CLOSED-FORM SOLUTIONS TO DYNAMIC STOCHASTIC CHOICE PROBLEMS*

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Abstract

This paper introduces a parametric class of Kreps Porteus preferences that yield closed form solutions to dynamic stochastic choice problems. These preferences are applied to a simple stochastic macroeconomic model which relaxes the representative agent assumption. This example is designed to illustrate one of the many possible ways in which these preferences may be useful to both theoretical and applied researchers.

1. Introduction

The purpose of this paper is to introduce a parametric class of preferences that should be useful in a wide range of theoretical and applied situations. These preferences are representative of the choices of a decision maker whose behavior obeys the Kreps Porteus axioms (1978), henceforth KP. The axioms characterize sequential rational choice in situations where the passage of time may directly influence decisions and they are referred to by Kreps and Porteus as temporal von Neumann-Morgenstern (VNM) preferences.

The literature on temporal choice under uncertainty is extensive although, to date, most applications to economic problems have directly applied the von Neumann-Morgenstern axioms to intertemporal consumption sequences. Luce and Raiffa (1957) provide an excellent discussion of the VNM axioms in an atemporal context and the survey paper by Machina (1982) is a good introduction to the possible ways in which one might relax these assumptions.

The key papers, on which my own contribution is founded, are by Kreps and Porteus (1978, 1979;1, 1979;2) who provide an axiomatic foundation and a representation theorem for the functional specification that I suggest below. Related papers by Selden (1978, 1979) deal with a two period analysis of the same problem, and an alternative axiomatization and a treatment of the infinite horizon case is found in Epstein (1986). Related work and a discussion of the Kreps Porteus structure can be found in Chew and Epstein (1987;1, 1987;2).

2. General Discussion

I shall be exclusively concerned with the problem faced by a mortal consumer who must make a finite sequence of savings decisions when the future is uncertain. In the standard representation of this problem one assumes that rational choice is characterized by the solution to a dynamic programming problem of the following type:

$$P \begin{cases} 1) & \underset{\{\hat{c}_t\}_{t=0}^{T}}{\text{Max}} \quad \underset{t}{E} \sum_{t=0}^{T} \beta^t u(c_t) \\ 2) & \text{s.t. } a_{t+1} = \tilde{R}_t a_t + \tilde{\omega}_t - c_t \qquad t=0,1,\ldots,T . \\ 3) & a_0 = \bar{a}_0 . \\ 4) & a_{T+1} \geq 0 . \end{cases}$$

The function $u \equiv \sum_{t=0}^{T} \beta^t u(c_t)$ may be interpreted as a von Neumann-Morgenstern utility index defined over the space of consumption sequences $\{c_t\}_{t=0}^{T}$ where the consumption set is taken to be $\mathbb{R}_+^T$. The tildas over the variables $\tilde{R}_t$ and $\tilde{\omega}_t$ are used to denote the assumption that they are random variables and the interpretation of the sequence of constraints (2) is that the individual receives endowments $\{\tilde{\omega}_t\}_{t=1}^{T}$ which may be invested in a single risky asset. The asset a_t is assumed to pay a gross return $\tilde{R}_t$ and in general I shall allow for the possibility that the sequences $\{\tilde{\omega}_t\}_{t=1}^{T}$, $\{\tilde{R}_t\}_{t=1}^{T}$ are jointly distributed random variables that may take values in $\mathbb{R}_+^{2T}$. The expectation operator that appears in equation (1) has the interpretation of an expectation taken over the joint probability distribution of $\{\tilde{\omega}_s,\tilde{R}_s\}_{s=t+1}^{T}$ conditional on the realizations of $(\tilde{\omega}_s,\tilde{R}_s)$ for all $s \leq t$.

A solution to P is represented by a number, $\hat{c}_0$, and a sequence of functions $\hat{c}_t : \mathbb{R}_+^{2(t-1)} : \rightarrow \mathbb{R}_+$, $t=1,\ldots T$, where $\hat{c}_t$ is interpreted as a contingent plan. It represents a list of actions, one for every possible realization of past values of $\tilde{\omega}$ and $\tilde{R}$, which the consumer proposes to undertake in period t.

Stated in this way, this problem is a direct application of expected utility theory which has a distinguished history dating back to Bernouilli. But the application of expected utility theory to the choice of intertemporal consumption sequences makes no reference to the temporal nature of the consumer's problem. The axioms of atemporal expected utility theory are typically justified by an appeal to simple thought experiments in which it is suggested that a violation of one or other of the von Neumann-Morgenstern axioms would be irrational; the discussion of the Allais paradox in Raiffa (1970, page 80 ff) is a good example of this approach. But in a temporal context these discussions are not as compelling and the Kreps and Porteus framework provides a rationalization of a violation of the VNM axioms that can be traced explicitly to the sequential nature of decisions.

3. The Relationship to von Neumann and Morgenstern

Kreps and Porteus provide two alternative axiomatizations of their approach. One set of axioms views choice as a sequence of decisions, each of which is made by an individual whose one step ahead preferences obey the VNM axioms. The sequence of VNM rational choices is wedded together with a time consistency axiom. KP also provide a second formulation which is equivalent to an axiomatization of choice over the set of intertemporal consumption sequences. For the sake of completeness I provide a brief description of this second formulation.

To describe the axioms it is necessary to introduce some notation. Let $h_t \equiv \{c_0, c_1, \ldots, c_t\}$ be a consumption <u>history</u>. Let $p_t(h_t)$ be a joint conditional probability distribution over <u>future</u> consumption sequences $\{c_s\}_{s=t+1}^{T}$ conditional on the history that has occurred being h_t. Let P_t be the set of all conditional distributions $p_t(h_t)$, and P be the set of all joint distributions over sequences $\{c_s\}_{s=0}^{T}$. An element of P_t is, therefore, a conditional distribution of an element of P for some realization of a consumption history h_t.

The key difference between the KP and VNM representations hinges on the timing of the resolution of uncertainty. Imagine standing at time zero, and choosing between two elements of $P_t(h_t)$ for some $t > 0$. One is being asked to rank alternative consumption lotteries each of which contains identical consumption sequences up until time t, but possibly different distributions over consumption from $t+1$ up to T. Now think of mixing any two of these conditional distributions by flipping a coin which comes up heads with probability α and tails with probability $(1-\alpha)$ <u>but flip the coin at date $k < t$.</u> This new mixture is also an element of $P_t(h_t)$ which will be denoted $(k,t;p,p')$ when p and p' are elements of $P_t(h_t)$. A decision maker whose preferences admit an expected utility representation over intertemporal consumption lotteries must be indifferent to the timing of the coin flip in the experiment described above. A KP individual may, on the other hand, prefer either early or late resolution of uncertainty.

The following three axioms characterize KP choice:

A1. There exists a complete transitive ordering, $\succsim$, over the elements of P.

A2. The relation $\succsim$ is continuous on P.

A3. If $p,p' \in P_t(h_t)$ satisfy $p \succ p'$ then $(t,\alpha;p,p'') \succ (t,\alpha;p',p'')$ for all $\alpha \in (0,1)$ and $p'' \in P_t(h_t)$.

The key axiom is (A3) which is a temporal version of the independence of irrelevant alternatives. KP show (1978, Theorem 2, page 195) that axioms A1, A2 and A3 imply that there exists a sequence of ordinal utility functions and a sequence of value functions which is defined recursively by maximizing the value of utility over period t consumption choices. Unlike the standard VNM approach, the one period ordinal utility function need not be linear in probabilities. The relationship with von Neumann-Morgenstern preferences is given by the following axiom which, in conjunction with the other three axioms, implies the existence of a single VNM utility index over intertemporal consumption lotteries.

A4. For all t, h_t, $\alpha \in [0,1]$ and $p, p' \in P_t(h_t)$,
$(t, \alpha; p, p') \sim (t-1, \alpha; p, p')$.

This implies that the difference between KP and VNM preferences hinges solely on the issue of preference for, or indifference to, the timing of the temporal resolution of uncertainty.

4. The Value Function Approach

In this section I introduce the idea that KP preferences are representable by a sequence of value functions that may be non-linear in probabilities. This discussion sets the stage for the parametric form which is introduced in section 5.

Von Neumann-Morgenstern preferences have held center stage in macroeconomics for at least twenty years in spite of their intractability in many applications. For example, consider problem P that was introduced above. It is known that one may recursively define a sequence of value functions $\{v_t(a_t)\}_{t=0}^{T}$, where v_t represents the value of the optimal program to a decision maker who owns assets a_t at date t. This sequence is defined by the formulae

$$5) \qquad v_T(a_T) \equiv u(\tilde{R}_T a_T + \tilde{\omega}_T)$$

$$6) \qquad v_t(a_t) \equiv \max_{c_t}\{u(c_t) + \beta E_t\{v_{t+1}(a_{t+1})\}\}$$

$$7) \qquad \text{s. t.} \quad a_{t+1} = \tilde{R}_t a_t + \tilde{\omega}_t - c_t, \qquad t=0,1,\ldots,T-1 .$$

A great deal is known about the general properties of the functions $v_t(\cdot)$, and for special cases one may obtain closed form solutions for the optimal decision rules. For example, if one is willing to restrict attention to the case of multiplicative uncertainty (random interest but deterministic endowments) then closed form solutions are attainable to the class of preferences that are usually referred to[1] as constant relative risk aversion; i.e. $u(c_t) = \frac{1}{\rho} c_t^{\rho}$. On the other hand with only additive uncertainty (random endowment but deterministic interest rates) one can solve the quadratic case. But the general case of random interest <u>and</u> random endowments does not admit a closed form solution except in the trivial situation when $u(\cdot)$ is an affine function. In this case the agent's preferences are linear, not only across states of nature, but also through time.

If, on the other hand, one is willing to drop the assumption of indifference to temporal resolution (axiom A4) then the weaker axiom set A1-A3 implies that the choice of intertemporal consumption sequences admits a value function representation where the value functions are defined recursively as follows:[2]

$$8)\qquad v_T(a_T) \equiv u_T(\tilde{R}_T a_T + \tilde{\omega}_T)$$

$$9)\qquad v_t(a_t) \equiv \max_{c_t} u(c_t, \underset{t}{E}[v_{t+1}(a_{t+1})])$$

$$10)\qquad \text{s. t.}\quad a_{t+1} = \tilde{R}_t a_t + \tilde{\omega}_t - c_t, \qquad t=0,1,\ldots,T-1 .$$

Equation (9) differs from the VNM approach (equation (6)) in that v_t is non-linear in the expectation operator E_t. This generalization would appear to complicate the problem and make things more, rather than less, difficult. However, by choosing $u(\cdot)$ correctly one can find a class of decision problems that yield closed form solutions in a wide variety of cases.

5. Parametric Forms

Let the functions u_T and v_T be defined as follows:

$$11)\qquad u_T = c_T$$

$$12)\qquad v_T = \max_{c_T} \underset{T-1}{E}[c_T]$$

$$13)\qquad \text{s.t.}\qquad c_T \le \tilde{R}_T a_T + \tilde{\omega}_T .$$

Now define the sequence of functions $\{v_t\}_{t=0}^{T-1}$ by the recursive rule described in equation (9) where $u(\cdot) : \mathbb{R}_+^2 \to \mathbb{R}_+$ is given by

$$14)\qquad \begin{cases} u(x,y) \equiv (\alpha x^{\rho} + \beta y^{\rho})^{\frac{1}{\rho}}, & \rho \neq 0 \\ u(x,y) \equiv x^{\alpha} y^{\beta} & \text{if } \rho = 0 , \end{cases}$$

and $u(\cdot)$ is subject to the parametric restrictions

$$15)\quad \begin{cases} \alpha + \beta = 1 , \\ \alpha , \beta \geq 0 , \\ \rho \leq 1 . \end{cases}$$

Before providing explicit functional forms for the sequence of value functions $\{v_t\}_{t=1}^{T}$ and for the decision rules that determine $\{c_t\}_{t=0}^{T}$ it helps to introduce some additional notation. Define the functions F and $G : \mathbb{R}_+ \to \mathbb{R}_+$

$$16)\quad \begin{cases} F(x) = (x^{\frac{\rho}{1-\rho}}\beta^{\frac{1}{1-\rho}} + \alpha^{\frac{1}{1-\rho}})^{\frac{1-\rho}{\rho}} , & \rho \neq 0 \\ F(x) = \alpha^{\alpha}\beta^{\beta}x^{\beta}, & \rho = 0 \end{cases}$$

$$17)\quad \begin{cases} G(x) = \dfrac{\alpha^{\frac{1}{1-\rho}}}{(\beta^{\frac{1}{1-\rho}}x^{\frac{\rho}{1-\rho}} + \alpha^{\frac{1}{1-\rho}})} , & \rho \neq 0 \\ G(x) = \alpha , & \rho = 0 . \end{cases}$$

The decision rule for consumption is most conveniently expressed in terms of two variables that resemble a compounded interest rate and a human wealth term. However, this analogy is not exact since the "human wealth" variable involves the parameters of the function $u(\cdot)$. More precisely, define the sequences of variables $\{Q_t\}_{t=0}^{T}$, $\{h_t\}_{t=0}^{T}$ as follows

18) $F(Q_T) = 1$.

19) $Q_t = \underset{t}{E}[\tilde{R}_{t+1} F(Q_{t+1})] \qquad t=1,\ldots,T-1$.

20) $h_T = \tilde{\omega}_T$

21) $h_t = \omega_t + \underset{t}{E}[\frac{h_{t+1} F(Q_{t+1})}{Q_t}] \qquad t=1,\ldots,T-1$.

Further, let w_t be given by

22) $w_t \equiv \tilde{R}_t a_t + h_t \qquad t=0,1,\ldots,T$.

One may think of the term w_t as "perceived wealth" because of the analogous role that it plays to market wealth in the non-stochastic case. w_t consists of the market value of physical assets, plus the subjectively discounted value of future endowments, h_t , where the subjective discount factors are embodied in the updating rules (19) and (21). Given these definitions it is easy to check that the decision rules $\{c_t\}_{t=0}^{T}$ and the value functions $\{v_t\}_{t=0}^{T}$ are given by

23) $c_t = G(Q_t) w_t$,

24) $v_t = F(Q_t) w_t$, $\qquad t=0,1,\ldots,T$.

The system of equations (18-22) gives explicit rules for determining the values of the variables Q_t and w_t in terms of the conditional moments of the joint endowment/return process $\{\tilde{\omega}_s, \tilde{R}_s\}_{s=t+1}^{T}$. One may therefore summarize the behavior of an agent with preferences of this type by keeping track of two rather simple functional equations, (19) and (21). It is to be hoped that this rather simple form may prove useful in a variety of contexts, one of which is discussed in section 7.

6. Some Possible Objections

I would like to raise two issues that may be foreseen by the alert reader. The first of these deals with the consistency of intertemporal plans if agents do not have VNM preferences. This issue has arisen in the literature on estimating lifecycle consumption equations. Some authors have noted that preference specifications which are typically applied to data are unable to distinguish the co-efficient of relative risk aversion from the intertemporal elasticity of substitution. This has led to work by Hall (1985) and by Zin (1986) who use non VNM specifications of choice in an attempt to separate attitudes to risk from preferences over the timing of consumption plans. One should note, however, that some care must be taken when adopting this approach since some specifications of choice under uncertainty may lead to inconsistent decision rules in the absence of commitment.

This issue was first raised by Strotz (1978) in the context of preferences that change through time and it has since been discussed by a number of authors including Pollak (1968) and Donaldson and Johnsen (1985). One way around this problem involves imposing consistency of plans by forcing the decision maker to play a game with his own future incarnations.[3] In this approach the agents' preferences over consumption sequences may be inconsistent, but this inconsistency will be recognized and corrected for by an optimal plan. Although this approach may be fruitful one might prefer to take the stand that consistency should be axiomatic. Kreps and Porteus take this latter route by providing an alternative characterization of their approach under which time consistency is one of the basic axioms. The preference structure described in section 5, therefore, generates decision rules that are fully time consistent.

A second issue that may be of interest to the reader involves the particular attitude of the decision maker to temporal resolution of uncertainty that is implied by the functional form suggested above. One may show that if the function $u(C,E[V])$ is convex (concave) in its second argument then the individual prefers early (late) resolution. Since the function described in equation (14) is concave in its second argument, the preferences that I have described always imply a preference for late resolution. How is one to interpret this property? One may come up with plausible illustrations of arguments under which one may favor either early <u>or</u> late resolution. (See, for example, the discussion in Chew and Epstein (1987)). It is important to recognize, however, that the preference for early or late resolution refers to distributions of <u>consumption</u> sequences and not to distributions of exogenous uncertain events. In particular, this means that early resolution of uncertainty is of no use in aiding optimal plans. Bearing this in mind, there is a sense in which preference for late resolution may be interpreted as a kind of self insurance. The individual described above has a positive rate of time preference. That is, if ω_t and R_t are non-stochastic then the marginal rate of substitution between consumption in adjacent periods is tilted towards the present for all α, β, ρ. By putting off the resolution of uncertainty, the individual is able to attain a (temporary) insurance over uncertain events. Since the decision maker weights the present more heavily than the future this ability to postpone resolution will always have positive value.

7. An Application to Macroeconomics

In this section I introduce a very simple general equilibrium model which is designed to illustrate one of the possible uses that other researchers may have for the parametric structure that I have introduced in this paper.

For the last couple of decades there has been a growing interest in the idea that macroeconometric work should be given microeconomic foundations. One of the key assumptions of this approach to macroeconomics is the idea that the stochastic disturbances that arise in applied work should be matched with theoretical constructs in the form of shocks to the underlying preferences and technologies of a model economy. In practice, this methodology has led applied researchers to adopt a representative agent assumption as a practical solution to certain problems that arise when one attempts to implement the approach. In particular, the representative agent assumption provides a solution to the problems that are posed by the difficulty of finding VNM preferences that can be aggregated in simple ways. But, as I shall illustrate below, this solution does not come without cost.

Let us suppose that one wishes to model an economy in which a single agent, Robinson Crusoe, lives for T periods and has VNM preferences of the type that were described in section 2. Robinson Crusoe receives a random endowment each period which he may invest in a "fruit tree" technology that yields a random return. For concreteness, assume that the endowment sequence and the return sequence are independently distributed random variables with an exogenously specified distribution.

The assumptions that I have described have a number of observable implications, one of which has been pursued in some depth in a series of papers by Hall (1978), Flavin (1981) and others.[4] This literature observes that, if aggregate consumption can be described by the solution to an individual's optimizing problem, then the data should obey a stochastic Euler equation of the following kind:

$$25) \qquad u'(C_t) = E_t[\beta R_{t+1} u'(C_{t+1})] .$$

where C_t represents aggregate consumption. However, attempts to estimate equations of this form, using time series data, are typically rejected rather forcefully.

There are many possible reasons why this approach might be expected to fail, one of which arises directly from potential difficulties with the representative agent assumption. The following alternative assumptions are suggested by work on non-stochastic economies by Blanchard (1985) and by Weil (1986). The assumptions made by Blanchard and Weil cannot be directly applied to stochastic economies, if one assumes VNM preferences, because of the difficulties of finding closed form solutions that can be easily aggregated. The preferences that I have discussed above, however, simplify matters considerably.

Assume that, instead of a representative agent economy, one has a new generation of individuals born in each period. To make aggregation simple, assume that each generation faces the same terminal date, T, irrespective of date of birth. Generations born later, therefore, have shorter life spans. Agents are identical, except for date of birth, and each individual makes intertemporal choices according to the decision rules described in section 5. Under these assumptions one obtains a rather simple closed form relationship that links aggregate consumption between periods. For the case of $\rho = 0$ (the KP equivalent of logarithmic utility), this relationship takes the form

$$26) \quad C_t = \frac{1}{(\gamma-\alpha)} E_t[C_{t+1}F(Q_{t+1})] + \frac{(\gamma-1)}{(\gamma-\alpha)} \alpha Y_t$$

where Y_t is aggregate income and γ is the population growth factor.[5] The variable Q_{t+1} is the subjective discount factor which is found recursively using the updating rule described above by equation (19). This discount factor is the same for all individuals because of the assumption of a common terminal date.

Notice that only for the case of no growth, $\gamma = 1$, does equation (26) resemble an individual's Euler equation. In all other instances one would expect that income might have some explanatory power if one were to run regressions that were based on a representative agent model. Campbell and Deaton (1987) report that one of the ways the the Euler equation model is violated is attributable to unaccountable additional explanatory power that is found when income growth is added to consumption growth regressions. It therefore seems plausible that the above model may prove useful in applied research in which one may hope to rescue equilibrium methodology by dropping the representative agent approach.

Conclusion

The model that I have presented is designed to illustrate one of the many potential applications of the parametric specification presented in this paper. There are presumably other instances in which closed form solutions may be useful to both the theorist and to the applied researcher. To the theorist the ability to find simple examples may stimulate the genesis of general theorems. To the applied researcher the ability to find convenient parametrizations of aggregative models should facilitate the confrontation of theory with evidence. This paper represents a small contribution to both areas.

Appendix 1

This appendix provides a sketch of the proof that the closed form solution to the value function described in the text is valid. The proposed solution for v_t is given by

$$\text{A1)}\qquad v_t = F(Q_t)w_t \, .$$

Taking expectations of v_t at $t-1$ using the identity (22) and the asset accumulation rule one obtains

$$\text{A2)}\qquad \underset{t-1}{E}(v_t) = \underset{t-1}{E}[\tilde{R}_t F(Q_t)]\,[R_{t-1}a_{t-1} + w_{t-1} - c_{t-1}] + \underset{t-1}{E}[h_t F(Q_t)]$$

which simplifies, using definitions (19) and (21), to

$$\text{A3)}\qquad \underset{t-1}{E}(v_t) = Q_{t-1}(w_{t-1} - c_{t-1}) \, .$$

By substituting (A3) into equation (9) and using the functional form (14) for $u(\cdot)$ one obtains the first order conditions

$$\text{A4)}\qquad \alpha c_{t-1}^{\rho-1} - Q_{t-1}\rho\beta[Q_{t-1}(w_{t-1} - c_{t-1})]^{\rho-1} = 0$$

which may be rearranged to give the functional form (23) using the definition of $G(\cdot)$ given in equation (17). By substituting the solution for c_{t-1} at a maximum (equation (23)), into the function $u(\cdot)$, one attains the expression

A5) $\quad v_{t-1} = F(Q_{t-1})w_{t-1}$.

This establishes that if (A1) is a correct representation of the value function at t , then it is also correct at $t-1$. One completes the proof by establishing that v_{T-1} is described by (A1) given the definition of v_T in equation (12).

Footnotes

1. Since the parameter ρ also governs intertemporal elasticity of substitution this terminology is often confusing. In the Kreps Porteus case, risk aversion and intertemporal elasticity of substitution are not the same thing.

2. Throughout this paper I maintain the assumption of payoff history independence. In general, KP preferences at a point in time may depend on the entire history experienced by the consumer.

3. This is, essentially, the strategy that Pollak (1968) refers to as "sophisticated" planning. Larry Epstein has suggested this strategy (in a private communication) in relation to preferences that generalize Kreps and Porteus by dropping time consistency.

4. See also the papers by Zeldes (1984) and by English, Ichimura and Wilcox (1987).

5. Equation (26) is derived by defining aggregate human wealth and aggregate physical wealth, and finding the equations of motion of these aggregate variables. These are given by the formulae:

$$A_{t+1} = \tilde{R}_t A_t + \tilde{\omega}_t n_t - C_t$$

$$H_t = n_t \tilde{\omega}_t + \frac{1}{\gamma} E_t\left[\frac{H_{t+1} F(Q_{t+1})}{Q_t}\right] .$$

where n_t is the number of individuals alive at date t. Equation (26) follows after some simple algebra by recognizing that if everyone is identical (except for their ownership of physical wealth), then (23) may be applied to aggregate variables C_t, H_t and A_t, where A_t is aggregate physical wealth; i.e.,

$$C_t = G(Q_t)(\tilde{R}_t A_t + H_t) .$$

For the case $\rho = 0$, this simplifies since $G(Q_t) = \alpha$.

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