Bayesian Learning and the Optimal Investment Decision of the Firm

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This paper is circulated for discussion purposes only and its contents should be considered preliminary.

Introduction

This paper is about learning. It illustrates how in a two period allocation problem with uncertainty in each period, an economic agent's decisions are influenced by the knowledge that he is able to learn about the uncertainty. The time periods are linked through the learning process of the economic agent.

The problem to be analysed is that faced by a firm deciding whether or not to invest in a new technology or production process, whose returns are not known with certainty. Because of the two period environment, the firm is able to experiment with the new process in the first period, and observe the results before making another investment decision at the beginning of the second. Given the opportunity for learning, how will this affect the decision of the firm in the first period?

Arrow (1962) examined the implications of incorporating learning into an economic growth model. However he imposes the assumption that there exists a learning time trend. Learning, defined as the acquisition of knowledge, is the product of experience which is measured by cumulative gross investment. It is assumed that the greater the investment, the greater is the productivity of the new capital goods. In his model, Arrow is imposing the assumption that increased knowledge always leads to increased productivity; an example of a typical black box: knowledge goes in and increased productivity comes out. We redefine learning as the acquisition of information; with this information a firm is able to judge which is the most productive of a number of processes, and then chooses the most productive. We could call this method learning by sampling rather than learning by doing. Hence, although we may still observe the fact that increased knowledge or information has led to increased productivity or the most productive process, this alternative approach highlights the decision problem facing the firm: the choice of technique. We are now looking inside the black box.

By assuming that there is an element of uncertainty in the environment, represented by a distribution function, whose parameters are unknown; the definition of learning as the acquisition of information means that the learning process can be modelled using the statistical techniques of Bayesian analysis.

Kohn and Shavell (1974) consider the problem of sequential decision making. They define the problem when the distribution function has known parameters as static. There is no possibility for learning in the static case. Any other type of sequential decision problem is adaptive. Bayesian learning about an unknown parameter of the distribution is an example of the adaptive case.

In two illuminating articles in 1974, Kihlstrom uses Bayes' rule to solve the consumer problem of maximising utility in a single period, when product quality is a random variable with an unknown distribution. The consumer is allowed to sample before purchasing the good, and in experiencing the sample gains information about product quality which enables him to update his subjective distribution function.

Grossman, Kihlstrom and Mirman (1977) provide a more general approach to adaptive sequential decision problems. They explicitly recognise the two period nature of the problem: sampling occurs in the first period, before a final decision is made in the second. The solution is found by dynamic programming, using the technique of backward induction. The agent maximises the second period pay-off function; for every possible realisation of the random, variable from the first period. These optimal values are weighted by the probability that are random variable will take on any particular value. The agent then maximises his first period pay-off function, which includes an indirect pay-off function for the second period.

The paper presented here can be regarded as a specific example of the general framework proposed by Grossman et al. It is related both in format and technique to a short paper by Cyert, DeGroot and Holt (1978); however, we shall provide more detailed comparative static results and also give an explanation of a paradoxical result that they obtain.

The major result of the papers cited above, and reproduced here, is that the presence of a learning mechanism and the opportunity for sampling leads to a higher optimal value of the decision variable in the first period than otherwise, due to experimentation. This result can be contrasted with the static two period models of Sandmo (1970) and Modigliani and Drèze (1972), where provided certain conditions are satisfied, the presence of uncertainty in the second period leads to a lower optimal value of the decision variable in the first period, since there is no experimentation.

The objective function

The firm has a choice between two production techniques a and b . Each process yields a return which contributes to the total profit of the firm, Π :

$$\Pi = \Pi^a + \Pi^b$$

The size of the return to each process is a constant mark up of the quantity of the technique in production, i.e. the price cost margin is constant, so that the profit function is linear in the quantity of capital. It is further assumed that the isoquants have fixed coefficients, so that once the firm decides on the level of capital, the quantity of labour is determined. This assumption means that the firm only has the levels of capital as its decision variables.

Let K_t^a and K_t^b be the quantities of processes a and b in production at time t . Process a is termed "old" and process b , "new" . The problem facing the firm is to decide upon the proportion of the two processes to be used in production. The issue is complicated by the returns to the new process not being known with certainty. A random variable represents the returns to the new process. This random variable can be thought to account for the uncertain and unknown nature of the productivity of the new process, its marginal cost or the price/demand for the final product. In contrast the returns to the old process are known with certainty. This situation would typically reflect the decision problem facing a firm deciding whether or not to adopt a new production technology. This new technology can be envisaged as being divisible into small units. The returns per unit of the new process are the random variables. Thus the returns from an investment of K^b units will be

$$\sum_{j=1}^{K^b} \tilde{\varepsilon}_j$$

The random variables $\tilde{\varepsilon}_j > 0$ 1/ are independently and identically distributed. The expected returns can be written $E(\varepsilon)K^b$.

The restrictive form of the production technology is to enable us to identify the impact of learning mechanism in the purchase of new capital equipment. It is an interesting, but different question to ask what the impact of a new item of capital will have on the amount of labour employed. We could allow the labour-capital decision to enter the model by imposing a learning by doing assumption onto the labour force. But to an extent the effect of new capital on future employment prospects is encompassed in the random variable, $\tilde{\varepsilon}$.

The firm maximises the expected discounted stream of profits over two periods, with respect to the quantities of the two types of capital in each period:

$$EV_1 = E\Pi_1 + \frac{1}{1+i} E\Pi_2$$

where $$\Pi_t = K_t^a + \sum_{j=1}^{K_t^b} \tilde{\varepsilon}_j$$

and i is the social rate of discount which is assumed equal to the market rate of interest. Taking expected values:

$$E\Pi_t = K_t^a + E(\varepsilon)\ K_t^b$$

The objective at the start of the first period is

$$EV_1 = \max_{K_1^a, K_1^b} \left\{ \sum_{t=1}^{2} \frac{1}{(1+i)^{t-1}} (K_t^a + E(\varepsilon) K_t^b) \right\}$$

The form of the profit function means that in a single period problem K_t^a and K_t^b are perfect substitutes for each other. An isoprofit line can be defined as a locus of combinations of K_t^a and K_t^b that earn a specified level of profit, and will be a downward sloping straight line.

In the first period the firm holds subjective beliefs about its uncertain environment, $\tilde{\varepsilon}$, represented by a prior distribution function. The firm makes a decision (K_1^a, K_1^b) and consequently observes the realisation of a sequence of the random variables. It then forms a likelihood function from this sample of realised values; that is, the firm asks itself from which particular distribution is it likely that this observed sample originated. Armed with this experience, the firm modifies its initial beliefs using Bayes' rule:

Posterior distribution $\propto$ Prior distribution x Likelihood function.

The firm obtains an updated set of subjective beliefs for use in the second period, represented by the posterior distribution.

Most importantly, the firm is hypermetropic (long sighted)[2/]. It knows at the outset that this learning process will occur. It is aware that any decision it makes in the first period may have implications for the decision problem in the second period.

Assuming away any indivisibilities, the firm is allowed to purchase quantities of the two processes it desires. If the firm purchases one unit of the new process, then it will observe the realisation of the random variable, and not only will it reap the pay-off from the process, but it will also draw some inference about the unknown distribution of the returns on which the likelihood function is based. The larger is the sample, the more confident is the firm that the likelihood function describes the true distribution, and a more confident inference on the form of the unknown distribution can be made.

To solve the maximisation problem, the method of dynamic programming is used. The maximum value function EV_1 will have two parts: expected profits in the first period and an indirect profit function for the second

$$EV_1 = \max_{K_1^a,\, K_1^b} \left\{ K_1^a + E(\varepsilon) K_1^b + \frac{1}{1+i} EV_2 \right\} \qquad (1)$$

The Uncertainty

Suppose the density function of the random variable, $\tilde{\varepsilon}$, is known to be normal, with unit variance[3/] but the value of the mean is unknown. However the firm holds subjective beliefs about the value of the mean described by a prior normal distribution. Let the value of the unknown mean be z, and the subjective density of z is $N(\mu_0, \frac{1}{\phi})$, where μ_0 is the mean and ϕ is the firm's degree of precision in its expectation of the mean of z being μ_0. The degree of precision is the inverse of the variance. The more confident is the firm that μ_0 is the true mean, so the higher is ϕ ; and the higher is ϕ , the lower is the variance. The firm installs a certain quantity, K_t^b of the new process, and then observes the total returns from this process. Of course this is really a single observation, but could be viewed as a series of observations and the firm can easily compute the average return per unit of the new technology. Let $\bar{z}$ be the average return, then when K_t^b units of the process are installed

$$\bar{z} = \frac{1}{K_t^b} \sum_{j=1}^{K_t^b} \tilde{\varepsilon}_j$$

and $\tilde{\varepsilon}$, as above, has mean z and unit variance. So $\bar{z}$ will also be a normal random variable with mean z and variance $\frac{1}{K_t^b}$, and will lie in the range $(0, \infty)$. The value taken by $\bar{z}$ will be a "sufficient statistic" to provide information on the true mean of ε , see DeGroot (1970).

Thus the firm has some initial belief about the unknown mean z , given by its prior density function $(\mu_0, \frac{1}{\phi})$, it then observes a sample of returns $j = 1, \ldots K_t^b$ from which it computes the sample mean $\bar{z}$, from which it then forms a likelihood function. Combining the likelihood function and the prior distribution, we obtain the posterior distribution of the unknown parameter z . The posterior will be normal, with mean and variance given below.[4/]

$$\mu_{\bar{z}} = \frac{\mu_0 \phi + \bar{z} K_t^b}{\phi + K_t^b}$$

$$\sigma^2_{\bar{z}} = \frac{1}{\phi + K_t^b} .$$

Thus the mean of the posterior distribution, is simply a weighted average of the initial beliefs μ_0 and the sample mean $\bar{z}$.

In the first period, in order to maximise its objective the firm will use the expectations operator based on the prior distribution over the unknown parameter z . In the final period it will use the posterior distribution, which is normal with parameters $\mu_{\bar{z}}$ and $\sigma^2_{\bar{z}}$. But looking from the first period, before a decision has yet been made, the sample is still unobserved, so $\bar{z}$ is itself a random variable, which will need to be integrated out. The marginal distribution of $\bar{z}$ is normal with mean μ_0 and variance $\frac{1}{\phi} + \frac{1}{K_t^b}$. 5/

The constraint

To complete the problem suppose that there is a financial limit on the amount of investment in each period. In order to suppress the effect of capital accumulation on the model, we shall initially assume that the rate of depreciation on existing capital stock is one. This means that both processes need replacing at the end of each period. This may seem an unreal condition to impose on a model attempting to represent investment decisions; the reason is that we wish to isolate the dynamic implications of information accumulation between periods, and incorporating capital accumulation will only disguise the results.

Initially, there will be a limited amount of cash available for investment in each period. The firm has an investment budget which must be spent

$$M_t = K_t^a + rK_t^b \qquad (2)$$

where r is the relative price of the new process. The firm is not allowed to invest any saved funds in another period; the funds must be spent in the respective period. This is replaced in the next budget constraint where the funds available can be spread over two periods, and if any unspent funds from the first period are invested at an interest rate i then

$$(1 + i)\left[M - (K_1^a + rK_1^b)\right] = K_2^a + rK_2^b \qquad (3)$$

<u>The solution to the model</u>

In the final period, there is no future, so there is no profit to be gained in further information. The firm will either produce using the old process or the new one. The firm maximises expected profits in the second period.

$$E\Pi_2 = \max_{K_2^a,\, K_2^b} \left\{ \int_0^\infty (K_2^a + zK_2^b) f(z|\mu_{\bar{z}}, \sigma_{\bar{z}}^2)\, dz \right. \qquad (4)$$

This yields optimal values for K_2^a , K_2^b which are substituted back, so that $E\Pi_2$ becomes an indirect profit function. But these optimal values depend upon $\bar{z}$, which is itself a random variable, when viewed from the first period. So the firm computes optimal value schedules for every possible value of $\bar{z}$, and then weights the resulting indirect profit functions by the probability of observing a particular $\bar{z}$. If the marginal distribution of $\bar{z}$ is $N(\mu_0, \frac{1}{\phi} + \frac{1}{K_1^b})$, [5/] where K_1^b is the amount of new capital installed in the first period, then

$$EV_2 = \int_0^\infty \left[\max_{K_2^a,\, K_2^b} E\Pi_2 \right] f(\bar{z}|\mu_0, \frac{1}{\phi} + \frac{1}{K_1^1})d\bar{z} \qquad (5)$$

Substituting these results into equation (1) the objective can be written out in full as:

$$EV_1 = \max_{K_1^a,\, K_1^b} \left\{ \begin{array}{l} \int_0^\infty (K_1^a + zK_1^b) \; f\;(z|\mu_0, \frac{1}{\phi})\;dz \\ \\ + \frac{1}{1+i} \int_0^\infty \left[\max_{K_2^a,\, K_2^b} \int_0^\infty (K_2^a + zK_2^b) \; f\;(z|\cdot)dz \right] f(\bar{z}|\cdot)d\bar{z} \end{array} \right\} \qquad (6)$$

Suppose the budget constraint is given by equation (2), then substituting this into equation (6) we obtain

$$EV_1 = \max_{K_1^b} \left\{ \begin{array}{l} \int_0^\infty (M_1 - rK_1^b + zK_1^b) \; f\;(z|\cdot)dz \\ \\ + \frac{1}{1+i} \int_0^\infty \left[\max_{K_2^b} \int_0^\infty (M_2 - rK_2^b + zK_2^b) \; f\;(z|\cdot)dz \right] f(\bar{z}|\cdot)d\bar{z} \end{array} \right\} \qquad (7)$$

The solution can be found by the method outlined earlier. In the final period the firm maximises expected profits for all values of $\bar{z}$, as in equation (4) :

$$E\Pi_2 = \max_{K_2^b} \left\{ M_2 - rK_2^b + \mu_{\bar{z}} K_2^b \right\}$$

$$\frac{d(E\Pi_2)}{dK_2^b} = -r + \mu_{\bar{z}}$$

So $$K_2^b = \frac{M_2}{r} \text{ if } r \leq \mu_{\bar{z}} = \frac{\mu_0 \phi + K_1^b \bar{z}}{\phi + K_1^b}$$

$$= 0 \text{ if } r > \mu_{\bar{z}}$$

The optimal value of K_2^b depends upon the realised value of $\bar{z}$. For sufficiently high values of $\bar{z}$: $K_2^b = \frac{M}{r}$; for low values of $\bar{z}$: $K_2^b = 0$. The profit in the second period is determined by the production technique chosen.

$$\text{If } r \leq \mu_{\bar{z}} \Rightarrow K_2^b = \frac{M_2}{r} \Rightarrow V_2 = \frac{M_2}{r} \mu_{\bar{z}}$$

$$\text{if } r > \mu_{\bar{z}} \Rightarrow K_2^b = 0 \Rightarrow V_2 = M_2$$

The indirect expected profit function can be obtained by substituting the optimal decision variables as in equation (5) .

$$EV^2 = \int_{\bar{z}^*}^{\infty} \frac{M_2}{r} \left[\frac{\mu_0 \phi + K_1^b \bar{z}}{\phi + K_1^b} \right] f\left(\bar{z} \mid \mu_0 , \frac{1}{\phi} + \frac{1}{K_1^b}\right) d\bar{z} + \int_0^{\bar{z}^*} M_2 \, f(\bar{z} \mid \cdot) d\bar{z} \tag{8}$$

where $\bar{z}^*$ satisfies $r = \dfrac{\mu_0 \phi + K_1^b \bar{z}^*}{\phi + K_1^b}$

Equation (8) can be substituted into equation (7) to obtain a first period objective function in which the only decision variable is K_1^b , and the effect of K_1^b on the second period indirect profit function is explicitly recognised

$$EV_1 = \max_{K_1^b} \left\{ M_1 - rK_1^b + \mu_O K_1^b + \frac{1}{1+i} EV_2 \right\} \qquad (9)$$

First order conditions yield:

$$\frac{d(EV_1)}{dK_1^b} = -r + \mu_O + \theta \qquad (10)$$

$$\text{where} \quad \theta = \frac{1}{1+i} \cdot \frac{d(EV_2)}{dK_1^b}$$

We wish to show that the opportunity for learning will cause the firm to purchase more of the new process in the first period than otherwise. Firstly it is necessary to show that EV_2 is increasing in K_1^b .

Proposition 2 : EV_2 is a non-decreasing function of K_1^b

Thus, an increase in investment in the new process in the first period increases expected profits in the second. This comes about because the larger is K_1^b , the larger is the sample upon which the sample mean , is based. The larger is K_1^b the more confident is the firm that the sample mean is the true mean.

The first order conditions yield the following conditional demands:

$$\left.\begin{aligned} K_1^{b*} &= \frac{M_1}{r} && \text{if} \quad r < \mu_0 + \theta \\ 0 < K_1^{b*} &< \frac{M_1}{r} && \text{if} \quad r = \mu_0 + \theta \\ K_1^{b*} &= 0 && \text{if} \quad r > \mu_0 + \theta \end{aligned}\right\} \qquad (11)$$

In a single period model, or if the firm's outlook was myopic and did not recognise its ability to learn about its uncertain environment, the condition for $K_1^b = \frac{M_1}{r}$ would be $r \leqslant \mu_0$; however the additional positive term on the right-hand side of equation (11) means that it is more likely that the firm will purchase some of the new process.

We can now state Theorem 1 which is the initial result of this paper.

Theorem 1 : If the objective function is given by equation (7) and if K_1^{b**} is the value of the decision variable which maximises the objective in the non-adaptive case and K_1^{b*} is the optimal decision rule in the adaptive case, then $K_1^{b*} \geqslant K_1^{b**}$.

This result is the same as that obtained by Grossman et al, (Theorem 2, p. 538).

The reason that $K_1^{b*} \geqslant K_1^{b**}$ is that in the adaptive case not only does the firm obtain returns from the process, but also gains information about the distribution of these returns; information which will be used in the next period. Information and the new process are complementary. If the firm ignores this complimentarity its expected profits will be lower.

From equation (A2) it can be seen that $\frac{d(EV_2)}{dK_1^b}$ depends upon K_1^b, so that there is a possibility of an internal solution. It may well be that it is optimal for the firm to mix the two processes in the first period, whereas in a myopic model this would not be true. In either case in the final period when there is no further use for information, the firm will use either one process or the other.

What is the shape of the EV_1 function? This is stated as propositions 2 and 3.

Proposition 3 : The expected profit function EV_1 is convex in K_1^b at low values of K_1^b and concave at high values.

Proposition 4 : If $\mu_0 < r$ then

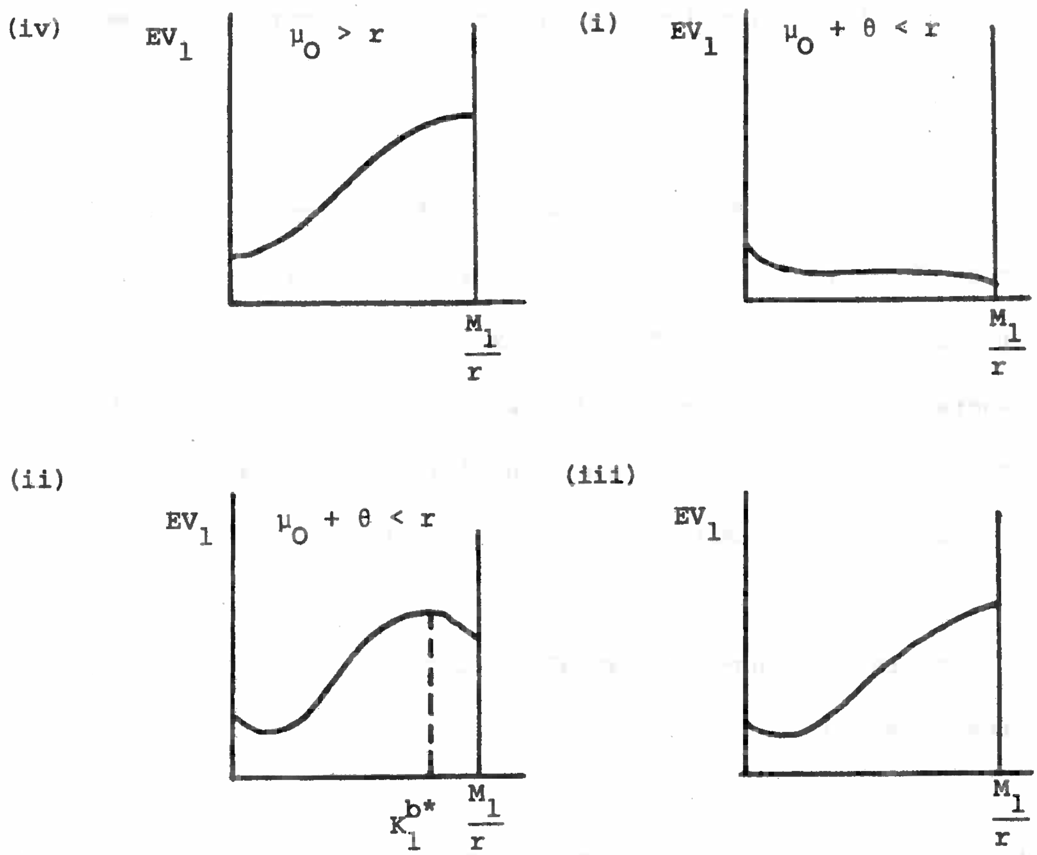
(i) EV_1 is everywhere a decreasing function of K_1^b

or (ii) EV_1 has a maximiser, K_1^{b*} in the interval $(0, \frac{M_1}{r})$

or (iii) EV_1 is decreasing in K_1^b at $K_1^b = 0$, but increasing at $K_1^b = \frac{M_1}{r}$

or (iv) if $\mu_0 \geq r$ then EV_1 is an increasing function of K_1^b.

These propositions can be illustrated in diagrams:



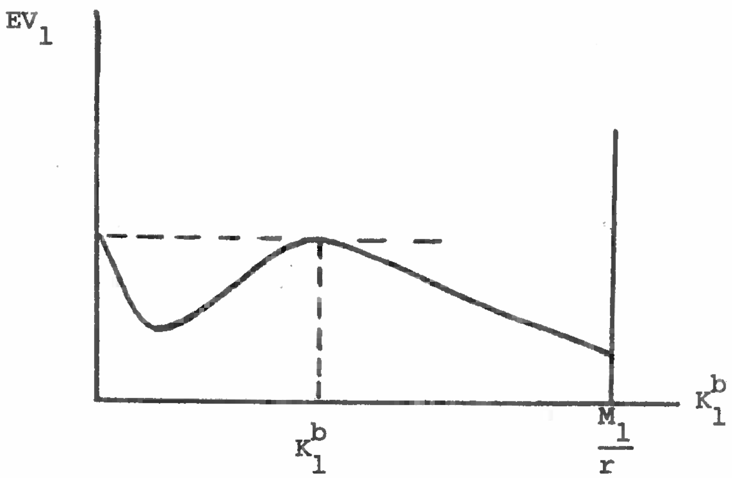
The most interesting case is where the interior solution is defined. The optimal policy for the firm in the first period is to mix the two processes. Whether the model yields an internal or corner solution depends upon the values of the five parameters: r, M_1, M_2, μ_o and ϕ.

As we have drawn the diagrams, we have assumed there is only one point of inflexion, however the second order conditions do not guarantee this, so there may be multiple peaks and troughs.

The effect of parameter values on the solution can be illustrated with reference to changes in the degree of precision. This is stated as Proposition 5:

> A firm with an objective function defined by equation (7) will have higher expected profits the more risky is the initial distribution. Risky being defined as a high initial variance, which is the inverse of ϕ .

That is, increased variance, increases expected profits. The more uncertain firms are about their environment, the greater the expected profits from information. (See Rothschild, 1974a, p. 691). The effect of a decrease in ϕ , on the diagrams above will be to pivot the EV_1 upwards around the point where $K_1^b = 0$; since at $K_1^b = 0$, ϕ does not enter the expected profit function. Thus we can imagine a particular situation where $\mu_o < r$ and for a sufficiently high value of ϕ , we would observe diagram (i) . Then as ϕ decreased, the curve would move upwards into a shape similar to diagram (ii) and finally to diagram (iii). Of interest is the case where the peak of the EV_1 function is equal to the value of the EV_1 function at $K_1^b = 0$.



The diagram implies that for a small change in ϕ , the optimal policy for the firm is suddenly to switch from using none of the new process to K_1^{b*} . It is not optimal for the firm to introduce the new process gradually. If it is worthwhile experimenting with the new process, the firm should move straight away to the optimal sample size.

It is interesting to note the difference between an increase in K_1^b and ϕ on the objective function, EV_1 . EV_2 is increasing in K_1^b , but decreasing in ϕ ; even though an increase in either parameter reduces the variance of the random variable in the second period.

Thus, the firm has an initial belief about the expected returns of the new process, represented by μ_0 and an associated variance $\frac{1}{\phi}$. Proposition 5 shows that the higher is the variance the higher is expected profits, since the greater the initial variance the greater the returns from information. K_1^b is the information component, an increase in K_1^b reduces the variance of the random variable in the next period, as well as putting a greater weight on the firm's experience relative to its initial beliefs.

Further properties of the model

Having stated the initial results, we shall now concern ourselves with some comparative statics and the implications of changing the budget constraint. How will changes in the parameters μ_0 , r , M_1 and M_2 affect K_1^{b*} ? The results will be confused by the changing sign of the second derivative given in Proposition 3. However, we know that if K_1^{b*} exist then the second order conditions hold.

Consider first the effect of a change in the mean on the demand for process b in the first period. What happens if the expected return on the new process rises?

Proposition 6 : K_1^{b*} is monotonically non-decreasing in μ_0 .

Thus, if the expected returns from the new process are higher, the firm will demand more of the new process because it is a better buy and further the firm will demand more information about it.

Reference to the first order conditions in equations (10) and (A3) indicate that the effect on K_1^b of an increase in either M_1 or M_2 is obviously positive.

We would hope that a positive price change would have a negative effect on the optimum demand for new equipment:

Proposition 7 : K_1^{b*} is monotonically non-increasing in r .

Finally, we are concerned with the effect of a change in the initial degree of precision on the quantity of K_1^b demanded. It turns out that for high values of ϕ , when the firm is very confident that the true mean return of the new process is μ_0 , then an increase in ϕ reduces K_1^{b*} . When ϕ is low, the firm has less confidence in the mean μ_0 being the true mean, then an increase in ϕ increases K_1^{b*} .

Proposition 8 : If $\phi \geq \frac{K_1^b}{2}$ then K_1^{b*} is monotonically non-increasing in ϕ. For some lower value of ϕ the reverse.

Now suppose the constraint changes to equation (3), then equation (8) can be rewritten:

$$EV_2' = \int_{\bar{z}^*}^{\infty} (1+i) \frac{[M - (K_1^a + rK_1^b)]}{r} \left[\frac{\mu_0\phi + K_1^b \bar{z}}{K_1^b + \phi}\right] f(\bar{z})d\bar{z}$$

$$+ \int_0^{\bar{z}^*} (1+i)[M - (K_1^a + rK_1^b)] \; f(\bar{z})d\bar{z} \qquad (12)$$

Then, if we continue with the assumption that the market rate of interest in the budget equation is the same as the social rate of discount, the firm faces the following objective function in the first period.

$$EV_1' = \max_{K_1^a,\, K_1^b} \left\{K^a + \mu_0 K_1^b + \frac{1}{1+i} EV_2'\right\} \qquad (13)$$

where EV_2' is given in equation (12). The firm now has two decision variables in the first period: K_1^a and K_1^b. It does not have to spend all of its budget by the end of the first period; in fact the less it spends the more it will have to invest when it is more certain of the distribution of returns. It can be shown that in this case the firm will not invest in the old process in the first period.

Proposition 9 : If the objective function is given by equation (13), then $K_1^{a*} = 0$.

This does not mean that the firm will never invest in the old process, only that it will not invest in it in the first period: it will not be optimal to mix the two processes. On the other hand this result does not mean that the firm will always sample, it may be the case that $K_1^{b*} = 0$. In which case, in the second period the firm will invest its entire budget in the old process, which is essentially a one period solution i.e. $K_2^a = M$. If the firm decides upon the old process, then it will choose that process for ever, since it can gain no new information about the returns to the new process, so it will never swop to the new process.[6/]

In a sense time has become the firm's decision variable; since we are living in a world of perfect markets and no installation or replacement costs, the firm is able to experiment with the new process for as long as is optimal, and then make a decision between the new and old processes.

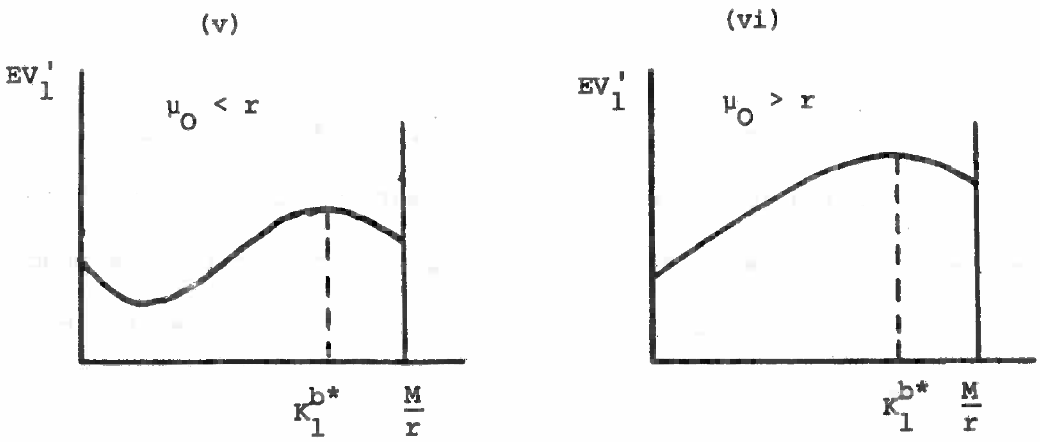
The decision not to insure against a poor run of returns from experimenting with the new process, by in part investing in the old process is crucially dependent upon the social discount rate being less than or equal to the market rate of interest. In a world of imperfect capital markets the opposite is likely to occur. In which case propostion 9 should be modified.

Proposition 9A : $K_1^{a*} = 0$, if and only if the social discount rate i_1, is less than or equal to the market rate of interest i_2 .

It can also be shown that it will never be optimal to spend the entire budget on the new process in period 1.

Proposition 10 : If the objective function is given by equation (13) then $K_1^{b*} < \frac{M}{r}$.

By evaluating the second derivatives, the shape of the EV_1 function can again be drawn.



Thus in diagram (vi) when $\mu_0 > r$, it will always be optimal to sample in the first period. Again, in diagram (v), when $\mu_0 < r$, the decision to sample will depend upon the relative values of the parameters.

A myopic firm would choose a value for K_1^b of either $\frac{M}{r}$ or 0 as $\mu_0 \gtrless r$. Thus if $\mu_0 > r$ the myopic firm chooses $K_1^{b**} = \frac{M}{r}$. However the adaptive firm chooses $K_1^{b*} < \frac{M}{r}$. Thus in this case the flexibility of the budget constraint means that

$$K_1^{b**} > K_1^{b*}$$

This result can be contrasted with Theorem 1 and is similar to the result obtained by Cyert, DeGroot and Holt. Because the budget constraint does not have to be exhausted at the end of each period, the firm will never use all of its budget to sample. However with static distributions there is no advantage in sampling; whatever the parameter values are at the beginning of the first period they will be unaltered by the beginning of the second. So the same decision will be made in each period.

Suppose that in order to encourage the firm to purchase the new process, as a marketing device, the price in the first period is less than the price in the second. How will this affect the demand for the new process?

Proposition 11 : If the objective of the firm is given by equation (A6) where $r_1 < r_2$, and if $\mu_0 > r_1$, then for sufficiently high value of $\bar{z}^*$, $K_1^{b*} = \frac{M}{r_1}$.

The prospect of an increase in price in the second period, means that provided the firm believes that the new process does have a high expected return, then the firm will not bother learning, but will spend its entire budget on the new process in the first period. This conclusion can be contrasted with Proposition 10.

Proposition 12: With different prices for the new process in each of the periods, the effect of a change in price in the second period on demand for the new process in the first may be either positive or negative.

Whether or not the effect is positive or negative depends upon whether the process is expected to yield high or low returns relative to its price.

If high returns are expected so that the firm expects to purchase the process after sampling, then a rise in the price in the second period will cause the firm to purchase more of the process in the first period, since it expects to purchase all of the new process at some stage anyway. The rise in r_2 induces intertemporal substitution, the firm purchases more of the new process in the first period instead of purchasing it at a higher price in the second.

If low returns are expected, then the firm is unlikely to purchase the new process at any stage. A rise in price in either of the periods will consequently have a normal negative effect.

Conclusion

The paper has shown that the firm will purchase not less of the new process under an adaptive framework than a static one. With no opportunity for learning, the firm will equate the value of the marginal product of capital to its cost. In the specific profit function considered here, the condition is that if the returns parameter is greater than the cost the new process should be purchased. If the inequality is reversed, the old process is best. However the opportunity for learning increases the value of a unit of capital in the first period, and consequently the firm purchases more of the new process. It was shown in Proposition 10 that a two period budget constraint will alter this conclusion. The flexibility of the constraint means that the first period can be defined as the investment in the new process.

The form of our objective function means that the results are dependent upon the parameters taking specific values. Further work would enable us to define the range of these parameter values. One interesting parameter is the degree of confidence ϕ ; as was shown, when the degree of confidence is high the firm will not bother to sample, but for a small change in ϕ , the firm will jump from a position of no sampling to the optimal sample size.

We have also shown, that with a flexible budget constraint, the firm will never purchase any of the old process in the first period, unless it is only going to purchase the old process over both periods. This result crucially depends upon there being perfect capital markets. Alternatively the assumption of a flexible budget constraint means that it is never optimal to use up the entire budget experimenting. However, different prices for the new process in each of the two periods may change this conclusion.

Appendix

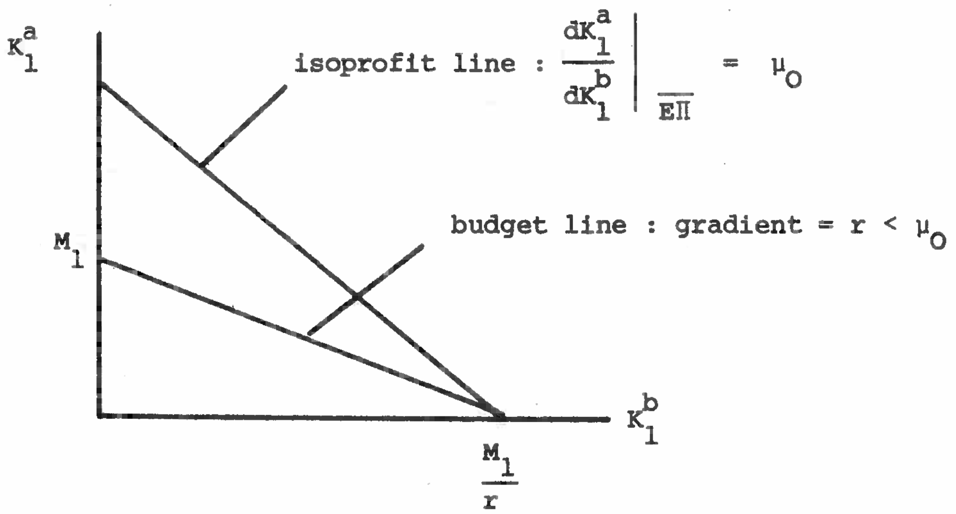
Proposition 1   Marginal distribution of $\bar{z} \sim N(\mu_0, \frac{1}{\phi} + \frac{1}{K_1^b})$

Proof: See DeGroot (1970), p. 263, Exercise 23.

Proof, Theorem 1   In a single period problem, F.O.C's for maximum profits are:

$$\frac{d(E\Pi_1)}{dK_1^b} = -r + \mu_0$$

$$\left.\begin{array}{lll} \text{If } r \leq \mu_0 & K_1^b = \frac{M_1}{r} \\ \phantom{\text{If }} r > \mu_0 & K_1^b = M_1 \end{array}\right\} \qquad \text{(A1)}$$



In the adaptive case, c.f. equation (A3) and (12), there is additional positive term θ , where

$$\theta = \frac{1}{1+i} \cdot \frac{dEV_2}{dK_1^b} > 0$$

There are now three conditional demand schemes:

(i) If $r \leq \mu_0$ $\quad K_1^{b*} = \frac{M_1}{r}$

$K_1^{b**} = \frac{M_1}{r}$

(ii) If $r \geq \mu_0$ but $r - \mu_0 \leq \theta$ $\quad K_1^{b*} = \frac{M_1}{r}$

$K_1^{b**} = 0$

(iii) If $r \geq \mu_0$ but $r - \mu_0 \geq \theta$ $\quad K_1^{b*} = 0$

$K_1^{b**} = 0$

Thus $K_1^{b*} \geq K_1^{b**}$, and it is condition (ii) that ensures the inequality sign.

<u>Proposition 2, Proof</u> The distribution of the random variable in equation (8) can be transformed into a standard normal distribution.

Define a new random variable $x : -\infty \leq x \leq +\infty$

$$x = \frac{\bar{z} - \mu_0}{\sqrt{\frac{1}{\phi} + \frac{1}{K_1^b}}}$$

Then
$$\frac{\mu_0\phi + K_1^b \bar{z}}{\phi + K_1^b} = \mu_0 + x \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \cdot \frac{1}{(K_1^b + \phi)^{\frac{1}{2}}}$$

and
$$x^* = (r - \mu_0)(\phi + K_1^b)^{\frac{1}{2}} \cdot \left(\frac{\phi}{K_1^b}\right)^{\frac{1}{2}}$$

Substituting in (8)

$$EV_2 = \frac{M_2}{r}\int_{x^*}^{\infty}\left[\mu_0 + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \frac{x}{(K_1^b + \phi)^{\frac{1}{2}}}\right] f(x|0,1)\,dx + M_2\int_{-\infty}^{x^*} f(x|\cdot)\,dx \tag{A2}$$

x^* contains K_1^b as an argument, but differentiating around an integral $\frac{dEV_2}{dK_1^b} = 0$.

So $$\frac{dEV_2}{dK_1^b} = \frac{M_2}{2r} \cdot \frac{\phi}{(K_1^b + \phi)^{3/2}} \cdot \frac{1}{(\phi K_1^b)^{\frac{1}{2}}} \int_{x^*}^{\infty} xf(x)\ dx > 0 \tag{A3}$$

<u>Proposition 3, Proof</u> Differentiate (A3) w.r.t. K_1^b

using $$\frac{dx^*}{dK_b^1} = -(r - \mu_0)\left(\frac{\phi}{K_1^b}\right)^{3/2} \cdot \frac{1}{(\phi + K_1^b)^{\frac{1}{2}}}$$

Then $$\frac{d^2(EV_2)}{d(K_1^b)^2} = \frac{M_2}{4r} \cdot \left(\frac{\phi}{K_1^b}\right)^{\frac{1}{2}} \cdot \frac{1}{K_1^b} \cdot \frac{1}{(\phi+K_1^b)^{3/2}}\left\{(r-\mu_0)^2 \frac{\phi^2}{K_1^b} \cdot f(x^*) - \frac{(\phi + 4K_1^b)}{(\phi + K_1^b)} \int_{x^*}^{\infty} xf(x)\,dx\right\}$$

which has a negative and a positive component. When $K_1^b = 0$; $x^* \to \infty$ and the negative term goes to zero. So, at low values of K_1^b, EV_1 is concave.

As K_1^b increases, the function becomes convex. As $K_1^b \to \infty$, the positive term disappears. However we are only interested in the range of K_1^b $(0, \frac{M_1}{r})$ and whether the point of inflexion has occurred before $\frac{M_1}{r}$, will depend upon the parameter values.

Proposition 4, Proof We know from Proposition 2 that $\theta > 0$

(iv) Thus if $\mu_0 \geq r$, then $\theta + \mu_0 > r$ so $\frac{dEV_1}{dK_1^b} > 0$. For (i)-(iii), $\mu_0 < r$ and sign of $\frac{dEV_1}{dK_1^b}$ will depend upon the size of $\frac{dEV_2}{dK_1^b}$ i.e. will the positive θ , be sufficient to outweigh the negative $(\mu_0 - r)$.

(i) If θ is small, such that $\mu_0 + \theta < r$ then $\frac{dEV_1}{dK_1^b} < 0$.

(iii) If θ is large, such that when $K_1^b = 0 \Rightarrow \theta = 0$, then $\frac{dEV_1}{dK_1^b} < 0$ but for high values of K_1^b , $\mu_0 + \theta > r$ then $\frac{dEV_1}{dK_1^b} > 0$.

(ii) However if θ takes on medium values, such that

At $K_1^b = 0 \Rightarrow \theta = 0$ then $\frac{dEV_1}{dK_1^b} < 0$.

At $K_1^b = \frac{M_1}{r}$; $\mu_0 + \theta < r$ then $\frac{dEV_1}{dK_1^b} < 0$.

Whichever effect dominates depends upon the parameter values.

Proposition 5, Proof Let an increase in ϕ represent an increase in the riskiness of the distribution. Differentiating equation (7) w.r.t. ϕ , where EV_2 is given by (A2) :

$$\frac{dEV_1}{d\phi} = \frac{1}{1+i} \cdot \frac{dEV_2}{d\phi} = - \frac{M_2}{r} \frac{(K_1^b + 2\phi)}{(K_1^b + \phi)^{3/2}} \cdot (\frac{K_1^b}{\phi})^{1/2} \cdot \frac{1}{\phi} \cdot \int_{x^*}^{\infty} xf(x)dx < 0$$

Proposition 6, Proof Implicitly differentiate equation (10)

$$\frac{dK_1^b}{d\mu_0} = \frac{-1}{\frac{d^2(EV_1)}{d(K_1^b)^2}} \left\{1 + \frac{1}{1+i} \cdot \frac{M_2}{2r} \cdot \frac{r - \mu_0}{(K_1^b+\phi)^{1/2}} \cdot (\frac{\phi}{K_1^b})^{3/2} f(x^*)\right\}$$

Proposition 7, Proof Implicitly differentiate equation (10) using equation (A3)

$$\frac{dK_1^b}{dr} = \frac{-1}{\frac{d^2(EV_1)}{d(K_1^b)^2}} \left\{-1 - \frac{1}{1+i} \cdot \frac{M_2}{2} \cdot \frac{\phi}{(K_1^b+\phi)^{3/2}} \cdot \frac{1}{(\phi K_1^b)^{1/2}} \left[\frac{1}{r^2} \int_{x^*}^{\infty} xf(x)dx + f(x^*)(r - \mu_0)(\phi + K_1^b) \frac{\phi}{K_1^b} \right] \right\}$$

Proposition 8, Proof Implicitly differentiate equation (10) using (A3)

$$\frac{dK_1^b}{d\phi} = \frac{-1}{\frac{d^2(EV_1)}{d(K_1^b)^2}} \left\{ \cdot \frac{1}{1+i} \cdot \frac{d^2(EV_2)}{dK_1^b \, d\phi} \right\}$$

where $$\frac{d^2(EV_2)}{dK_1^b d\phi} = \frac{M_2}{2r} \cdot \frac{1}{(\phi K_1^b)^{\frac{1}{2}}} \cdot \frac{1}{(K_1^b+\phi)} \left\{ \frac{K_1^b - 2\phi}{(\phi + K_1^b)^{3/2}} \int_{x^*}^{\infty} xf((x)\, dx - \frac{\phi}{K_1^b}(r - \mu_0)^2 (K_1^b + 2\phi) + f(x^*) \right\}$$

Thus if $\frac{K_1^b}{2} \leqslant \phi$ then $\frac{d^2(EV_2)}{dK_b^1 d\phi} < 0$

<u>Proposition 9, Proof</u> Rewriting equation (A2) using (3)

$$EV_2' = (1+i)\left[M-(K_1^a + rK_1^b)\right]\left\{\frac{1}{r}\int_{x^*}^{\infty}\left[\mu_0 + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \cdot \frac{x}{(K_1^b+\phi)^{\frac{1}{2}}}\right] f(x)\,dx + \int_{-\infty}^{x^*} f(x)\,dx\right\} \tag{A5}$$

F.O.C.

$$\frac{dEV_1'}{dK_1^a} = 1 - \frac{1}{r}\int_{x^*}^{\infty}\left[\mu_0 + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \frac{x}{(K_1^b+\phi)^{\frac{1}{2}}}\right] f(x)\,dx - \int_{-\infty}^{x^*} f(x)\,dx$$

But for values of x in the range $(x^*, +\infty)$

$$r < \mu_0 + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \frac{x}{(K_1^b + \phi)^{\frac{1}{2}}}$$

and $$\int_{x^*}^{\infty} f(x)\,dx + \int_{-\infty}^{x^*} f(x)\,dx = 1$$

So $$\frac{dEV_1'}{dK_1^a} < 0 \Rightarrow K_1^a = 0$$

<u>Proposition 10, Proof</u> Differentiate (13) w.r.t. K_1^b, using (A5) and note

$$\int_{-\infty}^{+\infty}\left[\mu_O + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \frac{x}{(K_1^b+\phi)^{\frac{1}{2}}}\right] f(x)\,dx = \mu_O$$

$$\frac{dEV_1'}{dK_1^b} = \frac{M-(K_1^a + rK_1^b)}{2r} \cdot \frac{\phi}{(K_1^b+\phi)^{3/2}} \cdot \frac{1}{(\phi K_1^b)^{\frac{1}{2}}} \int_{x^*}^{\infty} xf(x)\,dx$$

$$+ \int_{-\infty}^{x^*}\left[(\mu_O - r) + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \frac{x}{(K_1^b+\phi)^{\frac{1}{2}}}\right] f(x)\,dx$$

But for values of x in the range $(-\infty, x^*)$

$$r > \mu_O + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \frac{x}{(K_1^b+\phi)^{\frac{1}{2}}}$$

so at $K_1^b = \frac{M}{r}$

$$\frac{dEV_1'}{dK_1^b} = 0 + \int_{-\infty}^{x^*}\left[\mu_O + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \frac{x}{(K_1^b+\phi)^{\frac{1}{2}}} - r\right] < 0$$

$$\therefore \quad K_1^{b*} < \frac{M}{r}$$

<u>Proposition 11, Proof</u> Rewriting equations (13) and (A5)

$$EV_1' = \max_{K_1^b} \mu_O K_1^b + \int_{x^*}^{\infty} \frac{M - r_1 K_1^b}{r_2}\left[\mu_O + \left(\frac{K_1^b}{\phi}\right)^{\frac{1}{2}} \frac{x}{(\phi+K_1^b)^{\frac{1}{2}}}\right] f(x)\,dx$$

$$+ \int_{-\infty}^{x^*} (M - r_1 K_1^b) f(x)\,dx \qquad \text{(A6)}$$

where $x^* = (r_2 - \mu_O)(\phi + K_1^b)^{\frac{1}{2}}\left(\frac{\phi}{K_1^b}\right)^{\frac{1}{2}}$

$$\frac{dEV_1''}{dK_1^b} = \left[1 - \frac{r_1}{r_2}\right]\mu_0 + \frac{M-r_1K_1^b}{2r_2} \cdot \frac{\phi}{(K_1^b+\phi)^{3/2}} \cdot \frac{1}{(\phi K_1^b)^{1/2}} \int_{x^*}^{\infty} xf(x)dx$$

$$+ r_1 \int_{-\infty}^{x^*} \left[\frac{1}{r_2}\left\{\mu_0 + \left(\frac{K_1^b}{\phi}\right)^{1/2} \frac{x}{(\phi+K_1^b)^{1/2}}\right\} - 1\right] f(x)dx \qquad \text{(A7)}$$

As $x^* \to \infty$: $\int_{-\infty}^{x^*} \left[\mu_0 + \left(\frac{K_1^b}{\phi}\right)^{1/2} \frac{x}{(\phi + K_1^b)^{1/2}}\right] f(x)dx \to \mu_0$.

Then (A7) can be written $\frac{dEV_1''}{dK_1^b} = \mu_0 - r_1 > 0$

so $K_1^{b*} = \frac{M}{r_1}$.

<u>Proposition 12, Proof</u> Implicitly differentiate (A7):

$$\frac{d^2(EV_1'')}{dK_1^b dr_2} = \frac{r_1}{r_2^2}\mu_0 - \frac{M-r_1K_1^b}{2r_2^2} \cdot \frac{\phi}{(K_1^b+\phi)^{3/2}} \cdot \frac{1}{(\phi K_1^b)^{1/2}} \int_{x^*}^{\infty} xf(x)dx$$

$$- \frac{M - r_1K_1^b}{2r_2} \cdot \frac{r_2-\mu_0}{(K_1^b+\phi)^2} \cdot \left(\frac{\phi}{K_1^b}\right)^2 f(x^*)$$

$$- \frac{r_1}{r_2} \int_{-\infty}^{x^*} \left[\mu_0 + \left(\frac{K_1^b}{\phi}\right)^{1/2} \cdot \frac{x}{(K_1^b+\phi)^{1/2}}\right] f(x)dx \qquad \text{(A8)}$$

Divide (A7) by r_2 and substitute into (A8) .

$$\frac{d^2(EV_1'')}{dK_1^b dr_2} = \frac{\mu_0}{r_2} - \frac{r_1}{r_2} \int_{-\infty}^{x^*} f(x)dx - \frac{M - r_1K_1^b}{2r_2} \cdot \frac{r_2-\mu_0}{(K_1^b+\phi)^2}\left(\frac{\phi}{K_1^b}\right)^2 f(x^*)$$

Let $\int_{-\infty}^{x^*} f(x)dx = F(x)$

Then

$$\left.\begin{array}{l}\text{if } \mu_o > r_1 \\ \text{and } \mu_o > r_2\end{array}\right\} \Rightarrow \frac{dK_1^b}{dr_2} > 0$$

$$\left.\begin{array}{l}\text{if } \mu_o < r_1 F(x) \\ \text{and } \mu_o < r_2\end{array}\right\} \Rightarrow \frac{dK_1^b}{dr_2} < 0$$

But if $r_2 > \mu_o > r_1$ then $\dfrac{dK_1^b}{dr_2} \gtrless 0$

Notes:

1/ Random variables are denoted by a superscript "~" above the symbol.

2/ More correctly the firm is far-sighted.

3/ The variance could also be unknown, see DeGroot (1970) p. 169, but this would only complicate the problem. The variance is thus assumed constant with the mean being the only unknown parameter.

4/ For deviation of posterior mean and variance see DeGroot p. 167.

5/ See Proposition 1 in Appendix.

6/ Cf. Rothschild (1974b) p. 191
"If the machine whose pay-off probability is known is ever played, it will be played forever more."

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