Revised, December 1983

UNOBSERVED-COMPONENTS MODELS FOR SEASONAL ADJUSTMENT FILTERS

Peter Burridge and Kenneth F. Wallis

NO. 244

Time series models are presented for which the seasonal component estimates delivered by linear least squares signal extraction closely approximate those of the standard option of the widely-used Census X-11 program. Earlier work is extended by consideration of a broader class of models and by examination of asymmetric filters in addition to the symmetric filter implicit in the adjustment of historical data. Various criteria that guide the specification of unobserved-component models are discussed, and a new preferred model is presented. Other models generate filters that approximate X-11 rather well, explaining the wide acceptance of the X-11 method.

Key words: time series, seasonal adjustment, signal extraction, X-11 method, unobserved-components models, ARIMA models.

The computational assistance of Paul Fisher and the helpful comments of J.P. Burman, W.P. Cleveland and two referees are gratefully acknowledged. This research was supported by a grant from the Social Science Research Council.

1. INTRODUCTION

Procedures for the seasonal adjustment of economic time series used by official statistical agencies are often criticised for their ad hoc nature. While practical statisticians justify these procedures on intuitive and pragmatic grounds, pointing to their apparent success in satisfying the demands of users of statistics, theorists point to the absence of well-specified objectives and criteria of performance. Indeed, the very notions of the three components of a series that underly the practical procedures - trend-cycle, seasonal and irregular - are not well-defined.

Recently-developed alternative techniques rest on a more formal specification of the problem. Given stochastic models for the unobserved components, and a linear least squares criterion, classical signal extraction theory as described, for example, by Whittle (1963) can be applied to obtain an "optimal" estimate of the deseasonalized series. The seasonal adjustment problem was formulated in this way by Grether and Nerlove (1970), who assumed stationarity of the components and hence of the observed series. Subsequent work (Cleveland, 1972; Cleveland and Tiao, 1976; Box, Hillmer and Tiao, 1978; Burman, 1980; Hillmer and Tiao, 1982) has considered extensions to series that are adequately represented by models of the Box-Jenkins seasonal ARIMA class and so can be reduced to stationarity by differencing. A practical difficulty in implementing seasonal adjustment methods based on optimal signal extraction theory is that of specifying models for the unobserved components of the observed series, and while various suggestions have been made, the resulting methods have not yet been widely adopted by official statistical agencies.

In this paper we use the signal extraction approach to study the properties of a widely-used seasonal adjustment procedure, namely the Census Bureau's X-11 method (Shiskin et al., 1967), which we represent as a set of linear filters, as in Wallis (1982). These filters range from the one-sided moving average implicit in the preliminary adjustment of the current observation, through a number of asymmetric moving averages, to the symmetric moving average implicit in the adjustment of historical data. We present models whose optimal signal extraction filters virtually coincide with certain of these moving averages, and for which X-11 therefore provides the linear least squares estimate of the seasonally adjusted series. Our analysis extends that of Cleveland and Tiao (1976), who consider a truncated version of the symmetric X-11 filter, and Wallis (1981), who presents seasonal ARIMA models for the observed series for which the asymmetric X-11 filters minimize revisions in the seasonally adjusted series. Many statistical agencies run the X-11 program only once a year, and at that time project seasonal factors for the adjustment of the next twelve months' data, to be used as the data become available. Kenny and Durbin (1982) and Wallis (1982) have argued that this practice should be replaced by running the program every month, and it is not considered in the present paper.

The signal extraction interpretation of the X-11 filters is described in Section 2, and the models for which X-11 represents the optimal procedure in this sense are presented in Section 3. Section 4 contains concluding comments.

2. THE X-11 LINEAR FILTERS AND SIGNAL EXTRACTION MODELS

2.1 The X-11 linear filters

The X-11 procedure assumes that the observed monthly time series variable Y_t is made up of three unobserved components, namely the trend-cycle, seasonal and irregular components, denoted C_t, S_t and I_t respectively. We work with the additive decomposition

$$(2.1) \qquad Y_t = C_t + S_t + I_t ;$$

the program also includes a multiplicative option, which is essentially the same as (2.1) on taking logarithms. The seasonal adjustment problem is to obtain an estimate $\hat{S}_t$ of the seasonal component and subtract it from the original series, yielding the seasonally adjusted series

$$(2.2) \qquad Y_t^a = Y_t - \hat{S}_t .$$

The fact the X-11 program provides a decomposition of the original series into three estimated components, but we restrict attention to the seasonal component; we also neglect the option of graduating extreme values of the estimated irregular component. The program comprises a sequence of moving average or linear filter operations, whose net effect can be represented by a single set of moving averages. For a date sufficiently far in the past, the final or historical adjusted value $Y_t^{(m)}$ is obtained by application of the symmetric filter $a_m(L)$,

$$Y_t^{(m)} = a_m(L)\, Y_t = \sum_{j=-m}^{m} a_{m,j}\, Y_{t-j} ,$$

where L is the lag operator, and $a_{m,j} = a_{m,-j}$ for symmetry. For current and recent data this filter cannot be applied, and truncated, asymmetric filters are employed:

$$(2.3) \qquad Y_t^{(k)} = a_k(L)\, Y_t = \sum_{j=-k}^{m} a_{k,j}\, Y_{t-j}\,, \qquad k = 0,1,\ldots,m.$$

For the filter $a_k(L)$, the subscript k indicates the number of "future" values of Y entering the moving average; equivalently the superscript on Y indicates that $Y_t^{(k)}$ is the adjusted value of Y_t calculated from observations $Y_{t-m}, Y_{t-m+1}, \ldots, Y_t, \ldots, Y_{t+k}$. Thus $Y_t^{(o)}$ is the <u>first-announced</u> or <u>preliminary</u> seasonally adjusted figure. For the X-11 filters considered here the value of m is 84 (it is assumed that at the stage at which the program chooses a 9-, 13, or 23- term Henderson moving average to estimate the trend-cycle component, the 13-term average is chosen). The program is then represented as a set of 85 linear filters, in which respectively 0,1,...,84 "future" data-points appear. Although m=84 implies that seven years' past data are required to calculate a current seasonally adjusted value, the remote weights are very small.

A useful description of the filters is provided by the frequency response function

$$a_k(\omega) = \Sigma a_{k,j}\, e^{-i\omega j} = |\, a_k(\omega)\, |\, e^{i\theta_k(\omega)}\,,$$

the squared gain $|a_k(\omega)|^2$, or transfer function of the filter, representing the extent to which the contribution of the component of frequency ω to the total variance of the series is modified by the action of the filter. The weights and transfer functions of three X-11 filters of particular interest, $a_o(L)$, $a_{12}(L)$ and $a_{84}(L)$, are presented in Figures 1-3 where they act as benchmarks for the signal extraction filters described below. Further details of their calculation may be found in Wallis (1982), and a listing of the coefficients is available on request.

2.2 Linear least squares signal extraction

In the signal extraction literature the unobserved components introduced in (2.1) are treated as uncorrelated random processes, and the signal extraction problem is to "estimate" S_t, say, from observations on Y. The linear least squares approach to this problem is to construct a linear filter

(2.4) $$\hat{S}_t = f_k(L)\, Y_t$$

so as to minimise the mean square error $E(S_t - \hat{S}_t)^2$, where k again indexes the number of future observations available as of time t.

The classical theory assumes that the autocovariances of the unobserved components and hence of the observed variable are known. Both in theoretical work and in practical implementation this requirement has been met by postulating linear models for the components, and expressing the autocovariances as functions of those models' parameters. Accordingly, we consider component models of the following form

(2.5) $$\phi_c(L)\, C_t = \theta_c(L)\, u_t\ , \quad \phi_s(L)\, S_t = \theta_s(L)\, v_t\ , \quad I_t = w_t$$

where u_t, v_t and w_t are uncorrelated normally distributed white noise series. Particular specifications are achieved by imposing restrictions on the autoregressive and moving average lag polynomials in (2.5); these are discussed in the next section. For the moment we simply assume that each component is "parsimoniously parameterized", that is, that each of the pairs of polynomials $\{\phi_c(L), \theta_c(L)\}$ and $\{\phi_s(L), \theta_s(L)\}$ have no common factors, and also that the pair $\{\phi_c(L), \phi_s(L)\}$ has no common factor. Then the composite model for Y_t is

$$(2.6) \qquad \phi(L)\, Y_t = \beta(L)\, \varepsilon_t ,$$

where $\phi(L) = \phi_c(L)\, \phi_s(L)$ and $\beta(L)$ is obtained as the canonical factorization of the autocovariance generating function of $\phi(L)Y_t$, defined in terms of the parameters of (2.5), namely

$$(2.7) \quad \sigma_\varepsilon^2 |\beta(z)|^2 = \sigma_u^2 |\phi_s(z)\theta_c(z)|^2 + \sigma_v^2 |\phi_c(z)\theta_s(z)|^2 + \sigma_w^2 |\phi_c(z)\phi_s(z)|^2$$

where the notation $|h(z)|^2$ denotes $h(z)h(z^{-1})$.

The signal extraction filters depend, essentially, on the ratio of the second term on the right-hand side of (2.7) to the complete expression. Thus we have

$$(2.8) \qquad f_k(z) = \frac{\sigma_v^2\, \phi_c(z)\, \phi_s(z)}{\sigma_\varepsilon^2\, \beta(z)} \left[\frac{\phi_c(z^{-1})\, \theta_s(z)\, \theta_s(z^{-1})}{\beta(z^{-1})\, \phi_s(z)} \right]_{-k}$$

where $[h(z)]_{-k}$ denotes that part of $h(z)$ containing powers of z greater than or equal to $-k$, and as $k\to\infty$ we obtain the symmetric filter

$$(2.9) \qquad f_\infty(z) = \frac{\sigma_v^2\, |\phi_c(z)\, \theta_s(z)|^2}{\sigma_\varepsilon^2\, |\beta(z)|^2} .$$

These results, as treated extensively by Whittle (1963), rested initially on an assumption of stationarity and a semi-infinite sample $\{Y_\tau;\ \tau \le t + k\}$.

Cleveland and Tiao (1976) and Pierce (1979) have considered extensions to difference-stationary processes, and Burridge and Wallis (1983) show that the same filters can be obtained more generally as the steady-state Kalman filters under appropriate assumptions on initial conditions.

2.3 Model specification

In seeking unobserved-component models which lend a signal-extraction interpretation to the X-11 filters, we postulate various possible forms of the component models (2.5) and in each case choose particular parameter values by matching the corresponding signal extraction filters to the X-11 filters by numerical methods. Since the signal extraction filters are defined in (2.4) as filters that estimate the seasonal component, the correspondence we seek is

$$a_k(L) = 1 - f_k(L), \qquad (2.10)$$

where the $a_k(L)$ are X-11 filters defined in (2.3) and $f_k(L)$ are given, for particular component model specifications, by (2.8). In this section we discuss various criteria that guide the selection of such specifications.

First, preliminary investigation indicated the need to incorporate differencing and seasonal differencing operators, $(1 - L)$ and $(1 - L^{12})$ respectively, in accordance with the common empirical identification of the composite model (2.6) with the Box-Jenkins seasonal ARIMA model. Following Cleveland (1972), Box, Hillmer and Tiao (1978), Burman (1980) and Hillmer and Tiao (1982) we factor the seasonal difference operator as

$$(2.11) \qquad 1 - L^{12} = (1 - L)(1 + L + L^2 + \ldots + L^{11})$$
$$= (1 - L)\, U(L), \text{ say,}$$

and associate $(1 - L)$ and $U(L)$ with, respectively, the trend-cycle and seasonal autoregressive operators, $\phi_c(L)$ and $\phi_s(L)$. While the operator $(1 - L^{12})$ produces spikes in the pseudo-spectrum at frequencies $j\pi/6$, $j = 0,1,\ldots,6$, the factorization identifies the peak at the origin with the ordinary difference operator $(1 - L)$ and properly associates it with a nonseasonal component. This also conforms to the assumption of the previous section that $\phi_c(L)$ and $\phi_s(L)$ have no common factor: Pierce (1979) shows that application of a filter corresponding to (2.8) to a series in which signal and noise processes share a common unit root yields a signal estimate with unbounded variance, while in the Kalman filter context Burridge and Wallis (1983) show that conditions for the convergence of the signal extraction error variance exclude the existence of an unstable common factor, such a factor representing an "undetectable" case, in the state-space terminology.

A second consideration for the autoregressive specification arises from the fact that, in (2.10), $a_k(L)$ is a polynomial in L of finite degree whereas $f_k(L)$ is infinite. In seeking models for which the equivalence holds we ignore the remote coefficients of $f_k(L)$ and fit only the $m + k + 1$ non-zero coefficients of $a_k(L)$. In general, models with a predominantly autoregressive specification, so-called "bottom-heavy" models, generate very long signal extraction filters which are not well able to approximate the relatively rapid decline of the X-11 filter coefficients, especially at multiples of 12 lags. Thus, while in empirical modelling of observed time series one is often indifferent between enhancing an inadequate model on the autoregressive or the moving average side, in the present context there is a clear preference for "top-heavy", or moving-average-dominated specifications. Following Cleveland (1972) and Cleveland and Tiao (1976), the moving average operator $\theta_s(L)$ is written as a polynomial in L^{12}, as in the conventional Box-Jenkins seasonal specification, again ensuring a top-heavy specification.

Thirdly, while in practical time series analysis a three-component decomposition may be useful, identifying in particular an irregular component to assist the analysis of outliers, we note that this may not be possible in the present context. The difficulty is most easily seen by considering the denominator of the symmetric filter (2.9), given as the right-hand side of (2.7), and studying the first and third terms, which do not appear in the numerator of the filter. With $\phi_s(L) = U(L)$ as defined in (2.11), $\phi_c(L) = (1 - L)^d$ and $\theta_c(L)$ a polynomial of degree q with coefficients to be determined, then if $d \leq q$ multiple sets of values of these coefficients give the same filter, in each case being accompanied by appropriate variations in the variances σ_u^2 and σ_w^2. Equivalently we note that the sum of an ARMA (p,q) process and an independent white noise process has an ARMA (p,q) representation provided that $p \leq q$. Thus an emphasis on "top-heavy" or "balanced" models for C_t together with a white noise specification for I_t precludes separate identification of their models from the filter coefficients, and when this occurs we report a model for the composite nonseasonal element $N_t = C_t + I_t$. (The possibility of arbitrarily allocating white noise to the seasonal component, which arises in the empirical identification of components from a given composite model for an observed series, and which must be ruled out by an arbitrary "minimum variance" or "canonical" assumption (Box, Hillmer and Tiao, 1978; Hillmer and Tiao, 1982) does not occur in the present case, where a given filter determines the variance ratio $\sigma_v^2/\sigma_\varepsilon^2$.)

For each model specification of a given form, we find the parameter values that minimize the unweighted sum of squared differences between the X-11 coefficients and those of the truncated signal-extraction-based adjustment filter by numerical methods. That is, over the $m + k + 1$ coefficients of $a_k(L)$, we seek to make (2.10) hold as closely as possible in this sense. Our least squares criterion differs from those used by Cleveland (1972) and Cleveland and Tiao (1976) in two respects. First, the "X-11" filter used by Cleveland omits the intermediate adjustments of the estimated seasonal components (steps (c) and (g) in Wallis' description) and is truncated after 42 terms. Secondly, Cleveland and Tiao fit their model to this truncated seasonal filter and to the corresponding trend filter simultaneously, while we fit only the seasonal adjustment filter. Our results extend the work of these authors by estimating a wider class of models, and by fitting these to two of the asymmetric filters, $a_0(L)$ and $a_{12}(L)$, in addition to the symmetric filter $a_{84}(L)$. This enables us to test whether the same model emerges from the signal extraction interpretation of these filters, hence whether the X-11 filters are internally consistent in this sense.

3. ESTIMATION RESULTS

3.1 Introduction

It is clear from Figures 1-3 that the X-11 filters are dominated by their seasonal weights, and so models whose optimal signal extraction filters match these principal characteristics will tend to give a good overall fit. For example, the elementary model

$$(1 - L)\, C_t = u_t \;, \qquad U(L)\, S_t = v_t \;, \qquad I_t = w_t$$

with appropriately chosen innovation variance ratios achieves a residual sum of squares of 0.023 against the sum of squares of the $a_{84}(L)$ coefficients of 0.785. While we can thus say that it accounts for 97.1 per cent of the variation in the weights, we see below that slightly more elaborate models are able to reduce substantially the residual sum of squares.

Our calculations were organized as follows. Working with a three-component specification, we normalized on the irregular variance $(\sigma_w^2 = 1.0)$ and tested various seasonal and cyclical specifications, in each case calculating the values of the parameters, including σ_u^2 and σ_v^2, that give the best fit. As noted in section 2.3, the seasonal component models have $\phi_s(L) = U(L)$ and $\theta_s(L)$ a polynomial of degree Q in L^{12}. Preliminary investigations led to cyclical component models having $\phi_c(L) = (1 - L)^d$, since estimation of unrestricted autoregressions in general resulted in unit roots being found; $\theta_c(L)$ is a general polynomial of degree q. If $q \geq d$ the decomposition of the nonseasonal into cyclical and irregular components is not unique, as noted above, but this difficulty is avoided at the estimation stage by assigning a convenient value to the cyclical innovation variance.

Within the range of possibilities thus delineated, the choice of a final model is guided by the same considerations that conventionally apply in the empirical modelling of observed time series. A given model is extended if adding a further parameter achieves a substantial reduction in residual sum of squares, but if the additional parameter results only in a marginal improvement, a preference for simple, "parsimonious" models leads us back to the initial specification.

3.2 Models for $a_o(L)$, $a_{12}(L)$ and $a_{84}(L)$

The specification eventually chosen, for each of these three filters, has $d = 2$, $q = 2$ and $Q = 2$: reducing the order of either moving average specification substantially increases the residual sum of squares, while including a third parameter causes only a slight decrease. A closely competing specification has $d = 1$, $q = Q = 2$, but this results in a combined residual sum of squares some 10 per cent greater.

For the one-sided filter, $a_o(L)$, the preferred model is

$$\begin{aligned} U(L)\ S_t &= (1 + 1.00\ L^{24})\ v_t \\ (1 - L)^2\ N_t &= (1 - 1.43\ L + 0.70\ L^2)\ \eta_t \end{aligned} \qquad (3.1)$$

$\sigma_v^2/\sigma_\eta^2 = 0.018$, RSS = 0.041, original sum of squares = 0.931

The coefficients in $\theta_s(L)$ are estimated, but the coefficient of L^{12} is non-zero only in its fourth decimal place, and is not reported. The associated signal extraction adjustment filter, $\tilde{a}_o(L)$, and its transfer function are plotted in Figure 4. We find that, of the three filters considered, the goodness of fit is poorest in the one-sided case, principally because in the X-11 filter (Figure 1), the weight at lag 24 is greater than that at lag 12, while in the signal extraction filters such weights decline smoothly. This feature of X-11 is possibly due to the ad-hoc adjustment of end-weights of some of the component moving averages, which has been remarked upon elsewhere (Kenny and Durbin, 1982; Wallis, 1981).

For the asymmetric filter $a_{12}(L)$, the preferred model is

$$U(L)\ S_t = (1 + 0.33\ L^{12} + 0.99\ L^{24})\ v_t \qquad (3.2)$$

$$(1 - L)^2\ N_t = (1 - 1.55\ L + 0.82\ L^2)\ \eta_t$$

$$\sigma_v^2/\sigma_\eta^2 = 0.026, \quad RSS = 0.017, \quad \text{original sum of squares} = 0.717$$

The coefficients and transfer function of the associated filter $\tilde{a}_{12}(L)$ are plotted in Figure 5. A particular feature of the transfer function for the X-11 filter $a_{12}(L)$ (Figure 2) is that it does not appear to represent a step in a gradual transition from the one-sided to the symmetric filter: the characteristic oscillations between the seasonal frequencies are absent, the seasonal dips cover a rather wide frequency band, and the initial departure from 1 occurs at a rather lower frequency. It is interesting that the transfer function of $\tilde{a}_{12}(L)$ has the same appearance, thus these features seem to be associated with the lag-12 filter, even when it is an optimal signal extraction filter for a model of the same form as (3.1).

For the symmetric filter $a_{84}(L)$ the preferred model is

$$(3.3)\qquad \begin{aligned} U(L)\, S_t &= (1 + 0.71\, L^{12} + 1.00\, L^{24})\, v_t \\ (1 - L)^2\, N_t &= (1 - 1.59\, L + 0.86\, L^2)\, \eta_t \end{aligned}$$

$\sigma_v^2/\sigma_\eta^2 = 0.017$, RSS = 0.0036, original sum of squares = 0.785

It is in this case that the closest match of a signal extraction filter to an X-11 filter is achieved, and the coefficients and transfer function of $\tilde{a}_{84}(L)$ are plotted in Figure 6, for comparison with those of $a_{84}(L)$ in Figure 3.

The composite model for the observed series corresponding to the unobserved-components models (3.1) - (3.3) is

$$(3.4)\qquad (1 - L)(1 - L^{12})\, Y_t = \beta(L)\, \varepsilon_t\ ,$$

where the moving average operator, $\beta(L)$, is of degree 26. Its coefficients are given, for each of the three cases, in Table 1. While the higher-order coefficients are negligible, intermediate coefficients at lags 2-11 are not, thus the common multiplicative specification $(1 - \theta L)(1 - \Theta L^{12})$ can only be an approximation to $\beta(L)$. Simple specifications for the component models typically do not yield simple overall models! The three composite models are rather similar, and in broad outline also correspond to the model of Cleveland and Tiao (1976), although their decomposition into seasonal and nonseasonal components is somewhat different.

Comparing the component models, we note that the moving average specifications are more similar than might appear at first sight. Each moving average operator has a pair of complex conjugate roots, and expressing

TABLE 1

Coefficients of the composite moving average operator $\beta(L) = 1 + \beta_1 L + \ldots + \beta_{26} L^{26}$ for component models (3.1) - (3.3)

Lag	(3.1)	(3.2)	(3.3)
1	-.53	-.64	-.67
2	.28	.30	.29
3	.23	.23	.23
4	.23	.22	.22
5	.22	.22	.21
6	.22	.21	.20
7	.21	.20	.20
8	.21	.20	.19
9	.20	.19	.19
10	.19	.18	.18
11	.14	.13	.12
12	-.43	-.33	-.33
13	.45	.44	.45
14	-	.02	.02
15	-	-	-
16	-	-	-
17	-	-	-
18	-	-	-
19	-	-	-
20	-	-	-
21	-	-	-
22	-	-	-
23	-.01	-.01	-.01
24	.05	.05	.04
25	-.04	-.05	-.04
26	.01	.02	.01
$\sigma_v^2/\sigma_\varepsilon^2$	.012	.016	.011

\- denotes a coefficient less than 0.01 in absolute value these in polar form gives the modulus and angle as follows:

	seasonal MA		nonseasonal MA	
	modulus	angle	modulus	angle
(3.1)	0.999	0.50π	0.824	0.83π
(3.2)	0.997	0.45π	0.910	0.82π
(3.3)	0.999	0.38π	0.930	0.83π

To consider the impact of these differences in parameter values together with the observed variations in variance ratios, we calculate the residual sum of squares in fitting to each of the three X-11 filters the signal extraction filters associated with each of the estimated models (3.1) - (3.3), with the following results.

	X-11 filter		
Model	$a_o(L)$	$a_{12}(L)$	$a_{84}(L)$
(3.1)	0.0413	0.0236	0.0151
(3.2)	0.0547	0.0168	0.0043
(3.3)	0.0561	0.0175	0.0036

The "diagonal" element is the smallest in each column, representing the best-fitting model already chosen, but we see that the alternative estimated models fit a given filter relatively well, models (3.2) and (3.3) being rather close together and fitting better overall than model (3.1). The perceived differences between the estimated models indicate that, with a signal extraction interpretation, the X-11 filters are not internally consistent. Nevertheless these differences are seen to be slight, indicating that the inconsistency is not great, although the fact that $a_o(L)$ is the outlier among the three filters considered is a matter for concern, given that public attention is usually focussed most heavily on the first-announced seasonally adjusted figure, and that less attention is usually paid to subsequent revisions.

Our finding that models with $q = Q = 2$ and $d = 1$ are close competitors, at their best-fitting parameter values, to the models (3.1) - (3.3), and that much simpler models can also fit rather well, indicates a certain robustness of the X-11 filters in this context. This corresponds to the general acceptance of X-11 across a wide range of observed series. To investigate this further within the framework of our preferred model specification, we have studied the sensitivity of the residual sum of squares to parameter variation. With respect to the innovation variance ratio σ_v^2/σ_η^2 , increasing this lengthens the signal extraction filter and so worsens the approximation at longer lags. Since it is the rather small variance ratio that makes the higher-order coefficients in the composite moving average $\beta(L)$ negligible (Table 1), this suggests that the optimal signal extraction interpretation of X-11 does not hold up well for composite models with moving average specifications of higher order than the widely-used $(1 - \theta L)(1 - \Theta L^{12})$. With respect to the moving average parameters, we have considered variations in their values accompanied by compensating variations in the innovation variances that keep the variances of $U(L)\ S_t$ and $(1 - L)^2\ N_t$ in the same proportion. Then the goodness of fit is more sensitive to changes in the nonseasonal moving average parameters, so much so that the roots of the seasonal moving average operator can be driven quite close to zero without dramatically worsening the fit. Again this is a reflection of the variance ratio, and suggests that a wide range of seasonal component specifications can be well accommodated in X-11. Variations in the nonseasonal moving average parameters have a greater effect on the composite moving average $\beta(L)$, suggesting that the appropriateness of X-11 for a series adequately represented by the seasonal ARIMA $(0,1,1) \times (0,1,1)_{12}$ specification would depend on the similarity of its key parameters to those given in Table 1.

4. CONCLUSION

The models (3.1) - (3.3) represent data generation processes for which seasonal adjustment (at lags 0, 12 and 84 respectively) is accomplished virtually optimally, in a linear least squares sense, by the X-11 program. Our analysis extends previous work by considering asymmetric X-11 filters as well as the symmetric "historical" filter, and by searching over a wider class of models. Some divergence from previous results is also to be expected as we concentrate exclusively on the seasonal adjustment filter, unlike Cleveland (1972) and Cleveland and Tiao (1976), who include the trend filter in their fitting exercises.

The overall model for the observed series implied by these component models is similar in broad outline to that of Cleveland and Tiao, although there are differences in detail. In particular, since in our models β_{13} is greater in absolute value than β_{12}, the common multiplicative specification $(1 - \theta L)(1 - \Theta L^{12})$ is a less appropriate approximation. However greater differences are to be found in the specification of the components. The use of both trend and seasonal filters allows Cleveland and Tiao to specify a three-component model, whereas in the context of our preferred forms and fitting only the seasonal filter only a two-component specification is possible. Our seasonal component incorporates the autoregressive operator $U(L) = 1 + L + \ldots + L^{11}$, whereas theirs has the seasonal difference $1 - L^{12}$. While the process of specifying the components has some essentially arbitrary elements, our choice avoids associating a spectral peak at the origin with the seasonal component and, given that the non-seasonal autoregression includes the factor $(1 - L)$, ensures that the signal extraction error variance converges to a finite limit, both of which seem reasonable. Thus we suggest that the models (3.1) - (3.3) should replace that of Cleveland and Tiao as the "standard" interpretation of X-11 in a pure seasonal adjustment context. Since these models are not identical, and so the X-11 filters are not internally consistent in a signal extraction sense, strictly speaking, the particular model (3.1), (3.2) or (3.3) to be adopted as a justification of X-11 depends on whether one is considering the adjustment of current data, of one-year-old data, or of historical data.

When amendments or extensions to existing procedures are proposed, it is conventional to evaluate them by studying their effect on a sample of real or artificial time series. Our results enable us to predict the results of comparisons between model-based signal extraction methods of seasonal adjustment and X-11: for data well-described or generated by the models of section 3, little improvement will be observed. To demonstrate any advantage of a signal extraction method over X-11 it will be necessary to choose data of a considerably different form.

REFERENCES

BOX, G.E.P., HILLMER, S.C. and TIAO, G.C. (1978). Analysis and modeling of seasonal time series. In _Seasonal Analysis of Economic Time Series_ (A. Zellner, ed.), pp. 309-334. Washington, DC: Bureau of the Census.

BOX, G.E.P. and JENKINS, G.M. (1970). _Time Series Analysis, Forecasting and Control_. San Francisco: Holden-Day.

BURMAN, J.P. (1980). Seasonal adjustment by signal extraction. _Journal of the Royal Statistical Society_ A, 143, 321-337.

BURRIDGE, P. and WALLIS, K.F. (1983). Signal extraction in non-stationary series. Warwick Economic Research Paper No. 234.

CLEVELAND, W.P. (1972). _Analysis and Forecasting of Seasonal Time Series_. Ph.D. Dissertation, University of Wisconsin.

CLEVELAND, W.P. and TIAO, G.C. (1976). Decomposition of seasonal time series: a model for the Census X-11 program. _Journal of the American Statistical Association_, 71, 581-587.

GRETHER, D.M. and NERLOVE, M. (1970). Some properties of "optimal" seasonal adjustment. _Econometrica_, 38, 682-703.

HILLMER, S.C. and TIAO, G.C. (1982). An ARIMA-model-based approach to seasonal adjustment. _Journal of the American Statistical Association_, 77, 63-70.

KENNY, P.B. and DURBIN, J. (1982). Local trend estimation and seasonal adjustment of economic and social time series (with discussion). _Journal of the Royal Statistical Society_ A, 145, 1-41.

PIERCE, D.A. (1979). Signal extraction error in nonstationary time series. _Annals of Statistics_, 7, 1303-1320.

SHISKIN, J., YOUNG, A.H. and MUSGRAVE, J.C. (1967). The X-11 variant of the Census Method II seasonal adjustment program. Technical paper no. 15 (revised). Washington, DC: Bureau of the Census.

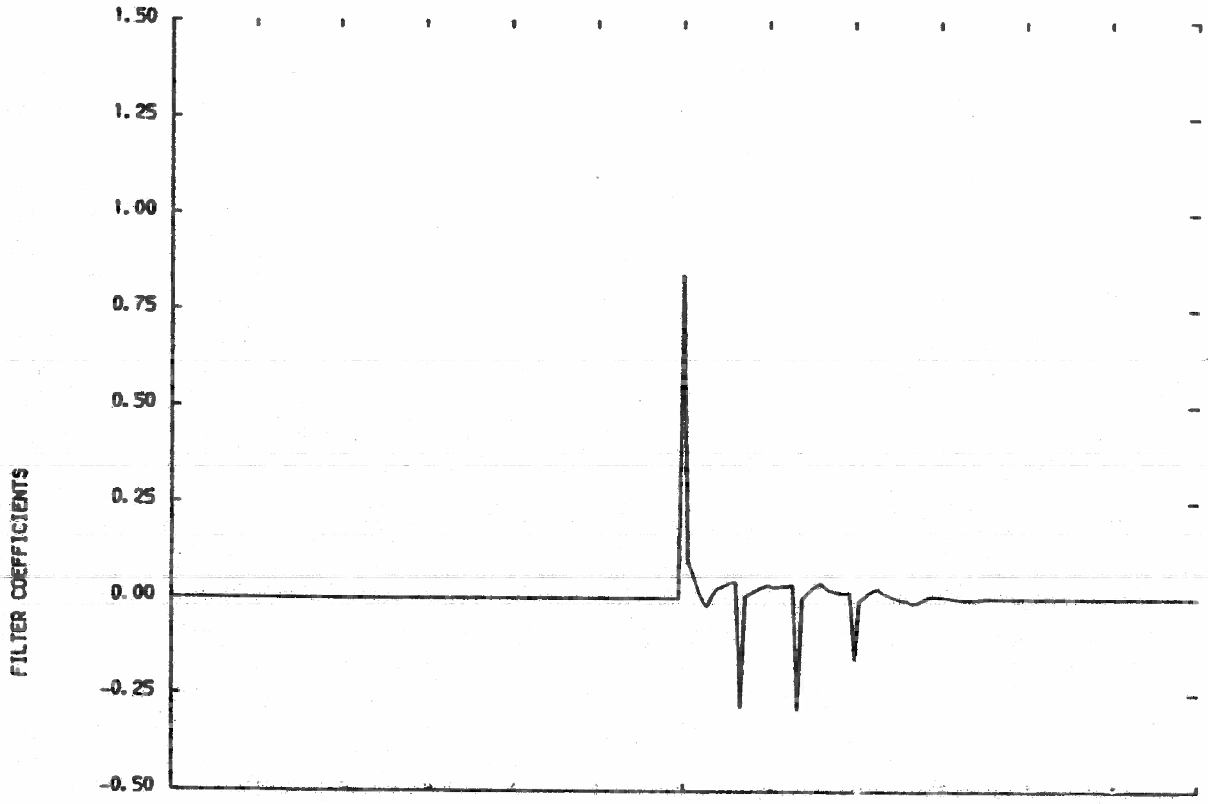
WALLIS, K.F. (1981). Models for X-11 and 'X-11-FORECAST' procedures for preliminary and revised seasonal adjustments. Presented at the ASA-CENSUS-NBER Conference on Applied Time Series Analysis of Economic Data, Washington, DC.

WALLIS, K.F. (1982). Seasonal adjustment and revision of current data : linear filters for the X-11 method. _Journal of the Royal Statistical Society_ A, 145, 74-85.

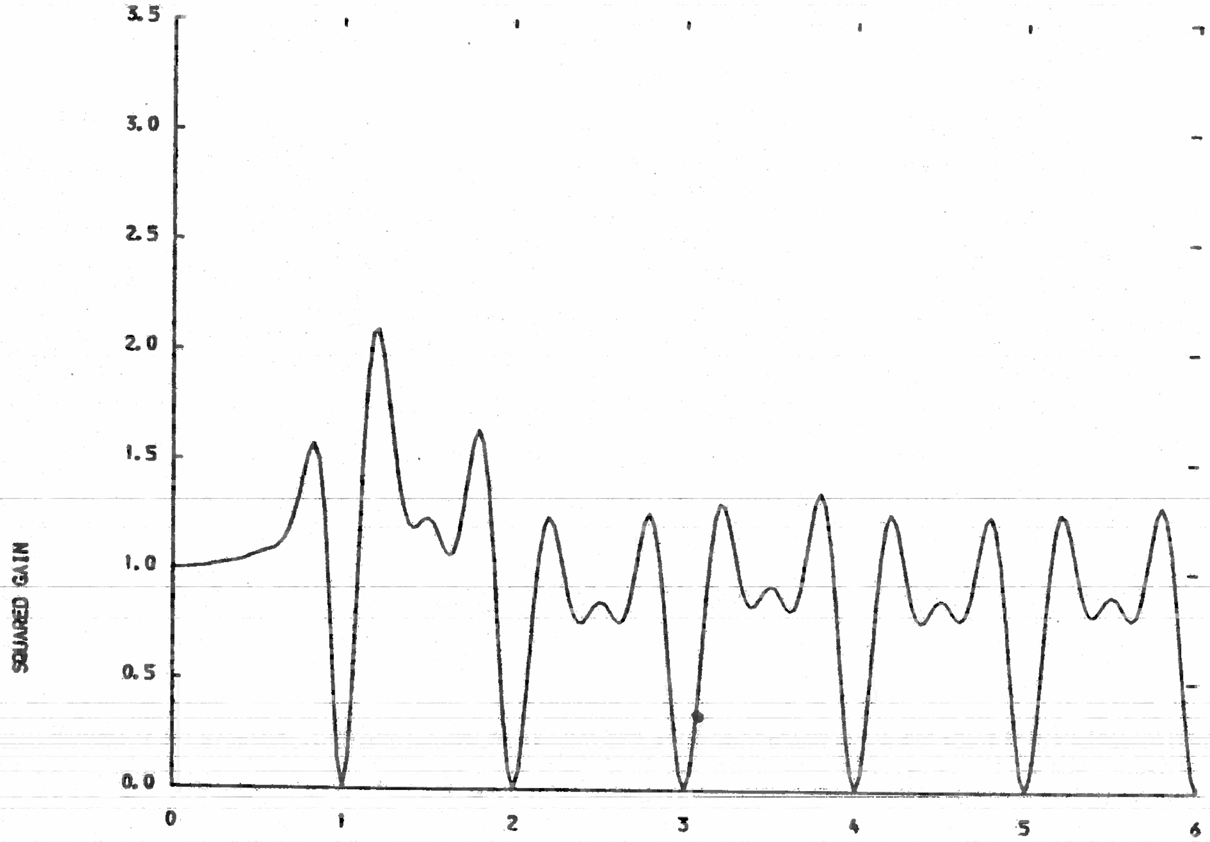
WHITTLE, P. (1963). _Prediction and Regulation by Linear Least-Square Methods_. London: English Universities Press.

LIST OF FIGURES

Figure 1

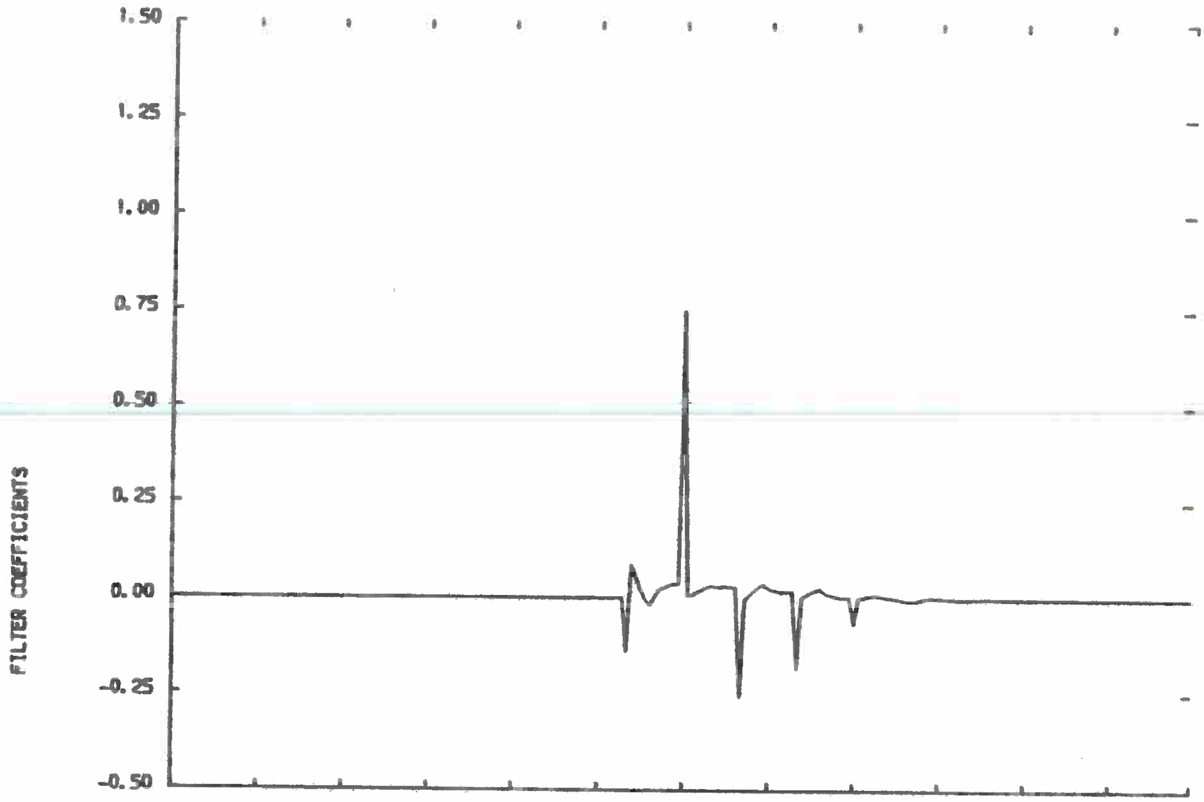


1.50
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
FILTER COEFFICIENTS



3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
SQUARED GAIN
0
1
2
3
4
5
6

Figure 2


1.50
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
FILTER COEFFICIENTS

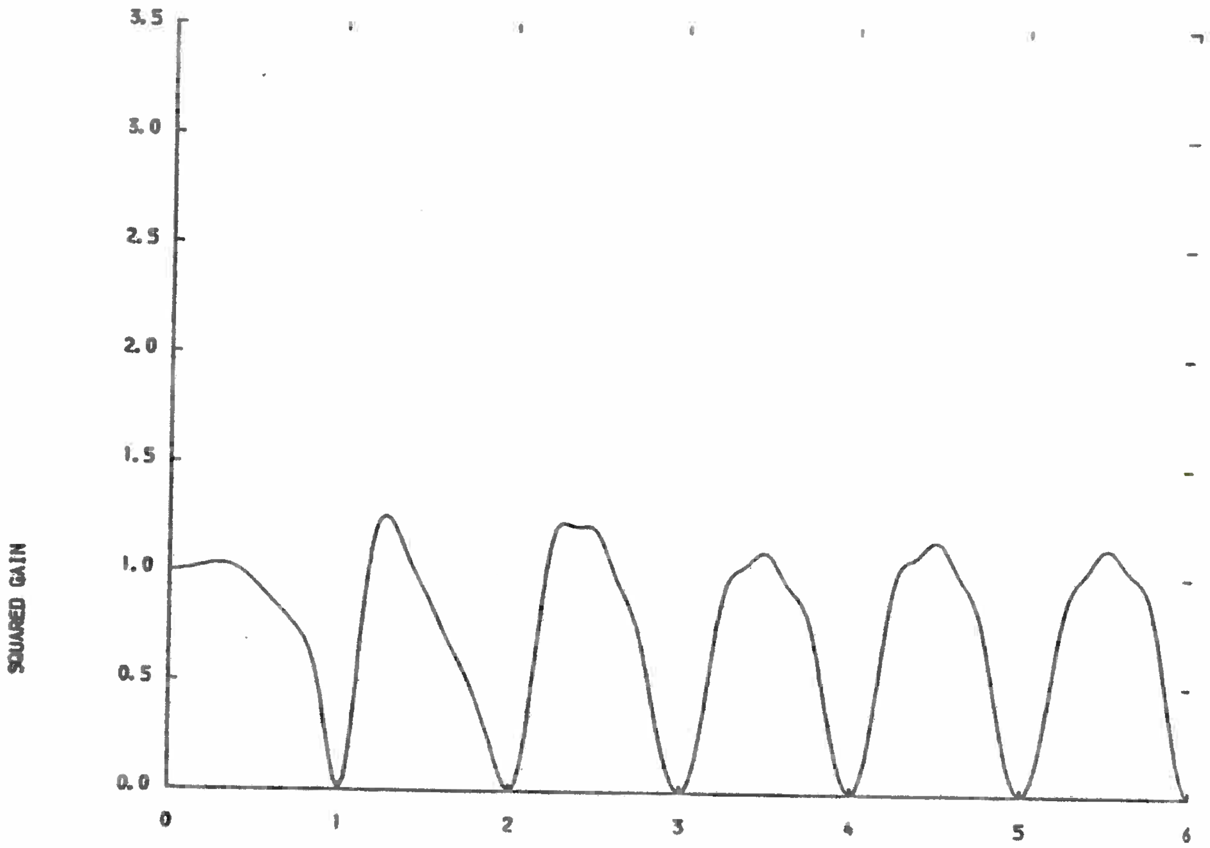
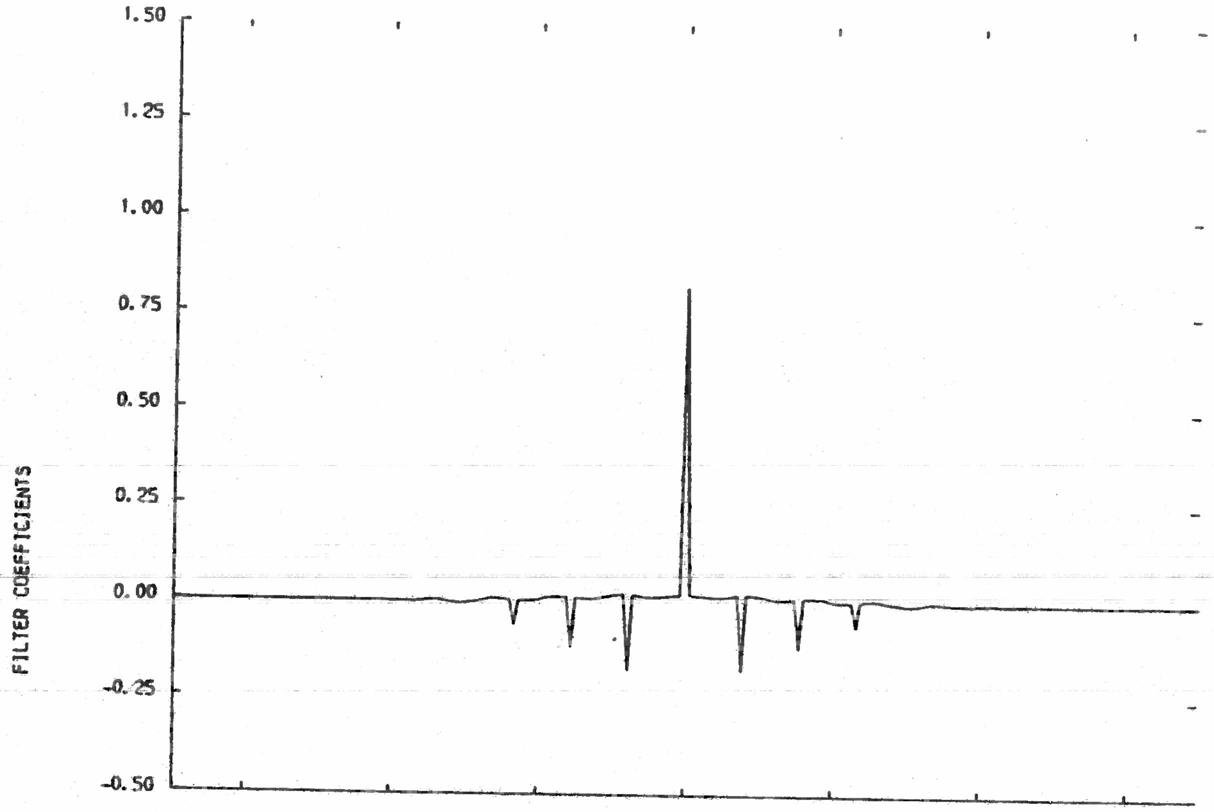
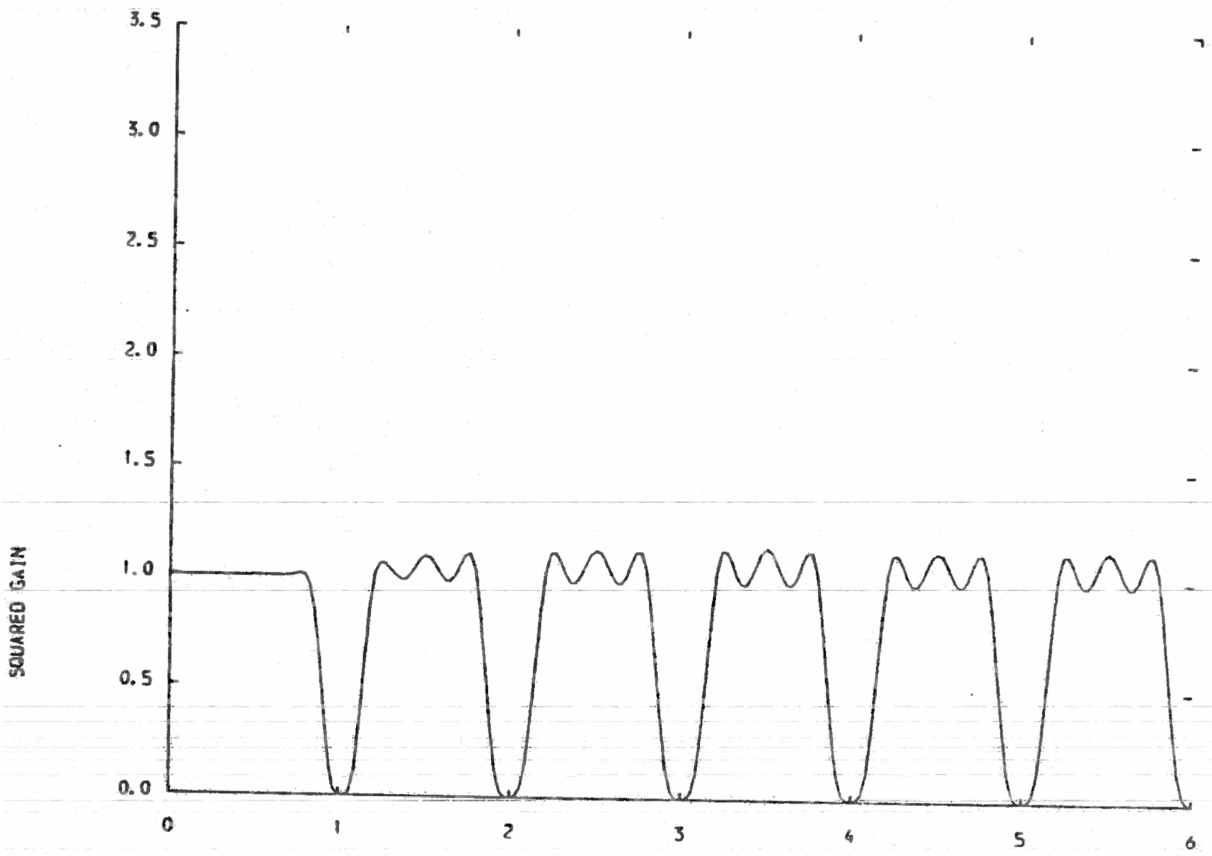

3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
SQUARED GAIN
0
1
2
3
4
5
6

Figure 3

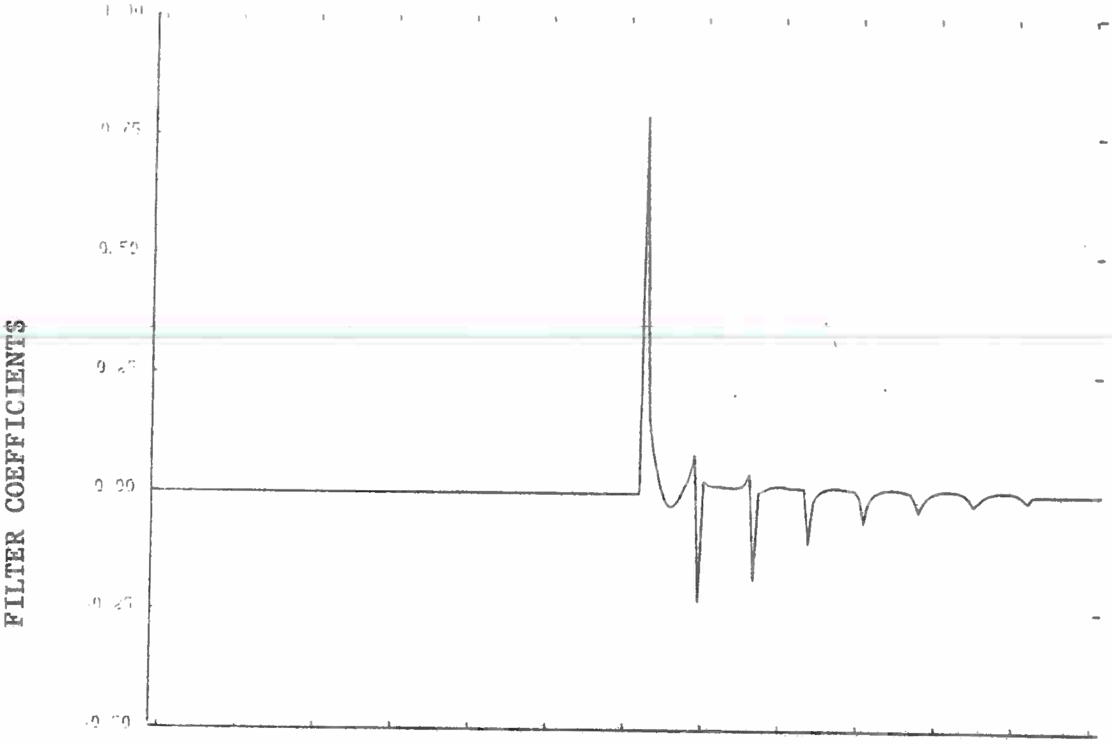


1.50
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
FILTER COEFFICIENTS

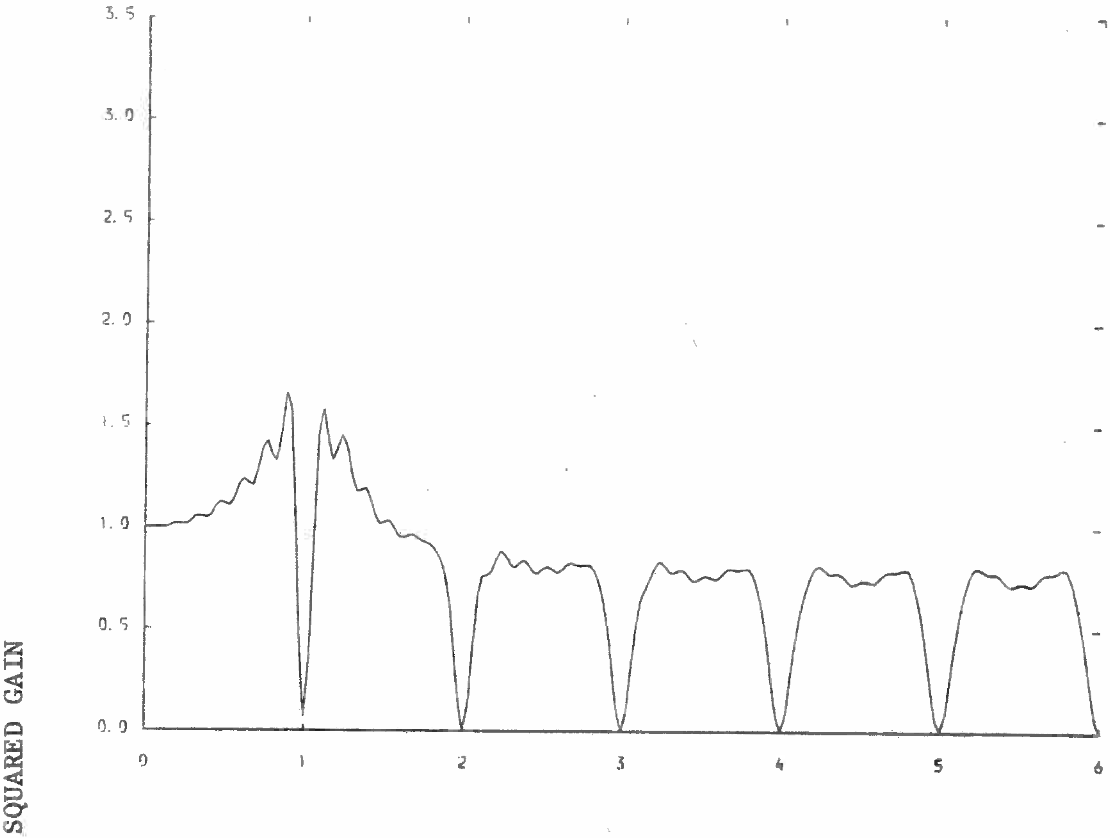


3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
SQUARED GAIN
0
1
2
3
4
5
6

Figure 4

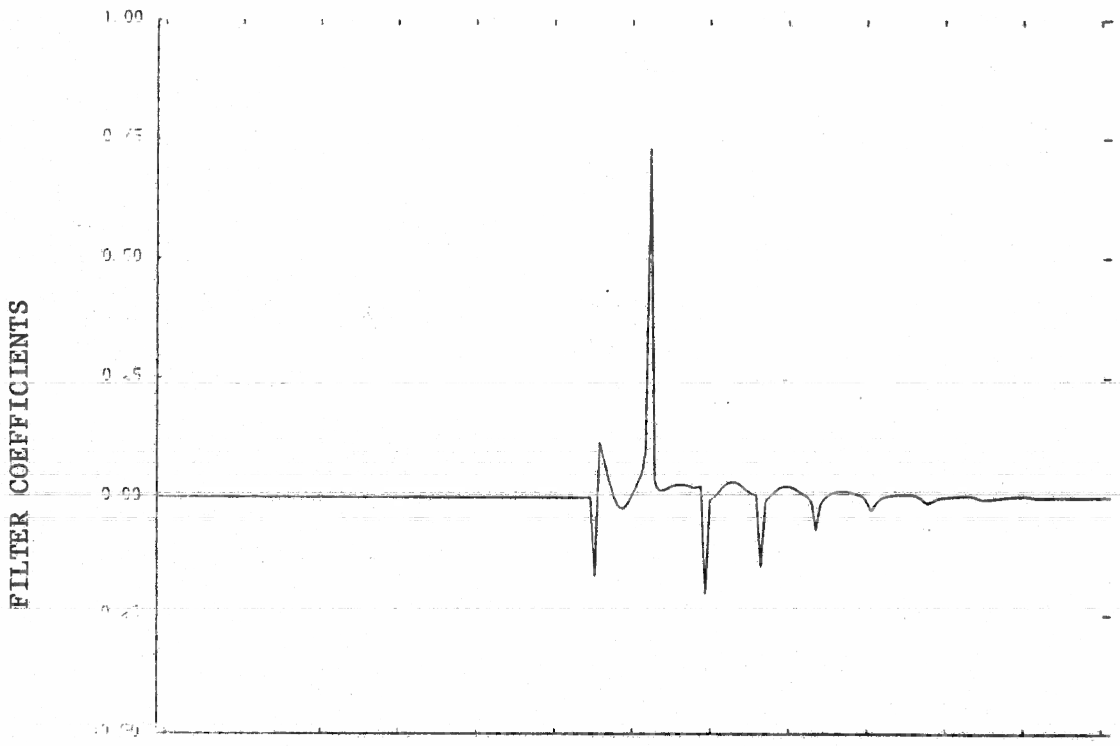


FILTER COEFFICIENTS
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50

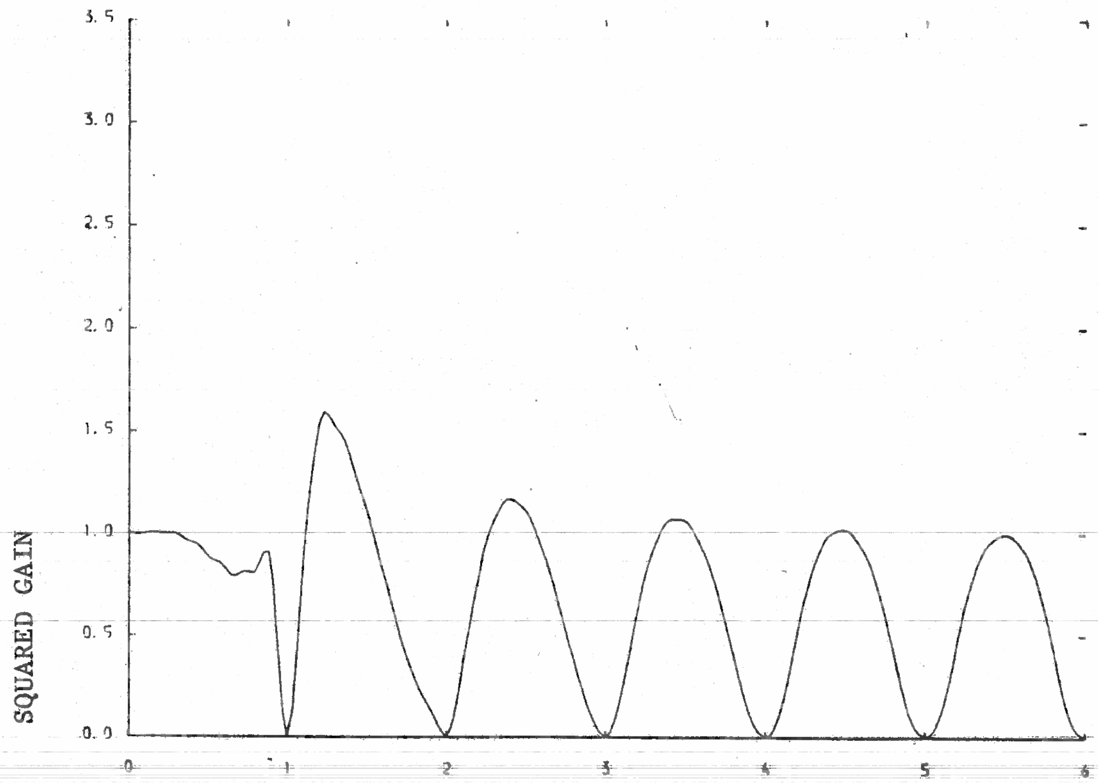


SQUARED GAIN
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
6

Figure 5

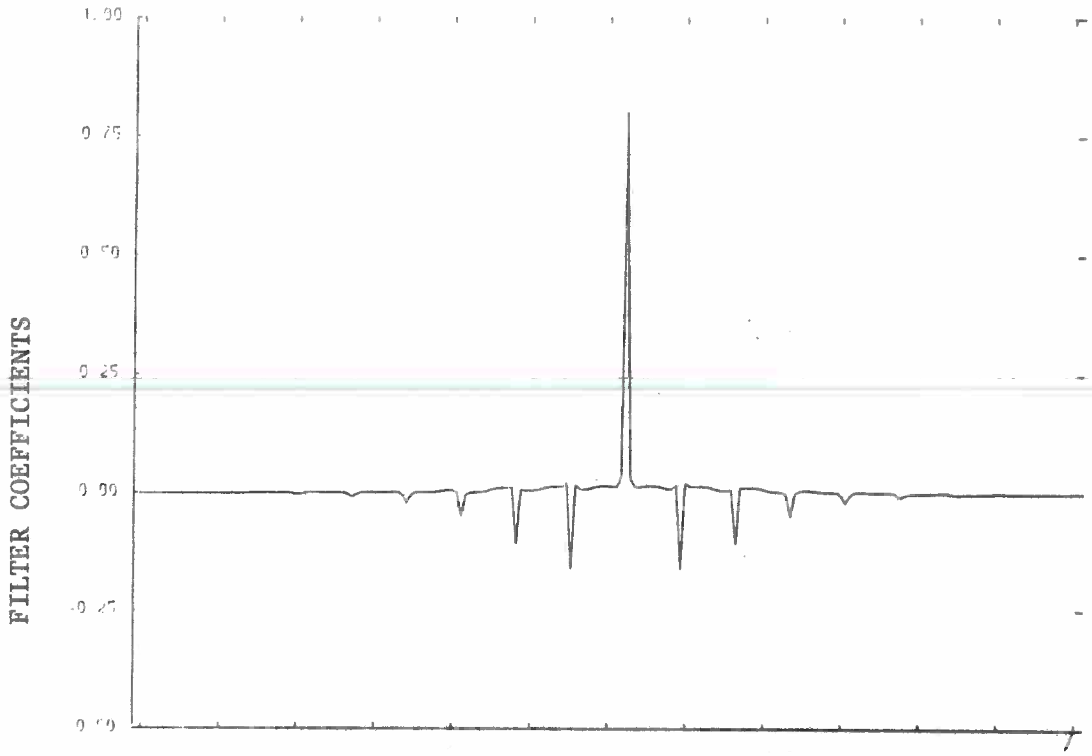


FILTER COEFFICIENTS
1.00
0.75
0.50
0.25
0.00

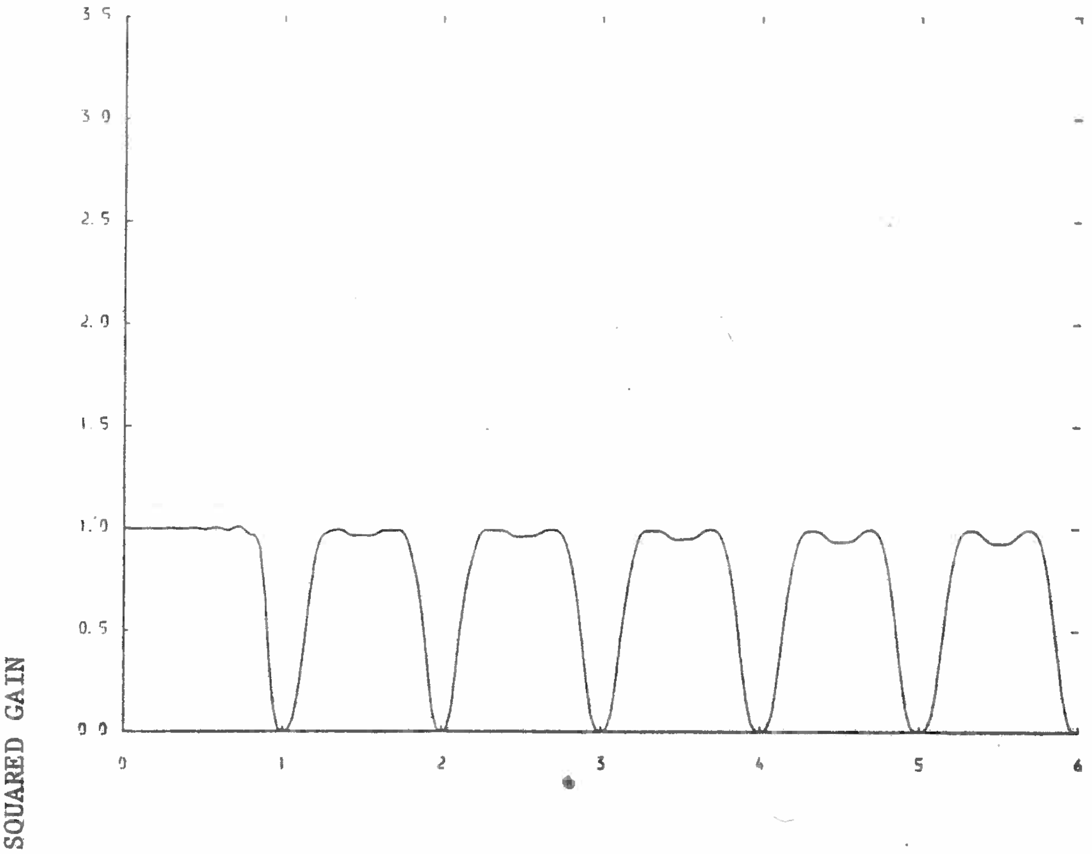


SQUARED GAIN
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
6

Figure 6



1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
FILTER COEFFICIENTS



3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
6
SQUARED GAIN