ON COURNOT-NASH EQUILIBRIA WITH
EXOGENOUS UNCERTAINTY*

Jan Aaftink [(1)], Norman Ireland [(2)], and Murat Sertel [(

NUMBER 210

# WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

UNIVERSITY OF WARWICK
COVENTRY

# ON COURNOT-NASH EQUILIBRIA WITH EXOGENOUS UNCERTAINTY*

Jan Aaftink [(1)], Norman Ireland [(2)], and Murat Sertel [(3)]

NUMBER 210

July 1982

(1) Department of Applied Mathematics, Twente, University of Technology, Netherlands
(2) Department of Economics, University of Warwick
(3) Department of Economics, Boğazici University, Istanbul

* Ireland and Sertel are grateful to the British Council (Academic Links and Interchange Scheme) for financial support.

# ON COURNOT-NASH EQUILIBRIA WITH EXOGENOUS UNCERTAINTY


## Abstract

A large literature has accumulated which examines how the optimal solution of an agent maximising the expectation of a real-valued function depending on a random parameter $p$ and the agent's behaviour $x$ reacts to perturbations in the first and second **moments** of $p$. In this literature $p$ is given exogenously, i.e. independent of $x$. We extend the theory in two aspects. First we allow there to be many agents and broaden our attention to regard market behaviour. Second, we allow $p$ to depend on the behaviours of the participating agents, for example when $p$ is a price vector relevant to an oligopoly. The method used is an extension of Ireland (2), in that an analogy is made between the effects on behaviour of uncertainty in $p$ and the effects of a change in $p$. We study, in particular, the Cournot solution with respect to perturbations in the first two moments of two types of parameter defining a linear demand for an industry - one parameter corresponding to ordinate intercept and the other to slope. This analysis immediately gives the old results as a corollary. We also apply the analysis to a cooperative of individuals where there is uncertainty in the return to communal work. In the applications we study the kind of simplifying assumptions necessary and the nature of the results.

## 1. Introduction

The effects of uncertainty on the optimal decisions of economic agents constitute a topic of considerable importance but commensurate difficulty. The approach of Rothschild and Stiglitz ((8), (9)) finds sufficient conditions for determining particular qualitative effects of uncertainty or increased uncertainty. There, sufficient conditions often involve third derivatives of agents' utility functions, and complexities in interpretation often prevent satisfactory results being obtained. Even if reasonable convincing conclusions can be reached, as in the theory of the competitive firm (see also Sandmo (10) and Ishii (6)), extensions to cases where the dimension of agents' decisions is more than one, or where the outcome to be determined is the result of a number of agents' decisions under uncertainty, are not generally productive.

One alternative approach (Ireland (2)) which has a restricted validity, but otherwise appears to offer promise of more fruitful results, is to consider the effects on an agent's behaviour of uncertainty in a parameter in relation to the effects on that agent's behaviour of a change in the same parameter under certainty. Such an approach uses an approximation which holds better the smaller the amount of uncertainty, and so resulting conditions are in essence necessary conditions for determining qualitative effects of uncertainty. Although not sufficient conditions (for more than small uncertainty) and therefore not implying perfectly general results, they do have two interesting and useful properties. Firstly, they indicate likely effects of uncertainty - and they definitely hold for sufficiently small uncertainty, and secondly they allow us to discard the complementary set of outcomes as <u>general</u> quali tative predictions. Thus if the prediction from our approximation is that less of commodity $i$ will be produced under uncertainty than under certainty, we conclude firstly that this is a likely general result and secondly that more to be produced under uncertainty does not always occur.

The restrictions involved in applying the approximation approach are that the utility of each economic agent is a function (or can be expressed as a function) of a single argument which is linear in the uncertain parameter(s) and strictly concave in the variable(s) over which the individual maximises the expected value of his or her utility.

In Section 2 below we set out a formal model of $m$ economic agents taking part in a game. We consider a Cournot-Nash equilibrium of agents' behaviour in the presence of uncertainty concerning one or more parameters in Section 3, and then present two illustrative applications of the analysis in Section 4. Conclusions and some remaining comments are in a final section. The more tedious mathematics are relegated to an Appendix.

## 2. A Formal Statement of the Model

We consider a game $\Gamma$ with a set of $M = \{1, . . ., m\}$ of players $i \varepsilon M$ , each equipped with a behaviour space $X_i$ . We define

$$X^i = \prod_{j \varepsilon M/\{i\}} X_j ,$$

$$X = \prod_{i \varepsilon M} X_i ,$$

denoting generic elements of $X_i$, $X^i$ and $X$ by $x_i$ , $x^i$ and $x = (x_i, x^i)$ , respectively $(i\varepsilon M)$ . Each player $i\varepsilon M$ is assumed to have a utility function $U_i : X \to R$ with a special form. In fact, for each $i\varepsilon M$ , we posit functions $f_i : X \to R$ and $g_i : X \to R^n$ , with $g_i(x) = (g_{i1}(x), \ldots, g_{in}(x))^T$ , and we set

$$U_i(x) = u_i(Y_i(x)) , \tag{2.1}$$

where

$$Y_i(x) = f_i(x) + [p(x)]^T \cdot g_i(x) . \tag{2.2}$$

The function $u_i : R \to R$ is assumed to be strictly increasing in $Y_i$.

Here $p(x) = (p_1(x), \ldots, p_n(x))^T \cdot \varepsilon R^n$ at each $x\varepsilon X$ , and $p(x)$ is a random vector which has the particular form:

$$p(x) = \bar{p}(x) + \alpha\{r + S\psi(x)\} \tag{2.3}$$

where

$$\bar{p}(x) = a + B\psi(x) \tag{2.4}$$

with $a = (a_1, \ldots, a_n)^T \varepsilon R^n$ , $B$ an $n \times n$ diagonal matrix whose diagonal is $b = (b_1, \ldots, b_n)^T \varepsilon R^n$ , $\psi = X \to R^n$ is a function $(\psi(x) = (\psi_1(x) \ldots, \psi_n(x))^T)$ , $\alpha \varepsilon R$ , $r = (r_1, \ldots, r_n)^T$ a random vector, and $S$ an $n \times n$ diagonal matrix with diagonal $s = (s_1, \ldots, s_n)^T$ , of random variables. The means of $r$ and $s$

are given by $E\{r\} = E\{s\} = 0$ , and we denote the covariance matrices $E\{r \cdot r^T\}$ by $V$ and $E\{s\,s^T\}$ by $W$ respectively. Furthermore, $r$ and $s$ are assumed to be independent random vectors, i.e. $E\{r_i\, s_j\} = 0$ , $i, j = 1, \ldots, n$ .

We assume for each $i \varepsilon M$ that $x_i$ is an $n_i$-dimensional Euclidean space $(n_i \geqslant n)$ , and that the functions $u_i$, $f_i$, $g_i$ are twice continuously differentiable, as is the function $\psi$ .

Each agent $i \varepsilon M$ is understood to maximise the expected utility $E\{u_i\}$ over $x_i \varepsilon X_i$ .

With these basic data, we will study solutions $\underline{x}(a, b, \alpha)$ of the game $\Gamma$ according to a Cournot-Nash (non-cooperative) solution. The solution $\underline{x} = \underline{x}(a, b, \alpha)$ is assumed to be locally isolated for all possible (relevant) values of $(a, b, \alpha)$ . In each case, our interest is in comparing the effect of $(a, b)$ on $\underline{x}$ with that of $\alpha$ . In the style of Ireland, (2), where nevertheless $p$ is independent of $x$ and $M = \{1\}$ , we are able to study (locally) the relations between

$$\tilde{\underline{x}}(\alpha) - \hat{\underline{x}}$$

on the one hand, and

$$\left.\frac{\partial \overset{\circ}{\underline{x}}(a,\tilde{b})}{\partial a}\right|_{a = \tilde{a}} \qquad \text{and}$$

$$\left.\frac{\partial \overset{\circ}{\underline{x}}(\tilde{a},b)}{\partial b}\right|_{b = \tilde{b}}$$

on the other, where $\overset{\circ}{\underline{x}}(a, b) = \underline{x}(a, b, 0)$ denotes the associated solution in the certainty case where $\alpha = 0$, $\underline{\tilde{x}}(\alpha) = \underline{x}(\tilde{a}, \tilde{b}, \alpha)$ denotes the associated solution in the case where $(a, b)$ is held fixed to $(\tilde{a}, \tilde{b})$ and $\underline{\hat{x}} = \underline{x}(\tilde{a}, \tilde{b}, 0)$ .

In the case of a single agent $(m = 1)$, the vector $p(x)$ can be interpreted as a vector of prices dependent on the agent's decisions $x$ , which may be employment levels of factors of production, amounts of commodities to be produced, marketing decisions, etc. The agent may have market power, i.e., prices dependent on the agent's behaviour, in the way defined above in (2.3) and (2.4). Then, $\underline{\tilde{x}}(\alpha) - \underline{\hat{x}}$ is the effect of uncertainty of magnitude $\alpha$ on optimal decisions and $\left.\frac{\partial \overset{\circ}{\underline{x}}(a, \tilde{b})}{\partial a}\right|_{a = \tilde{a}}$ and $\left.\frac{\partial \overset{\circ}{\underline{x}}(\tilde{a}, b)}{\partial b}\right|_{b = \tilde{b}}$ is the change in optimal decisions given $\alpha = 0$ (certainty) of changes in $a$ or $b$ evaluated at $(\tilde{a}, \tilde{b})$ . The effect of uncertainty on decisions is thus to be related to the effects of changes in $\bar{p}(x)$ due to changes in $a$ or $b$ . Thus, for instance, the effects of uncertainty concerning the intercept and slope of a linear price function $p(x)$ would be related to the effect of parametric changes in such a price function under certainty.

In the case of a number of agents $(m > 1)$, agent $i$ has control over the variables $x_i = (x_{i1}, \ldots, x_{in}, x_{i,n+1}, \ldots, x_{in_i})$ . The variables $(x_{i,n+1}, \ldots, x_{in_i})$ are supposed to have no influence on the other agents, i.e., we assume from the beginning $\frac{\partial f_j}{\partial x_{ik}} \equiv 0$ and $\frac{\partial g_j}{\partial x_{ik}} \equiv 0$ , $k = n + 1, \ldots, n_i$, $j \neq i$ . An equilibrium is found as the Cournot-Nash non-cooperative solution to the m-person game, where each agent $i$ is maximising his or her expected utility given $\underline{x}^i$ , that is the decisions of the other $(m - 1)$ agents. In the Cournot-Nash equilibrium agents are not affected by uncertainty in each other's behaviour: this is determined; rather, each agent is influenced by the uncertainty in the $p(x)$ function due to the random vectors $r$ and $s$.

## 3. The Approximation

The solution $\underline{x} = \underline{x}(a, b, \alpha)$ that we examine here is obtained by

$$\max_{x_i \varepsilon X_i} E\left\{u_i(Y_i(x_i, \underline{x}^i))\right\}, \quad i = 1, \ldots, m. \tag{3.1}$$

First order conditions give:

$$\frac{\partial}{\partial x_i} E\left\{u_i(Y_i(x_i, \underline{x}^i))\right\}\Big|_{x_i = \underline{x}_i} = 0, \quad i = 1, \ldots, m \tag{3.2}$$

As stated in Section 2, we wish to compare

$$\tilde{\underline{x}}(\alpha) - \hat{\underline{x}}$$

with

$$\left.\frac{\partial \overset{\circ}{\underline{x}}(a, b)}{\partial a}\right|_{a = \tilde{a}} \quad \text{and} \quad \left.\frac{\partial \overset{\circ}{\underline{x}}(\tilde{a}, b)}{\partial b}\right|_{b = \tilde{b}},$$

where $\tilde{\underline{x}}(\alpha) = \underline{x}(\tilde{a}, \tilde{b}, \alpha)$, $\overset{\circ}{\underline{x}} = \underline{x}(a, b, 0)$,

and $\hat{\underline{x}} = \tilde{\underline{x}}(0) = \overset{\circ}{\underline{x}}(\tilde{a}, \tilde{b}) = \underline{x}(\tilde{a}, \tilde{b}, 0)$.

By examining (3.2) when $(a, b) = (\tilde{a}, \tilde{b})$ we can obtain an implicit form

$$F(\tilde{\underline{x}}, \alpha) = -\alpha G(\tilde{\underline{x}}, \alpha) ,$$

where $F, G : R^{\sum_{i\varepsilon M} n_i} \times R \rightarrow R^{\sum_{i\varepsilon M} n_i}$.

Linear approximation of F and G around $(\tilde{\underline{x}}, \alpha) = (\hat{\underline{x}}, 0)$ gives (see Appendix) :

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot (\tilde{\underline{x}} - \hat{\underline{x}}) = \alpha^2 \cdot \rho_i(h_i(\hat{\underline{x}})) \cdot \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot V \cdot g_i(\hat{\underline{x}})$$

$$+ \alpha^2 \cdot \rho_i(h_i(\hat{\underline{x}})) \cdot \left\{\left[\frac{\partial \psi}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \mathrm{diag}(g_i(\hat{\underline{x}}))\right.$$

$$\left. + \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \mathrm{diag}(\psi(\hat{\underline{x}}))\right\} \cdot W \cdot g_i^{\psi}(\hat{\underline{x}}) , \quad i = 1, \ldots, m . \qquad (3.3)$$

where: $\rho_i(Y_i) = -\frac{u_i''(Y_i)}{u_i'(Y_i)}$, the coefficient of absolute risk aversion for the $i^{th}$ agent,

$$h_i(x) = f_i(x) + [\bar{p}(x)]^T \cdot g_i(x) ,$$

$$g_i^{\psi}(x) = \begin{bmatrix} g_{i1}(x) \cdot \psi_1(x) \\ \vdots \\ g_{in}(x) \cdot \psi_n(x) \end{bmatrix} ,$$

$$\mathrm{diag}(g_i(x)) = \begin{bmatrix} g_{i1}(x) & & 0 \\ & \ddots & \\ 0 & & g_{in}(x) \end{bmatrix} ,$$

$$\operatorname{diag}(\psi(x)) = \begin{bmatrix} \psi_1(x) & & \\ & \ddots & \\ & & \psi_n(x) \end{bmatrix} .$$

To find

$$\left.\frac{\partial \underset{\sim}{\overset{o}{x}}(a, \tilde{b})}{\partial a}\right|_{a = \tilde{a}} \quad \text{and} \quad \left.\frac{\partial \underset{\sim}{\overset{o}{x}}(\tilde{a}, b)}{\partial b}\right|_{b = \tilde{b}}$$

we set $\alpha = 0$ in (3.2) and implicitly differentiate with respect to $a$ and $b$ to obtain (3.4a) and (3.4b) :

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \frac{\partial \underset{\sim}{\overset{o}{x}}}{\partial a}(\tilde{a}, \tilde{b}) + \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T = 0 , \tag{3.4a}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \frac{\partial \underset{\sim}{\overset{o}{x}}}{\partial b}(\tilde{a}, \tilde{b}) + \left[\frac{\partial \psi}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \operatorname{diag}(g_i(\hat{\underline{x}}))$$
$$+ \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \operatorname{diag}(\psi(\hat{\underline{x}})) = 0 , \; i = 1, \ldots, m . \tag{3.4b}$$

Using (3.4a) and (3.4b) in (3.3) now gives the summary comparison we seek:

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot (\tilde{\underline{x}} - \hat{\underline{x}}) = -\alpha^2 \cdot \hat{\rho}_i \cdot \frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \left\{\frac{\partial \underset{\sim}{\overset{o}{x}}}{\partial a}(\tilde{a}, \tilde{b}) \cdot V \cdot g_i(\hat{\underline{x}})\right.$$
$$\left. + \frac{\partial \underset{\sim}{\overset{o}{x}}}{\partial b}(\tilde{a}, \tilde{b}) \cdot W \cdot g_i^{\psi}(\hat{\underline{x}})\right\} , \quad i = 1, \ldots, m . \tag{3.5}$$

where $\hat{\rho}_i = \rho_i(h_i(\hat{\underline{x}}))$ .

Equation (3.5) is the major result of this paper. It is a very general result relating to a set $M = \{1, \ldots, m\}$ of agents, each making many decisions and faced with $n$ prices or other parameter functions each with two kinds of uncertainty ( $r$ and $s$ ) . A general interpretation of (3.5) will not be attempted, but rather we present in the next section a number of examples and applications which provide useful insights concerning the nature of a Cournot-Nash equilibrium under exogenous uncertainty. In some of the applications we focus on an average or aggregate measure of decisions rather than individual level decisions. We also consider the case of all identical agents (at least in their behaviour under certainty, although they may have different attitudes to risk), and of just two agents. These kinds of assumptions are much used in discussions of Cournot-Nash equilibria in the existing literature.

## 4. Examples and Applications

### (a) Cournot Oligopoly

The model described in Section 2 has an obvious application in oligopoly theory. Consider $m$ firms each producing an identical product, the industry demand for which is represented by the stochastic inverse demand function with $n = 1$ :

$$p(x) = a - bQ + \alpha(r - sQ) ,$$

where $Q = \sum_{i=1}^{m} x_i$ and $(r, s)$ are stochastic. Each agent has the same subjective probability density function on $r$ , and similarly on $s$ . Random $r$ implies uncertain intercept and random $s$ uncertain slope of the inverse demand function.

Each firm maximises the (expected) utility from profit, so that the $i^{th}$ firm chooses $x_i$ given (true in equilibrium) assumptions about $\underline{x}^i$, ex ante of $(r, s)$ being revealed. Profit for the $i^{th}$ firm is

$$Y_i(x) = (a - bQ)x_i - c_i(x_i) + \alpha(r - sQ)x_i ,$$

so that: $f_i(x) = -c_i(x_i)$ , and

$$g_i(x) = x_i$$

in (2.2). We can immediately pick out the individual expressions defined in (3.3). We have

$$h_i(x) = -c_i(x_i) + \bar{p}(x)\cdot x_i \tag{4.1}$$

where

$$\bar{p}(x) = a - bQ .$$

Also: $\mathrm{diag}(\psi(x)) = -Q$ ,

$$g_i^{\psi}(x) = -Qx_i , \text{ and}$$

$$\mathrm{diag}(g_i(x)) = x_i .$$

We also have:

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right] = -\left[\tilde{b}, \ldots, \tilde{b}, \tilde{\phi}_i + \tilde{b}, \tilde{b}, \ldots, \tilde{b}\right], \tag{4.2}$$

where $\tilde{\phi}_i$ in the $i^{th}$ component of the m-vector is

$$\tilde{\phi}_i = c_i''(\hat{\underline{x}}_i) + \tilde{b}\,.$$

Thus (3.3) takes the simpler form:

$$-\left[\tilde{b}, \ldots, \tilde{b}, \tilde{\phi}_i + \tilde{b}, \tilde{b}, \ldots, \tilde{b}\right]\cdot(\tilde{\underline{x}} - \hat{\underline{x}})$$

$$= \alpha^2 \hat{\rho}_i V \hat{x}_i + \alpha^2 \hat{\rho}_i\left\{\hat{\underline{x}}_i + \hat{\underline{Q}}\right\} W \hat{\underline{Q}} \hat{\underline{x}}_i\,, \tag{4.3}$$

and (3.4a) and (3.4b) take the form:

$$-\left[\tilde{b}, \ldots, \tilde{b}, \tilde{\phi}_i + \tilde{b}, \tilde{b}, \ldots, \tilde{b}\right]\cdot\frac{\partial \overset{o}{\underline{x}}}{\partial a} + 1 = 0\,, \tag{4.4a}$$

$$-\left[\tilde{b}, \ldots, \tilde{b}, \tilde{\phi}_i + \tilde{b}, \tilde{b}, \ldots, \tilde{b}\right]\frac{\partial \overset{o}{\underline{x}}}{\partial b} + (\hat{\underline{x}}_i)^2 - \hat{\underline{Q}} = 0 \tag{4.4b}$$

The final approximation (3.5) is then:

$$\left[\tilde{b}, \ldots, \tilde{b}, \tilde{\phi}_i + \tilde{b}, \tilde{b}, \ldots, \tilde{b}\right]\cdot(\tilde{\underline{x}} - \hat{\underline{x}})$$

$$= \alpha^2 \hat{\rho}_i V \hat{\underline{x}}_i\left[\tilde{b}, \ldots, \tilde{b}, \tilde{\phi}_i + \tilde{b}, \tilde{b}, \ldots, \tilde{b}\right]\cdot\frac{\partial \overset{o}{\underline{x}}}{\partial a}$$

$$+ \alpha^2 \hat{\rho}_i W \hat{\underline{Q}} \hat{\underline{x}}_i\left[\tilde{b}, \ldots, \tilde{b}, \tilde{\phi}_i + \tilde{b}, \tilde{b}, \ldots, \tilde{b}\right]\cdot\frac{\partial \overset{o}{\underline{x}}}{\partial b}\,,$$

$$i = 1, \ldots, m \tag{4.5}$$

Equation (4.5) holds for all $i$, and is useful in a number of particular cases.

Case (i): Firms are identical under certainty but have different attitudes towards risk (i.e., $Y_i(\cdot) = Y_j(\cdot)$, all $i, j$, but different utility functions). Then adding equation (4.5) over all $i$ yields, using $\hat{\underline{x}}_i = \hat{\underline{x}}_j$, all $i, j$ so that $\hat{\underline{x}}_i = \frac{1}{m}\hat{\underline{Q}}$ (but $\tilde{\underline{x}}_i$ is not necessarily equal to $\tilde{\underline{x}}_j$), and $\tilde{\phi}_i = \tilde{\phi}$, all $i$:

$$(m\tilde{b} + \tilde{\phi})\,(\tilde{\underline{Q}} - \hat{\underline{Q}}) = -\alpha^2 V\,(m\tilde{b} + \tilde{\phi})\,\frac{\hat{\underline{Q}}}{m}\,\frac{\partial \mathring{\underline{Q}}}{\partial a}\,\frac{\sum_{i=1}^{m}\hat{\rho}_i}{m}$$

$$+\ \alpha^2 W\,\hat{\underline{Q}}(m\tilde{b} + \tilde{\phi})\,\frac{\hat{\underline{Q}}}{m}\,\frac{\partial \mathring{\underline{Q}}}{\partial b}\,\frac{\sum_{i=1}^{m}\hat{\rho}_i}{m},$$

or:

$$\tilde{\underline{Q}} - \hat{\underline{Q}} = -\alpha^2\left(V\,\frac{\partial \mathring{\underline{Q}}}{\partial a} - W\,\hat{\underline{Q}}\,\frac{\partial \mathring{\underline{Q}}}{\partial b}\right)\frac{\hat{\underline{Q}}}{m}\,\frac{\sum_{i=1}^{m}\hat{\rho}_i}{m}$$

Thus the proportionate reduction in industry output due to uncertainty in the intercept of the inverse demand function alone $(W = 0)$ is:

$$\frac{\tilde{\underline{Q}} - \hat{\underline{Q}}}{\hat{\underline{Q}}} = -\alpha^2\,V\,\frac{\partial \mathring{\underline{Q}}}{\partial a}\,\frac{1}{m}\,\frac{\sum_{i=1}^{m}\hat{\rho}_i}{m}, \tag{4.7}$$

and this depends on the amount of uncertainty $(\alpha^2 V)$, the response of industry output defined by a Cournot equilibrium to a parallel shift under certainty $(\frac{\partial \mathring{\underline{Q}}}{\partial a})$ and the average coefficient of absolute risk aversion $(\frac{1}{m}\sum_{i=1}^{m}\hat{\rho}_i)$. We would expect $\frac{\partial \mathring{\underline{Q}}}{\partial a} > 0$ and so if on average there is risk aversion $(\frac{1}{m}\sum_{i=1}^{m}\hat{\rho}_i > 0)$ then industry output will be less under uncertainty. Uncertainty concerning the intercept of the inverse demand function thus involves an industry output reduction analogous to that for a reduction in the intercept under certainty.

Note that if uncertainty is involved only in the slope of the inverse demand function, then for an (on average) risk averse industry such uncertainty produces a qualitatively similar response to an <u>increase</u> in $b$ under certainty. An increase in $b$, like a reduction in $a$, produces lower profits and less utility.

<u>Case (ii)</u>: Suppose $c_i''(\hat{x}_i) = \hat{c}$ all $i$, i.e., $\tilde{\phi}_i = \tilde{\phi}$, all $i$, and $\hat{\rho}_i\hat{x}_i = \hat{\theta}$, all $i$. In this case it is no longer assumed that outputs and responses to parameter changes under certainty will be the same for all $i$. Nevertheless, again add (4.5) over all $i$ and obtain

$$(m\tilde{b} + \tilde{\phi})(\tilde{\underline{Q}} - \hat{\underline{Q}}) = -\alpha^2\hat{\theta}\, V\,(m\tilde{b} + \tilde{\phi})\frac{\partial \overset{\circ}{\underline{Q}}}{\partial a} + \alpha^2\hat{\theta}\, W\,\hat{\underline{Q}}(m\tilde{b} + \tilde{\phi})\frac{\partial \overset{\circ}{\underline{Q}}}{\partial b} \qquad (4.8)$$

or:

$$\tilde{\underline{Q}} - \hat{\underline{Q}} = -\alpha^2\hat{\theta}\left[V\frac{\partial \overset{\circ}{\underline{Q}}}{\partial a} - W\,\hat{\underline{Q}}\frac{\partial \overset{\circ}{\underline{Q}}}{\partial b}\right], \qquad (4.9)$$

which is a result comparable with (4.7), so that the same interpretation of response to risk can be made if firms are not identical under certainty, but have the same slope of marginal cost function (c) and the same output-weighted coefficient of absolute risk aversion.

<u>Case (iii)</u>: Suppose we have only two firms $(m = 2)$. Define $\hat{c}_1 = c_1''(\hat{\underline{x}}_1) + 2\tilde{b}$ and $\hat{c}_2 = c_2''(\hat{\underline{x}}_2) + 2\tilde{b}$. Then (4.5) written out, gives for $i = 1$:

$$(\tilde{\underline{x}}_1 - \hat{\underline{x}}_1)(\hat{c}_1\hat{c}_2 - \tilde{b}^2) = -\alpha^2 V[\hat{\rho}_1\hat{\underline{x}}_1\hat{c}_1\hat{c}_2 - \hat{\rho}_2\hat{\underline{x}}_2\tilde{b}^2]\,\frac{\partial\mathring{\underline{x}}_1}{\partial a}$$

$$-\alpha^2 V[\hat{\rho}_1\hat{\underline{x}}_1 - \hat{\rho}_2\hat{\underline{x}}_2]\hat{c}_2\tilde{b}\,\frac{\partial\mathring{\underline{x}}_2}{\partial a} + \alpha^2 W(\hat{\underline{x}}_1 + \hat{\underline{x}}_2)[\hat{\rho}_1\hat{x}_1\hat{c}_1\hat{c}_2 - \hat{\rho}_2\hat{\underline{x}}_2\tilde{b}^2]\frac{\partial\mathring{\underline{x}}_1}{\partial b}$$

$$+\alpha^2 W(\hat{\underline{x}}_1 + \hat{\underline{x}}_2)[\hat{\rho}_1\hat{\underline{x}}_1 - \hat{\rho}_2\hat{\underline{x}}_2]\tilde{c}_2\tilde{b}\,\frac{\partial\mathring{\underline{x}}_2}{\partial b}\;, \qquad (4.10)$$

and for $i = 2$ :

$$(\tilde{\underline{x}}_2 - \hat{\underline{x}}_2)(\hat{c}_1\hat{c}_2 - \tilde{b}^2) = -\alpha^2 V[\hat{\rho}_2\hat{\underline{x}}_2 - \hat{\rho}_1\hat{\underline{x}}_1]\hat{c}_1\tilde{b}\;\frac{\partial\mathring{\underline{x}}_1}{\partial a}$$

$$-\alpha^2 V[\hat{\rho}_2\hat{\underline{x}}_2\hat{c}_1\hat{c}_2 - \hat{\rho}_1\hat{\underline{x}}_1\tilde{b}^2]\,\frac{\partial\mathring{\underline{x}}_2}{\partial a} + \alpha^2 W(\hat{\underline{x}}_1+\hat{\underline{x}}_2)[\hat{\rho}_2\hat{\underline{x}}_2-\hat{\rho}_1\hat{\underline{x}}_1]\hat{c}_1\tilde{b}\;\frac{\partial\mathring{\underline{x}}_1}{\partial b}$$

$$+\alpha^2 W(\hat{\underline{x}}_1 + \hat{\underline{x}}_2)[\hat{\rho}_2\hat{\underline{x}}_2\hat{c}_1\hat{c}_2 - \hat{\rho}_1\hat{\underline{x}}_1\tilde{b}^2]\,\frac{\partial\mathring{\underline{x}}_2}{\partial b}\quad . \qquad (4.11)$$

Suppose $\hat{\rho}_1\hat{\underline{x}}_1 = \hat{\rho}_2\hat{\underline{x}}_2 = \hat{\theta}$ , then the formulas for $(\tilde{\underline{x}}_1 - \hat{\underline{x}}_1)$ and $(\tilde{\underline{x}}_2 - \hat{\underline{x}}_2)$ become very much simpler, even though $\hat{c}_1$ is not necessarily equal to $\hat{c}_2$ :

$$\tilde{\underline{x}}_i - \hat{\underline{x}}_i = -\alpha^2 V\,\hat{\theta}\,\frac{\partial\mathring{\underline{x}}_i}{\partial a} + \alpha^2 W(\hat{\underline{x}}_1 + \hat{\underline{x}}_2)\,\frac{\partial\mathring{\underline{x}}_i}{\partial b}\;,\quad i = 1, 2\;. \qquad (4.12)$$

Again, the same interpretations are possible as in cases (i) and (ii), only here behaviour relates to a particular firm rather then the industry as a whole.

Alternatively, assume that, as in case (i), both firms are identical under certainty, only their attitudes towards risk differ. Then

$$\tilde{\underline{x}}_1 - \hat{\underline{x}}_1 = -\alpha^2 V \hat{\underline{x}}_i \frac{\partial \overset{o}{\underline{x}}_i}{\partial a} \left\{ \hat{\rho}_1 \frac{\hat{c}_1\hat{c}_2 + \hat{c}_2\tilde{b}}{\hat{c}_1\hat{c}_2 - \tilde{b}^2} - \hat{\rho}_2 \frac{\tilde{b}^2 + \hat{c}_2\tilde{b}}{\hat{c}_1\hat{c}_2 - \tilde{b}^2} \right\}$$

$$+ \alpha^2 W \hat{\underline{Q}} \hat{\underline{x}}_i \frac{\partial \overset{o}{\underline{x}}_i}{\partial b} \left\{ \hat{\rho}_1 \frac{\hat{c}_1\hat{c}_2 + \hat{c}_2\tilde{b}}{\hat{c}_1\hat{c}_2 - \tilde{b}^2} - \hat{\rho}_2 \frac{\tilde{b}^2 + \hat{c}_2\tilde{b}}{\hat{c}_1\hat{c}_2 - \tilde{b}^2} \right\},$$

where $i = 1$ or $2$. Now as $\hat{c}_1 = \hat{c}_2 = \hat{c}$ from the assumption of identical firms under certainty:

$$\tilde{\underline{x}}_1 - \hat{\underline{x}}_1 = -\alpha^2 \left[ \frac{\hat{c}}{\hat{c} - \tilde{b}} \hat{\rho}_1 - \frac{\tilde{b}}{\hat{c} - \tilde{b}} \hat{\rho}_2 \right] \cdot \left[ V \frac{\partial \overset{o}{\underline{x}}_i}{\partial a} - W\hat{Q} \frac{\partial \overset{o}{\underline{x}}_i}{\partial b} \right] \hat{\underline{x}}_i \qquad (4.13)$$

$$\tilde{\underline{x}}_2 - \hat{\underline{x}}_2 = -\alpha^2 \left[ \frac{\hat{c}}{\hat{c} - \tilde{b}} \hat{\rho}_2 - \frac{\tilde{b}}{\hat{c} - \tilde{b}} \hat{\rho}_1 \right] \cdot \left[ V \frac{\partial \overset{o}{\underline{x}}_i}{\partial a} - W \hat{Q} \frac{\partial \overset{o}{\underline{x}}_i}{\partial b} \right] \hat{\underline{x}}_i \qquad (4.14)$$

Adding (4.13) and (4.14) of course yields (4.6). Subtracting (4.14) from (4.13) yields a comparison of responses:

$$\tilde{\underline{x}}_1 - \tilde{\underline{x}}_2 = -\alpha^2 \frac{\hat{c} + \tilde{b}}{\hat{c} - \tilde{b}} (\hat{\rho}_1 - \hat{\rho}_2) \left[ V \frac{\partial \overset{o}{\underline{x}}_i}{\partial a} - W \hat{Q} \frac{\partial \overset{o}{\underline{x}}_i}{\partial b} \right] \hat{\underline{x}}_i , \qquad (4.15)$$

so that, assuming $(V \frac{\partial \overset{o}{\underline{x}}_i}{\partial a} - W \hat{Q} \frac{\partial \overset{o}{\underline{x}}_i}{\partial b}) > 0$ and $\frac{\hat{c} + \tilde{b}}{\hat{c} - \tilde{b}} > 0$, we have that the smaller output under uncertainty is produced by the firm with the higher coefficient of absolute risk aversion.

(b) Collective Firms and Private Plots

The application in (a) only considered one decision variable per agent ($n_i = 1$, all $i$), and yet one major aspiration of the procedure outlined in Section 3 was to allow consideration of multiple decision variables. Here we will show this possibility of applying the analysis to a simple model of a collective farm where individual worker $i \in \{1, . . ., m\}$ chooses the allocation of his time to work on the communal land ( $k_i$ hours), work on his private plot of land ( $\ell_i$ hours) and leisure ( $(E - k_i - \ell_i)$ hours) . For a discussion of collective farms see Bonin (1), Ireland and Law (3) and Oi and Clayton (7). The utility of an individual member is given by (2.1), where (2.2) is of the form:

$$Y_i = p \frac{\underline{q}(K)}{m} + z_i(\ell_i) - \beta_i(k_i + \ell_i) , \tag{4.16}$$

and $p = \bar{p} + \alpha r$ .

We write $K = \sum_{i=1}^{m} k_i$ , $L = \sum_{i=1}^{m} \ell_i$ , so that $\underline{q}(K)$ is total output of the communal plot which is sold to the State at a price $p$ per unit and then the revenue is distributed equally among all workers on the collective farm. (Other distributions rules will be considered briefly later). The individual obtains an income-equivalent return of $z_i(\ell_i)$ from the private plot and $-\beta_i(k_i + \ell_i)$ represents the cost in terms of leisure of working on both the communal and private plots. Non-negativity conditions on $k_i$, $\ell_i$ and $E - k_i - \ell_i$ are assumed to be satisfied at all interesting equilibria. The form of (4.16) is rather special but others can be found which still satisfy (2.2). Now

$$h_i(\cdot) = \bar{p} \frac{\underline{q}(K)}{m} + z_i(\ell_i) - \beta_i(k_i + \ell_i)$$

$$g_i(\cdot) = \frac{\underline{q}(K)}{m} ,$$

so that for just two individuals $(m = 2)$ , the left-hand-side of equation (3.5) stacked for $i = 1, 2$ is

$$\begin{bmatrix} (\frac{\bar{p}q''}{2} - \beta_1'') & -\beta_1'' & \frac{\bar{p}q''}{2} & 0 \\ -\beta_1'' & (z_1'' - \beta_1'') & 0 & 0 \\ \frac{\bar{p}q''}{2} & 0 & (\frac{\bar{p}q''}{2} - \beta_2'') & -\beta_2'' \\ 0 & 0 & -\beta_2'' & (z_2'' - \beta_2'') \end{bmatrix} \begin{bmatrix} \tilde{\underline{k}}_1 - \hat{\underline{k}}_1 \\ \tilde{\underline{\ell}}_1 - \hat{\underline{\ell}}_1 \\ \tilde{\underline{k}}_2 - \hat{\underline{k}}_2 \\ \tilde{\underline{\ell}}_2 - \hat{\underline{\ell}}_2 \end{bmatrix} \qquad (4.17)$$

where the components of the matrix are evaluated at $(k_1, \ell_1, k_2, \ell_2) = (\hat{\underline{k}}_1, \hat{\underline{\ell}}_1, \hat{\underline{k}}_2, \hat{\underline{\ell}}_2)$ . Describe the matrix in (4.17) as $\hat{H}$ , then the right-hand-side of the stack of equations (3.5) is

$$-\alpha^2 \frac{q(\hat{K})}{2} V \begin{bmatrix} \hat{\rho}_1 & & & 0 \\ & \hat{\rho}_1 & & \\ & & \hat{\rho}_2 & \\ 0 & & & \hat{\rho}_2 \end{bmatrix} \hat{H} \begin{bmatrix} \frac{d\overset{o}{\underline{k}}_1}{d\bar{p}} \\ \frac{d\overset{o}{\underline{\ell}}_1}{d\bar{p}} \\ \frac{d\overset{o}{\underline{k}}_2}{d\bar{p}} \\ \frac{d\overset{o}{\underline{\ell}}_2}{d\bar{p}} \end{bmatrix} \qquad (4.18)$$

Now simple results can be obtained in at least two cases.

Case (i): Let $\hat{\rho}_1 = \hat{\rho}_2 = \hat{\rho}$ . Then, provided $\hat{H}^{-1}$ exists,

$$\tilde{\underline{x}}_i - \hat{\underline{x}}_i = -\alpha^2 \frac{q(\hat{K})}{2} V \hat{\rho} \frac{d\overset{o}{\underline{x}}_i}{d\bar{p}} , \quad x = k, \ell \ ; \quad i = 1, 2 \qquad (4.19)$$

Thus for each variable and for each individual, the response to uncertainty in price from communal output is the same constant $\lambda = -\alpha^2 \frac{q(\hat{K})}{2} V \hat{\rho}$ of the respective adjustment to a change in $\bar{p}$ .

Case (ii): Now consider the agents as identical in the certainty situation (same $h(\cdot)$-function) but allow them to have different attitudes to risk $(\hat{\rho}_i)$. Then (4.10) can be written as

$$-\alpha^2 \frac{\underline{q}(\underline{\hat{K}})}{2} V \begin{bmatrix} \hat{\rho}_1 & & & 0 \\ & \hat{\rho}_1 & & \\ & & \hat{\rho}_2 & \\ 0 & & & \hat{\rho}_2 \end{bmatrix} \begin{bmatrix} \bar{p}\underline{q}'' - \beta'' & -\beta'' \\ -\beta'' & z'' - \beta'' \\ \bar{p}\underline{q}'' - \beta'' & -\beta'' \\ -\beta'' & z'' - \beta'' \end{bmatrix} \begin{bmatrix} \frac{d\mathring{\underline{K}}}{d\bar{p}} \\ \frac{d\mathring{\underline{L}}}{d\bar{p}} \end{bmatrix} \qquad (4.20)$$

as $\dfrac{d\mathring{\underline{k}}_1}{d\bar{p}} = \dfrac{d\mathring{k}_2}{d\bar{p}}$, etc. Now add the first and third rows and the second and the last rows of (4.17) and (4.20) to obtain

$$\begin{bmatrix} \bar{p}\underline{q}'' - \beta'' & -\beta'' & \bar{p}\underline{q}'' - \beta'' & -\beta'' \\ -\beta'' & z'' - \beta'' & -\beta'' & z'' - \beta'' \end{bmatrix} \begin{bmatrix} \tilde{\underline{k}}_1 - \hat{\underline{k}}_1 \\ \tilde{\underline{\ell}}_1 - \hat{\underline{\ell}}_1 \\ \tilde{\underline{k}}_2 - \hat{\underline{k}}_2 \\ \tilde{\underline{\ell}}_2 - \hat{\underline{\ell}}_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \bar{p}\underline{q}'' - \beta'' & -\beta'' \\ -\beta'' & z'' - \beta'' \end{bmatrix} \begin{bmatrix} \tilde{\underline{K}} - \mathring{\underline{K}} \\ \tilde{\underline{L}} - \mathring{\underline{L}} \end{bmatrix}$$

from (4.17), and

$$-\alpha^2 \frac{\underline{q}(\underline{\hat{K}})}{2} V (\hat{\rho}_1 + \hat{\rho}_2) \begin{bmatrix} \bar{p}\underline{q}'' - \beta'' & -\beta'' \\ -\beta'' & z'' - \beta'' \end{bmatrix} \begin{bmatrix} \frac{d\mathring{\underline{K}}}{d\bar{p}} \\ \frac{d\mathring{\underline{L}}}{d\bar{p}} \end{bmatrix}$$

from (4.20).

Thus, provided the inverse of

$$\begin{bmatrix} \bar{p}\underline{q}'' - \beta'' & -\beta'' \\ -\beta'' & z'' - \beta'' \end{bmatrix}$$

exists, we have

$$\begin{bmatrix} \underline{\tilde{K}} - \underline{\hat{K}} \\ \underline{\tilde{L}} - \underline{\hat{L}} \end{bmatrix} = -\alpha^2 \frac{\underline{q}(\hat{\underline{K}})}{2} V(\hat{\rho}_1 + \hat{\rho}_2) \begin{bmatrix} \dfrac{d\underline{\overset{o}{K}}}{d\bar{p}} \\ \dfrac{d\underline{\overset{o}{L}}}{d\bar{p}} \end{bmatrix} \tag{4.21}$$

Again, the relationship between $(\underline{\tilde{K}} - \underline{\hat{K}})$ and $\frac{d\underline{\overset{o}{K}}}{d\bar{p}}$ is the same as between $(\underline{\tilde{L}} - \underline{\hat{L}})$ and $\frac{d\underline{\overset{o}{L}}}{d\bar{p}}$, so that uncertainty in communal product price has an effect on total labour supplies analogous to a reduction in that price under certainty, if individuals are, on average, risk averse.

The application above can be generalised in a number of directions. Particularly $m > 2$ presents no problem. For case (ii), other distribution rules for communal plot revenue, such as according to individual labour input (see Ireland and Law (4) and (5)), and other specifications for (2.2) can be accommodated with no additional difficulty.

## 5. Conclusions and Extensions

We saw in Section 4 how, by appropriate assumptions and restrictions, equations (3.5) (all $i$) could be used to relate differences in individual or aggregate behaviour due to uncertainty in a parameter to responses to changes in that parameter under certainty. Consider one such result, equation (4.19). Suppose one posed the question, what difference in parameter value under certainty would produce (approximately) the same behaviour as the given amount of uncertainty, i.e., $\Delta p$ which solves

$$\hat{\underline{x}}_i(\alpha) = \overset{o}{\underline{x}}_i(\bar{p} + \Delta p) .$$

The answer is simply (for the result (4.19))

$$\Delta p = -\alpha^2 \frac{\underline{q}(\hat{\underline{K}})}{2} V \hat{p} , \tag{5.1}$$

and $\Delta p$ can be described as a behaviour-equivalent change.

A $\Delta p$ of opposite sign could be used as a compensation to maintain the same behaviour with the onset of uncertainty. Note, however, that this would overcompensate a risk averse individual in terms of his expected utility. An Arrow-Pratt risk premium such that utility is equivalent to that under certainty would be approximately one half of $\Delta p$ in (5.1).

We have concentrated our analysis on a Cournot-Nash non-cooperative solution. However, other possible solution concepts to the game $\Gamma$ might be applied and can be the subject for further research. One obvious one we should mention here is a cooperative solution. In the case of identical utility functions and cost functions, an oligopoly problem such as in application (a) in Section (4) reduces to maximising the expected utility of an average firm, with respect to all decision variables $x$ $(x_i \equiv x)$ . Then

$$\frac{\partial}{\partial x}\left[\frac{\partial h}{\partial x_i}(\hat{\underline{x}})\right]^T = \frac{\partial}{\partial x}\left[\frac{\partial h}{\partial x}(\hat{\underline{x}})\right]^T$$

forms a square non-singular matrix and immediately:

$$\tilde{\underline{x}} - \hat{\underline{x}} = -\alpha^2\hat{\rho}\left\{\frac{\partial \overset{o}{\underline{x}}}{\partial a} V\, g(\hat{\underline{x}}) + \frac{\partial \overset{o}{\underline{x}}}{\partial b} W\, g^{\psi}(\hat{\underline{x}})\right\} . \qquad (5.2)$$

It is likely that most other solution concepts to the game do not yield such tractable results as the Cournot-Nash solution. Confirmation of this is a matter for further research.

## References


(1) Bonin, J.P., "Work Incentives and Uncertainty on a Collective Farm", *Journal of Comparative Economics*, 1, (1977).

(2) Ireland, N.J., "The Analogy Between Parameter Uncertainty and Parameter Change", *Economics Letters*, 6, (1980).

(3) Ireland, N.J. and Law, P.J., "Private Plot Restrictions in a Collective Farm Model", *Canadian Journal of Economics*, 5, (1980).

(4) Ireland, N.J. and Law, P.J., "Efficiency, Incentives and Individual Labour Supply in the Labour-Managed Firm", *Journal of Comparative Economics*, 5, (1981).

(5) Ireland, N.J. and Law, P.J., "*The Economics of Labour-Managed Enterprises*", Croom Helm, London, forthcoming (1982).

(6) Ishii, Y., "On the Theory of the Competitive Firm under Price Uncertainty", *American Economic Review*, 67, (1977).

(7) Oi, W.Y. and Clayton, E.S., "A Peasant's View of a Soviet Collective Farm", *American Economic Review*, 58, (1968).

(8) Rothschild, M. and Stiglitz, J.E., "Increasing Risk I : a Definition", *Journal of Economic Theory*, 2, (1970).

(9) Rothschild, M. and Stiglitz, J.E., "Increasing Risk II : Its Economic Consequences", *Journal of Economic Theory*, 3, (1971).

(10) Sandmo, A., "On the Theory of the Competitive Firm under Price Uncertainty", *American Economic Review*, 61, (1971).

Appendix

The Cournot-Nash solution $\underline{x}(a, b, \alpha)$ to the game $\Gamma$ considered in Section 2 is characterised by the following set of equations:

$$\frac{\partial}{\partial x_i} E\left\{u_i(Y_i(x_i, \underline{x}^i))\right\} \Big|_{x_i = \underline{x}_i} = 0 , i = 1, . . ., m ,$$

or

$$E\left\{u_i'(Y_i(\underline{x})) . \frac{\partial Y_i}{\partial x_i}(\underline{x})\right\} = 0 , i = 1, . . ., m. \tag{A1}$$

By definition of $Y_i$ :

$$\frac{\partial Y_i}{\partial x_i} = \frac{\partial h_i}{\partial x_i} + \alpha \cdot r^T \cdot \frac{\partial g_i}{\partial x_i} + \alpha \cdot [g_i(x)]^T \cdot S \cdot \frac{\partial \psi}{\partial x_i} + \alpha \cdot [\psi(x)]^T \cdot S \cdot \frac{\partial g_i}{\partial x_i} ,$$

where $h_i(x) := f_i(x) + a^T \cdot g_i(x) + [\psi(x)]^T \cdot B \cdot g_i(x)$ .

So, (A1) becomes:

$$E\left\{u_i'(Y_i(\underline{x}))\right\} \cdot \left[\frac{\partial h_i}{\partial x_i}(\underline{x})\right]^T + \alpha \cdot E\left\{u_i'(Y_i(\underline{x})) \cdot \left[\frac{\partial g_i}{\partial x_i}(\underline{x})\right]^T \cdot r\right\} +$$

$$\alpha \cdot E\left\{u_i'(Y_i(\underline{x})) \cdot \left[\frac{\partial \psi}{\partial x_i}(\underline{x})\right]^T \cdot S \cdot g_i(\underline{x})\right\}$$

$$+ \alpha \cdot E\left\{u_i'(Y_i(\underline{x})) \cdot \left[\frac{\partial g_i}{\partial x_i}(\underline{x})\right]^T \cdot S \cdot \psi(\underline{x})\right\} = 0 , \quad i = 1, . . ., m. \tag{A2}$$

Now to calculate $\frac{\partial \underline{x}^o}{\partial a}$ and $\frac{\partial \underline{x}^o}{\partial b}$ , set $\alpha = 0$ in (A2) and obtain,

$$u_i'(h_i(\overset{\circ}{\underline{x}}))\cdot\left[\frac{\partial h_i}{\partial x_i}(\overset{\circ}{\underline{x}})\right]^T = 0\ ,\ i = 1, \ldots, m\ ,$$

or, since $u_i(\cdot)$ is strictly increasing:

$$\left[\frac{\partial h_i}{\partial x_i}(\overset{\circ}{\underline{x}}(a,\ b),\ a,\ b)\right]^T = 0\ ,\quad i = 1, \ldots, m\ . \tag{A3}$$

Notice that from (A3) we also have

$$\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T = 0\ ,\quad i = 1, \ldots, m\ , \tag{A4}$$

where it is understood that we suppress the notation of $\tilde{a}$ and $\tilde{b}$ as arguments of functions. Equation (A3) written out gives:

$$\begin{aligned}&\left[\frac{\partial f_i}{\partial x_i}(\overset{\circ}{\underline{x}}(a,\ b))\right]^T + \left[\frac{\partial g_i}{\partial x_i}(\overset{\circ}{\underline{x}}(a,\ b))\right]^T\cdot a\\ &+ \left[\psi(\overset{\circ}{\underline{x}}(a,\ b))\right]^T\cdot B\cdot\frac{\partial g_i}{\partial x_i}(\overset{\circ}{\underline{x}}(a,\ b))\\ &+ \left[g_i(\overset{\circ}{\underline{x}}(a,\ b))\right]^T\cdot B\cdot\frac{\partial g}{\partial x_i}(\overset{\circ}{\underline{x}}(a,\ b)) = 0,\ i = 1, \ldots, m.\end{aligned} \tag{A5}$$

Differentiation with respect to $a$ on both sides of (A5) at $a = \tilde{a}$, $(b = \tilde{b})$, yields:

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T\cdot\frac{\partial\overset{\circ}{\underline{x}}}{\partial a}(\tilde{a},\ \tilde{b}) + \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T = 0\ ,\ i = 1, \ldots, m. \tag{A6}$$

Differentiation with respect to $b$ on both sides of (A5) at $b = \tilde{b}$, $(a = \tilde{a})$, yields:

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}\,(\hat{\underline{x}})\right]^T \cdot \frac{\partial \underline{x}^o}{\partial b}\,(\tilde{a},\ \tilde{b}) + \left[\frac{\partial g_i}{\partial x_i}\,(\hat{\underline{x}})\right]^T \cdot \operatorname{diag}(\psi(\hat{\underline{x}}))$$

$$+ \left[\frac{\partial \psi}{\partial x_i}\,(\hat{\underline{x}})\right]^T \cdot \operatorname{diag}(g_i(\hat{\underline{x}})) \quad = \quad 0\ ,\ i = 1,\ .\ .\ .,\ m\ . \qquad \text{(A7)}$$

where

$$\operatorname{diag}(g_i(x))\ :=\ \begin{bmatrix} g_{i1}(x) & & 0 \\ & \ddots & \\ 0 & & g_{in}(x) \end{bmatrix} ,$$

and

$$\operatorname{diag}(\psi(x))\ :=\ \begin{bmatrix} \psi_1(x) & & 0 \\ & \ddots & \\ 0 & & \psi_n(x) \end{bmatrix} .$$

For determining $(\tilde{\underline{x}} - \hat{\underline{x}})$ , set $(a, b) = (\tilde{a}, \tilde{b})$ in (A2) and obtain:

$$E\left\{u_i'(Y_i(\tilde{\underline{x}},\ \alpha))\right\} \cdot \left[\frac{\partial h_i}{\partial x_i}\,(\tilde{\underline{x}})\right]^T = -\alpha \cdot \left[E\left\{u_i'(Y_i(\tilde{\underline{x}},\ \alpha)) \cdot \left[\frac{\partial g_i}{\partial x_i}(\tilde{\underline{x}})\right]^T \cdot r\right\}\right.$$

$$+ E\left\{u_i'(Y_i(\tilde{\underline{x}},\ \alpha)) \cdot \left[\frac{\partial \psi}{\partial x_i}\,(\tilde{\underline{x}})\right]^T \cdot S \cdot g_i(\tilde{\underline{x}})\right\}$$

$$\left. + E\left\{u_i'(Y_i(\tilde{\underline{x}},\ \alpha)) \cdot \left[\frac{\partial g_i}{\partial x_i}\,(\tilde{\underline{x}})\right]^T \cdot S \cdot \psi(\tilde{\underline{x}})\right\}\right]\ ,\ i = 1,\ .\ .\ .,\ m. \qquad \text{(A8)}$$

Define: $\quad F_i(\tilde{\underline{x}},\ \alpha) \quad = \quad E\left\{u_i'(Y_i(\tilde{\underline{x}},\ \alpha))\right\} \cdot \left[\frac{\partial h_i}{\partial x_i}\,(\tilde{\underline{x}})\right]^T ,$

$$F(\underline{\tilde{x}}, \alpha) := \begin{bmatrix} F_1(\underline{\tilde{x}}, \alpha) \\ \vdots \\ F_m(\underline{\tilde{x}}, \alpha) \end{bmatrix} .$$

Linear approximation of $F$ around $(\underline{\tilde{x}}, \alpha) = (\underline{\hat{x}}, 0)$ is given by :

$$F(\underline{\tilde{x}}, \alpha) \simeq F(\underline{\tilde{x}}, 0) + \frac{\partial F}{\partial x}(\underline{\hat{x}}, 0) \cdot (\underline{\tilde{x}} - \underline{\hat{x}}) + \frac{\partial F}{\partial \alpha}(\underline{\hat{x}}, 0) \cdot \alpha ,$$

or

$$F_i(\underline{\tilde{x}}, \alpha) = F_i(\underline{\tilde{x}}, 0) + \frac{\partial F_i}{\partial \mathbf{x}}(\underline{\hat{x}}, 0) \cdot (\underline{\tilde{x}} - \underline{\hat{x}}) + \frac{\partial F_i}{\partial \alpha}(\underline{\hat{x}}, 0), \qquad i = 1, \ldots, m.$$

Now,

$$F_i(\underline{\hat{x}}, 0) = u_i'(h_i(\underline{\hat{x}})) \cdot \left[\frac{\partial h_i}{\partial x_i}(\underline{\hat{x}})\right]^T = 0 ,$$

$$\frac{\partial F_i}{\partial x}(\underline{\tilde{x}}, \alpha) = \left[\frac{\partial h_i}{\partial x_i}(\underline{\tilde{x}})\right]^T \cdot E\left\{u_i''(Y_i(\underline{\tilde{x}}, \alpha)) \cdot \frac{\partial Y_i}{\partial x}(\underline{\tilde{x}}, \alpha)\right\}$$

$$+ E\left\{u_i'(Y_i(\underline{\tilde{x}}, \alpha))\right\} \cdot \frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\underline{\tilde{x}})\right]^T ,$$

$$\frac{\partial F_i}{\partial x}(\underline{\hat{x}}, 0) = u_i'(h_i(\underline{\hat{x}})) \cdot \frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\underline{\hat{x}})\right]^T ,$$

$$\frac{\partial F_i}{\partial \alpha}(\underline{\tilde{x}}, \alpha) = E\left\{u_i''(Y_i(\underline{\tilde{x}}, \alpha)) \cdot \frac{\partial Y_i}{\partial \alpha}(\underline{\tilde{x}}, \alpha)\right\} \cdot \left[\frac{\partial h_i}{\partial x_i}(\underline{\tilde{x}})\right]^T ,$$

$$\frac{\partial F_i}{\partial \alpha}(\underline{\hat{x}}, 0) = 0 ,$$

and thus

$$F_i(\underline{\tilde{x}}, \alpha) \simeq u_i'(h_i(\underline{\hat{x}})) \cdot \frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\underline{\hat{x}})\right]^T \cdot (\underline{\tilde{x}} - \underline{\hat{x}}) \quad , i = 1, \ldots, m. \tag{A9}$$

Linear approximation of the term in the square brackets on the right-hand-side of (A8) is the sum of the linear approximations of the three terms in these brackets.

Define: $$G_i^I(\underline{\tilde{x}}, \alpha) := E\left\{u_i'(Y_i(\underline{\tilde{x}}, \alpha)) \cdot \left[\frac{\partial g_i}{\partial x_i}(\underline{\tilde{x}})\right]^T \cdot r\right\} \quad , i = 1, \ldots, m.$$

Then we can rewrite $G_i^I(\underline{\tilde{x}}, \alpha)$ as:

$$\begin{aligned} G_i^I(\underline{\tilde{x}}, \alpha) &= E\left\{u_i'(Y_i(\underline{\tilde{x}}, \alpha)) \cdot \sum_{j=1}^{n} r_j \cdot \left[\frac{\partial g_{ij}}{\partial x_i}(\underline{\tilde{x}})\right]^T\right\} \\ &= \sum_{j=1}^{n} E\left\{u_i'(Y_i(\underline{\tilde{x}}, \alpha)) \cdot r_j\right\} \cdot \left[\frac{\partial g_{ij}}{\partial x_i}(\underline{\tilde{x}})\right]^T \quad . \end{aligned}$$

Further:

$$G_i^I(\underline{\hat{x}}, 0) = \sum_{j=1}^{n} u_i'(h_i(\underline{\hat{x}})) \cdot E\{r_j\} \cdot \left[\frac{\partial g_{ij}}{\partial x_i}(\underline{\hat{x}})\right]^T = 0 \; ,$$

$$\begin{aligned} \frac{\partial G_i^I}{\partial x}(\underline{\tilde{x}}, \alpha) &= \sum_{j=1}^{n} \left[\frac{\partial g_{ij}}{\partial x_i}(\underline{\tilde{x}})\right]^T \cdot E\left\{u_i''(Y_i(\underline{\tilde{x}}, \alpha)) \cdot \frac{\partial Y_i}{\partial x}(\underline{\tilde{x}}, \alpha) \cdot r_j\right\} \\ &\quad + \sum_{j=1}^{n} E\left\{u_i'(Y_i(\underline{\tilde{x}}, \alpha)) \cdot r_j\right\} \cdot \frac{\partial}{\partial x}\left[\frac{\partial g_{ij}}{\partial x_i}(\underline{\tilde{x}})\right]^T \quad , \end{aligned}$$

$$\frac{\partial G_i^I}{\partial x}(\underline{\hat{x}}, 0) = 0 \; ,$$

$$\frac{\partial G_i^I}{\partial \alpha}(\tilde{\underline{x}}, \alpha) = \sum_{j=1}^{n} E\left\{u_i''(Y_i(\tilde{\underline{x}}, \alpha)) \cdot \frac{\partial Y_i}{\partial \alpha}(\tilde{\underline{x}}, \alpha) \cdot r_j\right\} \cdot \left[\frac{\partial g_{ij}}{\partial x_i}(\tilde{\underline{x}})\right]^T ,$$

$$\frac{\partial Y_i}{\partial \alpha} = r^T \cdot g_i(\tilde{\underline{x}}) + [\psi(\tilde{\underline{x}})]^T \cdot s \cdot g_i(\tilde{\underline{x}}) ,$$

$$\frac{\partial G_i^I}{\partial \alpha}(\hat{\underline{x}}, 0) = \sum_{j=1}^{n} u_i''(h_i(\hat{\underline{x}})) \cdot E\left\{r^T \cdot g_i(\hat{\underline{x}}) \cdot r_j + [\psi(\hat{\underline{x}})]^T \cdot s \cdot g_i(\hat{\underline{x}}) \cdot r_j\right\} \cdot \left[\frac{\partial g_{ij}}{\partial x_i}(\hat{\underline{x}})\right]^T .$$

By definition: $E\left\{r_j \cdot r^T\right\} = V_{j*}$ ($j^{th}$ row of $V$) ,

$$E\left\{r_j \cdot s_k\right\} = 0 .$$

So,

$$\frac{\partial G_i^I}{\partial \alpha}(\hat{\underline{x}}, 0) = \sum_{j=1}^{n} u_i''(h_i(\hat{\underline{x}})) \cdot V_{j*} \cdot g_i(\hat{\underline{x}}) \cdot \left[\frac{\partial g_{ij}}{\partial x_i}(\hat{\underline{x}})\right]^T$$

$$= u_i''(h_i(\hat{\underline{x}})) \cdot \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \begin{bmatrix} V_{1*} \cdot g_i(\hat{\underline{x}}) \\ \vdots \\ V_{n*} \cdot g_i(\hat{\underline{x}}) \end{bmatrix}$$

$$= u_i''(h_i(\hat{\underline{x}})) \cdot \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot V \cdot g_i(\hat{\underline{x}}) .$$

Therefore:

$$G_i^I(\underline{x}, \alpha) \simeq \alpha \cdot u_i''(h_i(\hat{\underline{x}})) \cdot \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot V \cdot g_i(\hat{\underline{x}}) , \; i = 1, \ldots, m. \qquad \text{(A.10)}$$

Define: $G_i^{II}(\tilde{\underline{x}}, \alpha) := E\left\{u_i'(Y_i(\tilde{\underline{x}}, \alpha))\cdot\left[\frac{\partial\psi}{\partial x_i}(\tilde{\underline{x}})\right]^T\cdot S\cdot g_i(\tilde{\underline{x}})\right\}$ .

Then we can rewrite $G_i^{II}(\tilde{\underline{x}}, \alpha)$ as:

$$G_i^{II}(\tilde{\underline{x}}, \alpha) = E\left\{u_i'(Y_i(\tilde{\underline{x}}, \alpha))\cdot\sum_{j=1}^{n} s_j\cdot g_{ij}(\tilde{\underline{x}})\cdot\left[\frac{\partial\psi_j}{\partial x_i}(\tilde{\underline{x}})\right]^T\right\}$$

$$= \sum_{j=1}^{n} E\left\{u_i'(Y_i(\tilde{\underline{x}}, \alpha))\cdot s_j\right\}\cdot g_{ij}(\tilde{\underline{x}})\cdot\left[\frac{\partial\psi_j}{\partial x_i}(\tilde{\underline{x}})\right]^T .$$

Further:

$$G_i^{II}(\hat{\underline{x}}, 0) = \sum_{j=1}^{n} u_i'(h_i(\hat{\underline{x}}))\cdot E\left\{s_j\right\}\cdot g_{ij}(\hat{\underline{x}})\cdot\left[\frac{\partial\psi_j}{\partial x_i}(\hat{\underline{x}})\right]^T = 0 ,$$

$$\frac{\partial G_i^{II}}{\partial x}(\tilde{\underline{x}}, \alpha) = \sum_{j=1}^{n} g_{ij}(\tilde{\underline{x}})\cdot\left[\frac{\partial\psi_j}{\partial x_i}(\tilde{\underline{x}})\right]^T\cdot E\left\{u_i''(Y_i(\tilde{\underline{x}}, \alpha))\cdot\frac{\partial Y_i}{\partial x}(\tilde{\underline{x}}, \alpha)\cdot s_j\right\}$$

$$+ \sum_{j=1}^{n} E\left\{u_i'(Y_i(\tilde{\underline{x}}, \alpha))\cdot s_j\right\}\cdot\left[\left[\frac{\partial\psi_j}{\partial x_i}(\tilde{\underline{x}})\right]^T\cdot g_{ij}'(\tilde{\underline{x}}) + g_{ij}(\tilde{\underline{x}})\cdot\frac{\partial}{\partial x}\left[\frac{\partial\psi_j}{\partial x_i}(\tilde{\underline{x}})\right]^T\right]$$

$$\frac{\partial G_i^{II}}{\partial x}(\hat{\underline{x}}, 0) = 0 ,$$

$$\frac{\partial G_i^{II}}{\partial\alpha}(\tilde{\underline{x}}, \alpha) = \sum_{j=1}^{n} E\left\{u_i''(Y_i(\tilde{\underline{x}}, \alpha))\cdot\frac{\partial Y_i}{\partial\alpha}(\tilde{\underline{x}}, \alpha)\cdot s_j\right\}\cdot g_{ij}(\tilde{\underline{x}})\cdot\left[\frac{\partial\psi_j}{\partial x_i}(\tilde{\underline{x}})\right]^T$$

$$\frac{\partial G_i^{II}}{\partial\alpha}(\hat{\underline{x}}, 0) = \sum_{j=1}^{n} u_i''(h_i(\hat{\underline{x}}))\cdot E\left\{r^T\cdot g(\hat{\underline{x}})\cdot s_j + [\psi(\hat{\underline{x}})]^T\cdot S\cdot g_i(\hat{\underline{x}})\cdot s_j\right\}$$

$$\cdot g_{ij}(\hat{\underline{x}})\cdot\left[\frac{\partial\psi_j}{\partial x_i}(\hat{\underline{x}})\right]^T .$$

By definition: $E\left\{s_j \cdot s\right\} = W_{*j}$ ($j^{th}$ column of $W$) ,

$$E\left\{r_j \cdot s_k\right\} = 0 \ .$$

So,

$$\frac{\partial G_i^{II}}{\partial \alpha}(\hat{\underline{x}}, 0) = u_i''(h_i(\hat{\underline{x}})) \cdot \sum_{j=1}^{n} \left[\psi(\hat{\underline{x}})\right]^T \cdot E\left\{\begin{bmatrix} s_j s_1 g_{i1}(\hat{\underline{x}}) \\ \vdots \\ s_j s_n g_{in}(\hat{\underline{x}}) \end{bmatrix}\right\} \cdot g_{ij}(\hat{\underline{x}}) \cdot \left[\frac{\partial \psi_j}{\partial x_i}(\hat{\underline{x}})\right]^T$$

$$= u_i''(h_i(\hat{\underline{x}})) \cdot \sum_{j=1}^{n} \left[\psi(\hat{\underline{x}})\right]^T \cdot \mathrm{diag}(g_i(\hat{\underline{x}})) \cdot W_{*j} \cdot g_{ij}(\hat{\underline{x}}) \cdot \left[\frac{\partial \psi_j}{\partial x_i}(\hat{\underline{x}})\right]^T$$

$$= u_i''(h_i(\hat{\underline{x}})) \cdot \left[\frac{\partial \psi}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \mathrm{diag}(g_i(\hat{\underline{x}})) \cdot W \cdot \mathrm{diag}(g_i(\hat{\underline{x}})) \cdot \psi(\hat{\underline{x}}) \ .$$

Therefore:

$$G_i^{II}(\tilde{\underline{x}}, \alpha) \simeq \alpha \cdot u_i''(h_i(\hat{\underline{x}})) \cdot \left[\frac{\partial \psi}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \mathrm{diag}(g_i(\hat{\underline{x}})) \cdot W \cdot \mathrm{diag}(g_i(\hat{\underline{x}})) \cdot \psi(\hat{\underline{x}}) \ ,$$

$$i = 1, \ldots, m. \qquad \text{(A.11)}$$

Define: $$G_i^{III}(\tilde{\underline{x}}, \alpha) = E\left\{u_i'(Y_i(\tilde{\underline{x}}, \alpha)) \cdot \left[\frac{\partial g_i}{\partial x_i}(\tilde{\underline{x}})\right]^T \cdot S \cdot \psi(\tilde{\underline{x}})\right\} .$$

Then, proceeding in the same way as for $G_i^{II}(\tilde{\underline{x}}, \alpha)$, it follows that

$$G_i^{III}(\tilde{\underline{x}}, \alpha) \simeq \alpha \cdot u_i''(h_i(\hat{\underline{x}})) \cdot \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot \mathrm{diag}(\psi(\hat{\underline{x}})) \cdot W \cdot \mathrm{diag}(\psi(\hat{\underline{x}})) \cdot g_i(\hat{\underline{x}}) \ ,$$

$$i = 1, \ldots, m. \qquad \text{(A.12)}$$

Define: $\rho_i(Y_i) := -\dfrac{u_i''(Y_i)}{u_i'(Y_i)}$ ,

$$g_i^{\psi}(x) := \begin{bmatrix} g_{i1}(x)\cdot\psi_1(x) \\ \vdots \\ g_{in}(x)\cdot\psi_n(x) \end{bmatrix} .$$

Then, (A8) - (A12) combined gives:

$$F_i = -\alpha(G_i^{I} + G_i^{II} + G_i^{III}) ,$$

or

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T\cdot(\tilde{\underline{x}} - \hat{\underline{x}}) = \alpha^2\cdot\rho_i(h_i(\hat{\underline{x}}))\cdot\left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T\cdot V\cdot g_i(\hat{\underline{x}})$$

$$+ \alpha^2\cdot\rho_i(h_i(\hat{\underline{x}}))\cdot\left[\frac{\partial\psi}{\partial x_i}(\hat{\underline{x}})\right]^T\cdot\mathrm{diag}(g_i(\hat{\underline{x}}))\cdot W\cdot g_i^{\psi}(\hat{\underline{x}})$$

$$+ \alpha^2\cdot\rho_i(h_i(\hat{\underline{x}}))\cdot\left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T\cdot\mathrm{diag}(\psi(\hat{\underline{x}}))\cdot W\cdot g_i^{\psi}(\hat{\underline{x}}) .$$

$$= \alpha^2\cdot\rho_i(h_i(\hat{\underline{x}}))\cdot\left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T\cdot V\cdot g_i(\hat{\underline{x}}) + \alpha^2\cdot\rho_i(h_i(\hat{\underline{x}}))\cdot\left\{\left[\frac{\partial\psi}{\partial x_i}(\hat{\underline{x}})\right]^T\right.$$

$$\left.\cdot\mathrm{diag}(g_i(\hat{\underline{x}})) + \left[\frac{\partial g_i}{\partial x_i}(\hat{\underline{x}})\right]^T\cdot\mathrm{diag}(\psi(\hat{\underline{x}}))\right\}\cdot W\cdot g_i^{\psi}(\hat{\underline{x}}) , \quad i=1,\dots,m.$$

Then, with (A6) and (A7) :

$$\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot (\tilde{\underline{x}} - \hat{\underline{x}}) = -\alpha^2 \cdot \rho_i(h_i(\hat{\underline{x}})) \cdot \frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T \cdot$$

$$\cdot \left\{\frac{\partial \overset{o}{\underline{x}}}{\partial a} \cdot V \cdot g_i(\hat{\underline{x}}) + \frac{\partial \overset{o}{\underline{x}}}{\partial b} \cdot W \cdot g_i^{\psi}(\hat{\underline{x}})\right\} \quad , \; i = 1, \ldots, m \, .$$

which is equation (3.5).

Note that the dimensions of the terms in (3.5) are:

| | | |
|---|---|---|
| $\frac{\partial}{\partial x}\left[\frac{\partial h_i}{\partial x_i}(\hat{\underline{x}})\right]^T$ | : | $n_i \times \sum_{i=1}^{m} n_i$ |
| $\tilde{\underline{x}} - \hat{\underline{x}}$ | : | $\sum_{i=1}^{m} n_i \times 1$ |
| $\alpha, \; \rho_i(h_i(\hat{\underline{x}}))$ | : | $1 \times 1$ |
| $V, \; W$ | : | $n \times n$ |
| $g_i(\hat{\underline{x}}), \; g_i^{\psi}(\hat{\underline{x}})$ | : | $n \times 1$ |
| $\frac{\partial \overset{o}{\underline{x}}}{\partial a}, \frac{\partial \overset{o}{\underline{x}}}{\partial b}$ | : | $\sum_{i=1}^{m} n_i \times n$ . |