequence of economies. If assumption (A) holds, then for every $p_o \in W(\mu)$ there exists a sequence $p_n(\cdot)$ of random price equilibria satisfying, for some positive constants $d_o, d_1^k, (k = 1,2,\ldots)$

$$\text{Prob}\left(\left| p_n(\cdot) - p_o \right| > d_o \sum_k \frac{C_n^k}{N_n} \left((C_n^k)^{-1/2} (\log C_n^k)^{1/2} + E\zeta^k(\omega,p_o)\right)\right) \leq$$

$$1 - \prod_k \left(1 - d_1^k (C_n^k)^{-1/2} (\log C_n^k)^{-3/2}\right) \qquad (4)$$

Proof. Consider the equation from a first-order Taylor expansion

$$\sum_{j \in I(\mathscr{E}_n)} \zeta_j^k(\omega,p) = \sum_{j \in I(\mathscr{E}_n)} \zeta_j^k(\omega,p_o) + (p-p_o) \sum_{j \in I(\mathscr{E}_n)} \frac{\partial \zeta_j^k}{\partial p}(\omega,\bar{p})$$

where $|\bar{p} - p_o| \leq |p - p_o|$. (5)

A result due to von Bahr (1967) is now needed. Let $\{Z_n\}_{n \geq 1}$ be a sequence of i.i.d. random vectors each with mean μ and dispersion matrix V. Let Λ denote the largest eigenvalue of V. Then, if $E|Z_1|^s < \infty$ for some integer $s \geq 3$

$$\text{Prob}\left(|\bar{Z} - \mu| > n^{-1/2} ((s-1)\Lambda \log n)^{1/2}\right) \leq dn^{-(s-2)/2} (\log n)^{-s/2} \qquad (6)$$

where $\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i$ and d is bounded on any bounded set of values of Λ.

Since the $\zeta^k(\cdot,p)$ are by assumption i.i.d. random vectors for each k, we may apply this result to obtain, for some constants d_1^k, d_2^k, d_3^k, $k = 1,\ldots m$, and $s=3$,

$$\text{Prob}\left(\left| \frac{1}{C_n^k} \sum_j \zeta_j^k(\cdot,p) - E\zeta^k(\cdot,p) \right| > d_1^k (C_n^k)^{-1/2} (\log C_n^k)^{1/2}\right) \leq$$

$$d_2^k (C_n^k)^{-1/2} (\log C_n^k)^{-3/2} \qquad (7)$$

$$\text{Prob}\left(\left| \frac{1}{C_n^k} \sum_j \frac{\partial \zeta_j^k}{\partial p}(\cdot,p) - E\frac{\partial \zeta^k}{\partial p}(\cdot,p)\right| > d_3^k (C_n^k)^{-1/2} (\log C_n^k)^{1/2}\right) \leq$$

$$d_2^k (C_n^k)^{-1/2} (\log C_n^k)^{-3/2} \qquad (8)$$

Let us denote $(C_n^k)^{-1/2}(\log C_n^k)^{-3/2}$ by $f(C_n^k)$ and $(C_n^k)^{-1/2}(\log C_n^k)^{1/2}$ by $g(C_n^k)$. Then we may rewrite the inequalities

$$\text{Prob}\left(\left|\frac{1}{N_n}\sum_j \zeta_j^k(\cdot,p) - \frac{C_n^k}{N_n} E\zeta^k(\cdot,p)\right| > \frac{C_n^k}{N_n} d_1^k\, g(C_n^k)\right) \le d_2^k\, f(C_n^k) \tag{9}$$

$$\text{Prob}\left(\left|\frac{1}{N_n}\sum_j \frac{\partial\zeta_j^k(\cdot,p)}{\partial p} - \frac{C_n^k}{N_n} E\frac{\partial\zeta^k(\cdot,p)}{\partial p}\right| > \frac{C_n^k}{N_n} d_3^k\, g(C_n^k)\right) \le d_2^k\, f(C_n^k) \tag{10}$$

Because of our assumption that all agents are stochastically independent we observe that

$$\text{Prob}\left(\left|\frac{1}{N_n}\sum_j \sum_k \zeta_j^k(\cdot,p) - \sum_k \frac{C_n^k}{N_n} E\zeta^k(\cdot,p)\right| > d_1 \sum_k \frac{C_n^k}{N_n}\, g(C_n^k)\right) \le$$

$$1 - \prod_k (1 - d_2^k\, f(C_n^k)) \tag{11}$$

$$\text{Prob}\left(\left|\frac{1}{N_n}\sum_j \sum_k \frac{\partial\zeta_j^k(\cdot,p)}{\partial p} - \sum_k \frac{C_n^k}{N_n} E\frac{\partial\zeta^k(\cdot,p)}{\partial p}\right| > d_3 \sum_k \frac{C_n^k}{N_n}\, g(C_n^k)\right) \le$$

$$1 - \prod_k (1 - d_2^k\, f(C_n^k)) \tag{12}$$

where $d_1 = \max_k (d_1^k)$ and $d_3 = \max_k (d_3^k)$

Now if $p_o \in W(\mu)$ we know that

$$\lim_{N_n\to\infty} \sum_k \frac{C_n^k}{N_n} E\zeta^k(\omega,p_o) = 0 \tag{13}$$

Returning to (5) and summing over k, we observe that

$$\frac{1}{N_n}\sum_{(j,k)\in I(\mathscr{E}_n)} \zeta_j^k(\omega,p_o) + \frac{1}{N_n}(p - p_o)\sum_{(j,k)\in I(\mathscr{E}_n)} \frac{\partial\zeta_j^k}{\partial p}(\omega,\bar{p}) = 0 \tag{14}$$

will solve implicitly for $p \in W(\mathscr{E}_n^\omega)$.

Using (11) and (12), we see that on a set of probability at least

$\prod_k (1 - d_2^k f(C_n^k))$ we may rewrite (14)

$$\sum_k \frac{C_n^k}{N_n} E\zeta^k(\omega,p_o) + \alpha_n + (p - p_o)\left[\sum_k \frac{C_n^k}{N_n} E\frac{\partial\zeta_j^k}{\partial p}(\omega,\bar{p}) + \beta_n\right] = 0 \qquad (15)$$

where α_n, β_n are random matrices with norm less than $d_4 \sum_k \frac{C_n^k}{N_n} g(C_n^k)$, and

$d_4 = \max(d_1, d_3)$.

Assumption (A) assures us that we may solve a.e. for $(p - p_o)$ from (15)

$$(p - p_o) = \left[\sum_k \frac{C_n^k}{N_n} E\frac{\partial\zeta_j^k}{\partial p}(\omega,\bar{p}) + \beta_n\right]^{-1} \left[- \sum \frac{C_n^k}{N_n} E\zeta^k(\omega,p_o) + \alpha_n\right] \qquad (16)$$

Since $\left[\sum_k \frac{C_n^k}{N_n} E\frac{\partial\zeta_j^k}{\partial p}(\omega,\bar{p}) + \beta_n\right]^{-1}$ must have a bounded norm, it will be possible to find a positive constant d_o such that the solution to (16) satisfies

$$|p - p_o| \leq d_o \sum_k \frac{C_n^k}{N_n} (g(C_n^k) + E\zeta^k(\omega,p_o)) \qquad (17)$$

We have already observed that $p \in W(\mathscr{E}_n^{\omega})$. But by definition a random price equilibrium $p_n(\omega)$ is such that $p_n(\omega) \in W(\mathscr{E}_n^{\omega})$ for almost all ω. This establishes the result. Q.E.D.

Remark. The proof is an adaptation of a result in Bhattacharya and Ghosh (1978).

Discussion.

We will first provide support for our characterization of thr result in the simple case where there is only one type of random agent. Then we may rewrite (4) as

$$\text{Prob}\left(| p_n(\cdot) - p_o | > d_o N^{-1/2}(\log N)^{1/2} \right) \leq d_1 N^{-1/2}(\log N)^{-3/2} \qquad (18)$$

observing that $E\zeta(\omega,p_o) = 0$ in this case.[2]

It is clear that $N^{-1/2}(\log N)^{1/2}$ converges to zero by an application of L'Hôpital's rule. Although it converges to zero more slowly than $N^{-1/2}$, it converges to zero faster than $N^{-1/2+\varepsilon}$ for any $\varepsilon > 0$. To see this, observe that

$$\frac{(\log N)^{1/2}}{N^{1/2}} = \frac{(\log N)^{1/2} N^{-\varepsilon}}{N^{1/2-\varepsilon}} \qquad (19)$$

But for any $\varepsilon > 0$, the numerator in (19) can be shown to converge to zero, again by an application of L'Hôpital's rule. A similar argument may be used to establish that $N^{-1/2}(\log N)^{-3/2}$ converges to zero more slowly than $N^{-(1/2+\varepsilon)}$ for any $\varepsilon > 0$.

The result referred to in footnote (1) above shows in the general case, that

$$\lim_{n\to\infty} \Sigma \frac{C_n^k}{N_n} E\zeta^k(\omega,p_o) = 0 \qquad (20)$$

From this it follows that the expression in (20) is equal to zero for all n in the case where $\frac{C_n^k}{N_n}$ are constants independent of n. This is the case of a simple replica economy.

---

2 This follows from Theorem 4.1 of B - M, equation (6.24).

We note also that the convergence to zero of the expression

$$1 - \prod_k (1 - d_1^k (C_n^k)^{-1/2} (\log C_n^k)^{-3/2})$$

seems to be guaranteed only in the case where the set of measures $(\mu_1, \mu_2, \ldots)$ is finite, and $\lim_{n\to\infty} C_n^k = \infty$. In this respect the result is weaker than Theorem 4.1.(ii) of B - M, since they show that $p_n(\cdot)$ converges almost surely to p_o without the restrictions mentioned above. However, it is perhaps not surprising that a certain amount of efficiency is lost in applying a result on the speed of convergence of sample means of i.i.d. random variables to sums of such sample means across random variables which are not i.i.d.

References

B. von Bahr, On the central limit theorem in R_k , Ark. Mat. 7 (1967), 61-69.

R.N. Bhattacharya and J.K. Ghosh, On the validity of the formal Edgeworth expansion, Annals of Statistics 6 (1978), 434-451.

R.N. Bhattacharya and M. Majumdar, Random exchange economies, Journal of Economic Theory 6 (1973), 37-67.

W. Hildenbrand, Random preferences and equilibrium analysis, Journal of Economic Theory 3 (1971), 414-429.