# RATIONAL SPECULATIVE BUBBLES IN AN EXCHANGE RATE TARGET ZONE

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First Draft : June 1990
Revised : September 1990
Second Revised Version : October 1990

No. 370

**WARWICK ECONOMIC RESEARCH PAPERS**

DEPARTMENT OF ECONOMICS

UNIVERSITY OF WARWICK
COVENTRY

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ABSTRACT

The recent theory of exchange rate dynamics within a target zone holds that exchange rates under a currency band are less responsive to fundamental shocks than exchange rates under a free float, provided that the intervention rules of the Central Bank(s) are common knowledge. These results are derived after having assumed a priori that excess volatility due to rational bubbles does not occur in the foreign exchange market. In this paper we consider instead a setup in which the existence of speculative behaviour is a datum the Central Bank has to deal with. We show that the defense of the target zone in the presence of bubbles is viable if the Central Bank accommodates speculative attacks when the latter are consistent with the survival of the target zone itself and expectations are self-fulfilling. We show that the instantaneous volatility of exchange rates within a band is not necessarily less than the volatility under free float. There need not be a constant trade-off between the volatility of the change in the exchange rate and the volatility of the change in the interest rate differential. Fundamental-dependent bubbles can account for the excess response of the exchange rate to the fundamental. The relationship between the exchange rate and the interest differential need not be negative, even if the target zone is fully credible.

* We would like to thank Torsten Persson, Christopher Sims (who saved us from error on at least one occasion), the participants at the Warwick Summer Research Workshop on "Exchange Rates and Financial Markets", July 9-27, 1990, in particular Renzo Avesani, Giuseppe Bertola, Andy Caplin, Bernard Dumas, Marcus Miller, Rick van der Ploeg, Paul Weller and the participants at the International Finance Workshop at Yale, in particular Giancarlo Corsetti, Vittorio Grilli, Koichi Hamada and Nouriel Roubini. Responsibility for any remaining errors rests with the authors. Paolo Pesenti would like to acknowledge financial support from the Ente per gli Studi Monetari, Bancari e Finanziari L. Einaudi, Rome.

## I. Introduction

The recent literature on exchange rate target zones inspired by Krugman's seminal article on the subject (Krugman [1990]) represents a successful marriage of policy relevance and technical innovation. Formal target zones like the exchange rate arrangements of the European Monetary System and informal target zones between the U.S. dollar, the Japanese yen and the D–Mark since the middle of the last decade were a prominent feature of the exchange rate system of the Eighties and promise to be so for the Nineties. The technical modeling innovation consists in the application of the theory of regulated Brownian motion (see e.g. Malliaris and Brock [1982], Harrison [1985], Dixit [1988], Karatzas and Shreve [1988], Dumas [1989] and Flood and Garber [1989]) to the study of the behaviour of a floating exchange rate that is constrained by appropriate interventions not to stray outside some given range or target zone.

The literature on this subject is growing very fast. Among the many recent contributions are Krugman [1987], Krugman and Rotemberg [1990], Miller and Weller [1988a,b; 1989; 1990a,b,c], Miller and Sutherland [1990], Froot and Obstfeld [1989a,b], Bertola and Caballero [1989, 1990], Klein [1989], Svensson [1989; 1990a,b], Pesenti [1990], Delgado and Dumas [1990], Lewis [1990], Avesani [1990], Ichikawa, Miller and Sutherland [1990], and Smith and Spencer [1990].

A common element in all of these analyses is that a "fundamentals only" solution is chosen for the exchange rate: The exchange rate, a non–predetermined state variable, is expressed as a function of the current and expected future values of the fundamentals only. The literature in fact concentrates on models with a single fundamental. This fundamental is governed by regulated Brownian motion, that is by unregulated Brownian motion as long as the fundamental stays within the lower and upper intervention thresholds and by (intermittent) interventions that keep the fundamental within these thresholds whenever the unregulated Brownian motion threatens to take the fundamental outside the intervention limits (and thus the exchange rate outside its target zone). This permits the current value of the spot exchange rate to be expressed as a non–linear function of the current value of the fundamental only.

The first step in this argument, the decision to solve for the current value of the exchange rate as a function only of the current and expected future values of the fundamental, means that speculative bubbles are ruled out.

In dynamic linear rational expectations models the arguments for ruling out speculative bubbles are well–known, if not necessarily wholly convincing. In many of the most popular models the bubble processes are non–stationary.[1] These non–stationary bubble processes then lead to non–stationary behavior of state variables such as asset prices. Unbounded growth or decline in these state variables for constant values of the fundamentals is argued to violate, eventually, certain often only implicitly stated feasibility constraints.[2]

Blanchard [1979] has developed an example of a non–stationary (stochastic or deterministic) linear bubble that has a finite expected lifetime. The probability of the bubble surviving for a given period of time declines with the length of that period at a constant exponential rate and approaches zero asymptotically (see also Blanchard and Fischer [1989, chapter 5]). Although the bubble remains non–stationary and its expected value grows exponentially without bound, the finite expected lifetime of this bubble may make it less vulnerable to the standard critique of non–stationary bubbles. Miller and Weller [1990a] and Miller, Weller and Williamson [1989] use the Blanchard bubble to analyze exchange rate behaviour (both with and without a target zone) in a stochastic version of the Dornbusch overshooting model (Dornbusch [1976]).

Certain non–linear models also possess non–stationary bubble solutions. For instance, the deflationary and inflationary bubbles studied by Hahn [1982] and by Obstfeld and Rogoff [1983], although obtained in non–linear models, are also non–stationary. In more general non–linear models it it however quite common to find stationary bubbles.[3]

The exchange rate target zone model potentially provides a particularly happy non–linear breeding ground for speculative bubbles. Provided the target zone is credible, that is provided there is certainty that the interventions required to defend the target zone will take place, the exchange rate can neither rise nor fall without bound. There are infinitely many intervention rules that are compatible with the defense of a given target zone. For a rational speculative bubble to exist, the intervention rule must of course be consistent, both within the target zone and at the boundaries, with the behaviour of the (regulated) fundamentals. This paper develops a simple and intuitive intervention rule that encompasses as a special case the solution without a bubble, yet is general enough to permit a wide range of exogenous and fundamental–dependent deterministic or stochastic bubbles. With this intervention rule the arguments against bubbles lose their bite. A non–stationary bubble does not now imply non–stationary behaviour of the exchange rate. This argument is not new. As early as 1987 William H. Branson, in conversation and discussion, repeatedly raised the possibility that target zones might lead to "indeterminacy" of the exchange rate. This paper can be viewed as an analytical confirmation of his conjecture.

In the next Sections we exposit the simplest version of the credible target zone model and analyze its behaviour with or without speculative bubbles.

## II. A Target Zone Model With or Without Speculative Bubbles.

### (a) The Model.

Our analysis of the place and role of speculative bubbles in an exchange rate target zone requires the exchange rate to be a non–predetermined (forward–looking) state variable, but does not make further demands on descriptive realism. We therefore feel comfortable in using Occam's razor and following the bulk of the literature on target zones by casting the analysis in terms of the simplest possible linear intertemporal substitution equation for the exchange rate and a (continuous time) random walk with drift for the unregulated fundamental. (Notable departures from the cult of extreme simplicity are Miller and Weller [1988a,b; 1989; 1990a,b,c] who analyze stochastic versions of the richer Dornbusch overshooting model.)

The exchange rate process is given in equation (1).

$$(1) \qquad s(t)dt = f(t)dt + \alpha^{-1}E_t ds(t) \qquad \alpha > 0.$$

Here $s(t)$ is the natural logarithm of the spot nominal exchange rate and $f(t)$ the fundamental determinant of the exchange rate. $E_t$ is the mathematical expectation operator conditional on the information available at time t to the private sector and the regulator. Both $s(t)$ and $f(t)$ are assumed to be observable at time t, i.e. $E_t s(t) = s(t)$ and $E_t f(t) = f(t)$, and the structure of the model is known.

For the purpose of this paper, the economic interpretation of $f(t)$ is irrelevant except insofar as it affects the credibility of our assumption that, with certainty, the target zone will be successfully defended. A common interpretation of f is in terms of relative nominal money stocks minus relative real–income–related demands for real money balances, i.e.

$$f = m - m^* - k(y - y^*)$$

where $m$ is the logarithm of the home country nominal money stock, y the logarithm of home country real output and k is the (common) income elasticity of money demand. Starred variables denote foreign quantities. In this case $\alpha$ denotes the inverse of the (common) interest semi–elasticity of money demand.[4]

If the two national real outputs are governed by exogenous Brownian motion, intervention means changes in the relative money supplies brought about by monetary financing of public sector financial deficits, through open market operations or by unsterilized foreign exchange market intervention. As shown in Buiter [1989] such intervention policies will in general require adjustments to primary (non–interest) fiscal surpluses to be sustainable indefinitely from a technical point of view (i.e. in order to prevent either exhaustion of foreign exchange reserves or unbounded growth of the public debt). A proof that they are politically feasible and credible is beyond the scope of this paper.

In the absence of intervention, the (unregulated) fundamental $\bar{f}$ would follow a Brownian motion process with drift $\mu$ and instantaneous variance $\sigma_f$. Formally, it is

$$d\bar{f}(t) = \mu dt + \sigma_f dW_f(t) \qquad \sigma_f > 0.$$

$W_f(t)$ is a standard Wiener process ($dW_f \equiv \lim_{\Delta t \to 0} \sqrt{\Delta t}\ \tilde{n}$ where $\tilde{n}$ is independently and normally distributed with zero mean and unit variance).

We now allow for intervention at the upper and lower boundaries of the exchange rate target zone. The regulated fundamental is governed by

$$(2) \qquad df(t) = d\bar{f}(t) - dI_u + dI_\ell = \mu dt + \sigma_f dW_f(t) - dI_u + dI_\ell.$$

$I_u$ and $I_\ell$ are the cumulative interventions at the upper and lower bounds of the exchange rate target zone, given by $s_u$ and $s_\ell$ respectively with $-\infty < s_\ell < s_u < \infty$. A detailed specification of these interventions is given below.

(b) The Solution

The general solution of (1) is given in equation (3). B(t) is the value of the speculative bubble at time t.

(3) $$s(t) = \alpha \int_t^{\infty} E_t f(v) e^{-\alpha(v-t)} dv + B(t).$$

The bubble B(t) can be any stochastic or strictly deterministic process satisfying

(4) $$B(t)dt = \alpha^{-1} E_t dB(t).$$

Since s(t) is part of the information set at t, so is B(t). We restrict B, where it is stochastic, to follow a diffusion process. Except for the case of the "Blanchard bubble", all examples will in fact restrict it further to be a twice continuously differentiable function of Brownian motions.

Note that the authorities are assumed to be unable to intervene directly in the bubble process (and indeed in the exchange rate process). They can only regulate the fundamental f.

Equations (3) and (4) have an important implication, which can be brought out clearly by considering the case of a positive deterministic bubble ($B(t) > 0$, $B(v) = B(t)e^{\alpha(v-t)}$, $v \geq t$). In this case, as $t \to +\infty$, $B(t) \to +\infty$. With B becoming unbounded positive as $t \to \infty$, the solution for s can only remain bounded from above (and s can remain within a target zone with a finite upper value of s) only if the first term on the R.H.S. of (3) (the integral over all future time of discounted expected future values of f) becomes unbounded negative. This requires large negative future expected values of f and indeed, in general, ultimately (as $t \to \infty$) unbounded negative expected values of f. Our intervention rule and the other features of our proposed solution for s indeed have this property.

In the interior of the target zone, the general solution for s as a function of current f and current B is given by[5]

(5) $$s(t) = f(t) + B(t) + \frac{\mu}{\alpha} + A_1(t)e^{\lambda_1(f(t)-C(t))} + A_2(t)e^{\lambda_2(f(t)-C(t))}$$

with

$$\lambda_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 + 2\alpha\sigma_f^2}}{\sigma_f^2}.$$

$A_1$, $A_2$ and C are constant as long as s remains in the interior of the target zone, but can change when s is at its upper or lower boundary. In fact we have

(6) $$\begin{aligned} A_1(t) &= A_1(\tau) \\ A_2(t) &= A_2(\tau) \\ C(t) &= C(\tau) \end{aligned}$$

where $\tau$ is the date of the most recent visit prior to (or coincident with) t of the exchange rate s to one of the boundaries, that is

(7) $$\tau \equiv \text{Supremum}\{t' \leq t, \text{ such that } s(t') = s_u \text{ or } s(t') = s_\ell\}.$$

We first define $f_u^0$, $f_\ell^0$, $A_1^0$ and $A_2^0$ in equations (8a–d). $f_u^0$ is the value of f for which the solution graph for s as a function of f has a point of tangency with the upper boundary of the target zone ($s = s_u$) when $B = 0$; $f_\ell^0$ is the value of f for which the solution graph for s as a function of f has a point of tangency with the lower boundary of the target zone ($s = s_\ell$) when $B = 0$.

(8a) $$s_u = f_u^0 + \frac{\mu}{\alpha} + A_1^0 e^{\lambda_1 f_u^0} + A_2^0 e^{\lambda_2 f_u^0}$$

(8b) $$0 = \left(1 + \frac{\partial B(\tau)}{\partial f}\Big|_{\substack{f=f_u^0 \\ B=0}}\right)\left(1 + \lambda_1 A_1^0 e^{\lambda_1 f_u^0} + A_2^0 e^{\lambda_2 f_u^0}\right).$$

(8c) $$s_\ell = f_\ell^0 + \frac{\mu}{\alpha} + A_1^0 e^{\lambda_1 f_\ell^0} + A_2^0 e^{\lambda_2 f_\ell^0}$$

(8d) $$0 = \left(1 + \frac{\partial B(\tau)}{\partial f}\Big|_{\substack{f=f_\ell^0 \\ B=0}}\right)\left(1 + \lambda_1 A_1^0 e^{\lambda_1 f_\ell^0} + A_2^0 e^{\lambda_2 f_\ell^0}\right).$$

Equations (8a–d) are of course the familiar tangency or no–arbitrage conditions often (though inaccurately) referred to as "smooth pasting" conditions, that expected interventions in f should not change the value of the exchange rate. The term $\left(1 + \frac{\partial B(\tau)}{\partial f}\Big|_{\substack{f=f_u^0 \\ B=0}}\right)$ in (8b) and the term $\left(1 + \frac{\partial B(\tau)}{\partial f}\Big|_{\substack{f=f_\ell^0 \\ B=0}}\right)$ in (8d) reflect the fact that, as we shall see below, the bubble may be a function of the value of the unregulated fundamental. For exogenous bubbles, conditions (8a–d) lead to the standard *S–shaped* solution for the no–bubble case considered in the literature.

For the case $B(\tau) \geq 0$ and $s(\tau) = s_\ell$ and for the case $B(\tau) \leq 0$ and $s(\tau) = s_u$, we have

(9a) $$A_1(\tau) = A_1^0$$

(9b) $$A_2(\tau) = A_2^0$$

(9c) $$C(\tau) = -B(\tau)$$

These two–sided "smooth pasting" conditions define a horizontal displacement of the "zero bubble" graph (defined by equations (5) and (9a–c)) to the left if $B(\tau) > 0$ and to the right if $B(\tau) < 0$. We refer to the graph of equation (5) when $A_1 = A_1^0$, $A_2 = A_2^0$ and $C = -B(t)$ as the **two–sided** *bubble eater* of B(t). Equation (5) with $B(t) = C(t) = 0$ and equations (8a–d) can therefore be interpreted as representing the two–sided bubble eater for $B = 0$.

It is clear by inspection that if the bubble process is independent of the unregulated fundamental, the values of f at the tangencies of the two–sided bubble eater of $B(\tau)$ with the edges of the target zone (denoted by $f = f_u$ when $s = s_u$ and $f = f_\ell$ when $s = s_\ell$) are given by

(9d) $$f_u = f_u^0 - B(\tau)$$

(9e) $$f_\ell = f_\ell^0 - B(\tau).$$

For $B(\tau) \geq 0$ and $s(\tau) = s_u$, $A_1(\tau)$, $A_2(\tau)$ and $C(\tau)$ are defined by equations (10a–c), which characterize "one–sided smooth pasting" at the upper boundary. Note that this point of tangency is at the same value of f, $f_u^0$, for which the two–sided smooth pasting conditions for $B = 0$, given in equations (8a–d), yield a point of tangency of the two–sided bubble eater of $B = 0$ with the upper boundary, that is $f_u = f_u^0$. When $B(t) = B(\tau)$, equations (5), (6) and (10a–c) define the one–sided bubble eater for $B \geq 0$. (It coincides with the two–sided bubble eater when $B = 0$).

(10a) $$s_u = f_u^0 + B(\tau) + \frac{\mu}{\alpha} + A_1(\tau)\, e^{\lambda_1 f_u^0} + A_2(\tau)\, e^{\lambda_2 f_u^0}$$

(10b) $$0 = \left(1 + \left.\frac{\partial B(\tau)}{\partial f}\right|_{f=f_u^0}\right)\left[1 + \lambda_1 A_1(\tau)\, e^{\lambda_1 f_u^0} + \lambda_2 A_2(\tau)\, e^{\lambda_2 f_u^0}\right]$$

(10c) $$C(\tau) = 0$$

For this case we also define, for future reference:

(10d) $$f_u = f_u^0$$

For $B(\tau) \leq 0$ and $s(\tau) = s_\ell$, $A_1(\tau)$ and $A_2(\tau)$ are defined by equations (11a–d), which characterize "one–sided smooth pasting" at the lower boundary.

(11a) $$s_\ell = f_\ell^0 + B(\tau) + \frac{\mu}{\alpha} + A_1(\tau)\, e^{\lambda_1 f_\ell^0} + A_2(\tau)\, e^{\lambda_2 f_\ell^0}$$

(11b) $$0 = \left(1 + \left.\frac{\partial B(\tau)}{\partial f}\right|_{f=f_\ell^0}\right)\left[1 + \lambda_1 A_1(\tau)\, e^{\lambda_1 f_\ell^0} + \lambda_2 A_2(\tau)\, e^{\lambda_2 f_\ell^0}\right]$$

(11c) $$C(\tau) = 0$$

(11d) $$f_\ell = f_\ell^0$$

Note that this point of tangency is at the same value of $f = f_\ell^0$ for which the two–sided smooth pasting conditions for $B = 0$, given in equations (8a–d), yield a point of tangency with the lower boundary, that is $f_\ell = f_\ell^0$. When $B(t) = B(\tau)$, equations (5), (6) and (11a–d) define the one–sided bubble eater for $B \leq 0$.

We now detail the interventions at the upper and lower boundaries of the exchange rate target zone. From equations (5), (6) and (7) and the boundary conditions determining the values of $A_1(\tau)$, $A_2(\tau)$ and $C(\tau)$, given in equations (8), (9) (10) and (11), the exchange rate in the target zone can be written as

$$s(t) = g[f(t), B(\tau), i\,] + B(t)$$

$i = 1$ if $g(.,.,.)$ is defined through one–sided smooth pasting

$i = 2$ if $g(.,.,.)$ is defined through two–sided smooth pasting

The bubble at time t can be a function of the current unregulated fundamental $\bar{f}(t)$ and/or of other variables $z(t)$. Note that $z(t)$ could be a function of <u>past</u> or <u>expected future</u> values of the unregulated fundamental. We therefore write $B(t) = b(\bar{f}(t), z(t))$. The exchange rate in the target zone can therefore be written as

$$s(t) = g(f(t), B(\tau), i) + b(\bar{f}(t), z(t)) = h(f(t), B(\tau), i, z(t)),$$

where $\frac{\partial h}{\partial f} \equiv \frac{\partial g}{\partial f} + \frac{\partial b}{\partial \bar{f}}\frac{\partial \bar{f}}{\partial f}$.

For simplicity we restrict the analysis to those bubbles for which, even if $B(t)$ depends on $\bar{f}(t)$, the function $h(f, .\,)$, retains its familiar s shape with a unique interior minimum and a unique interior maximum. $f_u$ is therefore the value of f for which $s = s_u$ and the appropriate bubble eater has an interior maximum, that is $f_u$ is defined by $h(f_u, B(\tau), i, z(\tau)) = s_u$, $\frac{\partial h}{\partial f}(f_u, \cdot\,) = 0$ and $\frac{\partial^2 h}{\partial f^2}(f_u; \cdot\,) < 0$. Note that because of the bubble, s may reach $s_u$ at values of the fundamental different from $f_u$. These will be denoted $f_u^*$. Similarly, $f_\ell$ is defined by $h(f_\ell, B(\tau), i, z(\tau)) = s_\ell$,

$\frac{\partial h}{\partial f}(f_\ell, \cdot) = 0$ and $\frac{\partial^2 h}{\partial f^2}(f_\ell, \cdot) > 0$. Again, because of the bubble, s may reach $s_\ell$ at values of the fundamental different from $f_\ell$. These will be denoted $f_\ell^*$.

We consider the simplest possible example of a target zone. A plausible interpretation of the formal model is that there is a fully credible commitment that the value of the exchange rate will be contained within an exogenously given bounded range, that is $-\infty < s_\ell \leq s \leq s_u < \infty$. The interventions in the fundamental process that keep the exchange rate within this range occur only when the exchange rate is at the boundaries. Two distinct kinds of interventions are required to defend the target zone in the presence of a bubble.

The first is a pair of infinitesimal reflecting interventions, $-dI_u < 0$ at the upper boundary if $f = f_u$ and $dI_\ell > 0$ at the lower boundary of the target zone if $f = f_\ell$. These negative infinitesimal interventions at the upper boundary $s = s_u$ and positive infinitesimal interventions at the lower boundary $s = s_\ell$, correspond to the original scheme analyzed by Krugman [1990]. If interventions in the interior of the target zone ("intramarginal" interventions) were permitted, a pair of reflecting interventions of exogenously given finite magnitudes, $-\Delta I_u < 0$ and $\Delta I_\ell > 0$ could be allowed (see Flood and Garber [1989]).

The second class of interventions comprises interventions of finite magnitude, that are functions of the change in the magnitude of the bubble since the previous visit to the boundaries. They too occur only when the exchange rate is at the boundaries of the target zone. They always are in the opposite direction from the infinitesimal (type I) interventions at the same boundary.

We interpret these (type II) discrete interventions as the passive accommodation of rational speculative attacks at the boundaries of the target zone, that are consistent with the survival of the target zone. We refer to such stock–shift portfolio reshuffles as "sustaining" or "friendly" speculative attacks.

The behaviour of the regulated fundamental is given in equations (12a) to (12c). The conditions in (12a), (12b), (12c($\alpha$)) and (12c($\gamma$)) are the standard ones for regulated Brownian motion (see e.g. Harrison [1985, p. 80]). If $f = f_u$ when $s = s_u$, they characterize a reflecting barrier at $f = f_u$ (given by (12a), (12b) and (12c($\alpha$)). If $f = f_\ell$ when $s = s_\ell$, they characterize a reflecting barrier at $f = f_\ell$ given by (12a), (12b) and (12c($\gamma$)).

The other conditions are non–standard and are required for our analysis of rational bubbles in a target zone. If when $s = s_u$ we have $f = f_u^* \neq f_u$ (in our model this always means $f_u^* < f_u$), they characterize (discrete) accommodated speculative attacks from $f = f_u^*$ to $f = f_u$, followed by (infinitesimal) reflection at $f = f_u$ (given by (12a), (12b) and (12c($\beta$)). If when $s = s_\ell$ we have $f = f_\ell^* \neq f_\ell$ (in our model this always means $f_\ell^* > s_\ell$), they characterize accommodated speculative attacks from $f = f_\ell^*$ to $f = f_\ell$, followed by reflection at $f = f_\ell$ (given by (12a), (12b) and (12c($\delta$)). The reasons for these non–standard features will become clear below.

We restrict attention to starting states $s \in [s_\ell, s_u]$ and starting date $t = 0$ and define $t_-$ as the instant immediately before t, that is

$$t_- \equiv \lim_{\substack{\Delta t \to 0 \\ \Delta t > 0}} (t - \Delta t) \quad .$$

The regulated fundamental is formally defined as follows:

(12a) $$f(t) \equiv \mu\, t + \sigma_f \int_0^t dW_f - I_u(t) + I_\ell(t) \quad \text{for all } t \geq 0$$

(12b) $I^u$ and $I^\ell$ are constant if $s_\ell < s < s_u$.

(12c) $I^u$ and $I_\ell$ vary when $s = s_u$ or $s = s_\ell$:

($\alpha$) If $s = s_u$ and $f = f_u$ then $I^u$ is increasing and right continuous.

($\beta$) If $s(t\_) = s_u$ and $f(t\_) = f_u^*$, then $f(t) = f_u$ (that is $-\Delta I_u = f_u - f_u^* > 0$) and ($\alpha$) applies henceforth.

($\gamma$) If $s = s_\ell$ and $f = f_\ell$ then $I^\ell$ is increasing and right continuous.

($\delta$) If $s(t\_) = s_\ell$ and $f(t\_) = f_\ell^*$, then $f(t) = f_\ell$ (that is $\Delta I_\ell = f_\ell - f_\ell^* < 0$) and ($\gamma$) applies henceforth.

It will be apparent that, for any size bubble we can calculate exactly both the magnitudes of the passive or accommodating discrete interventions required to allow the target zone to survive and the values of the fundamentals at which they will occur.

In our rudimentary model, there is no possibility of the speculative bubble influencing the behaviour of the fundamental. A bubble in this model can only influence the exchange rate directly. If the process governing the unregulated fundamental included the exchange rate as an argument then, because the bubble affects the exchange rate, it will indirectly influence the behaviour of the fundamental. An interesting analysis of a model in which the bubble affects the exchange rate both directly and indirectly through the fundamental is analyzed by Miller–Weller [1990a] and by Miller, Weller and Williamson [1989].

Finally, the behaviour of the authorities at the edges of the target zone can be summarized as follows: *"Reflect when this is sufficient; accommodate when this is necessary.*

## III. Smooth Pasting a Bubbly Exchange Rate

Consider the case of an exogenous bubble, that is one that does not depend on $\bar{f}$. The bubble can be either deterministic or stochastic. The graphical representation of this case is shown in Figures 1 and 2.

Without loss of generality we start the system off with $A_1 = A_1^0$, $A_2 = A_2^0$ and $C = 0$. Together with equation (5) and $B(t) = 0$, this defines the two–sided bubble eater for $B = 0$ (which coincides with the two one–sided bubble eaters for $B = 0$). This is the familiar solution graph for the target zone without a bubble, shown as $s = g(f, 0, 2)$ in Figure 1.

First consider changes in f and in B that keep the exchange rate in the interior of the target zone. As long as $s_\ell < s < s_u$, changes in B will represent, for any given value of f, a **vertical** displacement equal to B from g(f, 0, 2). Now consider what happens when the bubble causes the exchange rate to reach one of the edges of the target zone. For concreteness we take a positive bubble. When $B = B_1 > 0$ and $f = f_u^{*1}$, the solution for the exchange rate, given by $s = g(f, 0, 2) + B_1$, equals $s_u$. The exchange rate reaches $s_u$, the upper limit of the target zone, at a value of f for which smooth pasting does not apply: $\frac{\partial g}{\partial f} > 0$ at $f = f_u^{*1}$. Following our intervention rule (12c$\beta$), there is a discrete change (increase) in f from $f_u^{*1}$ to $f_u$, the value of f for which there is a tangency of the appropriate bubble eater of $B = B_1$ with the upper boundary of the target zone. Since $s = s_u$ and $B > 0$, the appropriate bubble eater is the one–sided one defined in equations (5) (with $B = B_1$), (6) and (10a–d). The relevant value of $f_u$ is therefore $f_u^0$, the value of f for which smooth pasting at the upper boundary occurs in the absence of bubbles.

The intuition underlying the mechanism described above is rather simple. The government is credibly committed to infinitesimal reflecting interventions at the boundaries $s_\ell$ and $s_u$. The value of the fundamental at which these interventions at the margins occur are determined by the appropriate two–sided or one–sided "no–arbitrage" (or smooth pasting) boundary conditions. Consider again the case of a positive bubble $B_1$ shown in Figure 1. The value of the fundamental at which the exchange rate reaches, say, the upper boundary ($f_u^{*1}$ in Figure 1) is different from $f_u^0$ at which agents know a reflecting intervention will occur, except in the case of a zero bubble. Since $f_u^{*1}$ is less than $f_u^0$, the discrete (stock–shift) jump in the fundamental that takes the system to the reflection point $[f_u^0, s_u]$. can be interpreted as a friendly or sustaining speculative attack which is accommodated by the authorities. Given that $f_u^{*1}$ is to the left of $f_u^0$, it follows that

$$(13) \qquad E_t ds\Big|_{f=f_u^{*1}} = \alpha(s_u - f_u^{*1})dt > E_t ds\Big|_{f=f_u^0} = \alpha(s_u - f_u^0)dt.$$

Since the expected rate of change of s at $[f_u^{*1}, s_u]$ in Figure 1 is (positive and) higher than the negative expected rate of change of s at $[f_u^0, s_u]$, the shift from $f_u^{*1}$ to $f_u^1$ can be interpreted as an increase in the relative demand for home country money due to an increase of the rate of exchange rate appreciation or, if instantaneous uncovered interest parity holds, a decrease in the domestic–foreign interest rate differential.[6] The authorities obligingly accommodate this friendly or sustaining speculative attack. From a different point of view, a friendly speculative attack occurs only if private agents know that it will be accommodated, and exactly because of the passive accommodation the speculative attack sustains the target zone.[7]

Immediately following that stock–shift increase in f, the exchange rate satisfies $s = s_u = g(f, B_1, 1) + B_1$. If subsequent changes in f and in B cause the exchange rate again to reach the upper limit of the target zone, $s_u$, with a larger value of the bubble, say $s = s_u = g(f, B_1, 1) + B_2$ with $B_2 > B_1$ at $f = f_u^{*2}$ in Figure 1, another stock–shift increase in f will occur, from $f_u^{*2}$ to $f_u^0$, and the subsequent motion of the system will be with reference to the one–sided bubble eater of $B_2$, drawn in Figure 1 as $g(f, B_2, 1) + B_2$. As B increases the one–sided bubble eater at the upper boundary becomes more and more steep. In the limit, as $B \to \infty$ , the graph of the one–sided bubble eater at the upper boundary approaches a vertical line through $f_u^0$.

Each stock–shift increase in f from $f_u^*$ to $f_u^0$ occurs at a given value of the exchange rate: $s = s_u$. There are no arbitrage gains to be had from these open market operations. At $[f_u^0, s_u]$ infinitesimal reflecting interventions take place. This is consistent with the curvature of the one–sided bubble eater at that point, as this satisfies the one–sided smooth pasting conditions at the upper boundary. Note that if at time t the exchange rate reaches $s_u$ at $f = f_u^0$, that is when the current value of the bubble is the same as the value of B at the most recent previous boundary visit, the size of the friendly speculative attack is obviously zero, and only a reflecting intervention occurs. Finally, $[f_u^0, s_u]$ lies to the right of the $45^0$ line through the origin, that is at $[f_u^0, s_u]$ we have $s < f$. Since $E_t ds = \alpha(s-f)dt$ , the expected rate of change of the exchange rate at $[f_u^0, s_u]$ is negative. This must be the case if the commitment to defend the target zone (and therefore to stop the exchange rate from rising above $s_u$) is a credible one.

Now consider what happens when, still with $B > 0$, the exchange rate reaches the lower boundary $s_\ell$ of the target zone. In this case, after s reaches $s_\ell$ , say with $B = B_3 > B_2 > 0$ and $f = f_\ell^{*3}$ in Figure 2, the new boundary conditions determining the values of $A_1$, $A_2$ and C are (9a,b c), characterizing $g(f, B_3, 2) + B_3$, that is the two–sided bubble eater of $B_3$. The two–sided bubble eater of $B_3$ lies to the left of the two–sided bubble eater of $B_2$ (not shown). There is a stock–shift reduction in f , from $f = f_\ell^{*3}$ to $f = f_\ell^0 - B_3$. This again can be thought of as an accommodated friendly speculative attack that takes the system to the value of f at which, consistent with the policy rule, infinitesimal reflecting interventions are undertaken. At $[s_\ell, f_\ell^0 - B_3]$, infinitesimal reflecting interventions stop f from falling below $f_\ell^0 - B_3$.

Note that, since $[f_\ell^0, s_\ell]$ is to the left of the $45^0$ line through the origin, and $[f_\ell^0 - B_3, s_\ell]$ is to the left of $[f_\ell^0, s_\ell]$ for $B_3 > 0$, the expected rate of change of the exchange rate is positive at $[f_\ell^0 - B_3, s_\ell]$. This is as it should be if the defense of the target zone (and in particular the commitment not to let s fall below $s_\ell$) is credible.

This also makes it clear why the two–sided bubble eater cannot represent a correct solution when $s = s_u$ and $B > 0$ (or when $s = s_\ell$ and $B < 0$). For a sufficiently large positive value of B, the two–sided bubble eater defines a point of tangency with the upper boundary of the target zone that is to the left of the point $[s_u, s_u]$. (The value of f that generates a point of tangency of the two–sided bubble eater and the upper boundary is given by $f = f_u = f_u^0 - B$ when the bubble process is exogenous as in the case we are considering. Any value of B greater than $f_u^0 - s_u$ would put us to the left of the $45^0$ line through the origin.). Such a position will be characterized by a positive expected rate of change of the exchange rate and therefore cannot represent a point on the upper boundary of a credible target zone whose regulators are committed to prevent s from rising above $s_u$.

Our solution to shift f to $f_u^0$ whenever s reaches the upper limit of the target zone and B is positive, is not the only possible scheme. Any value of f strictly greater than $s_u$ is a possible candidate, as it can be "smooth pasted" credibly (albeit generally only one–sidedly). We chose our specific rule both because it is very simple and because it yields the same analysis as the traditional models for the case when there are no bubbles.

It is interesting to analyze why the value $f_u = s_u$ is incompatible with the credibility of the zone. The point $[s_u, s_u]$ lies on the $45^0$ line, that is the locus of coordinates [f,s] characterized by a zero expected rate of depreciation. If we impose the 'smooth pasting' conditions when $f = s = s_u$, (small) positive realizations of increments in the unregulated fundamental will be offset with infinitesimal reflecting interventions. Negative realizations of df would bring the system to the left of the $45^0$ line through the origin, that is to a point with a positive expected rate of change of s. This is of course inconsistent with a credible defense of the target zone. If private agents believed that the authorities were stabilizing the exchange rate at $[s_u, s_u]$, a "friendly" speculative attack of infinitesimal size would occur, bringing the system back to $[s_u, s_u]$. This amounts to stabilizing the fundamental at $f = s_u$. Since $E_t ds = \alpha(s - f)dt$ the exchange rate would effectively be expected to be fixed.

It is easily seen that an exchange rate cannot remain fixed (or be expected to remain fixed) for any finite interval of time if the fundamental is constant and there is a non–zero speculative bubble. Consider equation (3), for the special case in which f is kept constant at $s_u$ forever. The solution for the exchange rate is given by $s = s_u + B$ which implies that, unless $B = 0$, the exchange rate cannot be constant.

Staying with the positive bubble, as the bubble grows (and it will always grow exponentially in expectation), every time s reaches the upper boundary with a larger value of B, the system will be moved back to $[f_u^0, s_u]$ and onto steeper and steeper one–sided bubble eaters. Every time s reaches the lower boundary with a larger value of B, the system will be moved further to the left, to the point $[f_\ell^0 - B, s_\ell]$ and the two–sided bubble eater for B.

With a stochastic bubble, a reversal in the direction of the bubble can bring the current trajectory below the corresponding one–sided bubble eater. This is shown in Figure 2 . $g(f, B_2, 1) + \hat{B}_3$ lies at a vertical distance $\hat{B}_3 - B_2 < 0$ below the bubble eater $g(f, B_2, 1) + B_2$.[8] In this case the exchange rate can reach its lower limit $s_\ell$ at a value of f *greater* than $f_u^0$ (say, $\hat{f}_\ell^{*3}$ in Figure 2), rather than at a value of f *less* than $f_u^0$ as considered above ($f_\ell^{*3}$ when $B = B_3$ in Figure 2). This is possible because infinitesimal reflecting interventions take place only when the *exchange rate* hits one of the predetermined thresholds and not when the *fundamental* crosses any threshold value. An interesting implication is that the exchange rate can reach its upper limit at some time t and subsequently its lower limit at some time $t' > t$ without changes in the level of the bubble, that is $B(t) = B(t')$. In fact, this is obviously possible with a stochastic bubble which can change direction: A decline (or an increase after the initial decline determined by the reflecting intervention) in f within the time interval $[t, t']$ together with an initial decrease in B followed by an increase of corresponding size leads to the case mentioned above.

Consider now the case where B grows without bound, as it will do if the bubble is deterministic and its initial value is positive, and as it will do *in expectation* even if it is stochastic as long as its initial value is positive. Even then, the exchange rate will with positive probability (because of the unbounded support of the normal disturbances driving df inside the target zone) reach the lower boundary of the target zone. There the unbounded growth of B is translated into an unbounded decline in f (since $f_\ell = f_\ell^0 - B$).

Note that as B grows, the one–sided bubble eater tangent to the upper boundary grows steeper. It therefore takes, ceteris paribus, smaller and smaller declines in f from $f_u^0$ for s to reach the lower boundary $s_\ell$. Is there a paradox or inconsistency as $t \to \infty$ and $B(t) \to \infty$ ? The one–sided bubble eater becomes vertical and there would therefore be a discrete fall in s from $s_u$ to $s_\ell$ for even the smallest decline in f. Note, however, that the date at which this apparent arbitrage opportunity occurs is always infinitely far away into the future. With any positive discount rate, the present value today of that arbitrage opportunity would be zero.

## IV. A Surfeit of Bubbles

Consider again the general solution to equation (1) given in (3). The first term on the R.H.S. of (3) is commonly called the fundamental solution, $s^f$

$$s^f(t) = \alpha \int_t^\infty E_t f(v)\, e^{-\alpha(v-t)} dv. \tag{14}$$

Consider equation (15) below, which is a special case of equation (5).

$$s = f + \frac{\mu}{\alpha} + A_1 e^{\lambda_1 f} + A_2 e^{\lambda_2 f}. \tag{15}$$

In the terminology of McCallum [1983] this is the minimal state solution: it involves only the state variable(s) and the state variables enter in a "minimal" way.

The first two terms on the R.H.S of (15) are the fundamental solution for the unregulated fundamental. All of the R.H.S. of (15), for some given (non–zero) values of $A_1$ and $A_2$, is the fundamental solution for the regulated fundamental.

For the unregulated fundamental, non–zero values of $A_1$ and/or $A_2$ would permit what Froot and Obstfeld [1989c] call intrinsic bubbles, within the class of functions expressing s(t) as a function of current fundamentals (f(t)) only.

For the regulated fundamental, the existence of intrinsic bubbles would require that f enters the solution for the exchange rate in the interior of the target zone through terms other than the ones appearing (with $A_1$ and $A_2$ determined by (9a–c), (10a–d), or (11a–d)) on the R.H.S. of equation (15) and with no other variable(s) included. Such intrinsic bubbles are clearly impossible.

The literature has generally cast its discussion of bubbles for our class of models in terms of the behaviour of s as a function only of current f for different values of $A_1$ and $A_2$. For example, in the case of a freely floating exchange rate, unless $A_1 = 0$, the deviation of s from f will become infinite as f grows without bound (since $\lambda_1 > 0$) and unless $A_2 = 0$, the deviation of s from f will become infinite as f falls without bound (since $\lambda_2 < 0$). $A_1 = A_2 = 0$ rules out only intrinsic bubbles. Many kinds of exchange rate bubbles are possible under a floating rate even if $A_1 = A_2 = 0$. With such bubbles, s can deviate from f (for any given value of f) by eventually unbounded amounts.

It is recognized in the literature (Froot and Obstfeld [1989b] are an example), that the choice of $A_1$ and $A_2$ only restricts the relationship between s and f, but that state variables other than f may enter the solution for s through bubbles. In general, bubbles can introduce one or more additional state variables into the solution for s.

Subject to the constraint that these bubbles obey (4), they can cause s to deviate in almost any conceivable manner from g(f, .), for any values of $A_1$ and $A_2$.

What is not, we believe, recognized as clearly is that s can depend on f in ways other than given by g(f, ·) (or the R.H.S. of (15)), provided s also depends (through B) on at least one other state variable. We show below that the bubble at time t, B(t) can be a function of the unregulated fundamental $\bar{f}(t)$, provided it also depends on some other state variable (or state variables) z(t), with $\frac{\partial z(t)}{\partial \bar{f}(t)} = 0$, which ensures that $E_t dB = \alpha B dt$. This implies that within the target zone, dB can be a function of df. z(t) can (but need not) be a function exclusively of past and/or anticipated future values of the unregulated fundamental. We call this class of generalized intrinsic bubbles "fundamental–dependent" bubbles.

(a) <u>Bubbles that don't burst</u>

We begin by considering speculative bubbles, both deterministic and stochastic, that don't collapse. Examples of non–collapsing bubble processes satisfying (4) include the simple exogenous deterministic process given in (16), where $B(0) = B_0$ is an arbitrary initial value for the B process.

(16) $$B(t) = B_0 e^{\alpha t}.$$

An example of a simple exogenous (backward–looking) stochastic bubble satisfying (4) is given in equations (17a,b), with $B_0$ arbitrary.

(17a) $$dB(t) = \alpha B(t)dt + \sigma_b(.)dW_b$$

(17b) $$B(t) = \int_0^t e^{\alpha(t-v)}\sigma_b dW_b(v) + e^{\alpha t}B_0.$$

The parameter $\sigma_b$ is the instantaneous variance of the bubble process. Note that $\sigma_b$ could be a function of s, f, t or any other set of variables without this affecting the fact that equation (17a) satisfies (4). For instance, if $\sigma_B$ is a positive constant, the bubble can "change sign". With geometric Brownian motion ($\sigma_B$ proportional to B), sign reversals of the bubble are ruled out, although B can change direction.

Important for our analysis, the bubble process can itself be a function of current, past or anticipated future values of the unregulated fundamental $\bar{f}$. One example is given in equations (18a,b) below. In what follows the variable $z(t)$ is either strictly deterministic or follows a diffusion process. The $z(t)$ process is a twice continuously differentiable function of its arguments. The key defining property of $z(t)$ is that it does not vary with $\bar{f}(t)$, i.e. $\frac{\partial z(t)}{\partial \bar{f}(t)} = 0$. For the example given below, the z process given in equation (18b) (a backward–looking deterministic process) generates, together with equation (18a), a bubble process that satisfies equation (4).

(18a) $$B(t) = k\bar{f}(t) + z(t) \qquad k \neq -1$$

(18b) $$z(t) = \int_0^t e^{\alpha(t-v)}[k(\alpha\bar{f}(v) - \mu)]dv + e^{\alpha t}z_0$$

$z_0$ is an arbitrary initial value for the z process. Note that, given the boundary conditions (8b,d), (9a,b), (10b) or (11b), $A_1$ and $A_2$ are indeterminate if $k = -1$.

More generally, equation (18a) will hold provided z satisfies

(19) $$E_t dz(t) = [k(\alpha\bar{f}(t) - \mu) + \alpha z(t)]dt.$$

The z process given in (19) (and therefore the B process itself) can be forward–looking, backward–looking (or a linear combination of the forward–looking and backward–looking solutions), it can be strictly deterministic or stochastic.

An example of a forward–looking solution for z (now treated as a non–predetermined variable) which satisfies equations (18a) and (19) (and of course equation (4)) is given in equation (20).

(20) $$z(t) = -\int_t^\infty e^{\alpha(t-v)} E_t k(\alpha\bar{f}(v) - \mu)dv + R(t).$$

Here R(t) is any strictly deterministic or stochastic process which satisfies $E_t dR(t)dt = \alpha R(t)dt$. It can be predetermined or non–predetermined. One possible choice for R(t) would be the process given on the right–hand side of (18b). This would make the bubble B(t) in (18a) a function of current, past and anticipated future values of the unregulated fundamental.

Another interesting bubble process involving $\bar{f}$ is the following:

(21a) $$B(t) = \bar{f}(t)z(t)$$

(21b)
$$dz(t) = \bar{f}(t)^{-1}[\alpha\bar{f}(t) - \mu]z(t)dt + \sigma_z dW_z \qquad \bar{f} \neq 0$$
$$z(t) = 0 \qquad \bar{f} = 0,\ \mu \neq 0$$
$$dz(t) = \sigma_z dW_z \qquad \bar{f} = 0,\ \mu = 0$$

($dW_z$ and $dW_f$ are assumed to be contemporaneously independent).

An example of a class of rational bubbles that has the property that $\frac{\partial B(t)}{\partial \bar{f}(t)}$

depends on $\bar{f}(t)$ is given in (22a–b).

(22a) $$B(t) = \bar{f}(t)^2 + z(t)$$

(22b) $$E_t dz(t) = [\alpha(\bar{f}(t)^2 + z(t)) - 2\bar{f}(t)\mu - \sigma_f^2]dt.$$

A simple backward–looking deterministic solution for $z$ that satisfies (22b) is

(23) $$z(t) = \int_0^t e^{\alpha(t-v)}[\alpha\bar{f}(v)^2 - 2\bar{f}(v)\mu - \sigma_f^2]dt + e^{\alpha t}z_0.$$

If B can depend on the regulated value of the fundamental as well as on the unregulated fundamental, we can generate the very unusual bubble given in equation (24) below.

(24) $$B(t) = -g(f(t), B(\tau),i) + k\bar{f}(t) + z(t).$$

This bubble is consistent with the law of motion (4), provided that (25) holds:

(25) $$E_t dz(t) = \{-\alpha[g(f(t),.) - k\bar{f}(t) - z(t)] + [g'(f(t),.) - k]\mu + \tfrac{1}{2}g''(f(t),.)\sigma_f^2\}dt$$

Equation (26) is an example of a simple stochastic process for $z$ that satisfies equation (25)

(26) $$z(t) = \int_0^t e^{\alpha(t-v)}\Big[\{-\alpha[g(f(v),.) - k\bar{f}(v)] + [g'(f(v),.)-k]\mu + \tfrac{1}{2}g''(f(v),.)\sigma_f^2\}dv + \sigma_z dW_z(v)\Big]dv + e^{\alpha t}z_0.$$

While this bubble is consistent with the law of motion (4) within the target zone, it does not permit the boundary conditions to be satisfied. With this process for B(t) the exchange rate is given by

$$s(t) = k\tilde{f}(t) + z(t). \tag{27}$$

Note that in this case s increases or decreases linearly with $\tilde{f}$ (if $k \neq 0$), thus making "smooth pasting" boundary conditions impossible to apply. When $k = 0$, s(t) is independent of $\tilde{f}(t)$, although z(t) of course depends on past values of $\tilde{f}$ (and of f).

(b) Bubbles that burst

Blanchard [1979] provides an interesting example of a rational speculative bubble whose expected duration (time until the moment a collapse occurs) is finite. It is most easily motivated in discrete time. Equation (28) is the discrete analog of equation (1).

$$s_t = \alpha(1+\alpha)^{-1}f_t + (1+\alpha)^{-1}E_t s_{t+1}. \quad \alpha > 0 \tag{28}$$

The solution of (28) conditional on current and expected future values of $f_t$ is

$$s_t = \alpha(1+\alpha)^{-1}\sum_{i=0}^{\infty}(1+\alpha)^{-i}E_t f_{t+i} + B_t \tag{29}$$

where $B_t$ satisfies

$$E_t B_{t+1} = (1+\alpha)B_t. \tag{30}$$

Consider the following process for B

(31) $B_{t+1} = (1-\pi)^{-1}(1+\alpha)B_t + \epsilon_{t+1}$ with probability $1-\pi$

$B_{t+1} = \hat{\epsilon}_{t+1}$ with probability $\pi$

where $0 \leq \pi \leq 1$ and

(32) $$E_t\epsilon_{t+1} = E_t\hat{\epsilon}_{t+1} = 0.$$

It is easily checked that (31) and (32) satisfy (30). Equations (31) and (32) define a bubble (stochastic if $\epsilon_t$ or $\hat{\epsilon}_t$ is a random variable) for which there is a constant probability of collapse, $\pi$, each period. If a collapse occurs in period t+1, the exchange rate returns to its fundamental value if $\hat{\epsilon}_{t+1} = 0$. More generally, if $\hat{\epsilon}_{t+1}$ is random, the expected post–collapse value of the exchange rate is its fundamental value, but the realized value of the exchange rate can differ from this expectation by a zero mean forecast error. If the bubble collapses in period t+1 and $\hat{\epsilon}_{t+1}$ is non–zero, a new bubble starts in period t+1 which follows a law of motion like (31) and (32), but possibly with a different value of $\pi$ and different (although still zero mean) random disturbance terms $\epsilon$ and $\hat{\epsilon}$.

In continuous time the same idea is captured as follows. $\pi$ now is the constant instantaneous probability of collapse of the bubble. Conditional on the bubble not having collapsed by time t, the probability of the bubble lasting till time $t + \Delta t$ is $e^{-\pi\Delta t}$ $(\Delta t \geq 0)$. We consider the following process for B. As before, $W_b$ and $\hat{W}_b$ are standard Wiener processes. Conditional on the bubble having lasted till time t, we have

(33) $$B(t+\Delta t) = e^{(\alpha+\pi)\Delta t}B(t) + \sigma_b \int_t^{t+\Delta t} e^{(\alpha+\pi)(t+\Delta t-v)}dW_b(v)$$

with probability $e^{-\pi\Delta t}$

$$B(t+\Delta t) = \hat{\sigma}_b \int_t^{t+\Delta t} e^{(\alpha+\pi)(t+\Delta t-v)}d\hat{W}_b(v)$$

with probability $1 - e^{-\pi\Delta t}$.

It follows that the expected rate of change of B per unit time over the interval $[t, t+\Delta t]$ is

(34) $$\frac{E_t B(t+\Delta t) - B(t)}{\Delta t} = \frac{(e^{\alpha\Delta t} - 1)}{\Delta t} B(t).$$

In the limit as $\Delta t \to 0$ this becomes $E_t dB(t) = \alpha B(t)dt$, as required. While the bubble lasts, the instantaneous rate of change of the bubble is given by

$$dB(t) = (\alpha + \pi)B(t)dt + \sigma_b dW_b(t).$$

While the expected rate of change of the bubble is $E_t dB = \alpha Bdt$, the expected rate of variation of B conditional on the bubble surviving the next instant, is given by $(\alpha + \pi)Bdt$.

Let $t_c$ be the (random) time when the bubble collapses. If the value of B at the time of collapse, $B(t_c) = \hat{\sigma}_b \int_t^{t_c} e^{(\alpha+\pi)(t_c-v)} d\hat{W}_b(v)$ is non–zero, a new bubble will start at $t_c$, possibly with a different value of $\pi$, different white noise processes

$dW_b$ and $d\hat{W}_b$ and different values for $\sigma_b$ and $\hat{\sigma}_b$.

A Blanchard bubble is perfectly compatible with our target zone model. When it collapses there will be a discrete change in the level of the exchange rate, which lies on the trajectory appropriate to the new (post–collapse) bubble. Since this change is unexpected, no arbitrage opportunities arise.

(c) Target zones with fundamental–dependent bubbles

If the value of the bubble at time t is a function of the contemporaneous value of the unregulated fundamental, then within the target zone the response of the exchange rate to the fundamental will be affected relative to the case of an exogenous bubble (and, a–fortiori, relative to the no–bubble case). This is straightforward, since in this case $\frac{\partial \bar{f}}{\partial f} = 1$. As long as $\frac{\partial B(t)}{\partial f(t)} \neq -1$, the tangency conditions (8b,d), (9a,b), (10b) and (11b) determine the values of $A_1$ and $A_2$ although $f_u$ and $f_\ell$ determined by the two–sided bubble eater differ from the values obtained in equations (9d–e) in the case of an exogenous bubble.

## V. Properties of the Exchange Rate and the Interest Rate Differential in the Target Zone.

By Ito's lemma, the exchange rate in the target zone is a diffusion process with stochastic differential

(35) $$ds(t) = [g_f(f,.)\mu + \tfrac{1}{2}g_{ff}(f,.)\sigma_f^2]dt + g_f(f,.)\sigma_f dW_f + dB.$$

Let $Var_t x(t')$ denote the variance of $x(t')$ conditional on the information available at time t, that is $Var_t x(t') \equiv E_t[x(t') - E_t x(t')]^2$. The instantaneous variance of the bubble process is given by $\sigma_B^2(f, \bar{f}, s, B, \cdot)$. Since the only restriction on the bubble process is that $E_t dB = \alpha B dt$, it is certainly permissible for the instantaneous variance of the bubble process to depend on $f$, $\bar{f}$, $s$, $B$ or other variables.

As before, recognizing that B may depend on $\bar{f}$, we write

$$B = b(\bar{f},z) \text{ with } \frac{\partial z}{\partial \bar{f}} \equiv 0.$$

Using self–explanatory notation we obtain

$$\text{(36a)} \qquad E_t ds(t) = (g_f+b_f)\mu + b_z E_t dz(t) + \frac{1}{2}\left[(g_{ff}+b_{ff})\sigma_f^2 + b_{zz}\sigma_z^2 + 2b_{zf}\rho_{fz}\sigma_f\sigma_z\right]dt$$

and

$$\text{(36b)} \qquad Var_t ds(t) = \left[(g_f+b_f)^2\sigma_f^2 + b_z^2\sigma_z^2 + 2(g_f + b_f)b_z\rho_{fz}\sigma_f\sigma_z\right]dt.$$

where $\rho_{fz}$ denotes the instantaneous coefficient of correlation between f and z.

Note that the responsiveness of the exchange rate to the current fundamental within the target zone is given by $g_f + b_f$. A fundamental–dependent bubble might therefore be able to explain over–reactions of the exchange rate to fundamentals.

For a bubble process to be consistent with a target zone under the specified intervention rule, it must be the case that, for any size bubble, the boundary conditions characterize tangency points that are local maxima (minima) if the tangency is at the upper (lower) boundary. For a bubble that is consistent with the law of motion in (4) but does not permit smooth pasting according to (8b,d), (9a,b), (10b) and (11b), we refer to the bubble in (18a,b) with $k = -1$ or the bubble given in

(24) and (25).

Note that the instantaneous conditional variance of the change in the exchange rate no longer in general goes to zero as the boundaries of the target zone are approached. As the exchange rate approaches the boundary, the variance of the exogenous component of the bubble need not go to zero nor need $g_f + b_f$ tend to zero.

At points on the boundary where smooth pasting, whether one–sided or two–sided, occurs however, the conditional variance of the instantaneous change in the exchange rate is equal to zero. Take the two–sided smooth pasting conditions at the upper boundary, reproduced here for the case of an exogenous bubble:

$$s = s_u = \frac{\mu}{\alpha} + f_u^0 + B + A_1^0 e^{\lambda_1(f_u^0+B)} + A_2^0 e^{\lambda_2(f_u^0+B)}$$

$$0 = 1 + \lambda_1 A_1^0 e^{\lambda_1(f_u^0+B)} + \lambda_2 A_2^0 e^{\lambda_2(f_u^0+B)}$$

At a two–sided smooth pasting point on the upper boundary, the effect of dB on ds is multiplied by $1 + \lambda_1 A_1^0 e^{\lambda_1(f_u^0+B)} + \lambda_2 A_2^0 e^{\lambda_2(f_u^0+B)}$, which equals zero.

The boundary conditions at the one–sided smooth pasting point on the upper boundary in the case of an exogenous bubble are

$$s = s_u = \frac{\mu}{\alpha} + f_u^0 + B + A_1 e^{\lambda_1 f_u^0} + A_2 e^{\lambda_2 f_u^0}$$

$$0 = 1 + \lambda_1 A_1 e^{\lambda_1 f_u^0} + \lambda_2 A_2 e^{\lambda_2 f_u^0}$$

Now $A_1$ and $A_2$ adjust to keep the boundary conditions satisfied when B varies at the boundary, that is $dB = -e^{\lambda_1 f_u^0} dA_1 - e^{\lambda_2 f_u^0} dA_2$ . At the boundaries the positive variance of dB (in the case where B is stochastic) fails to be translated into a positive variance of ds because of the endogenous adjustment of $A_1$ and $A_2$.

If, in the spirit of Obstfeld–Rogoff and Diba–Grossman, we accept the proposition that with a freely floating exchange rate bubbles cannot occur, the conditional expectation and variance of the change in the exchange rate are (letting $\check{s}$ denote the exchange rate under a free float):

$$E_t \frac{d\check{s}(t)}{dt} = \mu$$

and

$$Var_t \frac{d\check{s}(t)}{dt} = \sigma_f^2.$$

Trivially, since bubbles can exist in a target zone, the instantaneous conditional variance of changes in the exchange rate may be less under free floating than under a target zone. If uncovered interest parity holds, it must be the case that the interest differential $\delta$ is given by

$$\delta(t) \equiv i(t) - i^*(t) = E_t\, ds(t)/dt.$$

It follows that the instantaneous conditional mean and variance of the change in the interest differential in the target zone are given by

(37a) $$E_t\frac{d\delta(t)}{dt} = \alpha\Big[(g_f+b_f-1)\mu + b_z F_t dz + \tfrac{1}{2}((g_{ff}+b_{ff})\sigma_f^2 + b_{zz}\sigma_z^2 + 2b_{zf}\rho_{fz}\sigma_f\sigma_z)\Big]$$

(37b) $$Var_t\frac{d\delta(t)}{dt} = [\alpha(g_f+b_f-1)]^2\sigma_f^2 + (\alpha b_z)^2\sigma_z^2 + 2\alpha^2(g_f+b_f-1)b_z\rho_{fz}\sigma_f\sigma_z.$$

As noted in Svensson [1989], if $\sigma_z = 0$ (and therefore also $\rho_{fz} = 0$) equations

(36b) and (37b) imply that (denoting the conditional standard deviation by $SD_t$)

$$SD_t(\frac{ds}{dt}) - \alpha^{-1}SD_t(\frac{d\delta}{dt}) = \sigma_f.$$

This finding of a constant trade–off between the instantaneous variability of the change in the exchange rate and the instantaneous variability of the change in the interest differential is no longer automatically valid when there is a bubble, as can be verified by inspection of (36b) and (37b).

The conditional instantaneous mean and variance of the change in the interest differential under a free float, $d\check{\delta}$, are:

$$E_t d\check{\delta}(t) = 0$$

and

$$Var_t d\check{\delta}(t) = 0.$$

With or without a bubble, the free float therefore always delivers more stability in changes in the interest differential than does the target zone.

The relationship between the level of the exchange rate and the interest differential in the target zone is found by noting that

(38a) $$\frac{\partial \delta}{\partial f} = \alpha(g_f + \frac{\partial B}{\partial f} - 1)$$

and

(38b) $$\frac{\partial s}{\partial f} = g_f + \frac{\partial B}{\partial f}\,.$$

It follows that

(39) $$\frac{\partial s}{\partial \delta} = \frac{g_f + \frac{\partial B}{\partial f}}{\alpha(g_f + \frac{\partial B}{\partial f} - 1)}.$$

When there is no bubble and the range of permissible values of f is $f_\ell \leq f \leq f_u$ (that is the values of f corresponding to the upward–sloping part of the bubble eater for B = 0) we have, since $0 \leq g_f < 1$ for $f_\ell \leq f \leq f_u$, a downward–sloping relationship between the level of the exchange rate and the interest differential. This relationship can be qualitatively affected by the bubble in two ways. First, if the current value of the bubble is dependent on the current value of the fundamental, it need no longer be true that $g_f + \frac{\partial B}{\partial f}$ is between 0 and 1 everywhere within the target zone. More interestingly, even if the bubble is exogenous, the relation between exchange rate and interest rate differential within the target zone need no longer be negative everywhere.

Consider again the case of a system which starts off with a zero bubble. The relation between exchange rate and interest rate differential is implicitly defined by

$$s(t) \equiv g[s(t) - \alpha^{-1}\delta(t), B(\tau), i] + B(t).$$

We shall write this locus as $s(t) = j[\delta(t), B(\tau), B(t), i]$. For the case B = 0 the relationship between s and $\delta$ is the familiar backward S–shaped downward–sloping one shown in Figure 3 as as $j[\delta(t), 0, 0, 2]$. The tangency points have coordinates $[\alpha(s_u - f_u^0), s_u]$ and $[\alpha(s_\ell - f_\ell^0), s_\ell]$. A positive value of the current bubble shifts the locus upward and to the right along the ray with slope $\alpha^{-1}$. One such locus, for $B_1 > 0$, is shown as $j[\delta(t), 0, B_1, 2]$ in Figure 3. For values of $\delta$ greater than $\alpha(s_\ell - f_\ell^0) + \alpha B_1$ the relationship between s and $\delta$ is a positive one.

Now consider the case where the the exchange rate hits the upper boundary $s_u$ when $\delta = \delta_u^{*1} \equiv \alpha(s_u - f_u^{*1})$ and $B = B_1$ as in equation (13). The system now shifts onto the one–sided bubble eater of $B_1$, given by $j[\delta(t), B_1, B_1, 1]$ in Figure 3. For small values of B this bubble eater is downward–sloping. For a sufficiently large value of B, however, the one–sided bubble eater is downward–sloping only in the neighbourhood of the upper boundary, but becomes upward–sloping for lower values of s as shown in Figure 3 for $j[\delta(t), B_2, B_2, 1]$ when $B_2 > B_1$.

Harrison [1985, pp. 89–92] shows that the steady state or unconditional density function $\pi$ of Brownian motion f with instantaneous variance $\sigma_f^2$ and drift $\mu$ which is regulated on the close interval $[f_\ell, f_u]$ is given by:

$$\pi(f) = [f_u - f_\ell]^{-1} \qquad \text{if } \mu = 0 \text{ (the uniform density)}$$

$$\pi(f) = \frac{\theta e^{\theta f}}{e^{\theta f_u} - e^{\theta f_\ell}} \qquad \text{if } \mu \neq 0 \text{ (truncated exponential density)}$$

$$\theta = 2\mu\sigma_f^{-2}.$$

Without a bubble it follows that, since $s = g(f)$ is a strictly increasing function of f with continuous first derivative on the open interval $(f_\ell, f_u)$ and the implicit (inverse) function $f = g^{-1}(s)$ therefore exists on the target zone (except at the end points), the steady state density function of s, $\varphi(s)$ say, is given by:

$$\varphi(s) = \frac{\pi(g^{-1}(s))}{g'(g^{-1}(s))} \qquad s_\ell < s < s_u.$$

Svensson [1989] shows that the resulting distribution of exchange rates is U–shaped.

With a bubble, however, the long–run behaviour of the exchange rate is quite different. First of all, one of the state variables, the bubble, is non–stationary and unregulated. It therefore does not possess a steady state distribution. The other state variable, f, is of course non–stationary both when unregulated and when regulated.

## VI. Bubbles, target zones and intervention rules: what gives when they are inconsistent?

Our view of what happens to speculative bubbles under a credible target zone is rather different from that of Miller, Weller and Williamson [1989] (henceforth MWW). The formal analysis in their paper is cast in terms of Blanchard bubbles, but would apply a–fortiori to the non–collapsing bubbles we considered. MWW put their argument succinctly:

> *If all market participants believe that, when the exchange rate hits the edge of the band, the authorities will defend the rate by sudden intervention (designed to produce a jump in the rate), this must cause the bubble to burst. But this is inconsistent with the existence of a bubble in the first place: If all know this collapse is certain to occur at time t, all wish to sell the currency at t–ϵ. But then collapse will occur at t–ϵ. Repeating this argument we find that collapse must occur at time zero that is, all such bubbles are "strangled at birth".*
>
> (*Miller, Weller and Williamson* [*1989*]).

It should be recognized that in this paper we have changed the intervention rules followed by the authorities from what was assumed by MWW and by the other contributors to this literature. In addition to the standard reflection policies, we allow the accommodation by the authorities of sustaining speculative attacks. The intervention policies considered by MWW and in the rest of this literature, do not have this feature. It should therefore come as no surprise that from different assumptions we reach different conclusions.

Given the assumption that the only form of intervention is regulation at the boundaries through infinitesimal reflecting operations, the conclusion that this configuration is inconsistent with rational bubbles follows. One common response to this inconsistency is to treat the bubble as a *residual*. The reasoning goes as follows: at a point like $[f_u^{*1}, s_u]$ in Figure 1 there would be a discontinuous anticipated fall in the value of $s$ from $s_u$ to $\hat{s}$, the value of the exchange rate at $f = f_u^{*1}$ on the bubble eater for $B = 0$, given by $s = g(f, 0, 2)$. This violates the no–arbitrage condition.

What is presented in the previous paragraph as a "real–time" event, the *fall* of the exchange rate from $s_u$ to $\hat{s}$, can in fact be no more than the designation of $\hat{s}$ as the only value of the exchange rate reached when $f = f_u^{*1}$, that is consistent with the intervention rule. Equivalently, $B \equiv 0$ is the only value of the bubble consistent with the intervention rule. If the system really were (somehow) at $[f_u^{*1}, s_u]$, there would be an inconsistency. We cannot deduce how the exchange rate would behave starting from an inconsistent position.

Note that it is not just the assumption of a credible exchange rate target zone that produces the inconsistency, but the combination of a credible target zone and a particular intervention rule that can only support the target zone if no bubble exists. Our alternative intervention rule, which has a discrete (stock–shift) increase in $f$ at $[f_u^{*1}, s_u]$ from $f_u^{*1}$ to $f_u^0$ and a corresponding movement of the equilibrium from $[f_u^{*1}, s_u]$ to $[f_u^0, s_u]$, consistently combines a credible target zone and rational bubbles.

An obvious question would seem to be: Why would the authorities adopt a rule that permits the presence of bubbles rather than a rule that precludes their existence?

Putting the question this way prejudges the answer, because it assumes that, given an inconsistency between having a credible target zone, a rational bubble and a particular intervention policy, the inconsistency must be resolved by dropping the bubble, rather than by dropping the target zone or the intervention policy. We prefer to view the existence of a bubble as a datum: A bubble either exists or it doesn't, and the authorities must design their intervention policy accordingly, if they wish to support the target zone.

A particular intervention rule (e.g. infinitesimal reflecting interventions at the boundaries without accommodation of friendly speculative attacks) may of course be inconsistent with having a credible target zone in the presence of bubbles. This implies, in our view, a collapse either of the target zone or of the intervention rule, but not of the bubble, which is not an object of choice at some initial date, not even a collective one. Given the existence of a bubble, a credible target zone requires a policy rule specification that permits the bubble and the target zone to coexist. Of course it must also be capable of handling the no–bubble case . Our rule is an example of such a consistent policy.[9]

## VII. Conclusions

The theory of exchange rate behaviour within a target zone as developed in the recent literature holds that exchange rates under a currency band regime are less responsive to fundamental shocks than exchange rates under free float, provided that the intervention rules of the Central Bank(s) are common knowledge. Moreover, there always exists a trade–off between the instantaneous volatility of changes in the exchange rates and changes in the interest rate differential, independently of the size of the band and the degree of credibility of the target zone. These results are derived after having assumed *a priori* that "rational excess volatility" (due to so called *rational bubbles*) does not occur in the foreign exchange market. Implicitly or explicitly, it is assumed that speculative bubbles are incompatible with the existence and the persistence of a credible target zone, so that they never materialize.

We consider instead a setup in which the existence of speculative behaviour is a *datum* the Central Bank has to deal with. We show that there is no incompatibility between the existence of a target zone and the presence of rational bubbles. Rather, there are intervention rules that should be followed by the Central Bank when speculative bubbles arise, and these same rules include as a special case the traditional policies for defending an exchange rate band when speculative bubbles do not occur.

In the standard model, the defense of a target zone is guaranteed by intermittent variations in domestic credit (say through open market operations) and/or non sterilized foreign market interventions that take place when the exchange rate hits one of the limits of the band. For instance, when the exchange rate reaches the upper boundary, the stock of foreign reserves (or the stock of domestic credit) is reduced in order to prevent the exchange rate from rising further.

Heuristically, these operations can be characterized as infinitesimal *reflecting* interventions. Although the size of the reflecting operations may be quantitatively limited, the induced expectations stabilize the exchange rate even before the upper or lower limit is reached.

We show that in the *cum bubble* setup reflecting interventions are insufficient. In fact, when speculative bubbles arise, the companion phenomena of speculative attacks on the target zone must occur as well. These speculative attacks are stock–shift reshuffles of private agents' financial portfolios led by rational expectations of changes in the rate of exchange rate depreciation. As an example, if the rate of appreciation of the exchange rate were suddenly expected to increase, rational private agents would increase their demand for home country money and/or decrease their demand for foreign country money due to a decrease in the domestic–foreign interest rate differential.

We show that the defense of the target zone in the presence of bubbles is guaranteed if the Central Bank *accommodates* speculative attacks when the latter are *friendly*, that is when they are consistent with the survival of the target zone itself (given the Central Bank's rule). This intuitive policy rule is compatible with self–fulfilling expectations: a friendly attack occurs only if agents know that it will be accommodated, and exactly because of the passive accommodation the speculative attack *sustains* the target zone. Hence, the policy rule is summarized by the maxim "Reflect when this is sufficient; accommodate when this is necessary".

Many of the conclusions reached in the existing literature do not appear sufficiently robust when speculative bubbles are considered as well.

First, it is not true anymore that the instantaneous volatility of exchange rates within a target zone is always less than the instantaneous volatility of exchange rate under free float, provided that, for familiar reasons, in free float no speculative bubble arises.

Second, the finding of a constant trade–off between the instantaneous variability of the change in the exchange rate and the instantaneous variability of the change in the interest rate differential is no longer automatically valid when there is a bubble.

Third, the presence of fundamental–dependent bubbles may account for the excessive responsiveness of the exchange rate to the current fundamental within the target zone.

Fourth, the standard theory characterizes a stable *negative* relation between the level of the exchange rate and the expected rate of depreciation of the exchange rate (equal to the interest rate differential if uncovered interest parity holds). According to this model therefore, the higher the exchange rate, the lower the interest rate differential. Moreover the interest rate differential is always negative in the neighbourhood of the upper boundary of the exchange rate band; the opposite result holds in the neighbourhood of the lower boundary. The failure of this regularity to show up convincingly in the data for the EMS, has been rationalized with perceptions of an increased likelihood of realignments as the exchange rate approaches the edges of the band and similar ways of undermining the credibility of the target zone. We show that even when the target zone is fully credible, the presence of rational bubbles (even exogenous ones) is sufficient to reverse the relationship between the exchange rate and the interest differential.

# NOTES

[1]Consider a dynamic linear rational expectations model with constant coefficients that has the usual saddlepoint configuration (as many predetermined variables, $n_1$ say, as stable characteristic roots and as many non–predetermined variables, $n_2$ say, as unstable characteristic roots). Transform the system to canonical variables, by diagonalizing it or by using Jordan's canonical form. Group together the $n_2$ dynamic equations containing the canonical variables whose homogeneous equations are governed by the unstable roots. The general solution for these non–predetermined canonical variables can contain an $n_2$–dimensional bubble process that must satisfy the homogeneous equation system of the $n_2$ non–predetermined canonical variables. Bubble processes will therefore be non–stationary (in expectation). The state variables of the model, which are linear combinations of the canonical variables, will, if there is a bubble, be non–stationary. If there are more non–predetermined state variables than unstable roots, stationary bubbles will of course be possible even in linear models.

[2]Similar arguments can be made for non–stationary bubbles (and the non–stationary behavior of key endogenous variables frequently associated with non–stationary bubbles) in certain non–linear models. For instance, in overlapping generations (OLG) models of a competitive economy with a single perishable commodity and with non–interest–bearing outside money as the only store of value, rational deflationary bubbles (i.e. non–stationary bubbles with the price level declining to zero) have been shown to be infeasible. Take for instance the case of a two–period OLG model with a constant population in which the nominal money stock is constant,

all money is held by the old and only the young receive a constant bounded endowment stream of the single commodity. If the price level were to fall without bound, real cash balances would be increasing without bound and the demand for the commodity by the old generation would increase without bound. Since the supply of the good each period is bounded above by the sum of the endowments of the young, sooner or later the demand for the commodity must outstrip the supply. A sequence of money prices falling to zero can therefore not be rational expectations equilibrium (see e.g. Hahn [1982, p. 10]).

Unless very strong restrictions are placed on the private utility functions, rational inflationary bubbles (with the price level increasing without bound despite a constant nominal money stock) can exist in such an economy. The sequence of rising prices would, in the limit, drive the real value of money to zero. The steady state to which such a model converges is that of a non–monetary economy. To rule out inflationary bubbles as well, Obstfeld and Rogoff [1983], in an infinite–lived representative agent model, imposed a political–technological restriction on the terminal value of money: the government fractionally backs the currency by guaranteeing a minimal real redemption value for money. Even if private agents are not completely certain that they can redeem their money in any given period, this suffices to rule out speculative hyperinflations. While this assumption on government behaviour seems quite ad–hoc, it is often cited as the second blade of the scissors that cut the lifeline of non–stationary rational speculative bubbles, both deflationary and inflationary. Diba and Grossman [1988] also derive sufficient conditions for ruling out non–stationary inflationary bubbles.

[3] Sargent and Wallace's "Unpleasant Monetarist Arithmetic" model (Sargent and Wallace [1981]) can exhibit stationary rational bubbles. Even a first order deterministic non–linear difference equation may exhibit various kinds of periodic

solutions or chaotic behavior (see e.g. Benhabib and Day [1981, 1982] and Grandmont [1985]). Stationary bubbles are easily generated by such models (Azariadis [1981], Azariadis and Guesnerie [1986], Chiappori and Guesnerie [1988], Woodford [1987], Farmer and Woodford [1986]). Second order deterministic non–linear differential equations can generate limit cycles and third order non–linear differential equations can exhibit chaotic behavior. Again, such models can support stationary rational speculative bubbles.

[4]This interpretation of f comes from the two–country mini–model outlined below. All variables except interest rates are in natural logarithms. m is the home country nominal money stock, p the home country price level, y home country real GDP, i the home country short nominal interest rate and s the spot price of foreign exchange. Corresponding foreign country variables are starred.

$$m - p = ky - \lambda i, \quad k, \lambda > 0 \qquad \text{(Home monetary equilibrium)}$$

$$m^* - p^* = ky^* - \lambda i^* \qquad \text{(Foreign monetary equilibrium)}$$

$$p = p^* + s \qquad \text{(Purchasing Power Parity)}$$

$$E_t ds(t) = [i(t) - i^*(t)]dt \qquad \text{(Uncovered Interest Parity)}$$

From this very simple model we obtain the following relation:

$$s(t)dt = [m(t) - m^*(t) - k(y(t) - y^*(t))]dt + \lambda E_t ds(t).$$

This corresponds to equation (1) with $\lambda = \alpha^{-1}$ (the interest semi–elasticity of money demand) and $f = m - m^* - k(y - y^*)$.

[5]Equation (5) is of course equivalent to the perhaps more familiar form

$$s(t) = f(t) + B(t) + \frac{\mu}{\alpha} + \bar{A}_1(t)\, e^{\lambda_1 f(t)} + \bar{A}_2(t)\, e^{\lambda_2 f(t)}$$

where $\bar{A}_1$ and $\bar{A}_2$ are constant as long as s is in the interior of the target zone, but can change when s is at one of its boundaries. For many bubbles, including the exogenous ones, the representation given in (5) is attractive because it can bring out clearly the horizontal shift of the *two–sided bubble eater* (defined below) when the magnitude of the bubble varies.

[6]While the expected rate of change of s at the reflection points on the upper boundary is always less than that at points like $[f_u^*, s_u]$, the expected rate of change at $[f_u^*, s_u]$ can, for small bubbles, be negative.

[7]The notion of a friendly or sustaining speculative attack is related to the "sustaining" money demand by arbitrageurs in one of the solutions to the "gold standard paradox" proposed by Buiter and Grilli [1989]. See also Krugman and Rotemberg [1990].

[8] For reference we also draw $g(f, \hat{B}_3, 2) + \hat{B}_3$, the two–sided bubble eater of $\hat{B}_3$ in Figure 2. It lies to the right of the two–sided bubble eater of $B_2$ (not shown).

[9] A separate argument that might be made against bubble equilibria takes aim at its most striking feature: the possibility of very frequent interventions at the edges

of the zone.

Consider for simplicity a deterministic bubble that starts from a positive initial value. It might be argued that the interventions in the fundamental that are required to offset such a non–stationary bubble are not sustainable because, as the magnitude of the bubble grows over time, interventions at the lower boundary can be expected to become larger and larger. Finite international reserves (assuming these are the intervention medium) are bound to be exhausted in due course.

It should be noted that a very similar argument can be made even if there is no bubble, as long as there is positive drift in the fundamental. While the expected value of the fundamental grows linearly (at a constant rate $\mu$) rather than exponentially as the bubble does, the drift of the unregulated fundamental will also in finite time cause any finite stock of reserves to be exhausted with probability one. Indeed, even without drift in the fundamental (and without a bubble), a stochastic fundamental process driven by Brownian motion will bring any finite stock of reserves down to any positive lower bound in finite time with probability one (see Buiter [1989]).

The problems of international reserve exhaustion created by the bubble are of course less severe if the bubble is a Blanchard bubble, which has finite expected duration. Note also that in the most common interpretation of our model, the fundamental f stands for relative home country money supply minus relative real income – related money demand. Taking relative nominal money stocks as the object of regulation, there is nothing in the logic of the model that requires international reserves to be used to regulate the money stocks. Domestic credit expansion (whether reflecting open market operations or monetary financing of government budget deficits) can achieve the same monetary objectives.

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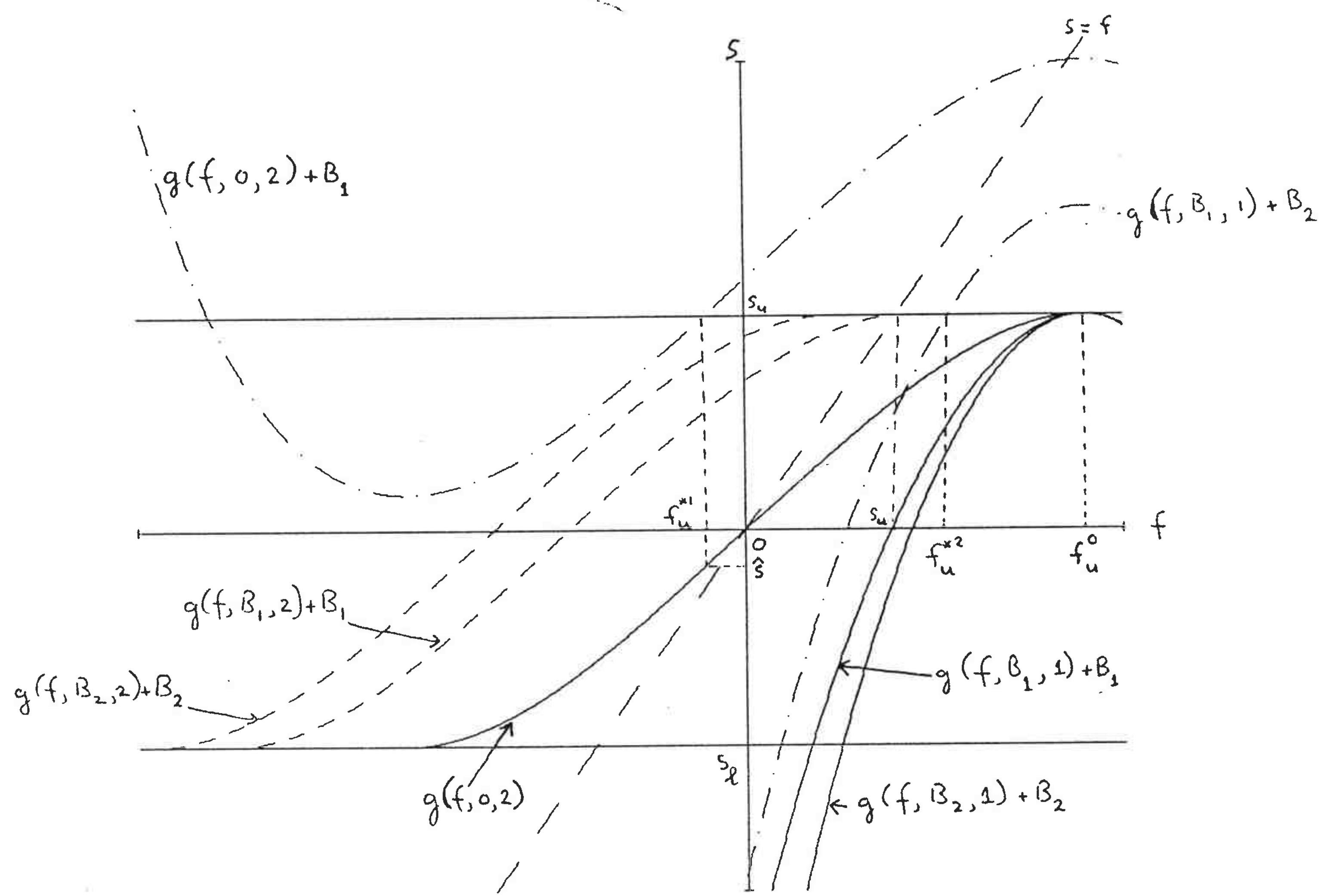
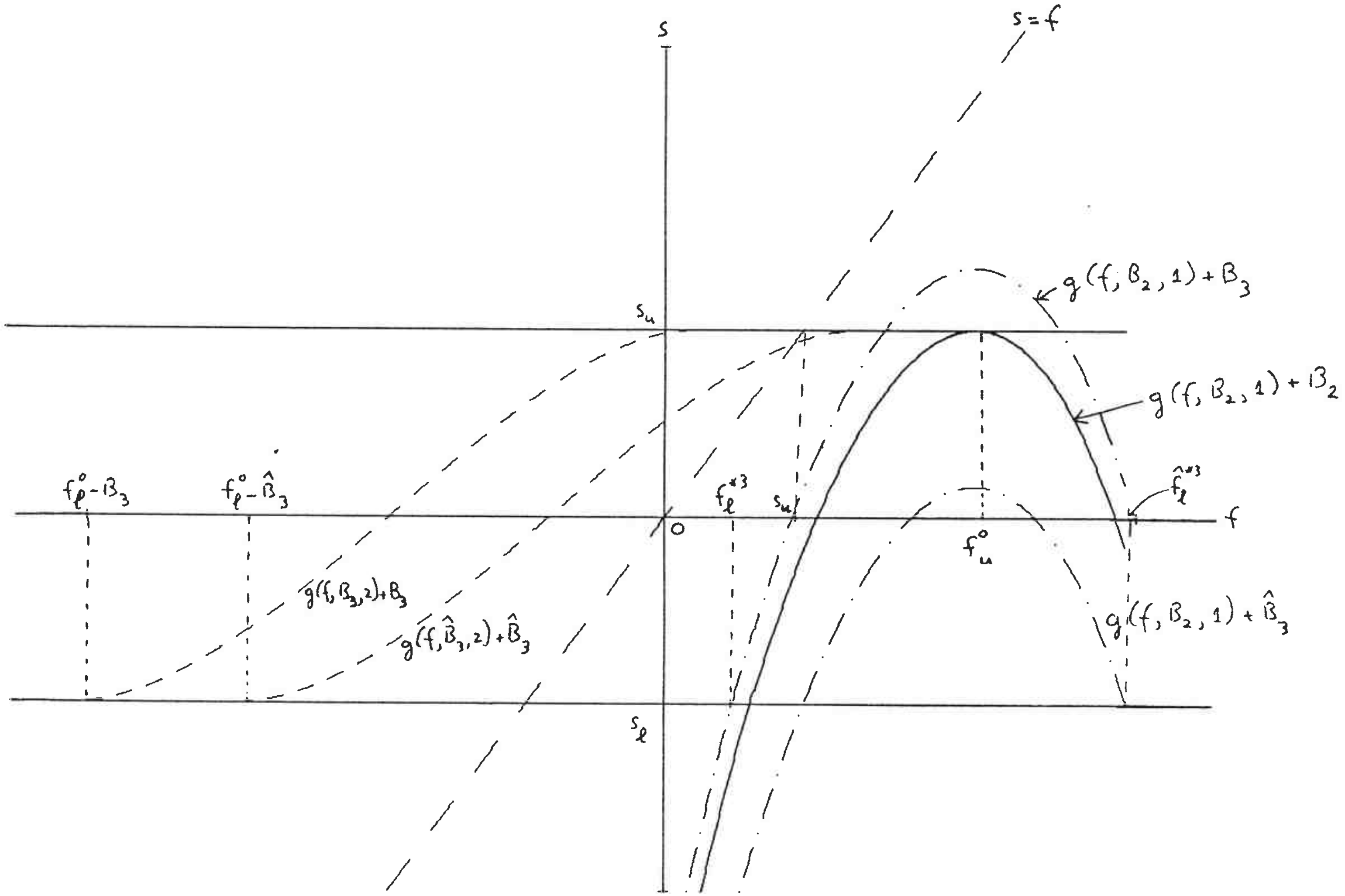
Figure 1

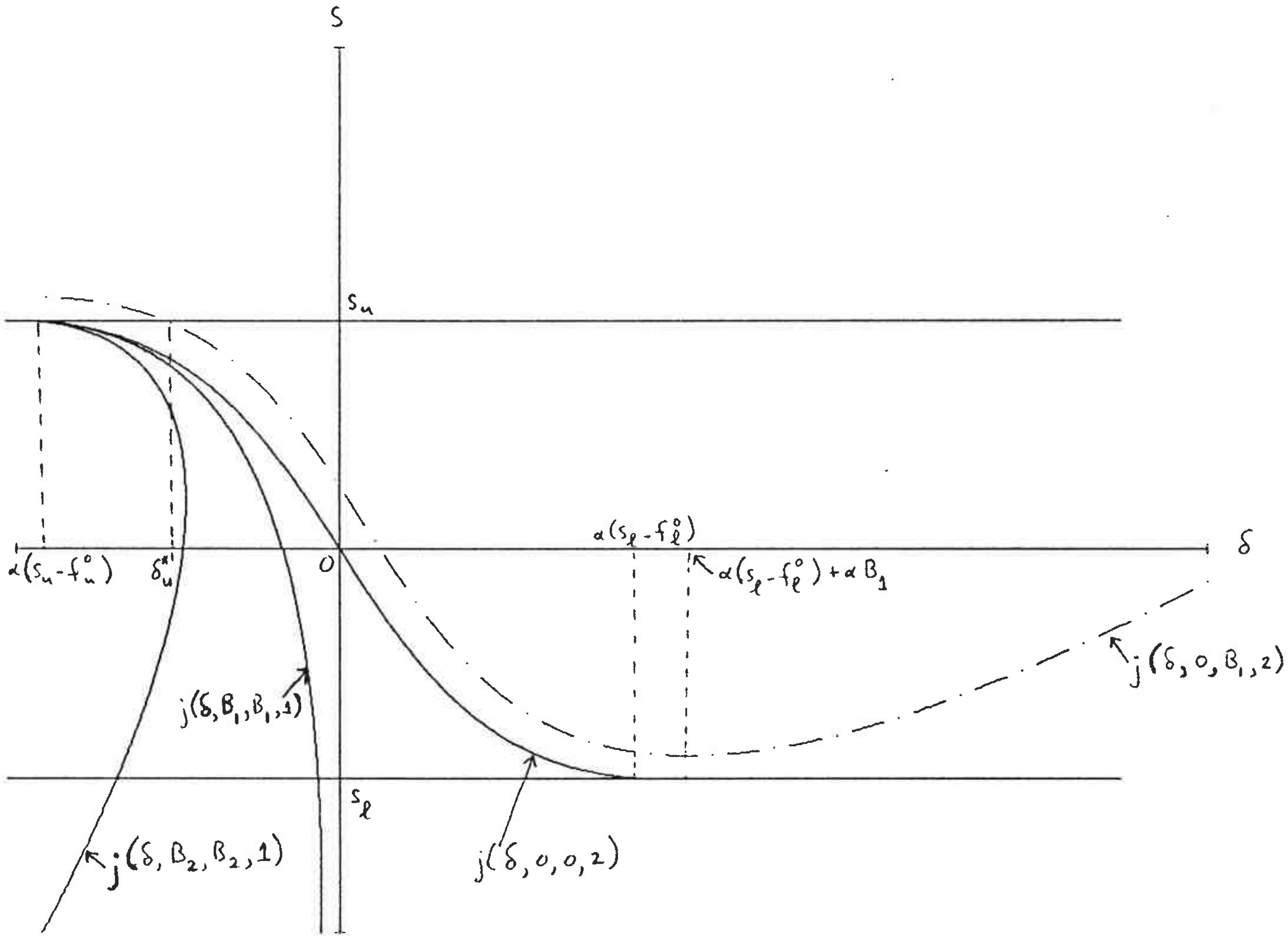

s
f
s = f
g(f, 0, 2) + B₁
g(f, B₁, 1) + B₂
$s_u$
$s_u$
$f_u^{*1}$
0
$\hat{s}$
$f_u^{*2}$
$f_u^{0}$
g(f, B₁, 2) + B₁
g(f, B₂, 2) + B₂
g(f, B₁, 1) + B₁
$s_\ell$
g(f, 0, 2)
g(f, B₂, 1) + B₂

# Figure 2



s
s = f
g(f, B2, 1) + B3
g(f, B2, 1) + B2
f
0
s_u
s_l
f_u^o
g(f, B2, 1) + B̂3
g(f, B3, 2) + B3
g(f, B̂3, 2) + B̂3
f_l^o - B3
f_l^o - B̂3

# Figure 3


s
$s_u$
$s_\ell$
δ
0
$\alpha(s_u - f_u^0)$
$\delta_u^*$
$\alpha(s_\ell - f_\ell^0)$
$\alpha(s_\ell - f_\ell^0) + \alpha B_1$
$j(\delta, 0, B_1, 2)$
$j(\delta, B_1, B_1, 1)$
$j(\delta, B_2, B_2, 1)$
$j(\delta, 0, 0, 2)$