SOLVING STOCHASTIC SADDLEPOINT SYSTEMS: A QUALITATIVE TREATMENT WITH ECONOMIC APPLICATIONS

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ABSTRACT

We examine the effect of introducing stochastic shocks into a linear rational expectations model with saddlepoint dynamics generated by a forward-looking asset price. We derive the fundamental differential equation governing the path of the asset price as a function of the "sluggish" variable. The equation does not admit of closed form solutions in general, but we provide a complete qualitative characterization of the solution paths which are symmetric about equilibrium. These are the relevant solutions to consider in the presence of symmetric boundary conditions.

We present two applications. The first analyzes how financial markets might react to the implementation of fiscal stabilization policy where public expenditures are only adjusted when GNP moves outside a threshold around a target level. Bond prices are perfectly flexible and move to satisfy an arbitrage condition. The second examines exchange rate behavior in the presence of a currency subject to a known realignment rule requiring an adjustment to monetary policy.

I. INTRODUCTION

Rational expectations models in which forward-looking financial markets correctly discount time-consuming processes of adjustment elsewhere in the economy (and expected future changes in government policy) have been widely applied to interpret macroeconomic phenomena. Dornbusch (1976), for example, proposed that flexible exchange rates might overshoot in response to monetary tightness, as financial markets forecast the high interest rates needed to adjust sluggish goods prices. Later, in considering stock price movements, Blanchard (1981) noted that whether news of fiscal expansion was good or bad for Wall Street would depend on the balance between the mark-up it implied for dividends and for interest rates. In a Tobin's q model of equity prices, Summers (1981) took discount rates to be fixed, and focused instead on effects of capital accumulation on (after tax) dividends and the price of stocks.

The essential tool of analysis used by these authors is a phase diagram whose saddlepoint structure reflects the conjuncture of the "no profit condition," as it applies to financial arbitrage and the slow dynamics of adjustment of non-financial variables (such as GNP and the capital stock) characteristic of these examples. The "stable branch" of the saddlepath is used to examine the process of adjustment towards equilibrium; but unstable trajectories are also used in considering anticipated future changes in exogenous conditions (cf. Wilson, 1979).

In something of a contrast to this deterministic treatment, modern finance theory has emphasized the role of stochastic elements, asset values being seen as the present discounted value of the stochastic processes driving dividends, etc. Another contrast with the economic examples we have cited (where dividends and/or interest rates are endogenous) is that finance theorists almost invariably adopt a partial equilibrium approach — so the parameters of the stochastic processes are independent of the asset prices they help to determine.

Recently, however, interesting analytical results have been obtained by applying just such a stochastic approach to problems in micro- and macroeconomics (under the simplifying assumption that agents are risk neutral). Thus, for an industry with lumpy costs of joining and leaving, Dixit (1988) shows explicitly how the trigger prices for entry and exit are related to the (exogenous) rate of diffusion of the Brownian motion process driving prices in the industry. And Krugman (1987,1988) derives an exact exponential formula showing how the prospect of intervention at the edges of a currency band will bias the exchange rate inside the band, assuming fundamentals follow an exogenous Brownian motion. By applying Krugman's approach to the Dornbusch model (1976), the current authors (1988) showed that the partial equilibrium restriction is not necessary — although one will typically need to use numerical methods to obtain exact results in a setting where the parameters of the stochastic processes are endogenous.

What is to be shown in this paper is that the essential qualitative features of symmetric solutions to linear stochastic simultaneous equation saddlepoint systems can be obtained analytically, without recourse to numerical methods. This is done in the section that follows. Two economic examples of this qualitative analysis are then given by way of illustration.

II. QUALITATIVE ANALYSIS OF LINEAR SADDLEPOINT SYSTEMS WITH WHITE NOISE DISTURBANCES

A. The Fundamental Differential Equation

The analysis of this paper is restricted to linear saddlepoint systems; the solutions, nevertheless, turn out to be nonlinear when white noise is added. The essential features are a stochastic differential equation describing the evolution of the "fundamental," x, and an arbitrage condition specifying the expected change in a forward-looking asset price, y. The only source of randomness is the white noise disturbance term entering the first equation, so the price of the asset is random only through its dependence on fundamentals.

To establish notation we write the system as

$$\begin{bmatrix} dx \\ \mathrm{E}\, dy \end{bmatrix} = \Gamma \begin{bmatrix} x\, dt \\ y\, dt \end{bmatrix} + \begin{bmatrix} \sigma\, dz \\ 0 \end{bmatrix} \tag{1}$$

where x is the sluggish ("state") variable,

y is the forward looking ("costate") variable,

z is a Brownian motion process with unit variance,

σ is a positive constant scaling factor,

Γ is a matrix of constant coefficients,

E is the expectations operator.

The elements of Γ are denoted by the first four characters of the Greek alphabet, as follows:

$$(2) \qquad \Gamma = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

so the characteristic equation is

$$(3) \qquad \lambda^2 - \text{trace}\ \Gamma + \Delta = \lambda^2 - (\alpha + \delta)\ \lambda + \Delta = 0$$

where λ denotes a root and $\Delta = \alpha\delta - \gamma\beta$ is the determinant of Γ. For the saddlepoint property, it is necessary that $\Delta < 0$.

It turns out that the eigenvectors and stationary loci of the system without noise (i.e., where $\sigma = 0$) play a central role in characterizing stochastic solutions. With an eye to the applications which follow, we will principally work with the case where these eigenvectors — the branches of the saddlepath — have slopes of opposite sign. (The qualitative analysis is essentially unchanged when the gradients have the same sign.) Without loss of generality, we assume that the requisite slopes are given by the following sign pattern of the coefficients of Γ, namely $\alpha < 0$, $\beta < 0$, $\gamma < 0$, $\delta > 0$. The gradient for the lines of stationarity for y and x follow immediately from (1), (2), setting $\sigma = 0$, as

$$\left.\frac{dy}{dx}\right|_{dy/dt\,=\,0} = -\gamma/\delta \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{dx/dt\,=\,0} = -\alpha/\beta,$$

as shown alongside the relevant angles in Figure 1. (Where σ is nonzero, these lines become loci of "expected" stationarity.) The slopes of the stable and unstable eigenvectors, denoted θ_s and θ_u, respectively, have the signs shown in the figure as $\theta_s = \gamma/(\lambda_s-\delta) > 0$ and $\theta_u = (\lambda_u-\alpha)/\beta < 0$.

The slopes of these eigenvectors are obtained as the roots of a quadratic equation in the parameters of Γ; and as this quadratic plays a role in the proofs that follow, it is given explicitly at this juncture. Let an eigenvector, normalized on its first element, be written as $\begin{bmatrix} 1 \\ \theta \end{bmatrix}$; it must, by definition, satisfy the condition that

(4) $$\Gamma \begin{bmatrix} 1 \\ \theta \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \theta \end{bmatrix}$$

or, in detail, (i) $\alpha + \beta\theta = \lambda$

(ii) $\gamma + \delta\theta = \lambda\theta$

where λ is a root of the characteristic equation of Γ. Together 4(i) and 4(ii) imply

(5) $$p(\theta) = \gamma + (\delta-\alpha)\,\theta - \beta\theta^2 = 0.$$

So $p(\theta)$ is the required quadratic expression whose zeros give the required slopes, θ_s, θ_u.

In the deterministic case, where $\sigma=0$, the two eigenvectors are the only paths connected to the equilibrium point at the origin (see Figure 1). But when the system is subject to stochastic shocks, there turns out to be an infinity of trajectories connected to the origin, whose qualitative features it is our intention to characterize. To do this we first derive the fundamental differential equation to be satisfied by any solution to (1), and then show how those which pass through the origin possess a symmetry which is essential to solutions of problems with symmetric boundary conditions (see Section III).

Let any particular solution be expressed as a function of x, so

$$(6) \qquad y = f(x).$$

Then Ito's Lemma implies that

$$(7) \qquad E\, dy = f'(x)\ \mathrm{E}\ dx + \frac{\sigma^2}{2} f''(x)dt$$

Rearrangement and substitution yields

$$(8) \qquad \frac{\sigma^2}{2} f''(x) = -f'(x)\ (\alpha x + \beta f(x)) \quad + \quad (\gamma x + \delta f(x))$$

the fundamental differential equation to be satisfied by any solution to the linear stochastic system.

The qualitative nature of those solutions whic' pass through the origin is shown in Figure 2. First we note that such solutions are strictly symmetric. (It follows directly from (8) that if $f(x)$ is a solution satisfying $f(0) = 0$, then $f(x) = -f(-x)$.) As mentioned earlier, other qualitative features evident in the figure depend essentially on the eigenvectors of the deterministic system — which are themselves stochastic solutions. Since the expected direction of movement in the stochastic case is characterized by the same stationary loci as for the deterministic case, it will be evident that Figure 2 is not a phase diagram. What it shows is the qualitative nature of the deterministic relationship which links the asset price to the fundamental in circumstances where the latter is randomly shocked by white noise and the system itself is subject to symmetric boundary conditions (whose exact nature will determine the relevant solution, as we show with examples later).

The notion that rational asset prices are deterministic functions of stochastic fundamentals, is, of course, central to much modern finance theory, e.g., the analysis of option pricing in Merton (1973). That feature of the stochastic saddlepoint systems which is peculiarly macroeconomic is that in general the process determining the "fundamentals" is not autonomous but depends on the asset price itself. (Thus, from (1), (2), $dx = \alpha x dt + \beta y dt + \sigma dz$ which, given a solution $y = f(x)$, becomes $dx = \alpha x dt + \beta f(x) dt + \sigma dz$ so that, unless $\beta = 0$, the diffusion process governing x will depend on the function $f(x)$.) It is precisely this "endogeneity of the fundamentals" which makes it impossible in general to find closed form solutions and leads one to a qualitative treatment.

We now derive the qualitative treatment implicit in Figure 2 in detail, proceeding from a local to a global analysis.

B. Local Analysis: Curvature Around Equilibrium

The curvature of paths in the neighborhood of the origin, here the equilibrium, may readily be established with reference to the quadratic expression $p(\theta)$. For if the function f'' is approximated to first order at the origin, we find that

$$(9) \qquad \frac{\sigma^2}{2} f''(x) \simeq \frac{\sigma^2}{2}(f''(0) + f'''(0)x)$$

$$= \frac{\sigma^2}{2} f'''(0)x$$

$$= [-f''(0)(\alpha \cdot 0 + \beta f(0)) - f'(0)(\alpha + \beta f'(0)) + \gamma + \delta f'(0)]x$$

$$= p(\theta)x$$

letting $\theta = f'(0)$.

On our assumptions, $p(\theta)$ must take the form illustrated in Figure 3(a), since $p(0) = \gamma < 0$, and $p(\theta)$ reaches a minimum at $\theta^* < 0$. This enables us to calculate the sign of f'' in a neighborhood of the origin, illustrated in Figure 3(b). As the angle $\tan^{-1}(\theta)$ goes from 0 to 360° and one proceeds counterclockwise round the origin, the sign of f'' changes with each crossing of an eigenvector and of the vertical axis. See Figure 3(b).

C. Global Analysis

Because of the symmetry of the solutions, it is only necessary to consider half the plane. Taking the right hand half-plane, where $x \geq 0$, we divide this into four regions as shown in Figure 4(a). Regions A and B lie on either side of the stable eigenvector and above the line, LL, of (expected) stationarity for x. Here the expected movement of x is towards zero. For regions C and D, which lie either side of the unstable eigenvector and below LL the expected movement of x is away from equilibrium. We consider regions in pairs. We first observe that, by a standard theorem on differential equations (see, for example, Birkhoff and Rota [1969], p. 152), the two initial conditions $f(0) = 0$ and $f'(0) = 0$ determine a unique solution to (8) in any compact convex region of the (x,y) plane; so the solution trajectories do not intersect other than at the origin.

1. Regions A and B. Rather than working directly with the function $f(x)$, it will be convenient to use the function $g(x)$ which measures the "distance" of f from the stable manifold of the deterministic system, i.e.,

(10) $$g(x) = f(x) - \theta_s x$$

where (i) $g' = f' - \theta_s$

(ii) $g'' = f''$.

While the slope of g differs from that of f by a constant, the convexity/concavity of the two functions is the same for any x. See Figure 4(b).

By substitution into (8), we find

$$\frac{\sigma^2}{2} g'' = -(g' + \theta_s)(\alpha x + \beta(g + \theta_s x)) + \gamma x + \delta(g + \theta_s x)$$

$$= -((\alpha + \beta\theta_s)x + \beta g)g' + p(\theta_s)x + (\delta - \theta_s\beta)g$$

and so

$$(11) \qquad \frac{\sigma^2}{2} g'' = -((\alpha + \beta\theta_s)x + \beta g)g' + (\delta - \theta_s\beta)g$$

as $p(\theta_s) = 0$.

To characterize paths in these regions it is necessary first to demonstrate that the terms multiplying g and g' are both positive. The former, $\delta - \theta_s\beta$, is positive as δ, $\theta_s > 0$, $\beta < 0$. As for the latter, $-((\alpha + \beta\theta_s)x + \beta g)$, we note that f lies above the LL locus in regions A and B; i.e., $f > -\frac{\alpha}{\beta} x$, so it follows that

$$g = f - \theta_s x > -\left(\frac{\alpha}{\beta} + \theta_s\right)x.$$

But then, since $\beta < 0$, $-((\alpha + \beta\theta_s)x + \beta g) > 0$.

Now in region A we consider all solutions satisfying $g'(0) > 0$. We wish to show that $g''(x) > 0$ for $x > 0$. Suppose this is not the case and let $\bar{x}$ be the smallest strictly positive value of x, such that $g''(\bar{x}) = 0$.

We know that g, g', and g'' are all strictly positive in a neighborhood of the origin. It must also be true from (11) that either $g(\bar{x}) \leq 0$ or $g'(\bar{x} \leq 0$. It follows, since g is continuous, that there must be $\hat{x} < \bar{x}$, satisfying $g''(\hat{x}) = 0$, contrary to hypothesis.

An exactly analogous argument establishes that $g''(x) < 0$ for $x > 0$ in region B, i.e., for all solution paths satisfying $\frac{\alpha}{\beta} - \theta_s < g'(0) < 0$. Thus, the curvature found in a neighborhood of the origin applies throughout each of these regions. Paths in A are strictly convex to the x axis, paths in B strictly concave (and, of course, on the boundary between the two regions, where $g' = \theta_s$, $g'' = 0$ for all $x > 0$).

2. <u>Regions C and D</u>. Although the expected movement of x is away from the current equilibrium in these regions, it is still possible to return to equilibrium by a sufficient sequence of appropriate shocks. Moreover, such trajectories are relevant to problems displaying "hysteresis," where the equilibrium itself is expected to shift when fundamentals diverge — as in the currency realignments example discussed below. So it is worth proceeding to characterize these paths.

Here it is convenient to define a function $h(x)$ to represent the "distance" of f from the unstable eigenvector, see Figure 4(C). So

(12) $$h(x) = f(x) - \theta_u x$$

where (i) $h' = f' - \theta_u$

(ii) $h'' = f''$.

By substitution we find,

$$(13) \qquad \frac{\sigma^2}{2} h'' = -((\alpha + \beta\theta_u)x + \beta h)h' + (\delta - \theta_u\beta)h$$

an analogue of equation (11) above. Both the terms multiplying h and h' can be signed as negative in regions C and D. As for the former, $\delta - \theta_u\beta$, we note that

$$p(\theta_u) = -\beta\theta_u^2 + (\delta - \alpha)\theta_u + \gamma = 0$$

so $\delta - \theta_u\beta = \frac{-\gamma}{\theta_u} + \alpha < 0$, as α, θ_u, $\gamma < 0$.

As for the former, the requirement that

$f < \frac{-\alpha}{\beta} x$ implies $h = f - \theta_u x < -\left(\frac{\alpha}{\beta} + \theta_u\right)x$ and so

$-(\beta h + (\alpha + \beta\theta_u) x) < 0$, given $\beta < 0$.

To examine the curvature of solutions near the unstable eigenvector, we begin in region D where all the solution trajectories are connected to the origin. By construction, the required curvature is also exhibited by the function h charted in the bottom panel of Figure 4. In what follows, we argue that any solution in the region has the qualitative characteristics shown by the curve labeled X_DX_D, i.e., it has a negative slope but is convex near the origin, reaches a minimum (here at x^*), and passes through a unique point of inflection (here at x^{**}) before approaching a limiting value as x increases without limit.

The property that $f'' > 0$ in the region near the origin establishes that $h''(x) > 0$ for paths starting from the origin with a negative slope, as shown by X_DX_D in the figure. The general characteristics of a solution path, as x moves away from the origin, are established step by step, as follows.

(a) <u>To show that there exists x^* satisfying $h'(x^*) = 0$</u>

Suppose not. Since $h'(0) < 0$, then, <u>ex hypothesi</u>, $h' < 0$ for all $x > 0$ and h is bounded away from zero as x increases, and is always negative. But then (13) implies that $h'' > 0$ and is bounded away from zero as x increases, which implies that h' must pass through zero, contrary to hypothesis.

(b) <u>To show that $h(x)$ is strictly convex over the range $0 \leq x \leq x^*$, and reaches a minimum at x^*</u>

That $h'' > 0$ over the range $0 \leq x \leq x^*$ follows from the fact that $h'(0) < 0$ and from the fact that the coefficients on h and h' in (13) are both negative. It follows from (a) that $h'(x^*)$ is a minimum.

(c) <u>To show that $h'(x) > 0$ for all $x > x^*$</u>

Since $h < 0$ for all $x > 0$ from the fact that $h'(0) < 0$ and the uniqueness theorem, at any point where $h' = 0$ it follows from (13) that $h'' > 0$. But if there existed any turning point after x^*, the first one would have to be a local maximum. So there can be no such points.

(d) <u>To show that there exists a unique point of inflection for $h(x)$ at $x^{**} > x^*$</u>

It follows from (c), and that h is bounded above by zero, and that $h''(x^*) > 0$ that there must exist some value of $x > x^*$ at which h'' passes through zero from above, i.e., $h''(x) = 0$, $h'''(x) < 0$. Let x^{**} be the smallest such value. Then, from (13)

$$\text{(14)} \qquad \frac{\sigma^2}{2} h'''(x^{**}) = -(\alpha+\beta\theta_u)h'(x^{**}) - \beta h'^2(x^{**}) + (\delta-\theta_u\beta)h'(x^{**}) < 0.$$

It follows from (14) that

$$\text{(15)} \qquad h'(x^{**}) < \frac{\delta - \alpha}{\beta} - 2\theta_u.$$

Suppose there exists another point of inflection at $x^{***} < x^{**}$. Then by assumption $h''(x^{***}) = 0$ and $h'''(x^{***}) > 0$. But since $h'' < 0$ for $x^{**} < x < x^{***}$ it follows from (15) that

$$h'(x^{***}) < \frac{\delta - \alpha}{\beta} - 2\theta_u.$$

But, using (14), this implies that $h'''(x^{***}) < 0$, which is a contradiction. Therefore the point of inflection at x^{**} is unique.

(e) <u>To show that h approaches a well-defined limit</u>

This follows immediately from (c) and the fact that h is bounded above by zero.

The arguments needed to characterise paths in region C are very similar to those above. The only point to note is that in region C some paths enter from region B, and it may be true that $h' < 0$ when they enter. So the analogue to (a) is a demonstration that there exists x^* satisfying $h'(x^*) \leq 0$, where we pick x^* to be the smallest value of x satisfying the inequality. Subsequent proofs then follow exactly as before, with suitable sign changes.

Note finally that so long as the saddlepoint structure of the deterministic model is preserved, the precise configuration of stable and unstable eigenvectors is unimportant for the qualitative characterization of solutions. But two special cases are worth remarking upon. First, if $\beta = 0$, then the unstable eigenvector coincides with the vertical axis, and the regions C and D disappear. Solution paths lying in the half-plane where $x \geq 0$ are globally concave or convex and all points of inflection (other than those at the origin) vanish. Second, if $\beta = \gamma = 0$, then the stable and unstable eigenvectors correspond to the horizontal and vertical axes, respectively. In this case, all turning points vanish.

III. ECONOMIC APPLICATIONS IN CLOSED AND OPEN ECONOMIES

Two illustrations of this qualitative analysis are provided here. First, we analyze how the bond market in a closed economy would react to fiscal stabilization policy which is applied discontinuously; and second, how the foreign exchange market might be expected to behave in the presence of an exchange rate band subject to stochastic realignment. In each case, we show how the particular solution path leading to equilibrium is determined by the (symmetric) boundary conditions of the problem.

A. Fiscal Policy With Thresholds And The Bond Market

To analyze how financial markets might react to the implementation of fiscal stabilization policy where public expenditures are only adjusted when GNP moves outside a threshold around target level, we make use of the familiar Hicksian IS/LM diagram, augmented here with the minimum of dynamics, as indicated in equations (A1) through (A4), below. (For simplicity, prices are taken as fixed, unlike the open economy case, considered next.)

Equation (A1) gives the condition for equilibrium in the money market; while equation (A2) indicates the determinants of aggregate demand, x. The latter depends positively on output, and negatively on the long run (real) interest rate, R. It is also affected by government intervention, indicated by the variable, g; this contracyclical intervention is, however, only applied outside the intervention limits $\underline{y} < \bar{y}$.

The sluggish adjustment of output, equation (A3), is subject to stochastic shocks. Bond prices are perfectly flexible, however, and satisfy the arbitrage condition $E(dR) = R(R-r)$ which is linearized around equilibrium $\hat{R}$ in equation (A4).

(A1) $m = -\lambda r + \kappa y$

(A2) $x = -\gamma R + \alpha y + g, \quad 0 < \alpha < 1$

where $g = 0, \quad \underline{y} < y < \bar{y}$

$g = -\beta y, \quad \bar{y} \le y \le \underline{y}, \quad \beta > 0$

(A3) $dy = \psi(x - y)dt + \sigma dz$

(A4) $E(dR) = \hat{R}(R - r)dt.$

Notation
- m the money supply
- y the level of real output
- x aggregate demand
- r the instantaneous short rate
- R the dividend yield on infinitely dated bonds.
- g fiscal intervention, applied contracyclically but subject to a threshold; see (A2) above.
- z Brownian motion process with a unit variance.

Before taking account of fiscal intervention, we note that, setting $g = 0$, the dynamics for adjustment are

(A5)
$$\begin{bmatrix} dy \\ EdR \end{bmatrix} = \begin{bmatrix} \psi(\alpha-1) & -\psi\gamma \\ \hat{R}\kappa\lambda^{-1} & \hat{R} \end{bmatrix} \begin{bmatrix} ydt \\ Rdt \end{bmatrix} + \begin{bmatrix} \sigma dz \\ 0 \end{bmatrix}$$

where we set $m = 0$, so that all variables are effectively measured as deviations from equilibrium. The negative determinant ensures saddlepoint dynamics, as shown in Figure 5. There the line of stationarity for the long bond rate appears as the LM curve, the locus of stationarity for output is the IS curve, and the eigenvectors, labelled as SS and UU, can be shown to have slopes of opposing sign, as drawn. Thus, in the absence of fiscal intervention, GNP and the long bond rate will be "diffused" along the stable eigenvector SS (with a finite asymptotic distribution which is Gaussian).

To take account of fiscal policy, we note first that in the absence of thresholds the effect of intervention would be simply to increase the absolute size of the top left-hand coefficient in (A5) changing it from $\psi(\alpha-1)$ to $\psi(\alpha-\beta-1)$, which would make the IS curve steeper and hence pivot the stable manifold down to S'S'. The effects of applying the policy subject to the thresholds, assumed to be equidistant from $\hat{y}$, can now be derived — by locating the symmetric solution to the first system which passes through the points F'F' where the intervention points cut the manifold S'S'. This solution ensures that the economy will lie on the second manifold, S'S', outside the threshold limits and also that asset prices will not jump when fiscal policy is activated at the threshold points. It is, of course, the enhanced probability of such intervention that bends the solution away from SS towards S'S' as output moves away from the target towards the limits of intervention. (What happens if the intervention points are <u>not</u> equidistant from the target, $\hat{y}$, is considered briefly in section IV, below.)

B. Realignment Rules For Exchange Rate Bands

In the previous example, reaching a GNP threshold acted as a trigger for an activist fiscal policy designed to speed a return to target GNP. In much the same way, reaching the edge of an exchange rate band may be the signal for a spirited defense of the currency, and, as Krugman has demonstrated in the paper we have cited above, the expectations of such action will lead to what he refers to as a "bias in the band." But reaching the edge may also be the occasion for a "realignment" of the band (and of the monetary policy to support it), and it is on the consequences of such realignment rules that we focus here, using a simplified version of the model in Dornbusch (1976, Appendix). The equations of the model and the variables used are as follows:

(B1) $$m = p + \kappa\bar{y} - \lambda i$$

(B2) $$y = -\delta(x + p - p^{*}) - \gamma i$$

(B3) $$dp = \phi(y - \bar{y})dt + \sigma dz$$

(B4) $$Edx = (i^{*} - i)dt$$

with notation

- m log of the money stock.
- p log of the price level.
- y log of GNP.
- $\bar{y}$ high employment GNP.

x log of the exchange rate (foreign currency price of domestic currency).

i instantaneous nominal interest rate.

z a Brownian motion process with unit variance parameter.

$\star$ indicates variables in "rest of world."

For simplicity the demand for money, on the right-hand side of equation (B1) is made to depend on exogenous high employment GNP, $\bar{y}$ — as well as the (endogenous) interest rate and price level. The determinants of aggregate demand, shown in (B2), are the real exchange rate and the (short-term) interest rate. In this open-economy model, the lag in the adjustment of output has been ignored, and attention focussed on the dynamics of price changes instead. Thus, as shown in (B3), the price level is adjusted in proportion to output and is also subject to a white noise disturbance term. (Since the money stock is assumed to be held constant between realignments, and since the realignments do not lead to systematic inflation, omitting any inflation expectations terms may be reasonably justified.) As for arbitrage in assets, it is assumed that the exchange rate adjusts so that its expected change equals international interest differentials, equation (B4).

Setting $p^{\star} = i^{\star} = m = \bar{y} = 0$ and substituting gives the autonomous dynamics of the price level and the exchange rate as

$$\text{(B5)} \qquad \begin{bmatrix} dp \\ Edx \end{bmatrix} = \begin{bmatrix} -\phi(\delta + \gamma\lambda^{-1}) & -\phi\delta \\ -\lambda^{-1} & 0 \end{bmatrix} \begin{bmatrix} pdt \\ xdt \end{bmatrix} + \begin{bmatrix} \sigma dz \\ 0 \end{bmatrix}$$

The eigenvectors are shown in Figure 6, where it can also be seen that the exchange rate is expected to be stable along the horizontal axis, while the locus of expected price stability is inclined at a greater angle than the 45° PPP line (along which the exchange rate moves *pari passu* with the price level leaving the real exchange constant).

A symmetric exchange rate band is shown, with upper and lower limits at $\bar{x}$ and $\underline{x}$, respectively, along with the two solutions we are to discuss. The first of these (labelled KK') is the unique path which becomes tangent to the edges of the currency band.[1] This is Krugman's solution, with the S-shaped curve exhibiting the bias for which he was able to obtain an exact solution (in the special case where fundamentals follow a pure random walk).

The second solution, labelled RR', shows how the exchange rate would behave if the market fully expects full accommodation of the price level movement as and when the rate reaches the edges of the band. (This will involve shifting the center of the band by exactly half the band width, and changing the money stock in proportion.) The expectation of such accommodation bends the trajectory away from K' all the way down to R', so the solution snakes around the PPP line, cutting it when the rate reaches the edges of the band.

Although the solution RR' lies much farther from the stable manifold than when no realignment was in prospect, nonetheless the expected price movement is always towards (the current) equilibrium. (When equilibrium is at E, for example, the price level is expected to fall as ER' lies above the LL schedule where (E(dp) = 0; and *vice versa*.) But this is not a necessary feature of all solutions to realignment problems, as can be shown by means of an example. Consider the case where $\gamma = 0$ and so, from (B5), the locus LL coincides with the 45o PPP line, as shown in Figure 7. Let the degree of accommodation now be more than 100%, so the percentage shift in the band — and in the money stock — which takes place when the edge of the band is hit exceeds half the band width. Then the solution must have a segment where the price level is expected to move away from the current equilibrium, as can be seen in the figure. In this case, a greater realignment is triggered by a given price deviation than with 100% accommodation (as the angle α reflecting the extent of accommodation is greater than 45o); and, just before the realignment, along the segment labelled FR', the expected movement of prices is away from equilibrium, i.e., $E(dp) > 0$.

The determinacy of the equilibrium price level (associated with a policy of fixing the money stock) is of course undermined by such realignments which allow the price level to follow a random walk (with the degree of realignment governing the rate of diffusion of the price level). Note, however, that while the price level may only be semi-stable in a global sense, there is nevertheless, at any point of time a local equilibrium towards which the price level has a general tendency to move (until and unless shocks build up sufficiently to lead to its revision).

IV. ASYMMETRIC BOUNDARY CONDITIONS: AN HEURISTIC APPROACH

Where the boundary conditions are not symmetric around equilibrium, then it is not to be expected that the solution should pass through the deterministic equilibrium. So the family of solutions analyzed heretofore will no longer be relevant. It is, nevertheless, possible that the nature of such asymmetric solutions can be seen by perturbing one of the symmetric boundaries, as is suggested by the following example using the closed economy model already described.

Let the two thresholds triggering fiscal policy action in the closed economy not be equidistant from target GNP. Specifically, let stimulatory action be taken more readily than the converse, i.e. $|\bar{y} - \hat{y}| < |\underline{y} - \hat{y}|$, as shown in Figure 8. As a first shot at the solution, consider the trajectories linking the points F and F' to the deterministic equilibrium at E. Notice, however, that these will meet at a kink at E. But stochastic processes smooth out such kinks, so one may conclude that the solution will pass above this equilibrium, as shown in Figure 8.

V. CONCLUSION

Recently, Avinash Dixit and Paul Krugman have found explicit analytical solutions for economic problems in which Brownian motion processes act as the trigger for discontinuous actions (exit/entry by firms, and official foreign currency intervention, respectively). These elegant, explicit solutions are not, however, available for stochastic variants of the saddlepoint model which has proved popular in macroeconomics.

In this paper it has been argued that a general qualitative treatment of such saddlepoint systems is nevertheless available given the regularity provided by linear behavioral equations and symmetric boundary conditions. The examples used suggest that such a qualitative approach may often prove more attractive than recourse to numerical techniques to obtain exact solutions for specific values of the parameters.

Footnotes

1. Once the edge of the band is reached, it is assumed here that monetary policy is assigned the task of keeping the rate there until such time as the fundamentals move the system back on to KK' (see Miller and Weller, 1988, for further discussion).

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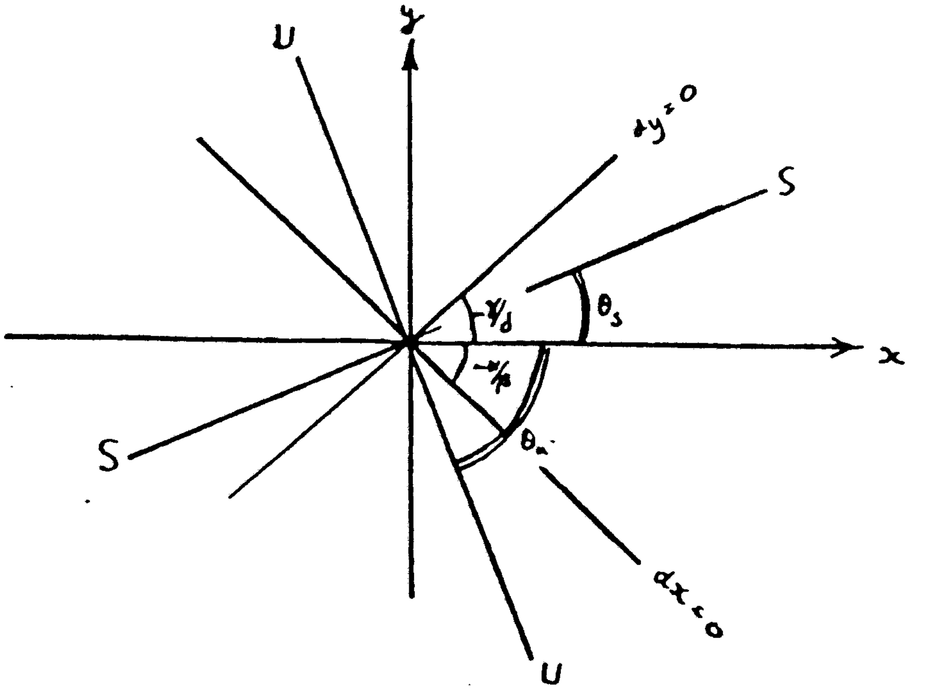


Figure 1. The saddlepoint of the deterministic system.

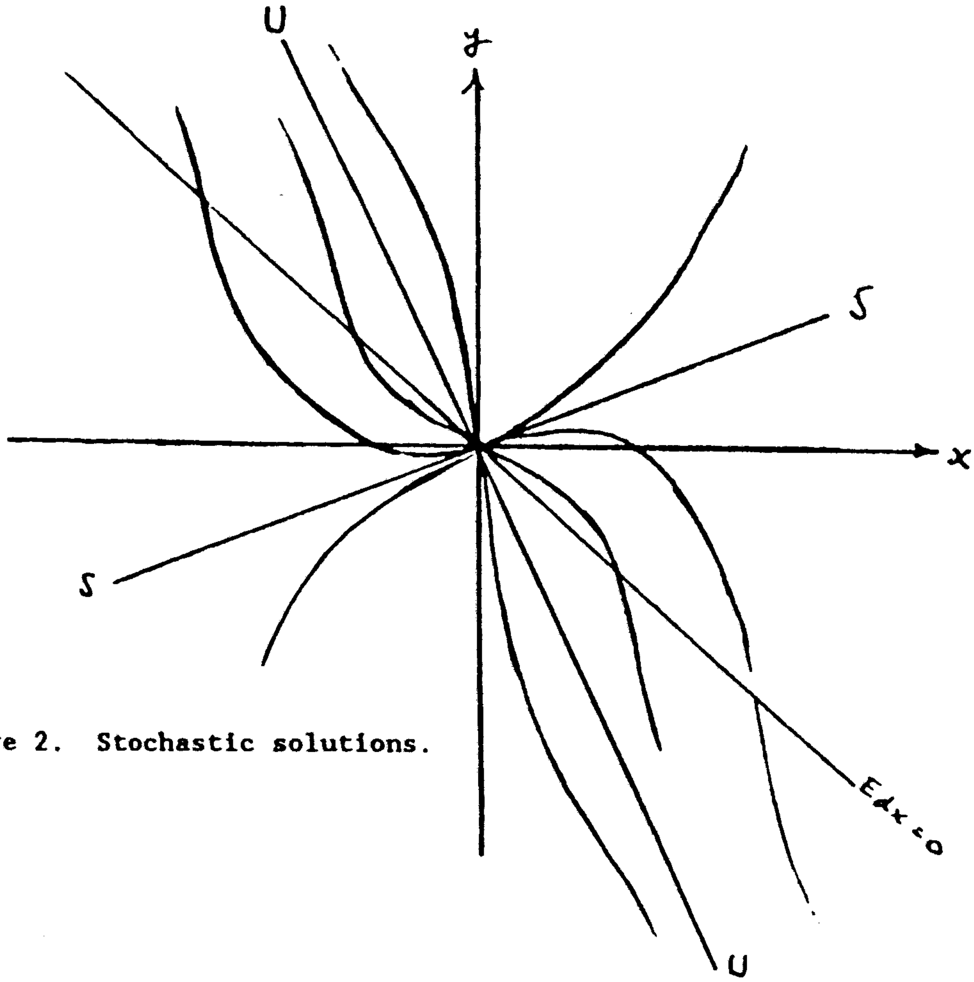
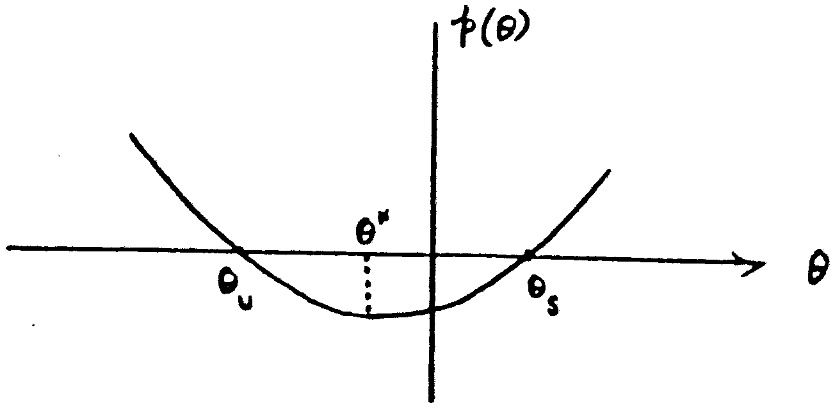
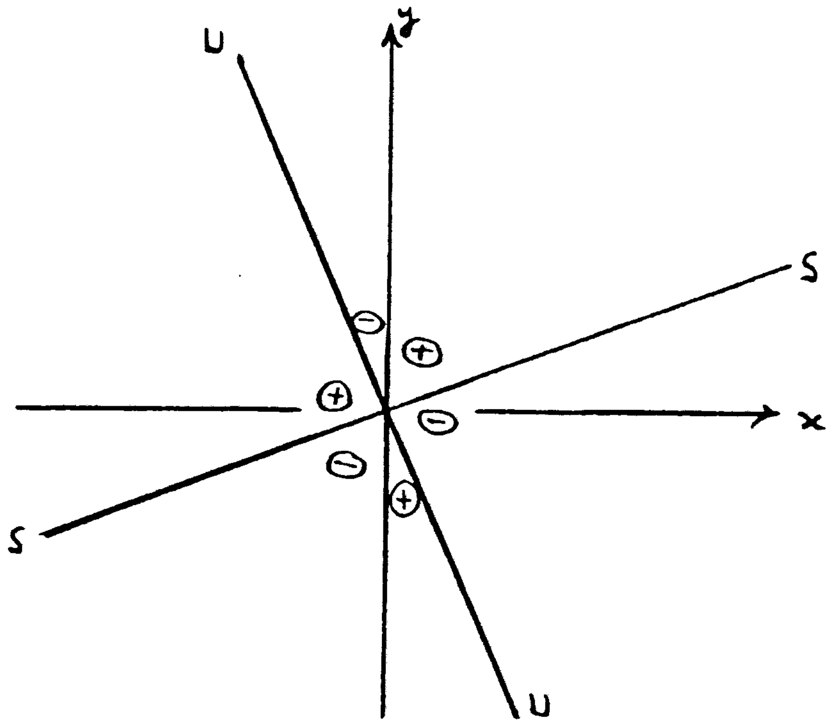


Figure 2. Stochastic solutions.

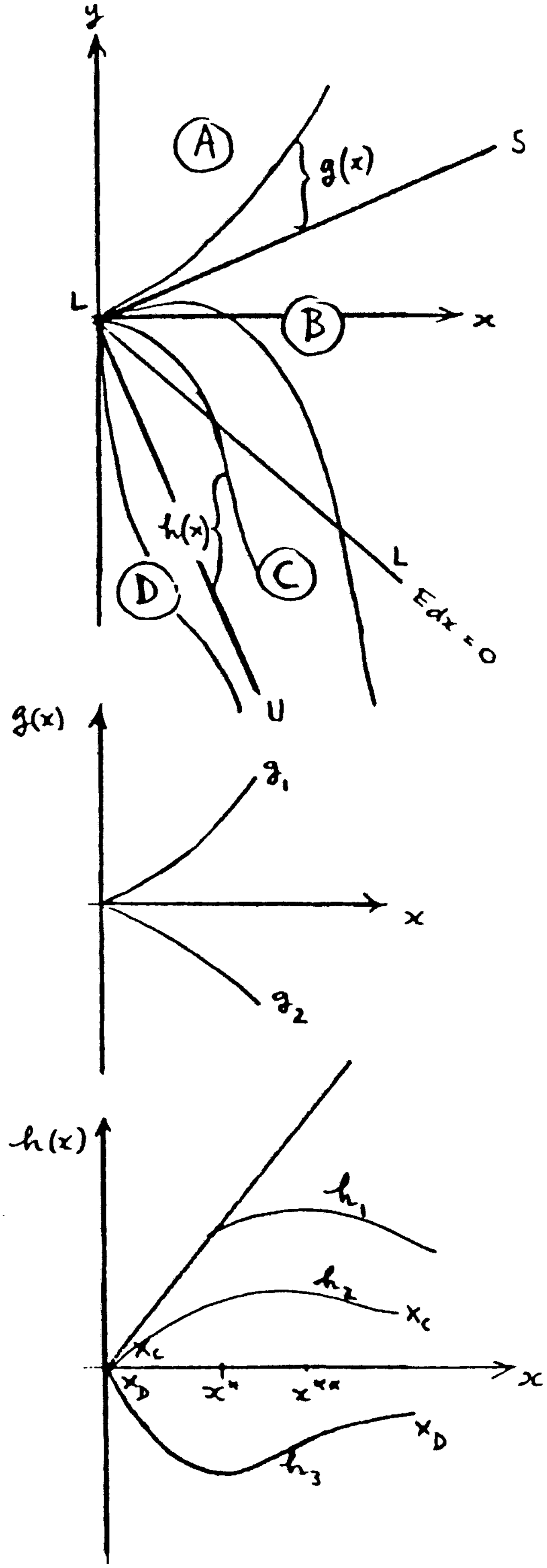


3(a)



3(b) the sign of f'' in a neighbourhood of the origin.

Figure 3. Local analysis.



4(a) the four regions

4(b) the function $g(x)$ for differing initial conditions

4(c) the function $h(x)$ for differing initial conditions

Figure 4. Global analysis.

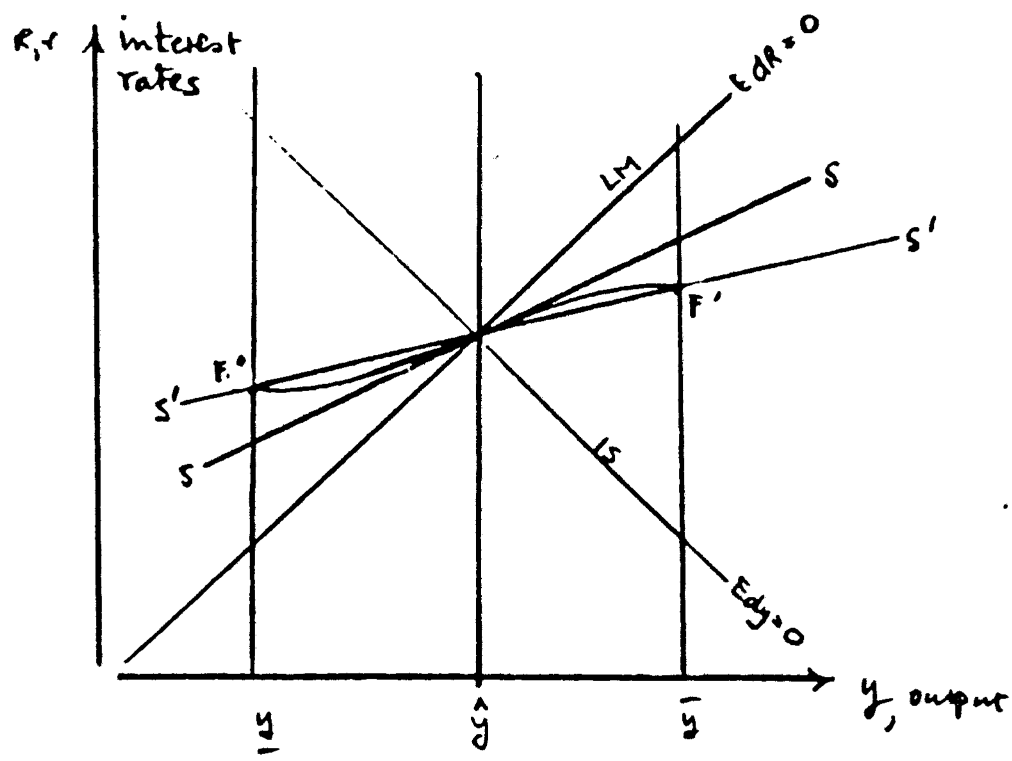


Figure 5. Stabilization policy with thresholds.

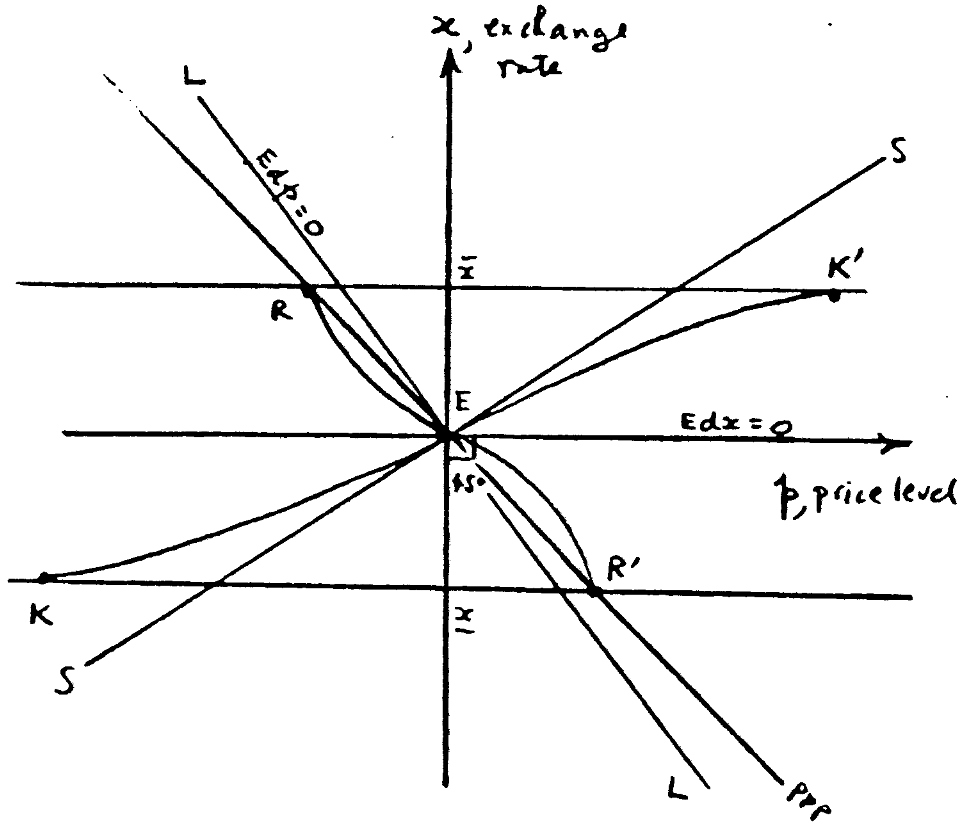


Figure 6. Currency bands: Krugman's bias and realignments.

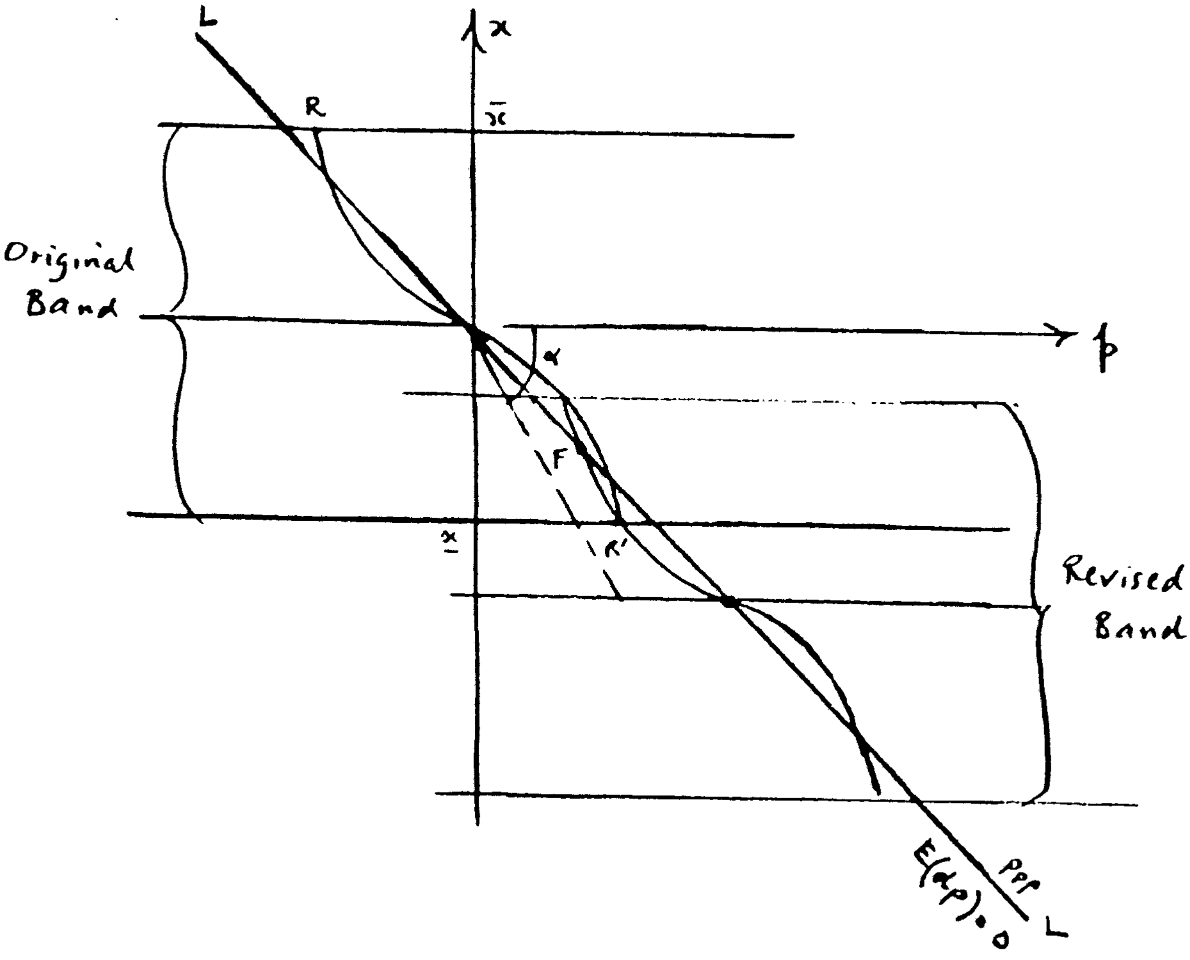


Fig. 7. An alternative realignment rule: more-than-full accommodation.

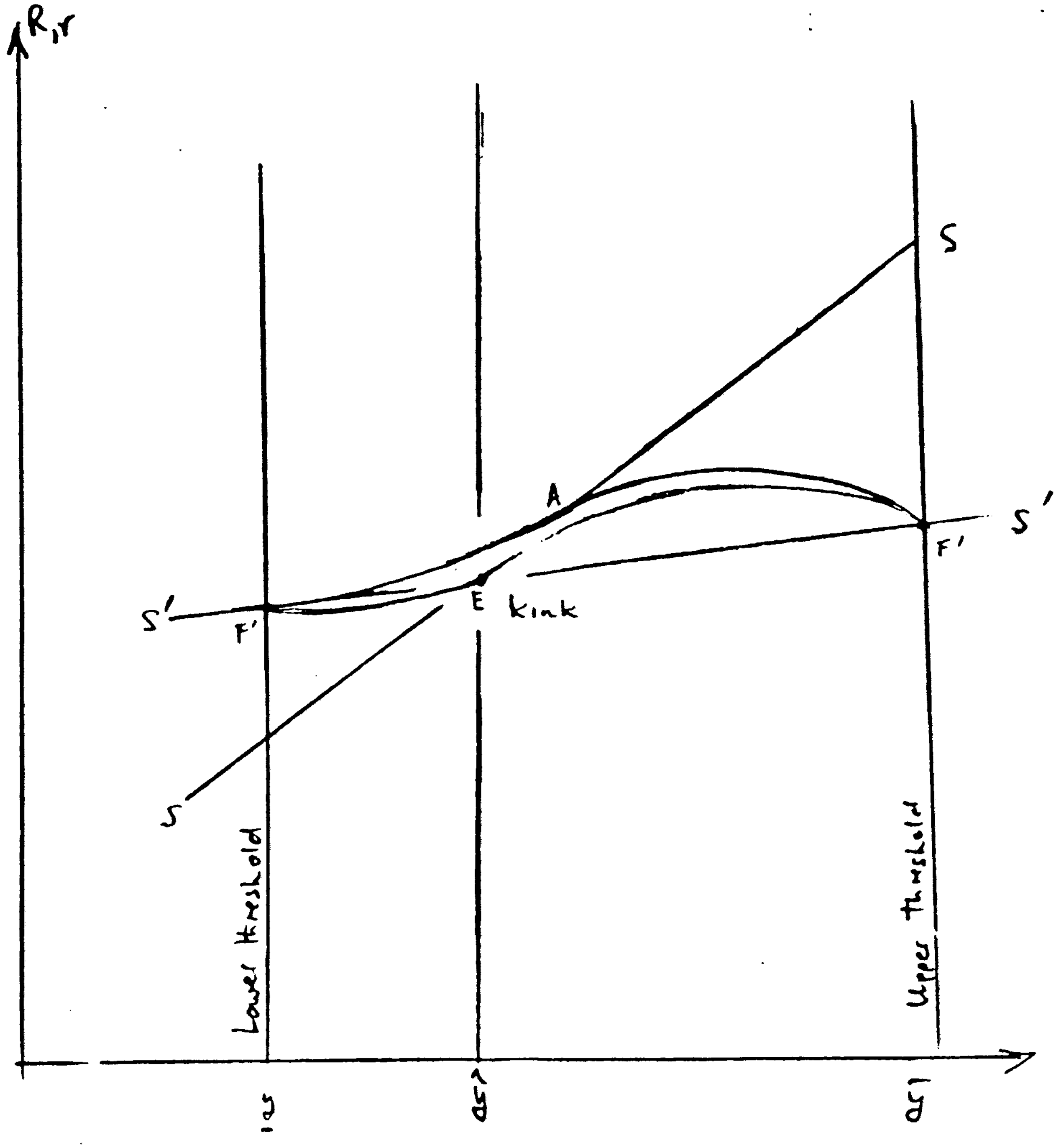


Figure 8   Asymmetric thresholds.