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RATIONAL EXPECTATIONS MODELS:

A General Method and Some Macroeconomic Examples

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SUMMARY

The paper presents a general solution method for rational expectations models that can be represented by systems of deterministic first order linear differential equations with constant coefficients. It is the continuous time adaptation of the method of Blanchard and Kahn. To obtain a unique solution there must be as many linearly independent boundary conditions as there are linearly independent state variables. Three slightly different versions of a well-known small open economy macroeconomic model were used to illustrate three fairly general ways of specifying the required boundary conditions. The first represents the standard case in which the number of stable characteristic roots equals the number of predetermined variables. The second represents the case where the number of stable roots exceeds the number of predetermined variables but equals the number of predetermined variables plus the number of "backward-looking" but non-predetermined variables whose discontinuities are linear functions of the discontinuities in the forward-looking variables. The third represents the case where the number of unstable roots is less than the number of forward-looking state variables. For the last case, boundary conditions are suggested that involve linear restrictions on the values of the state variables at a future date.

The method of this paper permits the numerical solution of models with large numbers of state variables. Any combination of anticipated or unanticipated, current or future and permanent or transitory shocks can be analysed.

SADDLEPOINT PROBLEMS IN CONTINUOUS TIME RATIONAL EXPECTATIONS MODELS: A GENERAL METHOD AND SOME MACROECONOMIC EXAMPLES.

(1) INTRODUCTION

This paper studies the solution of a class of rational expectations models that can be represented by systems of deterministic first order linear differential equations with constant coefficients. This class includes virtually all deterministic continuous time rational expectations models in the macroeconomic and open economy macroeconomic literature such as Sargent and Wallace (1973), Dornbusch (1976), Wilson (1979), Krugman (1979), Dornbusch and Fischer (1980) and Buiter and Miller (1981a,b). The method handles systems with state vectors of any dimension, n. As long as the forcing variables or exogenous variables do not "explode too fast", any combination of anticipated or unanticipated, current or future and permanent or transitory shocks can be analysed. Wilson's (1979) analysis of anticipated future shocks in systems where n=2 and Dixit's (1980) method for handling unanticipated current permanent shocks are special cases of the general method developed in this paper.

When the number of predetermined or backward-looking variables (n_1) equals the number of stable roots of the characteristic equation of the homogenous system and the number of non-predetermined, forward-looking or "jump" variables ($n-n_1$) equals the number of unstable roots, there is a natural way of specifying the n linearly independent boundary conditions that are required for a unique solution. This case is considered in Section 2. n_1 boundary conditions take the familiar form of initial conditions for the predetermined variables. The remaining $n-n_1$ boundary conditions are obtained from the terminal or transversality condition that the system should be "convergent". More precisely, if the particular solution of the system of equations exists and remains bounded for all time then the general solution of the system should remain bounded for all time. This transversality condition constrains the initial values of the $n-n_1$ non-predetermined variables to lie on the stable manifold; the influence of the $n-n_1$ unstable characteristic roots is neutralized.(1)(2)

If the system has "too many" unstable roots, i.e. if there are fewer stable roots than predetermined variables, no convergent solution exist for arbitrary initial values of the predetermined variables and the methods of this paper cannot be utilized. The case when there are more stable roots than predetermined variables is considered in Section 3. The transversality condition that the solution be convergent now no longer suffices to ensure a unique solution. Two examples are given in which economically sensible additional linear boundary conditions can be provided to guarantee uniqueness. One involves "backward-looking" variables that are nevertheless not predetermined. The other involves forward-looking variables "associated with" stable characteristic roots. Formally, all these models can be viewed as linear two-point boundary value problems with linear boundary conditions. The mathematical conditions for uniqueness are straightforward. The problem lies in the economic motivation of the boundary conditions. In ad-hoc macromodels this motivation can never be fully satisfactory.

(2) A continuous time version of the method of Blanchard and Kahn

The method presented in this Section is a straightforward continuous time adaptation of the solution method for linear difference models with rational expectations presented in Blanchard and Kahn (1980) and Blanchard (1980).

Consider the discrete time model of equations (1a) and (1b)

(1a) $$x_1(t+h)-x_1(t)=\alpha_{11}hx_1(t)+\alpha_{12}hx_2(t)+\alpha_{13}\left[E(x_1(t+h)\,|\,I(t))-x_1(t)\right]$$

$$\alpha_{14}\left[E(x_2(t+h)\,\Big|\,I(t))-x_2(t)\right]+\beta_1hz(t)$$

(1b) $$E(x_2(t+h)\,|\,I(t))-x_2(t)=\alpha_{21}hx_1(t)+\alpha_{22}hx_2(t)+\alpha_{23}\left[E(x_1(t+h)\,|\,I(t))-x_1(t)\right]$$

$$+\beta_2hz(t)$$

$x_1(t)$ is the n_1 vector of predetermined variables, $x_2(t)$ the $n-n_1$ vector of non-predetermined variables $(n\geq n_1)$. $z(t)$ is the k vector of exogenous or forcing variables. E is the mathematical expectation operator and I(t) the information set at the beginning of period t, conditioning the expectations formed in period t. $h\geq o$ is the length of the unit period. The predetermined variables $x_1(t+h)$ are functions only of variables known at time t, i.e. $E(x(t+h)|I(t))\equiv x(t+h)$, regardless of the realization of the variables in I(t+h) (see Blanchard and Kahn (1980, p1305). The non-predetermined variables $x_2(t+h)$ can be a function of any variable in I(t+h). I(t) includes all current and past values of x_1, x_2 and z as well as the true structure of the model given in (1a,b). It may include exogenous variables other than the "market fundamentals" (Flood and Garber (1980))[3] and future values of the exogenous variables; $I(t+h) \supseteq I(t)$.

The system (1a,b) can be represented by the more compact but equivalent system (1a',b'), provided the relevant matrix inverses in (2a-g) exist

(1a') $x_1(t+h)-x_1(t)=A_{11}hx_1(t)+A_{12}hx_2(t)+B_1hz(t)$

(1b') $E\left[x_2(t+h)\,|\,I(t)\right]-x_2(t)=A_{21}hx_1(t)+A_{22}hx_2(t)+B_2hz(t)$

where

(2a) $A_{11}=\Omega(I-\alpha_{13})^{-1}(\alpha_{11}+\alpha_{14}\alpha_{21})$

(2b) $A_{12}=\Omega(I-\alpha_{13})^{-1}(\alpha_{12}+\alpha_{14}\alpha_{22})$

(2c) $A_{21}=\alpha_{21}+\alpha_{23}A_{11}$

(2d) $A_{22}=\alpha_{22}+\alpha_{23}A_{12}$

(2e) $B_1=(I-\alpha_{13})^{-1}(\beta_1+\alpha_{14}\beta_2)$

(2f) $B_2=\beta_2+\alpha_{23}B_1$

(2g) $\Omega=\left[I-(I-\alpha_{13})^{-1}\alpha_{14}\alpha_{23}\right]^{-1}$

Dividing (1a') and (1b') by h and taking the limit as $h \to o$ yields:

(3a) $\frac{d}{dt}x_1(t)=A_{11}x_1(t)+A_{12}x_2(t)+B_1z(t)$

and, since $\hat{x}_2(t,t)=x_2(t)$,

(3b) $$\left.\frac{\partial \hat{x}_2(s,t)}{\partial s}\right|_{s=t} = A_{21}x_1(t)+A_{22}x_2(t)+B_2z(t)$$

Where for any variable y we use the notation $\hat{y}(s,t)\equiv E(y(s)|I(t))$,

$$\frac{d\,y(t)}{dt} \equiv \lim_{\substack{h\to o\\ h>o}} \left[\frac{y(t+h)-y(t)}{h}\right] \text{ and } \left.\frac{\partial \hat{y}(s,t)}{\partial s}\right|_{s=t} = \lim_{\substack{h\to o\\ h>o}} \left[\frac{\hat{y}(t+h,t)-\hat{y}(t,t)}{h}\right]$$

To solve (3a,b) we return to (1a') and (1b'), take expectations conditional on I(t), divide by h and take the limit as $h\to o$. This gives

(4a) $$\left.\frac{\partial \hat{x}_1(s,t)}{\partial s}\right|_{s=t} = A_{11}\hat{x}_1(s,t)\Big|_{s=t} + A_{12}\hat{x}_2(s,t)\Big|_{s=t} + B_1\hat{z}(s,t)\Big|_{s=t}$$

(4b) $$\left.\frac{\partial \hat{x}_2(s,t)}{\partial s}\right|_{s=t} = A_{21}\hat{x}_1(s,t)\Big|_{s=t} + A_{22}\hat{x}_2(s,t)\Big|_{s=t} + B_2\hat{z}(s,t)\Big|_{s=t}$$

We make the following assumptions:

(A1) $\hat{y}(s,t)\equiv y(s) \qquad s\leq t$

For s<t this means "perfect hindsight". For s=t it is the assumption of "weak consistency" made e.g. in Turnovsky and Burmeister (1977).

(A2) $\hat{z}(s,t)$ is a piecewise continuous function of s and t and $\hat{z}(s,t)$ is of exponential order for all t and for all $s\geq t$, i.e. for all t and for all $s\geq t$ there exist constant matrices C and α, C>o such that $|\hat{z}(s,t)|\leq Ce^{\alpha t}$. This assumption rules out explosive growth of the expectation of future values of z, held at time t.

Note that since for the predetermined variables

$E\left[x_1(t+h)\,|\,I(t)\right] \equiv x_1(t+h)$, we have in continuous time:

(5) $$\frac{dx_1(t)}{dt} \equiv \lim_{\substack{h\to o\\ h>o}} \left[\frac{x_1(t+h)-x_1(t)}{h}\right] = \lim_{\substack{h\to o\\ h>o}} E\left[\frac{x_1(t+h)-x_1(t)\,|\,I(t)}{h}\right] \equiv \frac{\partial \hat{x}_1(s,t)}{\partial s}\bigg|_{s=t}$$

The actual and the anticipated instantaneous rates of change of the predetermined variables coincide; equivalently:

$$\frac{\partial \hat{x}_1(s,t)}{\partial t}\bigg|_{s=t} \equiv \lim_{\substack{h\to o\\ h>o}} \left[\frac{E(x_1(t+h)\,|\,I(t+h)) - E(x_1(t+h)\,|\,I(t))}{h}\right] = 0$$

This is not in general true for the non-predetermined variables. Indeed we have

(6) $$\frac{dx_i(t)}{dt} \equiv \lim_{\substack{h\to o\\ h>o}} \left[\frac{x_i(t+h)-x_i(t)}{h}\right] = \lim_{\substack{h\to o\\ h>o}} \left[\frac{E\left(x_i(t+h)\,|\,I(t+h)\right)-E\left(x_i(t)\,|\,I(t)\right)}{h}\right]$$

$$= \lim_{\substack{h\to o\\ h>o}} \left[\frac{E\left(x_i(t+h)\,|\,I(t)\right)-E\left(x_i(t)\,|\,I(t)\right)}{h}\right]$$

$$+ \lim_{\substack{h\to o\\ h>o}} \left[\frac{E\left(x_i(t+h)\,|\,I(t+h)\right)-E\left(x_i(t+h)\,|\,I(t)\right)}{h}\right]$$

$$\equiv \frac{\partial \hat{x}_i(s,t)}{\partial s}\bigg|_{s=t} + \frac{\partial \hat{x}_i(s,t)}{\partial t}\bigg|_{s=t} \qquad i=1,2.$$

For x_2 the instantaneous rate at which expectations are revised, $\frac{\partial \hat{x}_2(s,t)}{\partial t}\Big|_{s=t,}$ will not be equal to zero at those instants at which "news" arrives. x_2 will therefore not in general be a continuous function of time: $\frac{\partial \hat{x}_2(s,t)}{\partial t}\Big|_{s=t}$ may well be unbounded at those instants that new information becomes available. Assumption A2 is convenient but perhaps too restrictive. Given A2, x_1 will be a continuous function of time. There are, however, quite reasonable models in which the instantaneous rate of change of x_1 can become unbounded because the value of z becomes unbounded at some point in time. Examples are Buiter and Miller (1981a,b). One of the predetermined state variables in their models is the real stock of money balances: $\ell(t) \equiv m(t)-p(t)$. m(t) and p(t) are the natural logarithms of the nominal money stock and the price level, respectively. In these "Keynesian" models p(t) is constrained to be a continuous function of time. Therefore, discrete discontinuous changes in the level of the nominal money stock at $t=t_o$ (which would occur e.g. if m(t) were a step function with a step at $t=t_o$) imply a discrete, discontinuous change in $\ell(t)$ at $t=t_o$; the instantaneous rates of change of m(t) and $\ell(t)$ are unbounded at $t=t_o$.

We can summarize (4a,b) compactly as follows:

(7) $$\frac{\partial \hat{x}(s,t)}{\partial s}\bigg|_{s=t} = A\, \hat{x}(s,t)\bigg|_{s=t} + B\, \hat{z}(s,t)\bigg|_{s=t}$$

where

(8a) $$x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(8b) $$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

(8c) $$B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

We assume that A can be diagonalized by a similarity transformation as in (9). Necessary and sufficient for this is that A have n linearly independent eigenvectors. A sufficient condition is that A have n distinct characteristic roots.

(9) $A=V\Lambda V^{-1}$ or $V^{-1}AV=\Lambda$

V is the nxn matrix whose columns are the right-eigenvectors of A. Λ is the diagonal matrix whose diagonal elements are the characteristic roots of A. A central assumption of this section is

(A3) A has n_1 characteristic roots with negative real parts (stable roots) and $n-n_1$ characteristic roots with positive real parts (unstable roots).

We now partition V, V^{-1} and Λ conformably, as follows:

(10a) $$V=\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

(10b) $$V^{-1}=\begin{bmatrix} (V^{-1})_{11} & (V^{-1})_{12} \\ (V^{-1})_{21} & (V^{-1})_{22} \end{bmatrix}$$

(10c) $$\Lambda=\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

Λ_1 is the $n_1 x n_1$ diagonal matrix whose diagonal elements are the stable roots of A and Λ_2 the $(n-n_1)x(n-n_1)$ diagonal matrix whose diagonal elements are the unstable roots of A. We also define

(11) $p=V^{-1}x$ or $x=Vp$.

Partitioning p conformably with V, V^{-1} and x we get

(12a) $$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} (V^{-1})_{11} & (V^{-1})_{12} \\ (V^{-1})_{21} & (V^{-1})_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

(12b) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

p_1 is an n_1 vector and p_2 an $n-n_1$ vector. Using (9) and (11), we can transform (7) into

(13) $$\frac{\partial \hat{p}(s,t)}{\partial s}\bigg|_{s=t} = \Lambda \hat{p}(s,t)\bigg|_{s=t} + V^{-1}B\hat{z}(s,t)\bigg|_{s=t}$$

or

(14a) $$\frac{\partial \hat{p}_1(s,t)}{\partial s}\bigg|_{s=t} = \Lambda_1 \hat{p}_1(s,t)\bigg|_{s=t} + \left[(V^{-1})_{11}B_1 + (V^{-1})_{12}B_2\right]\hat{z}(s,t)\bigg|_{s=t}$$

(14b) $$\frac{\partial \hat{p}_2(s,t)}{\partial s}\bigg|_{s=t} = \Lambda_2 \hat{p}_2(s,t)\bigg|_{s=t} + \left[(V^{-1})_{21}B_1 + (V^{-1})_{22}B_2\right]\hat{z}(s,t)\bigg|_{s=t}.$$

The forward-looking solution for $\hat{p}_2(s,t)$ as a function of s, holding t constant is

$$\hat{p}_2(s,t) = e^{\Lambda_2 s}K_2 - \int_s^{\infty} e^{\Lambda_2(s-\tau)}\left[(V^{-1})_{21}B_1 + (V^{-1})_{22}B_2\right]\hat{z}(\tau,t)d\tau \quad (4)$$

K_2 is an $n-n_1$ vector of arbitrary constants. For s=t this becomes

(15) $$\hat{p}_2(t,t) = e^{\Lambda_2 t}K_2 - \int_t^{\infty} e^{\Lambda_2(t-\tau)}\left[(V^{-1})_{21}B_1 + (V^{-1})_{22}B_2\right]\hat{z}(\tau,t)d\tau$$

Given assumption A2, the integral on the r.h.s. of (15) exists. (15) will only converge, however, if $K_2=0$. Imposing this transversality condition, (15) becomes

(16) $$\hat{p}_2(t,t)=-\int_t^{\infty} e^{\Lambda_2(t-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,t)d\tau$$

The weak consistency assumption (A1) implies that

$\hat{p}_2(t,t)=p_2(t)$. From (12a) we know that $p_2=(V^{-1})_{21}x_1+(V^{-1})_{22}x_2$

Therefore, provided $(V^{-1})_{22}$ has an inverse

(17) $$x_2(t)=-\left[(V^{-1})_{22}\right]^{-1}(V^{-1})_{21}x_1(t)-\left[(V^{-1})_{22}\right]^{-1}\int_t^{\infty} e^{\Lambda_2(t-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,t)d\tau$$

Equivalently, using (12b) we find that, provided V_{11} has an inverse,

$x_2=V_{21}\left[V_{11}\right]^{-1}x_1+\left[V_{22}-V_{21}\left[V_{11}\right]^{-1}V_{12}\right]p_2$. Since

$V_{22}-V_{21}\left[V_{11}\right]^{-1}V_{12}=\left[(V^{-1})_{22}\right]^{-1}$ (provided the inverse exists), (17) can also be written as

(17') $$x_2(t)=V_{21}\left[V_{11}\right]^{-1}x_1(t)-\left[(V^{-1})_{22}\right]^{-1}\int_t^{\infty} e^{\Lambda_2(t-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,t)d\tau.$$

The similarity between equation (17) or (17') and Blanchard and Kahn's equation (3) is immediately apparent. Here, as there, the current value of the non-predetermined variables depends on the current value of the predetermined variables and on current anticipations of all future values of the exogenous variables.

To find the solution for $x_1(t)$ we substitute (17) into (3a).

This yields

(18) $$\frac{d}{dt}x_1(t)=\left\{A_{11}-A_{12}\left[(V^{-1})_{22}\right]^{-1}(V^{-1})_{21}\right\}x_1(t)+B_1z(t)$$

$$-A_{12}\left[(V^{-1})_{22}\right]^{-1}\int_t^{\infty}e^{\Lambda_2(t-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,t)d\tau$$

From (9), (10a,b,c) and (8b) we find that

(19a) $$A_{11}=V_{11}\Lambda_1(V^{-1})_{11}+V_{12}\Lambda_2(V^{-1})_{21}$$

and

(19b) $$A_{12}=V_{11}\Lambda_1(V^{-1})_{12}+V_{12}\Lambda_2(V^{-1})_{22}$$

Therefore $A_{11}-A_{12}\left[(V^{-1})_{22}\right]^{-1}(V^{-1})_{21}=V_{11}\Lambda_1\left[(V^{-1})_{11}-(V^{-1})_{12}\left[(V^{-1})_{22}\right]^{-1}(V^{-1})_{21}\right]=V_{11}\Lambda_1\left[V_{11}\right]^{-1}$

Equation (18) therefore becomes

(20) $$\frac{d}{dt}x_1(t)=V_{11}\Lambda_1\left[V_{11}\right]^{-1}x_1(t)+B_1z(t)-A_{12}\left[(V^{-1})_{22}\right]^{-1}\int_t^{\infty}e^{\Lambda_2(t-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,t)d\tau$$

We choose the backward-looking solution for the predetermined variables $x_1(t)$. Therefore

(21) $$x_1(t)=V_{11}e^{\Lambda_1 t}\left[V_{11}\right]^{-1}K_1+\int_{-\infty}^{t}V_{11}e^{\Lambda_1(t-s)}\left[V_{11}\right]^{-1}\{B_1z(s)-A_{12}\left[(V^{-1})_{22}\right]^{-1}\int_{s}^{\infty}e^{\Lambda_2(s-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,s)d\tau\}ds$$ (5)

K_1 is an n_1 vector of arbitrary constants. We solve for this by using an initial condition for $x_1(t)$ at $t=t_o$, e.g.

(22) $$x_1(t_o)=\bar{x}_1(t_o)$$

The solution for $x_1(t)$ is then found to be

(23) $$x_1(t)=V_{11}e^{\Lambda_1(t-t_o)}\left[V_{11}\right]^{-1}\bar{x}_1(t_o)+\int_{t_o}^{t}V_{11}e^{\Lambda_1(t-s)}\left[V_{11}\right]^{-1}B_1z(s)ds$$
$$-\int_{t_o}^{t}V_{11}e^{\Lambda_1(t-s)}\left[V_{11}\right]^{-1}A_{12}\left[(V^{-1})_{22}\right]^{-1}\int_{s}^{\infty}e^{\Lambda_2(s-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,s)d\tau ds$$

or, using (19b)

(23') $$x_1(t)=V_{11}e^{\Lambda_1(t-t_o)}\left[V_{11}\right]^{-1}\bar{x}_1(t_o)+\int_{t_o}^{t}V_{11}e^{\Lambda_1(t-s)}\left[V_{11}\right]^{-1}B_1z(s)ds$$
$$-\int_{t_o}^{t}V_{11}e^{\Lambda_1(t-s)}\{\Lambda_1(V^{-1})_{12}\left[(V^{-1})_{22}\right]^{-1}+\left[V_{11}\right]^{-1}V_{12}\Lambda_2\}\int_{s}^{\infty}e^{\Lambda_2(s-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,s)d\tau ds$$

The similarity between (23) or (23') and Blanchard and Kahn's final form solution for $x_1(t)$ in their equation (4) is again immediately apparent. The value of the predetermined variables in period t depends on the initial condition $\bar{x}_1(t_o)$. The influence of the initial conditions vanishes as $t\to\infty$ since Λ_1 contains only the stable roots of A. The solution depends also on the actual values of the exogenous variables between time t_o and t. Finally it depends on all expectations, formed at any instant s between time t_o and t, of all values of the exogenous variables beyond s.

Dixit's formula

Consider the special case when the anticipated future values of z are all constant, i.e. $\hat{z}(\tau,t)=\bar{z}$, $\tau\geq t$. Equation (17) then simplifies to

$$x_2(t)=-\left[(V^{-1})_{22}\right]^{-1}(V^{-1})_{21}x_1(t)-\left[(V^{-1})_{22}\right]^{-1}\Lambda_2^{-1}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\bar{z}$$

Let $\bar{x}_2$ and $\bar{x}_1$ be the steady state values of x_2, respectively x_1, corresponding to $\bar{z}$. A little manipulation then shows that

$$(24)\quad x_2(t)-\bar{x}_2=-\left[(V^{-1})_{22}\right]^{-1}(V^{-1})_{21}(x_1(t)-\bar{x}_1)$$

or, using (17')

$$(24')\quad x_2(t)-\bar{x}_2=V_{21}\left[V_{11}\right]^{-1}(x_1(t)-\bar{x}_1).$$

These are the formulae obtained by Dixit (1980) for calculating the effect on the non-predetermined variables of previously unanticipated, immediate, permanent changes in the exogenous variables.

An Example

An example of the kind of model that fits the formal structure of this Section is the following generalization of a model by Dornbusch (1976). (See Buiter and Miller (1981a, 1981b) and Wilson (1979).

(25a) $m - p = ky - \lambda r$ $\qquad k, \lambda > o$

(25b) $y = -\gamma\left(r - \frac{\partial \hat{p}(s,t)}{\partial s}\Big|_{s=t}\right) + \delta(e-p)$ $\qquad \gamma, \delta > o$

(25c) $p = \alpha\omega + (1-\alpha)e$ $\qquad o \leq \alpha \leq 1$

(25d) $\frac{d\omega}{dt} = \phi y + \pi$ $\qquad \phi > o$

(25e) $\frac{\partial \hat{e}(s,t)}{\partial s}\Big|_{s=t} = r - r^*$

(25f) $\pi = \frac{dm^+}{dt}$

(25g) $\ell \equiv m - \omega$

(25h) $c \equiv e - \omega$

m is the nominal money stock, p the domestic price level, y real output, r the domestic nominal interest rate, e the exchange rate (domestic currency price of foreign currency) ω the money wage, π the underlying or "core" rate of inflation, r* the world interest rate. All variables except, r, r* and π are in logs. Equation (25a) is the LM curve, equation (25b) the IS curve. The price of domestic output is a mark-up on unit labour costs and unit import costs

(equation (25c)). The foreign currency price of imports is assumed constant. Through choice of units its logarithm equals zero. The augmented wage Phillips curve is given by equation (25d). The international interest differential is assumed to equal the expected rate of exchange depreciation (equation 25e). The underlying or core rate of inflation equals the right-hand side time derivative of the money stock:

$\frac{dm^+}{dt}(t) = \lim_{\substack{\tau \to t \\ \tau > t}} \frac{m(\tau - t)}{\tau - t}$. The money wage rate is treated as predetermined and is a continuous function of time, unlike the exchange rate. A convenient choice of state variables is $\ell \equiv m - \omega$ which is a measure of real liquidity and $c \equiv e - \omega$ which is a measure of competitiveness. c is a forward-looking jump-variable because of e. ℓ is predetermined. Except at those instants that m makes a discrete jump, it is a continuous function of time. We assume $\frac{d\,m(t)}{dt}$ to be constant in what follows so that $\frac{dm}{dt} = \frac{dm^+}{dt} = \mu$.

The state-space representation of the model is given in (26).

$$(26) \quad \begin{bmatrix} \frac{d\,\ell(t)}{dt} \\ \left.\frac{\partial\,\hat{c}(s,t)}{\partial s}\right|_{s=t} \end{bmatrix} = \frac{1}{\alpha\gamma(\lambda\phi - k) - \lambda} \begin{bmatrix} \phi\alpha\gamma & \phi\alpha(\lambda\delta - \gamma(1-\alpha)) \\ 1 & \alpha\delta(\phi\lambda - k) + \alpha - 1 \end{bmatrix} \begin{bmatrix} \ell(t) \\ c(t) \end{bmatrix} + \frac{1}{\alpha\gamma(\lambda\phi - k) - \lambda} \begin{bmatrix} \alpha\gamma\lambda\phi & -\phi\lambda\gamma(1-\alpha) \\ \lambda & \lambda + \gamma(k - \phi\lambda) \end{bmatrix} \begin{bmatrix} \mu \\ r(t) \end{bmatrix}$$

A necessary and sufficient condition for a stationary equilibrium of (26) (corresponding to constant values of the exogenous variables) to be a saddle-point (i.e. for the state matrix to have one stable and one unstable characteristic root) is $\alpha\gamma(\lambda\phi-k) - \lambda < o$. The interpreation of this condition is that, at a given level of competitiveness, an exogenous increase in aggregate demand raises output. The "saddlepath" for this model is upward-sloping in c-ℓ space.

We can apply the methods of this section to the model of equation (26). Note that $x_1 = \ell$, $x_2 = c$ and $z = \begin{bmatrix} \mu \\ r^* \end{bmatrix}$. The A and B matrices are given in (26). An initial condition is given for $\ell(t)$ at $t = t_o$. A graphical illustration of the effect of an unanticipated increase in the world interrest rate r* is given in Figure 1. The economy is assumed to be in steady-state equilibrium at E_1 for $t < t_o$. The new steady-state equilibrium corresponding to the higher value of r*, which has a higher value of c and a lower value of ℓ is at E_2. At $t = t_o$ a previously unanticipated increase in r* becomes part of the private agents' information sets. If the increase in r* occurs immediately (at $t=t_o$) the level of competitiveness jumps immediately to E_{12}. With ℓ predetermined this jump places it on the unique convergent trajectory S'S' through E_2. After the initial "jump depreciation", the real exchange rate gradually appreciates along S'S' to E_2. An anticipated future increase in r* at $t_1 > t_o$ causes an immediate jump depreciation to E'_{12}. This jump has to satisfy the condition that it places the system on that unstable trajectory (UU in Figure 1), drawn with reference to E_1, which will take it to the unique convergent trajectory S'S' trhough E_2 at $t = t_1$, that is at the moment that the foreign interest rate assumes its new higher value.

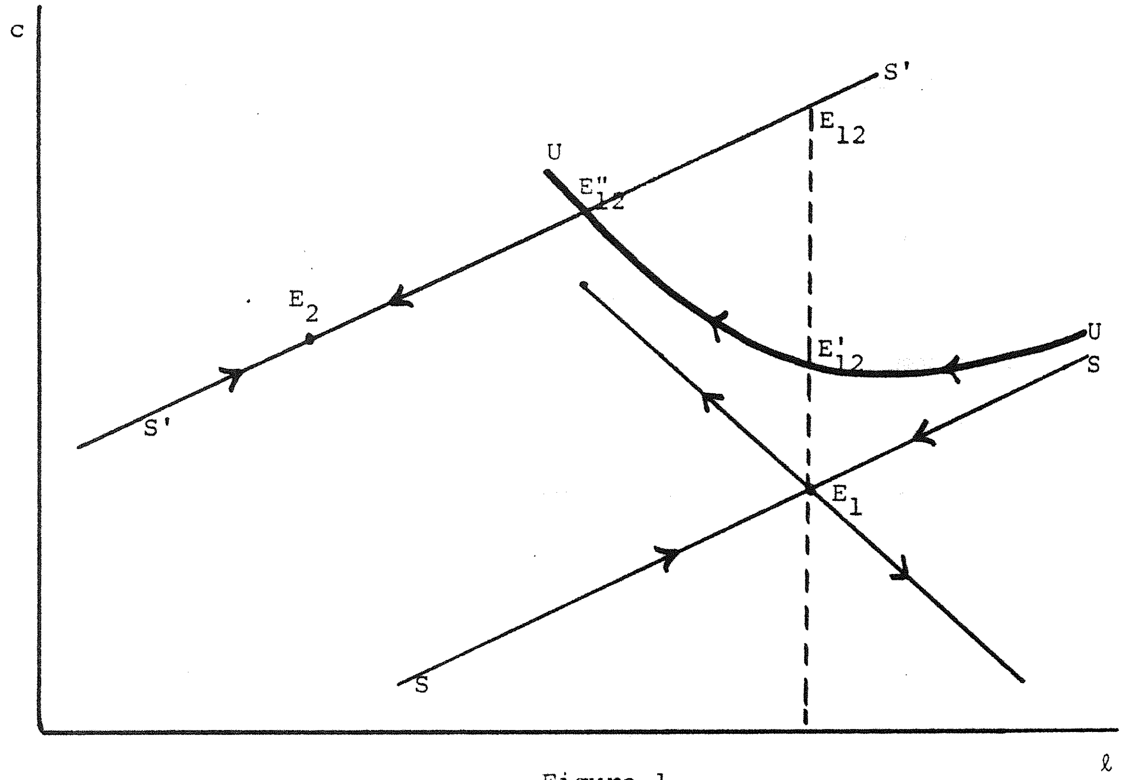

Figure 1

(3) The case of "too many" stable roots

(3a) "Backward-looking" but non-predetermined state variables

Consider the case where the matrix A has n_1 stable roots and $n-n_1$ unstable roots, but where there are only $n_1'<n_1$ predetermined variables. We first analyse the case where it is possible to identify, on economic grounds, n_1 state variables x_1 for which we choose a backward-looking solution as in (21). Of these n_1 backward-looking variables, n_1' are predetermined and will be denoted x_1'. The remaining n_1-n_1' are non-predetermined and are denoted x_1''. Thus

(27) $$x_1=\begin{bmatrix} x_1' \\ x_1'' \end{bmatrix}$$

Assume that at $t=t_o$ the following set of linear restrictions applies:

(28) $$F_1x_1''(t_o)+F_2x_1'(t_o)+F_3x_2(t_o)=f$$

F_1 is an $(n_1-n_1')x(n_1-n_1')$ matrix, F_2 an $(n_1-n_1')xn_1'$ matrix, F_3 an $(n-n_1')x(n-n_1)$ matrix and f an n_1-n_1' vector.

Provided F_1 is invertible (i.e. provided (28) represents n_1-n_1' independent boundary conditions) a unqiue convergent solution exists to the system (29a,b) with boundary conditions (30a,b)

(29a) $$\begin{bmatrix} \frac{d}{dt} & x_1'(t) \\ \frac{d}{dt} & x_1''(t) \end{bmatrix} = A_{11}\begin{bmatrix} x_1't) \\ x_1''(t) \end{bmatrix} + A_{12}x_2(t)+B_1z(t)$$

(29b) $$\frac{\partial}{\partial s}\hat{x}_2(s,t)\Big|_{s=t} = A_{21}\begin{bmatrix}x_1'(t)\\ x_1''(t)\end{bmatrix} + A_{22}x_2(t)+B_2z(t)$$

(30a) $$x_1'(t_o)=\bar{x}_1'(t_o)$$

(30b) $$x_1''(t_o)=-F_1^{-1}F_2x_1'(t_o)-F_1^{-1}F_3x_2(t_o)+F_1^{-1}f.$$

The solution is given by equations (30a,b) and

(17) $$x_2(t)=-\left[(V^{-1})_{22}\right]^{-1}(V^{-1})_{21}\begin{bmatrix}x_1'(t)\\ x_1''(t)\end{bmatrix}-\left[(V^{-1})_{22}\right]^{-1}\int_t^{\infty} e^{\Lambda_2(t-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,t)d\tau$$

(31) $$\begin{bmatrix}x_1'(t)\\ x_1''(t)\end{bmatrix} = V_{11}e^{\Lambda_1(t-t_o)}\left[V_{11}\right]^{-1}\begin{bmatrix}x_1'(t_o)\\ x_1''(t_o)\end{bmatrix} + \int_{t_o}^{t} V_{11}e^{\Lambda_1(t-s)}\left[V_{11}\right]^{-1}B_1z(s)ds$$
$$-\int_{t_o}^{t} V_{11}e^{\Lambda_1(t-s)}\left[\Lambda_1(V^{-1})_{12}\left[(V^{-1})_{22}\right]^{-1}+\left[V_{11}\right]^{-1}V_{12}\Lambda_2\right]\int_s^{\infty} e^{\Lambda_2(s-\tau)}\left[(V^{-1})_{21}B_1+(V^{-1})_{22}B_2\right]\hat{z}(\tau,s)d\tau ds$$

An example of a model that fits this format is found Buiter and Miller (1981b). It is obtained by making a fairly minor alteration to the model of equations (25a-h). The equation for the core rate of inflation (25f) is replaced by

(25f') $$\pi(t) = \eta\int_{-\infty}^{t} e^{-\eta(t-s)}\frac{dp(s)}{dt}ds \qquad \eta > o$$

This defines π as a backward-looking weighted average of current and past inflation rates with exponentially declining weights. We continue to treat ω and m (and therefore ℓ) as predetermined and continuous functions of time. Differentiating (25f') yields the familiar adaptive process

(25f") $$\frac{d\pi}{dt} = \eta\left(\frac{d}{dt}p - \pi\right)$$

From (25f') one can see that while π is backward-looking, it will not be a continuous function of time if p can make discontinuous jumps. From equation (25c) one can see that p will jump discontinously whenever e jumps discontinuously, if $\alpha < 1$. π can indeed be described as a "dependent" jump variable as it will jump if and only if p jumps. From (25f') or (25f") we derive:

(32) $$\pi(t) = \pi(t^-) + \eta(p(t) - p(t^-))$$

where $\pi(t^-) = \lim_{\substack{T \to t \\ T < t}} \pi(T)$ and similarly for $p(t^-)$.

The state-space representation of the model of equations (25 a,b,c,d,e,f", g and h) is:

(33) $$\begin{bmatrix} \frac{d\,\ell(t)}{dt} \\ \frac{d\,\pi(t)}{dt} \\ \left.\frac{\partial \hat{c}(s,t)}{\partial s}\right|_{s=t} \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \psi\alpha\gamma & \lambda+\alpha\gamma k & \psi\alpha(\lambda\delta-\gamma(1-\alpha)) \\ \eta(1-\alpha(1+\gamma\phi)) & \eta\lambda(1+\alpha(1+\gamma\phi)) & \eta[\alpha\phi(\gamma(1-\alpha)-\alpha\delta\lambda)-(1-\alpha)(1-\alpha(1-\delta k))] \\ 1 & \lambda & \alpha\delta(\phi\lambda-k)-(1-\alpha) \end{bmatrix} \begin{bmatrix} \ell(t) \\ \pi(t) \\ c(t) \end{bmatrix}$$

$$\Delta^{-1} \begin{bmatrix} \Delta & -\phi\lambda\gamma(1-\alpha) \\ 0 & \eta(1-\alpha)(\lambda+\gamma k) \\ 0 & \lambda+\gamma(k-\phi\lambda) \end{bmatrix} \begin{bmatrix} \mu \\ r^*(t) \end{bmatrix}$$

where $\Delta = \alpha\gamma(\phi\lambda-k) - \lambda < o$

For plausible values of the parameters of the model, the state matrix A of (33) will have two stable (complex conjugate) roots and one unstable root (see Buiter and Miller (1981b)). Yet there is only one predetermined variable, ℓ. We do, however, have three linearly independent boundary conditions which guarantee a unique solution for the model. First note that (32) can be written as

$$\pi(t) = \pi(t^-) + \eta(1-\alpha)(c(t)-c(t^-)) + \eta(\omega(t)-\omega(t^-))$$

Since $\omega(t)$ is a continuous function of time the last term vanishes and

(34) $$\pi(t) = \pi(t^-) + \eta(1-\alpha)(c(t)-c(t^-))$$

Using the notation of equations (27-31), $x_1' = \ell$, $x_1'' = \pi$ and $x_2 = c$.

Thus, starting the system off at $t=t_o$ one proceeds as follows. $\ell(t_o)$ is given by past history at $\bar{\ell}(t_o)$, say. Unless there is news at t_o (i.e. unless $I(t_o) \neq I(t_o^-)$), $\pi(t_o)$ will be equal to the historically given value $\pi(t_o^-)$. If there is "news" at t_o, $\pi(t_o)$ is determined using equations (30a), (30b) and (17) evaluated at $t=t_o$. Equation (34) is of the format of (28) or (30b). $c(t_o^-)$ is found by using (30a,b) evalued at $t=t_o^-$ and (17) evaluated at $t=t_o^-$. From t_o onwards, we treat $\pi(t)$ as predetermined until further "news" arrives, in which case (34) again becomes relevant. In the model of equation (33), an unanticipated permanent reduction in μ leads to an immediate "jump" appreciation of the real exchange rate, c, and a jump reduction in core inflation, π.

(3b) <u>Forward-looking state variables associated with stable characteristic roots</u>

Another small modification to the model of equations (25a-h) permits us to illustrate the class of models to be characterized and analysed in this subsection. The equation for the core rate of inflation (25f) is replaced by:

(25f''') $$\pi = \frac{\partial \hat{p}(s,t)}{\partial s}\Big|_{s=t}$$

This can be interpreted as perfect foresight or rational expectations in the labour market. We no longer treat the money wage rate as a continuous function of time. Both c and ℓ now are "jump" variables.

The state-space representation of the model of equation (25a-h) with (25f''') is

(35) $$\begin{bmatrix} \frac{d\ell(t)}{dt} \\ \frac{\partial}{\partial s}\hat{c}(s,t)\Big|_{s=t} \end{bmatrix} = \begin{bmatrix} \lambda^{-1} & -\left[\frac{(1-\alpha)(1-\alpha(1+\gamma\phi)+\alpha\delta k)+\alpha\delta\phi\lambda}{\lambda(1-\alpha(1+\gamma\phi))}\right] \\ 0 & \frac{-\phi\alpha\delta}{1-\alpha(1+\gamma\phi)} \end{bmatrix} \begin{bmatrix} \ell(t) \\ c(t) \end{bmatrix} + \begin{bmatrix} 1 & \frac{\lambda(1-\alpha(1+\gamma\phi))+\lambda\phi\gamma+k\gamma(1-\alpha)}{(1-\alpha(1+\gamma\phi))} \\ 0 & \frac{\phi\gamma}{1-\alpha(1+\gamma\phi)} \end{bmatrix} \begin{bmatrix} \mu \\ r^*(t) \end{bmatrix}$$

Note that the model has become recursive. The behaviour of c is completely independent of the behaviour of ℓ <u>except</u> for such interdependence as may be introduced via the boundary conditions. The two characteristic roots of the A matrix in (34) are λ^{-1} and $\frac{-\phi\alpha\delta}{1-\alpha(1+\gamma\phi)}$.

The sign of the latter is the sign of $-(1-\alpha(1+\gamma\phi))$. To interpret this condition we add a demand shock term d on the right-hand side of the IS equation (25b). A little manipulation then yields

$$y = -\frac{\gamma(1-\alpha)}{1-\alpha(1+\gamma\phi)} r^* + \frac{\alpha(1-\alpha)\delta}{1-\alpha(1+\gamma\phi)} c + \frac{(1-\alpha)}{1-\alpha(1+\gamma\phi)} d$$

For $o \leq \alpha < 1$, $1-\alpha(1+\gamma\phi)$ must be positive for an exogenous increase in demand to raise output at a given level of competitiveness. We assume this condition is satisfied. It implies that the characteristic root governing c is negative. Thus even though we have initial conditions for neither e nor ω (or neither c nor ℓ) there is one unstable and one stable root. Figure 2 depicts the response of c and ℓ to an unanticipated permanent increase in r*. The $\frac{d\ell}{dt} = o$ locus could be downward-sloping, but nothing essential hinges on that. With both c and ℓ free to jump in response to "news", the condition that c and ℓ remain bounded for bounded values of the exogenous variables μ and r* no longer suffices to select a unique solution trajectory. Consider an immediate unanticipated increase in r* at $t=t_o$. The new long run equilibrium is E_2. The initial position at t_o^- is assumed to be E_1. Any jump in ℓ and c which places the system anywhere on S'S' at $t=t_o$ satisfies the equations of motion and guarantees convergence to E_2. A plausible further restriction might be that the behaviour of this system with its two "forward-looking" variables should not be dependent on an "irrelevant" past. With the increase in r* occurring at t_o, when it is first anticipated, this would mean that the system jumps immediately to E_2, the new long-run equilibrium. If a previously unanticipated increase in r* is expected to occur at $t_1 > t_o$, any jump in c and that places the system on a divergent solution trajectory, drawn with reference to E_1, which will take it at $t = t_1$ to S'S', the convergent path through E_2, satisfies the equations of motion and converges to E_2. Three such divergent paths, UU, U'U' and U"U" are drawn in Figure 2. E_{12}, E'_{12} and E''_{12} are possible positions of c and ℓ at t_o. By analogy with the argument for the case of the immediate increase in r*, a case can be made for restricting the solution to an initial jump to E_{12} on UU', from where the system will arrive at E_2 when r* is actually increased, i.e. at $t = t_1$. (See Minford (1979).

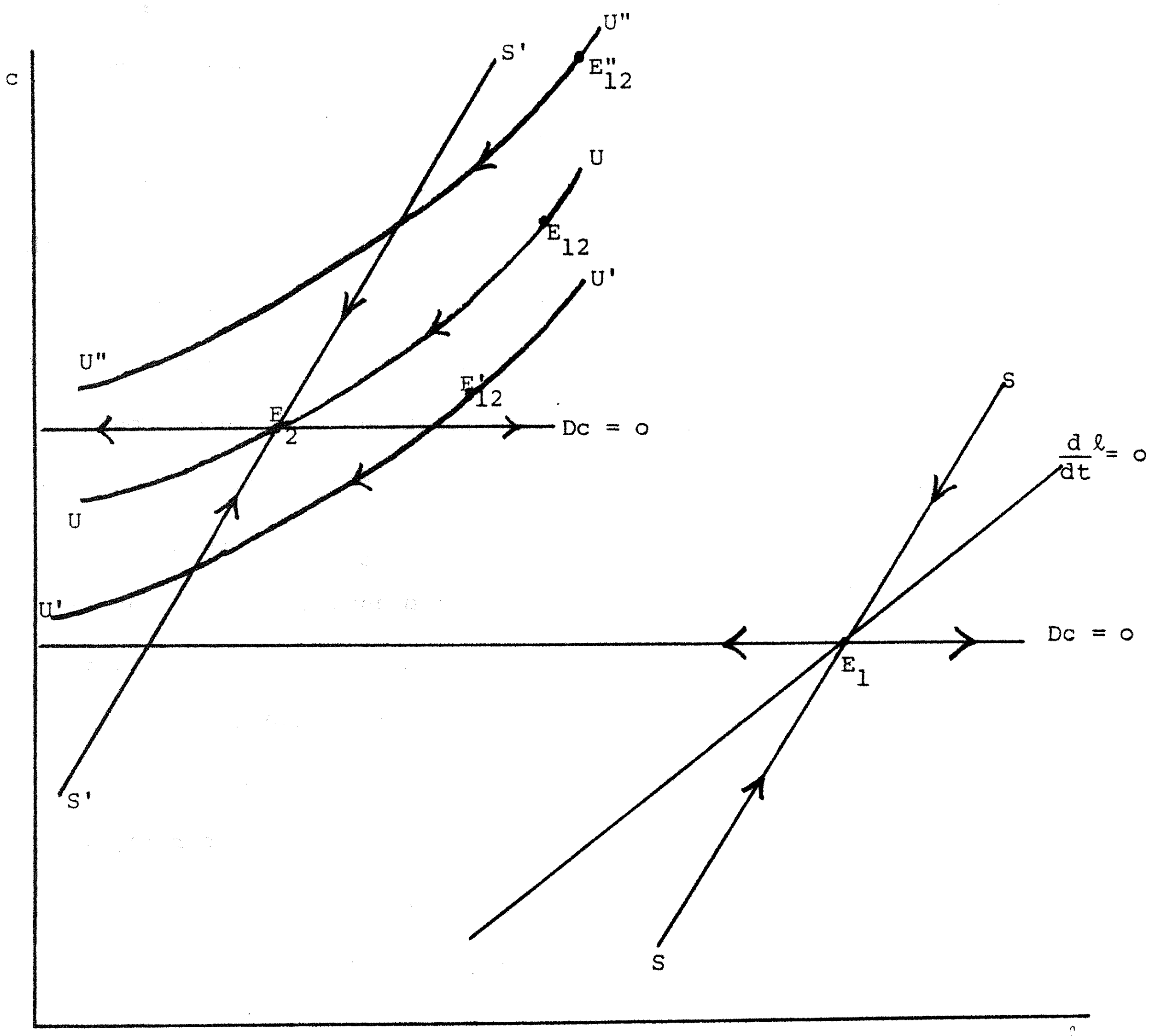

Figure 2

The proposed boundary conditions therefore take the form:

$$(3\ )\quad \begin{bmatrix} \ell(t_1) \\ c(t_1) \end{bmatrix} = - \begin{bmatrix} \lambda^{-1} & -\left[\dfrac{(1-\alpha)(1-\alpha(1+\gamma\phi)+\alpha\delta k)+\alpha\delta\phi\lambda}{\lambda(1-\alpha(1+\gamma\phi)}\right] \\ 0 & -\dfrac{\phi\alpha\delta}{1-\alpha(1+\gamma\phi)} \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} 1 & \dfrac{\lambda(1-\alpha(1+\gamma\phi))+\lambda\phi\gamma+k\gamma(1-\alpha)}{\lambda(1-\alpha(1+\gamma\phi))} \\ 0 & \dfrac{\phi\gamma}{1-\alpha(1+\gamma\phi)} \end{bmatrix} \begin{bmatrix} \mu \\ \bar{r}^* \end{bmatrix}$$

This class of boundary value problem can be solved using the method of adjoints.

The method of adjoints

We consider the model of equations (3a,b) over a time interval $t_o \leq t \leq t_1$ during which the information set does not change, i.e. $I(t)=I$, $t_o \leq t \leq t_1$. Over this interval, therefore, $\frac{\partial}{\partial s}\hat{x}_2(s,t)\Big|_{s=t} = \frac{d}{dt}x_2(t)$ and equations (3a,b) or (7) can be written as:

$$(37)\quad \frac{d}{dt}x(t)=Ax(t)+Bz(t). \qquad t_o \leq t \leq t_1$$

We now consider the two-point boundary value problem of equations (37) and (38)

$$(38)\quad Mx(t_o)+Nx(t_1)=r$$

Equation (38) gives n linear restrictions on the value of the state vector at two distinct dates.

Let $M \equiv \{\mu_{ji}\}$, $N = \{\nu_{ji}\}$ and $r = (\rho_1, \ldots, \rho_j, \ldots \rho_n)^T$ (ō)

$i, j = 1, 2, \ldots, n.$

We can therefore rewrite (38) as (38')

$$(38') \quad \sum_{i=1}^{n} \mu_{ji} x_i(t_o) + \sum_{i=1}^{n} \nu_{ji} x_i(t_1) = \rho_j \qquad j = 1, 2, \ldots, n .$$

x_i now denotes the i^{th} elements of x , $i = 1, 2, \ldots, n$.

Consider the adjoint system to (37).

$$(39) \quad \frac{d}{dt} s(t) = - A^T s(t)$$

We integrate the adjoint equations backward from $t = t_1$, once for each $x_i(t_1)$ in (38'), using as the terminal boundary conditions

$$(40) \quad s_i^{(j)}(t_1) = \nu_{ji} \qquad i, j = 1, 2, \ldots, n .$$

$s_i^{(j)}(t_1)$ is the i^{th} component at $t = t_1$ for the j^{th} backward integration of the adjoint equation. Thus, if ν_j^T denotes the transpose of the j^{th} row of N in equation (38), we have the solution

$$(41) \quad s^j(t) = e^{-(t-t_1)A^T} \nu_j^T \qquad j = 1, 2, \ldots, n .$$

Setting $t = t_o$ in (41) we obtain $s^{(j)}(t_o)$.

The fundamental identity for the method of adjoints is (see Roberts and Shipman [1972, pp. 17-22]):

$$(42) \quad \sum_{i=1}^{n} s_i^{(j)}(t_1)\, x_i(t_1) - \sum_{i=1}^{n} s_i^{(j)} x_i(t_o) = \int_{t_o}^{t_1} \sum_{i=1}^{n} s_i^{(j)}(t)\, b_i z(t)\,dt \qquad j = 1, 2, \ldots, n .$$

b_i is the i^{th} row of the matrix B .

Substituting for $s^{(j)}(t_1)$ from (40) into (42) and using (38') yields

$$\rho_j - \sum_{i=1}^{n} \mu_{ji}\, x_i(t_o) - \sum_{i=1}^{n} s_i^{(j)}(t_o)\, x_i(t_o) = \int_{t_o}^{t_1} \sum_{i=1}^{n} s_i^{(j)}(t)\, b_i z(t)dt$$

$$j = 1, 2, \ldots, n .$$

or

$$(43) \quad \sum_{i=1}^{n} [\mu_{ji} + s_i^{(j)}(t_o)]\, x_i(t_o) = \rho_j - \int_{t_o}^{t_1} \sum_{i=1}^{n} s_i^{(j)}(t)\, b_i z(t)dt$$

$$j = 1, 2, \ldots, n$$

Equation (43) constitutes a set of n equations in the n unknowns $x_i(t_o)$, i = 1, 2..., n. If they are linearly independent they will yield a unique solution for $x(t_o)$. Given the value of the entire state vector at $t = t_o$, equation (37) can be solved as a standard initial value problem. Its solution would be

$$(43) \quad x(t)=e^{A(t-t_o)}x(t_o)+\int_{t_o}^{t} e^{A(t-s)}Bz(s)ds, \quad t_o \leq t \leq t_1$$

However, in practical (i.e. numerical) applications, the true value of x at $t=t_o$ can only be approximated. Since A will in general possess unstable characteristic roots, any error in the calculation of $x(t_o)$ will be compounded as time passes. If there are unstable roots, it is therefore computationally superior, having calculated $x(t_o)$ using the method of adjoints, to use the solution method of equations (17) or (17') and (23) or (23'). Note that $\hat{z}(\tau,s)=z(\tau,t_o)=z(s)$ for $t_o \leq s \leq t_1$ when we apply this method. If the information set changes at $t=t_1$, we resolve the two-point boundary value problem. Equation (36) can be seen to be the special case of equation (38) with M=O.

(4) CONCLUSION

The paper presents a general solution method for rational expectations models that can be represented by systems of deterministic first order linear differential equations with constant coefficients. It is the continuous time adaptation of the method of Blanchard and Kahn. To obtain a unique solution there must be as many linearly independent boundary conditions as there are linearly independent state variables. Three slightly different versions of a well-known small open economy macroeconomic model were used to illustrate three fairly general ways of specifying the required boundary conditions. The first represents the standard case in which the number of stable characteristic roots equals the number of predetermined variables. The second represents the case where the number of stable roots exceeds the number of predetermined variables but equals the number of predetermined variables plus the number of "backward-looking" but non-predetermined variables whose discontinuities are linear functions of the discontinuities in the forward-looking variables. The third represents the case where the number of unstable roots is less than the number of forward-looking state variables. For the last case, boundary conditions are suggested that involve linear restrictions on the values of the state variables at a future date.

The method of this paper permits the numerical solution of models with large numbers of state variables. Any combination of anticipated or unanticipated, current or future and permanent or transitory shocks can be analysed.

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FOOTNOTES

(1) See Brock (1975) for a model in which these transversality conditions are derived from explicit optimizing behaviour by an infinite-lived consumer. The non-predetermined variables there have the interpretation of co-state variables in a dynamic optimization problem.

(2) The non-predetermined variables frequently are asset prices determined in efficient asset markets. Implicit arbitrage conditions rule out anticipated future jumps in these asset prices. Thus, except at those instants at which new information arrives, the non-predetermined variables are continuous functions of time. See Calvo (1977).

(3) We do not however, for reasons of space, consider solutions in which "extraneous" information plays a role.

(4) The exponential matrix e^C where C is an nxm matrix is defined by

$e^C \equiv \sum_{k=o}^{\infty} \frac{C^k}{k!}$. When C is a diagonal matrix

$$C=\begin{bmatrix} c_i & & & \\ & \ddots & & O \\ & & c_i & \\ & O & & \ddots \\ & & & & c_n \end{bmatrix} \text{ then } e^C \equiv \begin{bmatrix} e^{c_1} & & & \\ & \ddots & & O \\ & & e^{c_i} & \\ & O & & \ddots \\ & & & & e^{c_n} \end{bmatrix}.$$

(5) Using $e^{V_{11}\Lambda_1\left[V_{11}\right]^{-1}t} = V_{11}e^{\Lambda_1 t}\left[V_{11}\right]^{-1}$

(6) For any matrix Ω, Ω^T denotes the (complex conjugate) transpose of Ω.