Error correction models, co-integration and the internal model principle

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The paper considers the conditions under which a system of dynamic equations will be consistent with some prespecified multivariate equilibrium specification. Connections are also drawn between recent developments in the analysis of multivariate error correction models and the theory of co-integration and what is known as "The Internal Model Principle" in the design of robust linear feedback rules. Apart from clarifying and generalising our understanding of error correction systems and co-integration this approach suggests a relatively simple method for identifying the restrictions that determine whether a given dynamic system satisfies the conditions for being an error correction specification.

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I Introduction

This paper considers the following general question ; given a multivariate dynamic model written in vector autoregressive form ,

$$A(L)x_t = \epsilon_{1t} \quad , \tag{1}$$

under what conditions on the polynomial matrix A(L) will the model be consistent with some prescribed set of dynamic restrictions given by

$$B(L)x_t = e_t \quad , \tag{2}$$

Where $\{\epsilon_{1t}\}$ and $\{e_t\}$ are stationary, zero mean stochastic processes and B(L) is a polynomial matrix in the lag operator L.[(1)]

The most obvious case in which this question becomes relevant is when (1) represents a dynamic model incorporating short run responses and (2) represents an equilibrium specification for the model. In general however, we may also be interested in whether the transient response of a model satisfies certain characteristics such as common seasonal patterns among the variables. In addition there is no restriction in (2) that prevents us , in principle ,from considering the analysis of unstable models with roots strictly greater than unity. What makes the general question non trivial is , of course , that the dynamic characteristics of a system of equations can not be determined easily from an inspection of the properties of the constituent equations. For instance , the poles and zeros of a system as a whole may bear little relation to those of the individual equations given the dynamic cancellations that may arise in moving to a final form . Indeed it is the cancellation of unit roots within such a system that characterises co-integrating relationships among the variables in the vector x_t. Another example , of direct interest to this paper, is where error correcting behaviour may well be at work within the model although it is not obvious from the specification of any individual equation. What are the "error correction" restrictions for a system of equations ? For instance , do all the equilibrium error terms have to occur in each equation ?

[(1)] A(L) and B(L) may also be treated as rational polynomial matrices and the analysis below thereby extended to cover vector ARMA representations with relatively minor changes.

An alternative way of posing the question is to ask whether (1) contains as an "internal model" the specification given by (2). Similar considerations have led to what is known as "The Internal Model Principle" being put forward as a basic requirement in the design of robust linear feedback rules (see for instance, Wolovich (1974), Francis and Wonham (1975),(1976), Bengtsson (1977), Callier and Desoer(1982) and Wolovich and Ferreira (1979)).The objective in this case is to ensure that a feedback rule may be designed so as to be robust against a prescribed set of disturbances with some dynamic specification or alternatively for the model to "track" some desired dynamic reference trajectory. While our interest in econometrics does not lie in the design of such rules much of the recent literature on error correction models and co-integration amounts to asking such questions , *ex post*, of a given model.

What may not be immediately obvious is why the equilibrium specification (2) is apparently both dynamic and stochastic. We first demonstrate below why this approach does in fact allow considerable flexibility and is consistent with the standard characterization of equilibrium within the natural sciences and in other areas of economics. Although there are potentially important differences with the existing approach adopted within the co-integration literature in which equilibria are viewed as contemporaneous relations between economic variables that are all required to be of the same order of integration , typically I(1). Yoo (1986) has also recently attempted to generalise this restrictive notion of equilibrium employed in co-integration analysis and we shall compare his approach with ours in the analysis that follows.

We then describe the internal model principle and consider how the existence of an internal model may be checked through the determination of the right divisors for the (potentially rational) polynomial matrix A(L). A similar " common factor" condition has recently been noted by Davidson (1986) and this paper may be seen as a complementary analysis to the more mathematical discussion of similar issues by Davidson and also Johansen (1985). Section 4 makes the formal connection between the internal model notion and co-integration and draws parallels with the state space approach proposed by Aoki (1987a,1987b). In particular the conditions under which the co-integrating regression will involve dynamic transformations are determined and also how the degree of ambiguity in this dynamic specification may be partially eliminated by selecting a <u>minimal</u>

polynomial basis for the left null space of a particular polynomial matrix as a "co-integrating basis" for the model.

The right divisor property is not sufficient alone to ensure the existence of a multivariate error correction model and we consider in section 5 what further conditions are required for the system case after first briefly reviewing the equivalent results for single equation error correction models. A number of examples are then given to demonstrate the utility of the internal model approach. One observation that follows from the examples is that although an equation or system may be expressed in error correction form involving both differences and levels of variables it may still not satisfy the "error correction property" in that the equilibrium error vector is an I(0) process being both stationary and having zero mean.

The orientation we adopt is that of an applied researcher who has at hand an estimated econometric model of the form (1) and wants to consider whether this system of equations is consistent with a set of dynamic restrictions (2) potentially representing equilibrium behaviour. The arbitrary specification of these restrictions reflects the possibility that he may want to consider several different assumptions corresponding perhaps to different growth properties of the data. Since the model at hand , (1), has already been estimated without necessarily following the model building strategy proposed in Engle and Granger (1987) the researcher may be completely uncertain as to the potential equilibrium characteristics of the model. The initial stimulus for this work came , in fact , by way of a question from a colleague who was working on a small subsystem of a large macro-econometric model of the UK economy. The original model had not been constructed with any explicit regard to the degree of integration or co-integrating properties of the data and the question was raised of how we could determine *ex post* whether or not the subsystem satisfied any error correction properties. So no assumptions as to the empirical co-integration properties of the data are assumed in what follows. One major objective was then to see how the methodology used in Salmon (1982), which essentially considers the same form of *ex post* analysis could be extended to error correction systems.

2: The Specification of "Equilibrium" Behaviour

We start by justifying why the equilibrium specification is apparently both dynamic and stochastic.

An example will demonstrate several of the issues to be raised below. Consider the following two equation system

$$\Delta y_1 + y_1 + y_2 = e_1 \tag{3i}$$

$$\Delta y_1 + \Delta y_2 \quad = e_2 \tag{3ii}$$

which are normalised by the properties of the errors , e_1 and e_2, which are both assumed to be zero mean , stationary , stochastic processes, in other words they are both integrated of order zero , I(0). Clearly several different configurations for the stochastic properties of y_1 and y_2 are possible. They could for instance both be I(1) or I(0) processes but a more interesting case arises when they are both I(2). The equations then represent two distinct co-integrating relations between y_1 and y_2 . If we consider the second equation , given that e_2 is I(0) implies that (y_1+y_2) is an I(1) process if y_1 and y_2 are individually I(2). Applying a first difference transformation to (3i) we find,

$$\Delta^2 y_1 + \Delta(y_1+y_2) = \Delta e_1 \tag{4}$$

or

$$\Delta^2 y_1 + e_2 = \Delta e_1 \tag{5}$$

and hence

$$\Delta^2 y_1 = \Delta e_1 - e_2 \ , \tag{6}$$

confirming that y_1 is I(2) . Then from 3(ii) we see that

$$\Delta^2 y_2 = \Delta e_2 - \Delta e_1 + e_2 \tag{7}$$

which in turn confirms that y_2 is I(2).

Thus we see from 3(ii) that the variables y_1 and y_2 are CI(2,1) but also since Δy_1 is I(1) and (y_1+y_2) is I(1) and from 3(i) their sum is I(0) we find that the variables Δy_1 and (y_1+y_2) are CI(1,1).

So several co-integrating relations may exist in a model that involve the <u>same</u> variable in different dynamic transformations. Moreover within a <u>single</u> co-integrating relation (eg. (3i)) a variable (y_1) may occur in both levels and a first difference form. The approach developed by Granger and Engle has concentrated on the CI(1,1) case and attention has been focussed on contemporaneous equilibrium relations between economic variables. The difficulty demonstrated by the example lies in that prior to the analysis some form of dynamic normalisation has to be undertaken. Granger and Engle achieve this by requiring, *ab initio*, that all variables entering into the analysis are of the same order of integration. However this would be difficult to achieve in the present example unless Δy_1 existed as a different economic variable, such as the obvious relation between stock and flow variables. *Ex post*, having found the equilibrium relation, (3i), it is obviously reasonable to define Δy_1 as a new variable but coming to the problem from the point of view of the data analyst with observations on just y_1 and y_2, it is not clear that (3i) would be found and only (3ii) would be put forward as the equilibrium relation between y_1 and y_2. The dangers of using an incomplete set of equilibrium conditions are well known in economics and certainly if the resulting partial equilibrium specification were used in an error correction specification, following the model building approach put forward in Engle and Granger(1987), then the resulting model may be seriously inadequate.

One response is that whenever a variable is found to be I(2) then a nested search should also be made over all potentially co-integrating relations that involve first differences of the variable in question. The approach we follow in this paper is to allow the equilibrium relations to be described, as in (2), by a general dynamic system in the basic economic variables. Notice that this approach does not deny a stationary equilibrium between appropriately defined state variables, but just recognises that it may be difficult to determine empirically what the appropriate state variables should be. The relevant state variables for a given problem should appear naturally from a full specification of the economic theory underlying the problem. Rationalising a dynamic model by writing it in state space form provides a similar dynamic normalisation to that suggested by Granger and Engle, moreover the state vector will frequently involve dynamic transformations of the same variable, consider Newton's Law's of motion where the state vector consists of position, velocity and acceleration. Given the dynamic transformations inherent in the definition of a state vector it is natural to define an equilibrium in terms of stationarity of the state of the model. So the apparent dynamics in the "equilibrium" specification, (2), are principally intended to enable the selection of the appropriate state variables for the economic model at hand and to relax the prior normalisation required in the Engle-Granger approach. Yoo(1986) adopts an approach that suggests economic equilibria may involve non-instantaneous relations between (state) variables and while there may be some instances perhaps due to temporal aggregation or other timing considerations in which this situation may arise it is not the position taken in this paper. Aoki (1987a, 1987b) has recently provided an important alternative view of co-integration which exploits the state space formulation of dynamic models explicitly.

3: The Internal Model Principle

Any linear dynamic system may be decomposed into its fundamental modal characteristics and as such contains within its overall dynamic response some linear combination of these internal modes. Our particular interest lies with steady state behaviour but we may also be interested in whether a model contains some frequency specific behaviour . In the design of robust feedback rules some information has to be provided as to the nature of the dynamic characteristics of the disturbances a rule is likely to face . Similarly if the feedback rule is to be designed so as to "track" some reference path then the dynamic characteristics of this path have to be specified and built into the design of the rule . The Internal Model Principle simply states this intuitive idea; Bengtsson(1977) puts it as follows;

> *"it is both necessary and sufficient for output regulation to take place that the open loop path, consisting of the plant and the compensator in cascade, contains a suitably defined internal model of the environment."*

This same idea was reflected in Salmon (1982) where it was shown that the appropriate specification of a scalar error correction model depended on the dynamic characteristics of the equilibrium towards which the error correction model adjusted . The development in that paper was based on the notion of model type which does not generalise satisfactorily to multivariable error correction systems. A more appropriate extension is based on the determination of whether the potential equilibrium behaviour is contained as an internal model within our dynamic system.

There are several different routes to the internal model principle , one based on skew symmetric polynomials (see Wolovich and co-authors) and another based on the determination of right divisors of a polynomial matrix. We follow the analysis of Bengtsson (1977) below that requires several definitions before we can define the conditions for the existence of an internal model.

1. The order of a rational matrix $T(z^{-1})$, written as $\partial(T(z^{-1}))$ is defined as the sum of the degrees of the denominator polynomials in the Smith-McMillan form of $T(z^{-1})$.

2. A right divisor of a polynomial matrix $P(z^{-1})$ is a polynomial matrix $R(z^{-1})$ such that $P(z^{-1})=P_1(z^{-1})R(z^{-1})$ for some polynomial matrix $P_1(z^{-1})$.

3. A left matrix fraction description (MFD) of a rational polynomial matrix $T(z^{-1})$ is a pair of polynomial matrices $D(z^{-1})$ and $N(z^{-1})$ such that $T(z^{-1})=D(z^{-1})^{-1}N(z^{-1})$. A right MFD would be $T(z^{-1})=N(z^{-1})D(z^{-1})^{-1}$.

4. A rational matrix $T(z^{-1})$ can be written uniquely, through partial fraction expansions, as the sum of two strictly proper rational matrices $T(z^{-1})_+$ and $T(z^{-1})_-$ and a polynomial matrix $T(z^{-1})_p$ as

$$T(z^{-1}) = T(z^{-1})_+ + T(z^{-1})_- + T(z^{-1})_p \tag{8}$$

where the poles of $T(z^{-1})_+$ are all on or outside the unit circle (in a region $\mathbb{C}^+$) and those of $T(z^{-1})_-$ are all strictly inside the unit circle (in a region $\mathbb{C}^-$). The complex variable z defines the z transform or generating function of a discretely indexed variable such that the stable region is given by $\mathbb{C}^-$.

5. A rational matrix $T(z^{-1})$ having all its poles within $\mathbb{C}^-$ can be expressed as $T(z^{-1})_+ = 0$ and is said to be stable with respect to $\mathbb{C}^-$.

6. Let $T(z^{-1})$ and $W(z^{-1})$ be arbitrary rational matrices. Then $T(z^{-1})$ is said to contain an internal model of $W(z^{-1})$ if

$$\partial(T(z^{-1})) = \partial[T(z^{-1})\ W(z^{-1})] \tag{9}$$

Theorem 2. of Bengtsson then shows that if $T(z^{-1})$ and $W(z^{-1})$ are two arbitrary rational polynomial matrices with the same number of rows such that $T(z^{-1})=D_1(z^{-1})^{-1}N_1(z^{-1})$ and $W(z^{-1})= D_2(z^{-1})^{-1}N_2(z^{-1})$ are minimal MFDs then $T(z^{-1})$ contains $W(z^{-1})$ as an internal model if and only if $D_2(z^{-1})$ is a right divisor of $D_1(z^{-1})$.

The application of these results to our original problem is quite straight forward and intuitive. If we equate $T(z^{-1})$ with the rational polynomial matrix $A(z^{-1})^{-1}I$ and $W(z^{-1})$ with $B(z^{-1})^{-1}I$ [(2)], we see that for (1) to contain (2) as an internal model it is necessary and sufficient for $B(z^{-1})$ to be a right divisor of $A(z^{-1})$. In other words we require that

$$A(z^{-1}) = A_1(z^{-1})B(z^{-1})\ . \tag{10}$$

So if $B(z^{-1})$ is an internal model of $A(z^{-1})$ we may write

$$A(L)x_t = \epsilon_{1t} \tag{11}$$

as

$$A_1(L)B(L)x_t = \epsilon_{1t} \tag{12}$$

but given that $B(L)x_t = e_t$, equation (12) just describes the dynamic adjustment in the equilibrium error e_t as

$$A_1(L)e_t = \epsilon_{1t} \tag{13}$$

[(2)] If the original problem had been set up in vector ARMA form then the identity matrices in these MFDs would be replaced by the appropriate moving average polynomial matrices.

Given that $B(z^{-1})$ contains our model of equilibrium with the specified growth characteristics in the variables appearing as unit roots in $B(z^{-1})$ we will also need to ensure that $A_1(z^{-1})_+=0$, in other words that the polynomial matrix $A_1(z^{-1})$ has no unstable roots if adjustment towards the equilibrium specified by $B(z^{-1})$ is to be guaranteed. Moreover for the equilibrium error to be an I(0) process we need to ensure not only that it is stationary but that it has a zero mean. We shall refer to a specification that satisfies these later two conditions on the equilibrium error as having the "error correction property" for the given equilibrium or internal model specification, $B(L)x_t = e_t$.

If $B(z^{-1})$ were not a right divisor of $A(z^{-1})$ so that

$$A(z^{-1}) = A_1(z^{-1})B(z^{-1}) + Q(z^{-1}) \tag{14}$$

for some polynomial matrix $Q(z^{-1})$ we find the equilibrium error dynamics are determined by

$$A_1(L)e_t + Q(L)x_t = \epsilon_{1t} \tag{15}$$

and although it may be that $A_1(L)_+=0$ the non-stationary forcing term, $Q(L)x_t$, would in general prevent convergence to the equilibrium.

Kailath(1980) and Forney(1975) amongst others have considered the general conditions under which the potentially rational polynomial matrix $A(L)$ may factorise as described above for a given specification of $B(L)$. One route following Bengtsson is to check on the orders of the $A(L)$ and $B(L)$ matrices as described in 6 above but since we are only concerned with stable solutions for $A_1(L)$ the following development from Kailath (pps 462–464) applies. To consider the general case let $A(L)$, $A_1(L)$ and $B(L)$ be $(N\times p)$, $(N\times r)$ and $(r\times p)$ rational polynomial matrices where $B(L)$ is assumed to be of full column rank, $p \leq r$. We may think of the problem as one with r potential equilibrium conditions (not necessarily independent) for a model with N relations connecting p variables in x_t. Then if and only if

$$\upsilon_{\alpha}^{(p)}\left[B(z^{-1})\right] = \upsilon_{\alpha}^{(p)}\begin{bmatrix} B(z^{-1}) \\ A(z^{-1}) \end{bmatrix} \tag{16}$$

there will be a solution for $A_1(z^{-1})$ with no poles at α. Where $\upsilon_{\alpha}^{(p)}[W(z^{-1})]$ represents the p'th order valuation of $W(z^{-1})$ at α which is the difference between the number of poles and zeros of $W(z^{-1})$ at α. To find if a stable solution then exists this condition may be easily checked for all potential poles outside the unit circle by calculating the Smith–McMillan form for the two matrices above. The interpretation of condition (16) is fairly straight forward simply requiring that $A(z^{-1})$ contains no further "uncancelled" unstable poles than those accounted for in the equilibrium specified by $B(z^{-1})$. Note that the condition applies regardless of whether $A(L)$ and $B(L)$ are polynomial or rational polynomial matrices. Having determined whether or not a solution exists it may be relatively easily constructed by finding a left inverse for $B(L)$ giving $A_1(L)$ as

$$A_1(L) = A(L)B^{-L}(L) \tag{17}$$

If $B(L)$ does not have full column rank as assumed above then there will be no solution for $A_1(L)$ unless $A(L)$ is in the column range space of $B(L)$, ie. if and only if;

$$\text{rank}\left[B(z^{-1})\right] = \text{rank}\begin{bmatrix} A(z^{-1}) \\ B(z^{-1}) \end{bmatrix} \tag{18}$$

Notice that the apparent simplicity of the right divisor property does in fact hide a potentially complicated set of parametric restrictions in $A_1(L)$ that depend on the equilibrium specification. These restrictions are then the error correction restrictions for the multivariable system .

Given the essentially arbitrary structure of the stable $A_1(L)$ matrix it can be seen that there may in fact be a number of internal models all of which achieve the specified equilibrium but with differing transient adjustments . So if $A_1(L)$ itself has a right divisor so that

$$A_1(L)= A_2(L)A_3(L) \tag{19}$$

with $A_2(L)$ stable, we may write the original relation as

$$A(L) = A_2(L)B_1(L)= A_2(L)[A_3(L)B(L)] \tag{20}$$

implying a degree of non–uniqueness in the definition of an internal model . Since $B(L)$ is contained in $B_1(L)$ the original equilibrium specification is automatically satisfied by $B_1(L)$. This is really saying nothing more than that certain types of equilibrium behaviour are implied by others. For instance , given that the internal model will typically include the unstable modes or growth characteristics differencing an already stationary zero mean process will still lead to a stationary zero mean process. Adding a further equilibrium condition to an existing set provides another example and this approach could be used to investigate sequentially which part of a multivariate equilibrium condition causes a failure of the error correction property.

More fundamentally since a given equilibrium condition may be expressed in a number of different forms there is a lack of uniqueness in the factorization that corresponds to the potential lack of identification in error correction models noted by Granger (1986). Given some nonsingular (unimodular) transformation ,$T(L)$, it may be possible to express the right division condition as

$$A(L) = A_1(L)T(L)[T^{-1}(L)B(L)] \tag{21}$$

while maintaining stability of the transient or disequilibrium adjustment. This lack of uniqueness may be partially removed by the adoption of a minimal polynomial basis condition as we shall see in the following section but there would still remain a degree of ambiguity in the equilibrium specification in general that requires further identifying restrictions to be imposed before we could determine a unique factorization of $A(L)$.

4: Internal models and co-integration

Three related approaches to co-integration have been identified in the existing literature; the original "Granger" formulation that concentrates on the implied singularity of polynomial matrices (see Engle and Granger(1987)), the common trends idea of Stock and Watson (1986) and the state space approach suggested by Aoki (1987a, 1987b). The Internal Model concept simply provides a further alternative view of the same inter-relationships between economic variables and one that offers a relatively easy method of identifying both whether a system of equations is co-integrated for a given specification of an equilibrium and also a route for directly analysing the error correction restrictions in a model.

Co-integration implies that a set of individually non-stationary economic variables may appear in stationary linear combinations (or some more general transformation) of these variables. This result comes about through the cancellation of common factors in the dynamic interactions between the variables. In particular although multivariable transfer functions may in general contain poles and zeros at the same frequency without cancellation , co-integration implies that such cancellation does occur in such a way that the dynamic order of the system, measured by the McMillan degree or equivalently the order of the minimal state space representation, is reduced. The difficulty in analysing such systems as mentioned earlier lies in that the poles and zeros of the system as a whole will not necessarily be apparent from an inspection of the individual relations. The simplest way to then proceed is to express the system in Smith-McMillan form(see Kailath (1980) or Vidyasagar(1985) for example), a canonical form in which the system poles and zeros may be directly observed.

If we follow the approach put forward by Granger the vector of N non-stationary (integrated I(1)) variables , xt, may be expressed in a stationary purely non-deterministic moving average form as

$$\Delta x_t = C(L)\epsilon_t \tag{22}$$

for which co-integration implies that the rank of C(1) is N-r. This rank reduction at the zero frequency in turn implies the presence of r unit roots in the moving average polynomial matrix C(L). Expressing C(L) in Smith-McMillan form yields

$$C(L) = U_1(L)M(L)U_2(L) \tag{23}$$

where $U_1(L)$ and $U_2(L)$ are (non-unique) unimodular matrices and $M(L)$ is a diagonal matrix of the form $\mathrm{diag}\{\epsilon_i(L)/\psi_i(L)\}$ holding the zeros and poles of the (in general rational) polynomial matrix $C(L)$. In the present case we may factor $M(L)$ as

$$M(L) = D(L)\begin{bmatrix} I_{N-r} & 0 \\ 0 & \Delta I_r \end{bmatrix}$$

where the second matrix has extracted the unit roots. Since unimodular matrices are non-singular at all frequencies we may invert $U_1(L)$ and rewrite (22) as

$$U_1^{-1}(L)\Delta x_t = D(L)\begin{bmatrix} I_{N-r} & 0 \\ 0 & \Delta I_r \end{bmatrix} U_2(L)\epsilon_t \tag{24}$$

The common factors (unit roots) can now be clearly seen in the last r relations of this system. Cancelling in both the autoregressive and moving average parts we find

$$\begin{bmatrix} \Delta I_{N-r} & 0 \\ 0 & I_r \end{bmatrix} U_1^{-1}(L)x_t = D(L)U_2(L)\epsilon_t \tag{25}$$

or alternatively

$$U_2^{-1}(L)D^{-1}(L)\begin{bmatrix} \Delta I_{N-r} & 0 \\ 0 & I_r \end{bmatrix} U_1^{-1}(L)x_t = \epsilon_t \tag{26}$$

which is in the vector autoregressive form of equation (1)

$$A(L)x_t = \epsilon_t$$

It can also be clearly seen from the structure of $A(L)$ that the rank of $A(1)$ will in general be r rather than N. As shown in Yoo (1986) and reported in Engle(1987) it is now relatively easy to rewrite this co-integrated system in error correction form. Since $D(L)$ has full normal rank we may simply combine $U_2(L)$ and $D(L)$ to form another unimodular matrix $U_2^*(L)=D(L)U_2(L)$ and factoring $U_2^{*-1}(L)$ and $U_1^{-1}(L)$ conformably with the diagonal matrix containing the unit roots in the autoregressive polynomial we find

$$\begin{bmatrix} U_2^{*1}(L) & \gamma(L) \end{bmatrix} \begin{bmatrix} \Delta I_{N-r} & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} U_1^1(L) \\ \alpha'(L) \end{bmatrix} x_t = \epsilon_t \tag{27}$$

or

$$W(L)\Delta x_t = -\gamma(L)\alpha'(L)x_{t-1} + \epsilon_t \tag{28}$$

$$W(L)\Delta x_t = -\gamma(L)z_{t-1} + \epsilon_t \tag{29}$$

adopting the standard notation that the equilibrium error be written $z_t = \alpha'(L)x_t$. Notice that the co-integrating matrix ,$\alpha'(L)$, in this case is a polynomial matrix in L ; polynomial co-integrating vectors (PCIV) are discussed extensively in Yoo(1986).

The analysis in Engle and Granger (1987) which considers static or instantaneous co-integrating relationships is then seen to be a special case of a more general analysis that turns on the null space structure of polynomial matrices. In the standard case the cancellation of unit roots implies the singularity of $C(1)$ and co-integrating vectors are found by determining the basis for the left null space of $C(1)$. In general when we may be concerned with behaviour at other frequencies we need to determine the minimal polynomial basis of the null space of $C(L)$ at some frequency ,say λ_0. Forney (1975) and Kailath (1980) describe how such a null space may be characterised.

We are interested in the set of all rational (N×1) vectors $\{f(z^{-1})\}$ such that

$$f(z^{-1})C(z^{-1})=0$$

in other words the left null space of $C(z^{-1})$. Depending on the rank deficiency of $C(z^{-1})$ at the frequency λ_0 the dimension of this null space will be $N-r_{\lambda_0}$. If this were a vector space over real numbers , as in the static co-integration analysis, this dimension would be sufficient but since we are considering a vector space over rational polynomials we need further structure to characterise the polynomial basis. This additional structure is provided by what are known as the left minimal indicies of $C(z^{-1})$. As in the case of a vector space over real numbers any linearly independent set of vectors, $\{f(z^{-1})\}$, that span the appropriate space will provide a basis but it turns out that in the case of polynomial matrices that if each of these linearly independent polynomial vectors are chosen to have minimal degree then any polynomial basis will have the same set of degrees for the elements $\{f_i(z^{-1})\}$ in the spanning set. This unique set of indices for the left null space of $C(z^{-1})$ is called the left minimal indices of $C(z^{-1})$ and any corresponding spanning set of polynomial vectors is called a minimal polynomial basis for the left null space of $C(z^{-1})$.

From the decomposition of $C(z^{-1})$ into Smith-McMillan form given above we can see that

$$U_1^{-1}(z^{-1})C(z^{-1}) = M(z^{-1})U_2(z^{-1}) \tag{30}$$

and so any rows of $U_1^{-1}(z^{-1})$ that lead to zero rows on the right hand side of this expression form a left null space for $C(z^{-1})$ which will in general be a polynomial matrix. However typically we would expect $C(z^{-1})$ and hence $M(z^{-1})$ to be of full normal rank in which case the right hand side would only contain zero rows when evaluated at particular frequencies and hence the co-integrating basis would be a real matrix implying a fixed set of co-intgrating relations and hence a fixed relationship between the variables at every frequency. If $C(z^{-1})$ (and hence $M(z^{-1})$) were not of full rank then a polynomial matrix would represent the "co-integrating" basis implying that the set of economic variables were "co-integrated" at all frequencies albeit with different " co-integrating" relationships at each frequency. The set of left minimal indices indicate when this situation exists as they would take non-zero values. Given that it is unusual for $C(z^{-1})$ not to be of full normal rank in economic models would then suggest that static co-integrating regressions would be the norm. However the discussion of minimality shows that there may well be spanning vectors of the left null space of $C(z^{-1})$ that have higher degree than zero but if the minimal indices are zero then an instantaneous or static co-integrating relation will suffice. For instance if a static co-integrating relation were implied from this analysis then it would be equally legitimate to employ the same relation with all variables lagged to some common degree or possibly some common stable filter on all the variables as a co-integrating regression. Imposing the minimality condition thus removes, to a considerable extent , the ambiguity in dynamic specification but it should be noted that since there may still be a number of minimal polynomial bases the lack of identification noted by Granger (1986) that arises fundamentally from the non-uniqueness of the unimodular matrices used to derive the Smith-McMillan form of C(L) still remains.

An example may help to clarify the preceding argument; Engle and Yoo (1987) have considered the following bivariate model

$$(1-L)\begin{bmatrix} x_t \\ y_t \end{bmatrix} = (1-.4L)^{-1}\begin{bmatrix} 1-.8L & .8L \\ .1L & 1-.6L \end{bmatrix}\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \tag{31}$$

with vector AR representation,

$$\begin{bmatrix} 1-.6L & -.8L \\ -.1L & 1-.8L \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \tag{32}$$

The Smith-McMillan form of C(L) is given by the central matrix in the following decompositon

$$C(L) = \begin{bmatrix} 1-.8L & 0 \\ .1L & (1-.8L)^{-1} \end{bmatrix}\begin{bmatrix} (1-.4L)^{-1} & 0 \\ 0 & (1-L) \end{bmatrix}\begin{bmatrix} 1 & .8L(1-.8L)^{-1} \\ 0 & 1 \end{bmatrix}$$

$$\text{with } U_1^{-1}(L) = \begin{bmatrix} (1-.8L)^{-1} & 0 \\ -.1L & (1-.8L) \end{bmatrix}$$

and so we find

$$U_1^{-1}(L)C(L) = \begin{bmatrix} (1-.4L)^{-1} & .8L(1-.8L)^{-1}(1-.4L)^{-1} \\ 0 & (1-L) \end{bmatrix}$$

In this case C(L) is of full normal rank and so the co-integrating basis will be a real matrix thereby implying a fixed relationship between x_t and y_t at all frequencies. When we evaluate $U_1^{-1}(L)C(L)$ at the zero frequency we can see that the last row of $U_1^{-1}(L)$ provides the co-integrating vector (1, -2) and the rank of the co-integrating basis at z=1 is unity. The corresponding Error Correction form of the model is then easily derived as

$$(1-L)\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -0.4 \\ 0.1 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \epsilon_t \qquad (33)$$

To connect this analysis with the notion of an internal model we can see that if $B(L)x_t$ is to represent a set of equilibrium conditions amongst the xt then $B(z^{-1})$ must lie in the null space of $C(z^{-1})$. In other words $B(z^{-1})$ must also form a co-integrating basis. In the previous example the only internal model is that given by $x_t=2y_t$. In general most time series models will be more easily expressed in vector AR form rather than the moving average form used above so it would be convenient if we were able to check whether $B(z^{-1})$ fell into the null space of $C(z^{-1})$ using only the vector AR representation. From the development at the begining of this section we can see that $A(z^{-1})$ and $C(z^{-1})$ are related in the following manner

$$A(z^{-1}) = U_2^{-1}(z^{-1})\bar{M}(z^{-1})U_1^{-1}(z^{-1})$$

given

$$C(z^{-1}) = U_1(z^{-1})M(z^{-1})U_2(z^{-1})$$

where $\Delta I_N = \bar{M}(L)M(L)$.

So by taking the Smith-McMillan form of $A(L)$ directly we can find $U_1(z^{-1})$ (or a non-singular transformation of it). Hence we can check directly from the vector AR form whether or not $B(L)$ is spanned by $U_1(z^{-1})$ and hence whether or not $B(L)$ is a valid internal model of $A(L)$.

5: Error correction models and co-integration

To demonstrate the utility of the internal model approach we shall derive various results for the conditions for scalar error correction models from the principle before considering the multivariate case. We start by briefly reviewing the existing results for the single equation case (using the notation from Salmon (1982)).

5.1: Scalar ECM results

We consider the standard error correction form in which the "target" or equilibrium behaviour is described by a model corresponding to (2) ,

$$b(L)x_t^* = \epsilon_{2t} \tag{34}$$

Then writing the model in error correction form

$$x_t = a(L)(x_t^* - x_t)+\epsilon_t \tag{35}$$

where $a(L)=a_1(L)/a_*(L)$ is a rational polynomial in the lag operator L with $a_1(L)$ and $a_*(L)$ relatively prime and therefore (35) provides an irreducible representation. The disturbances ϵ_{2t} and ϵ_t are assumed to be stationary zero mean processes. The transfer function between the equilibrium error , $e_t=(x_t^* - x_t)$, and the "target" variables is given simply as

$$e_t = \frac{a_*(L)}{[a_1(L)+a_*(L)]} x_t^* - \frac{a_*(L)}{[a_1(L)+a_*(L)]} \epsilon_t \tag{36}$$

Three conditions are required to ensure the error correction property in a single equation (apart from the zero mean assumption on the disturbances)

i) The internal model or factor condition :
$a_*(L)$ must factorise such that $a_*(L)= \bar{a}(L)b(L)$

ii) Stability condition :

roots of $[a_1(z^{-1})+a_*(z^{-1})]$ must lie strictly inside the unit circle.

and

iii) Identifiability/controllability condition :

none of the zeros of $a(L)$ are also roots of $b(L)$

Conditions (i) and (ii) were considered in Salmon(1982) and Osborn (1986) . The third condition has not been discussed before perhaps because in the scalar case it will usually be trivial but since it is not a trivial issue in the multivariable case we shall introduce it now. The condition has in fact already been covered above by the assumption that the polynomials $a_1(L)$ and $a_*(L)$ are relatively prime . Consider the following factorisations of the non prime polynomials $a_*(L)$, $a_1(L)$ and $b(L)$, where we assume that the zeros of both $\theta(L)$ common factors are unstable;

$$a_*(L)=\theta_1(L)\theta_2(L)\bar{a}_*(L) \tag{37}$$

$$a_1(L)=\theta_1(L)\bar{a}_1(L) \tag{38}$$

$$b(L)=\theta_1(L)\theta_2(L) \tag{39}$$

so that the transfer function from ϵ_{2t} to e_t is then given by

$$\frac{\theta_1(L)\theta_2(L)\bar{a}_*(L)}{\theta_1(L)\{\theta_2(L)\bar{a}_*(L)+\bar{a}_1(L)\}\theta_1(L)\theta_2(L)} \tag{40}$$

It can then be seen that the unstable factor $\theta_1(L)$ will remain after the obvious cancellations . The presence of the common factor in the numerator of $a(L)$ and in $b(L)$ leads to a "hidden mode" ,$\theta_1(L)$, that can't be eliminated from the resulting dynamic specification. The relative prime condition ensures, together with the factor condition, that $a_1(L)$ and $a_*(L)\theta(L)$ will be coprime and hence any state space realisation of the transfer function would in this case be both stabilisable and detectable and it is not surprising that under these conditions the corresponding transfer function will deliver adjustment to the target.

5.2 The internal model in the scalar case

The internal model approach has already been used above with the model written in ECM form to ensure the exact cancellation of the unstable roots in the target variables. We now show, for the scalar case, how the ECM restrictions themselves may be derived by applying the internal model principle to the autoregressive form.

We take the standard first order ADL model considered by Davidson, Hendry, Srba and Yeo (1978),

$$y_t = \alpha_1 y_{t-1} + \beta_1 x_t + \beta_2 x_{t-1} + \epsilon_{1t} \tag{41}$$

and consider the conditions under which this model implies a constant proportional relationship between y and x in equilibrium when x is itself constant at a value k. Writing the model in autoregressive form

$$\begin{bmatrix} 1-\alpha_1 L & -(\beta_1+\beta_2 L) \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \epsilon_{1t} \tag{42}$$

where the desired equilibrium is specified as

$$B(L) \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} e_{1t} \\ \bar{e}_{2t} \end{bmatrix} \text{ where } B(L) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{43}$$

with e_{1t}, zero mean stationary and $\bar{e}_{2t}$ stationary but with mean equal to the constant k. Following the internal model principle we need to see if B(L) is a right divisor of A(L) and then find the conditions under which e_{1t} is an I(0) process. In this case by simple calculation we can see that B(L) is a right divisor with

$$A_1(L) = \left[1-\alpha_1 L \quad 1-\beta_1-(\alpha_1+\beta_2)L\right] \tag{44}$$

So given this specification of $A_1(L)$ the equilibrium error dynamics for y_t will be given by

$$(1-\alpha_1 L)e_{1t} = -[1-\beta_1 -(\alpha_1 + \beta_2)L](k + e_{2t}) + \epsilon_{1t} \tag{45}$$

where e_{2t} has a zero mean. The stability condition that $|\alpha_1| < 1$ is obvious but more importantly the standard ECM restriction, $1-\alpha_1-\beta_1-\beta_2=0$, drops directly out of the requirement that the right hand side must have a zero mean if the equilibrium error is to be $I(0)$.

What would happen if the equilibrium specification included some dynamics on x_t? Consider for instance

$$B(L) = \begin{vmatrix} 1 & -1 \\ 0 & 1-\gamma L \end{vmatrix} \tag{46}$$

in which case we find the equilibrium error is determined by

$$(1-\alpha_1 L)e_{1t} + \frac{(1-\alpha_1 L-\beta_1-\beta_2 L)}{1-\gamma_1 L}(k + e_{2t}) = \epsilon_{1t} \tag{47}$$

clearly if $|\gamma_1| < 1$ then multiplying out the denominator will just leave a stable adjustment together with the ECM restriction and a moving average error ensuring that e_{1t} is $I(0)$. If however $|\gamma_1| \geq 1$ then e_{1t} can not be $I(0)$ even if the ECM restriction is satisfied. So beyond the restriction on γ_1 that we would have expected nothing is altered in internal model analysis by the fact that there are additional dynamics in the equilibrium specification for x_t. Similarly it can be seen that the coefficient of proportionality between x_t and y_t in equilibrium can be non-unity without affecting the analysis

What happens if we take a second order ADL and consider a linear growth equilibrium for x_t and again look for a proportional equilibrium with y_t ? In this case we have,

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \beta_1 x_t + \beta_2 x_{t-1} + \beta_3 x_{t-2} + \epsilon_{1t} \qquad (48)$$

with

$$B(L) = \begin{bmatrix} 1 & -1 \\ 0 & 1-L \end{bmatrix} \qquad (49)$$

as the equilibrium specification. Solving the right divisor problem again we find

$$(1-\alpha_1 L-\alpha_2 L^2)e_{1t} + \frac{1-\alpha_1 L-\alpha_2 L^2-\beta_1-\beta_2 L-\beta_3 L^2}{1-L}(k + e_{2t}) = \epsilon_{1t} \qquad (50)$$

In this second order case it is now possible for the numerator in the second term to factorise in such a way as to cancel out the effect of the unit root in the denominator and still leave an error correction restriction. A possibility that wasn't available in the first order case in with γ=1. What is needed then for e_{1t} to be I(0) is that the factorisation

$$1-\alpha_1 L - \alpha_2 L^2 - \beta_1 - \beta_2 L - \beta_3 L^2 = \gamma(1-L)^2 \qquad (51)$$

holds , which implies the usual ECM restriction, $1=\alpha_1+\alpha_2+\beta_1+\beta_2+\beta_3$. Only ADL(2,2) forms obey this restriction will then satisfy the error correction property.

One final comment can be made on this single equation analysis before turning to the multivariable case. As long as the equilibrium error dynamics are stable in any of the cases above it is possible to eliminate a constant offset through the introduction of an intercept in the original dynamic model. As long as the intercept then assumes the value of the offset, for instance $(1-\beta_1 - \alpha_1 - \beta_2)k$ in equation (34) it is possible for e_{1t} to be I(0) without the error correction restrictions holding. However as noted by Pagan (1985) this form of adjustment would be highly non-robust to changing data sets delivering new estimates whereas the error correction restrictions are obviously robust.

5.3: The Multivariate Case

The notion of model type used in Salmon (1982) for the scalar case does not extend naturally to the multivariable case since no single degree of integration would necessarily best describe a system's ability to converge on a multivariable equilibrium in which the individual variables followed different growth patterns. The internal model principle dose lead to three conditions for the multivariate case analogous to those found in the scalar case.

Let x_t and x_t^* be N vectors and write the multivariable anologue of (35) as

$$x_t = A(L)(x_t^* - x_t) + \epsilon_{1t} \tag{52}$$

where now we assume that $A(L)$ is a rational polynomial matrix with a left matrix fraction description $A(z^{-1}) = [A_*(z^{-1})]^{-1} A_1(z^{-1})$ in which $A_1(z^{-1})$ and $A_*(z^{-1})$ are left coprime. The transfer function between the vector disequilibrium error process e_t and the target may then be written as

$$e_t = [A_*(L) + A_1(L)]^{-1} A_*(L) x_t^* + [A_*(L) + A_1(L)]^{-1} A_*(L) \epsilon_{1t} \tag{53}$$

If we characterise the multivariable target behaviour as

$$B(L) x_t^* = \epsilon_t \tag{54}$$

then a natural generalisation of the univariate conditions will ensure the cancellation of the dynamic modes in the characteristic polynomial of $B(L)$ by the numerator polynomial matrix in (53)(the right divisor property).

Sufficient conditions for a multivariate error correction system are then (assuming $\{\epsilon_{1t}\}$ and $\{\epsilon_t\}$ are zero mean stationary processes) that:

i) $B(z^{-1})$ is a right divisor of $A_*(z^{-1})$ (55)

ii) The roots of $\det[A_*(z^{-1}) + A_1(z^{-1})]$ lie strictly within the unit circle

iii) The transmission zeros of $A(z^{-1})$ are not also zeros of $\det[B(z^{-1})]$ and $A_1(z^{-1})$ is of full normal rank.

[Formal proofs of similar conditions to these for the design problem can be found ,for instance , in Callier and Desoer(1982), see in particular Theorems 31 and 60 ,pages 201-204).]

The critical operational difference as we move from the univariate to the multivariate case lies in the problem of how to achieve the cancellation of the target dynamics in the multivariable case. The fact that the poles and zeros of the individual equations may bear little relation to the poles and zeros of the overall multivariable transfer function ensures that the issue is not completely straight forward. The dynamic interactions within the endogenous variables of a model may well lead to a number of intermediate cancelations of common factors, whether they be unit roots or more general frequency decompositions, and hence it will be difficult to determine from a superficial examination of the multivariate transfer function whether an equilibrium dynamic specification is implied.

Condition (i) generalises the factor condition of the scalar case that led to the classification of the type of the adjustment mechanism , in the present multivariate case we need methods to check on the right divisors of $A_*(z^{-1})$. Condition (ii) delivers the stability condition and condition(iii) ensures that no "hidden modes" remain after the internal model has taken out the nonstationary dynamics in the equilibrium behaviour. Condition (iii) is actually stronger than is required for our problem, what is formally needed is that $B(z^{-1})$ and $A_1(z^{-1})$ be right coprime. This condition is satisfied if

$$\text{rank } M(z^{-1}) = \begin{bmatrix} B(z^{-1}) \\ A_1(z^{-1}) \end{bmatrix} = N \tag{56}$$

for all λ in $\mathbb{C}$ the complex plane. Since $A_1(L)$ is a polynomial matrix it will have full normal rank if it has full algebraic rank except at isolated values of z^{-1}, which are known as the transmission zeros of $A(z^{-1})$. For instance the matrix $N_1(z^{-1})$ below has rank 2 throughout $\mathbb{C}$ and could then appear in a MFD of a transfer function with no transmission zeros where as the matrix $N_2(z^{-1})$ has rank 1 when z^{-1} takes the values 0 and −2 and would appear in a MFD with two transmission zeros,

$$N_1(z^{-1}) = \begin{bmatrix} z^{-1} & 0 & z^{-1}+1 \\ 0 & z^{-1}+1 & z^{-1} \end{bmatrix} \qquad N_2(z^{-1}) = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}+2 \end{bmatrix}$$

Clearly if $A_1(z^{-1})$ is of full normal rank then the rank of the $2N{\times}N$ matrix $M(z^{-1})$ may fall beneath N when $\det[B(z^{-1})]$ has a root that coincides with a transmission zero of $A(z^{-1})$. The significance of this coprime condition being satisfied is there will be a completely controllable and observable realisation of the original MFD or transfer function ensuring that no unstable hidden modes will appear in $\det[A_*(z^{-1}) + A_1(z^{-1})]$.

5.2.1: Some examples

To demonstrate the utility of the internal model approach to analysing systems we shall use the following model of consumption and liquid asset behaviour has been developed by Hendry and von Ungern-Sternberg (1981) .

We consider the relationships between the three variables , consumption , c_t, liquid assets of the personal sector, l_t, and personal disposable income y_t (where all variables are in logarithms) given by ,

$$\Delta c_t = \alpha\, \Delta y_t + \beta(y_{t-1} - c_{t-1}) + \gamma(y_{t-1} - l_{t-1}) + \epsilon_{1t} \qquad (57)$$
$$\Delta l_t = \delta(y_{t-1} - c_{t-1}) + \epsilon_{2t} \qquad (58)$$

together with a model for income growth which we assume initially to be

$$\Delta y_t = g + \epsilon_{3t} \tag{59}$$

The stochastic equilibrium relations we take for this model following Hendry and von Ungern–Sternberg (1980) are

$$\begin{aligned} c_t &= k_1 + y_t + e_{1t} \\ l_t &= k_2 + y_t + e_{2t}. \\ \text{and} \quad y_t &= g + y_{t-1} + e_{3t} \end{aligned} \tag{60}$$

Although the model is already superficially in ECM form we may be uncertain as to the exogenous growth assumptions used when the model was constructed and so we apply the internal model principle to the vector autoregressive form. For the moment we also assume that the constants k_1 and k_2 are taken into the equilibrium error terms e_{1t} and e_{2t}. Writing the model in autoregressive form we have,

$$\begin{bmatrix} 1-a_1L & -a_4L & -(a_2+a_3L) \\ -b_3L & 1-b_1L & -b_2L \\ 0 & 0 & 1-c_1L \end{bmatrix} \begin{bmatrix} c_t \\ l_t \\ y_t \end{bmatrix} = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{bmatrix} \tag{61}$$

representing $\qquad A(L)\mathbf{x}_t = \epsilon_t$

with the parameter restrictions
$a_1 = -\beta + 1 \qquad b_1 = 1 \qquad c_1 = 1$
$a_2 = \alpha \qquad b_2 = \delta$
$a_3 = \beta + \gamma - \alpha \qquad b_3 = -\delta$
$a_4 = -\gamma$

notice that $\qquad \Sigma a_i = 1 \qquad \Sigma b_i = 1 \qquad \Sigma c_i = 1$

Applying the internal model principle by taking the right divisor specified by the equilibrium conditions from $A(L)$ we find the equilibrium error dynamics are given by

$$\begin{bmatrix} 1-a_1L & -a_4L & \dfrac{\bar{a}(L)}{1-L} \\ -b_3L & 1-b_1L & \dfrac{\bar{b}(L)}{1-L} \\ 0 & 0 & \dfrac{\bar{c}(L)}{1-L} \end{bmatrix} \begin{bmatrix} e_{1t} \\ e_{2t} \\ \bar{e}_{3t} \end{bmatrix} = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{bmatrix} \tag{62}$$

where $\bar{a}(L) = 1-a_2-(a_1+a_3+a_4)L \equiv 1-\alpha-(1-\alpha)L = (1-\alpha)(1-L)$ (63)

$\bar{b}(L) = 1-(b_1+b_2+b_3)L \equiv 1-L$

$\bar{c}(L) = 1-c_1L \equiv 1-L$

and the equilibrium error $\bar{e}_{3t} = e_{3t} + g$.
For the original model to be an error correction system we need to ensure that e_{1t} and e_{2t} are both $I(0)$. Solving for e_{1t} and e_{2t} in terms of $\bar{e}_{3t}$ we find

$$e_{1t} = -\frac{[(1-b_1L)\bar{a}(L) + a_4L\bar{b}(L)]}{\psi(L)} \frac{(e_{3t}+g)}{1-L} + \epsilon^*_{1t} \tag{64}$$

and

$$e_{2t} = -\frac{[b_3L\,\bar{a}(L) + (1-a_1L)\,\bar{b}(L)]}{\psi(L)} \frac{(e_{3t}+g)}{1-L} + \epsilon^*_{2t} \tag{65}$$

where $\psi(L) = (1-a_1L)(1-b_1L) - a_4b_3L^2$

Given the factorisations for $\bar{a}(L)$, $\bar{b}(L)$ and $\bar{c}(L)$ from (63) the unit roots in the denominators of (64) and (65) cancel out implying that both $\{e_{1t}\}$ and $\{e_{2t}\}$ will be stationary (given $c_1=1$). Next we need to check whether they have zero means.

Since $b_1=1$ there is one further unit root in (64) but since $a_4=-\gamma$ we find that $E(e_{1t})=\gamma g$. Similarly since there are no further unit roots in (65) we find that $E(e_{2t})=[\delta(1-\alpha)+\beta]g$. So if our original equilibrium required that both these means should be zero then the model specification would not ensure that both $\{e_{1t}\}$ and $\{e_{2t}\}$ were I(0), in other words it would not be an ECM specification for the given equilibrium.

This may not be too surprising given that income is following a first order growth process and the highest order of integration in the model is also one. From the deterministic scalar analysis we would expect there to be a constant offset in this case and that is exactly what we find with the non-zero means for e_{1t} and e_{2t}. If we now reconsider the equilibrium specification and note that it may have included non-zero mean values of k_1 and k_2 then whether or not the model is an ECM turns on whether these values coincide with $E(e_{1t})=\gamma g$ and $E(e_{2t})=[\delta(1-\alpha)+\beta]g$. In the present example it is probably highly unlikely that specific values could be assigned to k_1 and k_2 and if an intercept were included in the equations at the time of estimation then essentially we find another case of what Pagan referred to as "intercept adjustment". Given that the model specification is of too low an order to guarantee that e_{1t} and e_{2t} are I(0) the presence of the intercept will compensate for the non zero mean over the sample period. In this case it would seem to add substantial flexibility to the problem if we were unsure of the "correct" values for k_1 and k_2 in equilibrium. On the other hand the implied ECM would not be robust to changing samples and re-estimation. Notice also that co-integration to I(0) requires the estimated equilibrium error to have zero mean. The importance of this requirement may easily be forgotten in applied work leading to error correction models that rely on intercept adjustment rather than being independent of such constants.

Given that the original model has been found to be of too low an order to quarantee the ECM property without the help of an intercept adjustment the natural question to ask is whether there is a higher order specification that will ensure the equilibrium errors are I(0) regardless of the presence of an intecept in the equations. We look at two alternatives that have been considered before in Davidson (1984) and Johansen(1985).

The first alternative is simply to include the level of liquid assets in the consumption function in place of the income/liquid asset equilibrium error term.

$$\Delta c_t = \alpha\, \Delta y_t + \beta(y_{t-1} - c_{t-1}) + \gamma l_t \tag{66}$$

Remember that we are making no assumptions as to the co-integration properties of the data which might suggest that liquid assets were I(1) where as the rest of the terms in this model are probably I(0), making the proposed model meaningless. Proceding as above to take the right divisor specified by the equilibrium from this new model we find the equilibrium error equations are given by

$$e_{1t} = -\frac{[(1-b_1 L)\bar{a}(L) + a_4 \bar{b}(L)]}{\psi(L)} \frac{[e_{3t}+g]}{1-L} + \epsilon^*_{1t} \tag{67}$$

and

$$e_{2t} = -\frac{[\, b_3 L\, \bar{a}(L) + (1-a_1 L)\, \bar{b}(L)]}{\psi(L)} \frac{(e_{3t}+g)}{1-L} + \epsilon^*_{2t} \tag{68}$$

with changes in the parameterisation for this model given by,

$$\begin{array}{lll} \bar{a}(L) = 1-\alpha-\gamma+(1-\alpha)L & \text{and} & a_1 = 1-\beta \\ \bar{b}(L) = 1-L & & a_2 = \alpha \\ \bar{c}(L) = 1-L & & a_3 = -\alpha+\beta \\ & & a_4 = \gamma \\ & & \Sigma a_i = 1+\gamma \end{array}$$

From equations (67) and (68) with these parameters we find that although e_{1t} is stationary , e_{2t} is not because $\bar{a}(L)$ no longer factorises ,a problem which intercept adjustment can't cure.

The second alternative based on an apparently trivial change is to consider a consumption function of the form,

$$\Delta^2 c_t = \alpha \Delta^2 y_t + \beta\Delta(y_{t-1} - c_{t-1}) + \gamma(y_{t-2} - c_{t-2}) + \epsilon^*_{1t} \tag{69}$$

In this case the changed parameter restrictions on the vector autoregression are given by

$$a_1 = 2-\beta$$
$$a_2 = -1+\beta-\gamma$$
$$a_3 = \alpha$$
$$a_4 = -2\alpha + \beta$$
$$a_5 = \alpha-\beta+\gamma \text{ and we see that } \Sigma a_i = 1$$

and solving once again for the error dynamics we find,

$$e_{1t} = -\frac{[(1-b_1L)\bar{a}(L)]}{\psi(L)} \frac{(e_{3t}+g)}{(1-L)} + \epsilon_{1t}^{**} \tag{70}$$

and

$$e_{2t} = -\frac{[b_3L\,\bar{a}(L) + (1-a_1L - a_2L^2)\bar{b}(L)]}{\psi(L)} \frac{(e_{3t}+g)}{(1-L)} + \epsilon_{2t}^{*} \tag{71}$$

where $\psi(L) = (1-a_1L - a_2L^2)(1-b_1L)$

The parameter restrictions in turn imply for this model that

$$\bar{a}(L) = 1-a_3-(a_1+a_4)L-(a_2+a_5)L^2$$
$$= (1-\alpha)(1-L)^2$$

and
$$\bar{b}(L) = 1-L$$
$$\bar{c}(L) = 1-L$$
$$1-a_1L - a_2L^2 = 1-(2-\beta)L-(-1+\beta-\gamma)L^2$$

substituting these into equations (70) and (71) we find that the additional unit root(s) in the numerator of (70) ensure that e_{1t} will have a zero mean without the help of intercept adjustment. However there is still only one unit root to be canceled in (71) so we now find $E(e_{2t}) = \gamma g$. Both equilibrium errors are stationary given the cancellation of the unstable dynamics. This relatively minor change between the last two models has clearly made a significant difference in their error correcting properties.

6 : Conclusions

The internal model principle and the use of right divisors determined by the equilibrium conditions would seem to generalise the approach based on model type and to provide a powerful ,yet simple framework for the analysis of error correction systems. The examples above have not exploited the time series properties of the data, in particular no use has been made of the notion of co-integration in determining whether a given dynamic specification was consistent with a given equilibrium specification. As such the method seems to provide an analysis of the structural properties of error correction systems and the suggestion is then that this approach might be usefully employed in conjunction with co-integration analysis. One of the basic advantages of the error correction formulation is that the restrictions apply whatever the underlying parameter values and as such provide a degree of robustness in the specification. It is however not possible to construct a robust specfication if the equilibrium is itself misspecified. The internal model principle requires that the equilibrium be prespecified in some way and the model is then constructed conditionally on that equilibrium specification.

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