MONETARY UNCERTAINTY IN DISCRETE-TIME
UTILITY-OF-MONEY MODELS

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**Monetary Uncertainty in Discrete-Time Utility-of-Money Models***

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## Abstract

In a discrete-time model, higher variability of the future money supply affects current macroeconomic variables when beginning-of-period money is the argument of utility, but not when end-of-period money is. The effect is through the existence of a precautionary money demand.



Keywords: monetary uncertainty, utility of money, precautionary demand

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## 1. Introduction

We show that, in a discrete-time model where money provides utility, higher variability of future monetary policy affects current macroeconomic variables when beginning-of-period (BOP) money is the argument of utility, but not when end-of-period (EOP) money is.[1] The reason is that the BOP specification implies the existence of a "precautionary" demand for money. To obtain this conclusion, "higher variability" must be appropriately defined: we show that the appropriate measure is an increase in variance about a given expected *inverse* monetary growth rate. We proceed to characterise the effects of uncertainty. When utility is separable, the condition for the current price level to fall is that the "coefficient of relative prudence" in money holding should exceed two. When utility is non-separable, which permits consumption and money to be complements, complementarity causes the current price level again to fall.

## 2. Two Specifications of a Simple Model

We consider a stochastic pure-exchange economy with competitive, instantaneously-clearing markets.[2] The nonstorable output is an independently and identically distributed exogenous variable. The representative agent faces the problem:

$$\text{maximise} \quad E_1\left( \Sigma_{t=1}^{\infty} \beta^t u(c_t, m_t) \right)$$

$$\text{subject to} \quad a_t = i_t \frac{p_t}{p_{t+1}} [a_{t-1} + y_t + \tau_t - c_t - [1 - \frac{1}{i_t}] \frac{M_t}{p_t}]$$

$a_{t-1}$ is real wealth (the value of money and nominal bonds at the end of period t-1, deflated by $p_t$), $i_t$ is the gross nominal interest rate, $\tau_t$ is the real value of a lump-sum transfer, and $M_t$ is money held at the end of period t. The two specifications differ according to the definition of $m_t$:[3]

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1 The clearest published example of an "uncertainty irrelevance" result for monetary policy is by Lucas and Stokey (1987; "Example 2"), who, however, use a cash-in-advance model. Gertler and Grinols (1982) obtain a "relevance" result in a *continuous*-time utility-of-money model, but this is due to making money transfers a subsidy to wealth.

2 The model is very close to those of Leeper (1991) and Sims (1992).

3 Both occur in the literature, e.g. LeRoy (1984a,b), Sargent (1987, Ch. 4), Leeper (1991), Canova (1992) use the EOP version; Danthine and Donaldson (1986), Danthine et al. (1987) use the BOP version

$m_t \equiv M_t/p_t$ EOP specification

$m_t \equiv M_{t-1}/p_t$ BOP specification

In economic terms, the important difference is that with the BOP specification the money holdings which (implicitly) facilitate goods purchases in period t must be chosen in period t-1, before $(p_t, y_t, \tau_t)$, and thus the the optimal $c_t$, are known. The risk of being caught short of cash may thus give rise to a "precautionary" demand for money, in addition to the pure "transactions" demand.[4] The first-order conditions for solving this problem are:

$$u_c(t) = \beta E_t( i_t \frac{p_t}{p_{t+1}} u_c(t+1) ) \tag{1}$$

$$[1-\frac{1}{i_t}]u_c(t) = u_m(t) \tag{2a}$$

$$[1-\frac{1}{i_t}]u_c(t) = \beta E_t( \frac{p_t}{p_{t+1}} u_m(t+1) ) \tag{2b}$$

$(u_c(t) \equiv \partial u(c_t,m_t)/\partial c_t$, etc.) where (2a) and (2b) apply to the EOP and BOP specifications, respectively.

First consider monetary uncertainty in the EOP model. Equilibrium in the economy can be summarised by the following difference equation in $m_t$:[5]

$$m_t u_c(y_t,m_t) - m_t u_m(y_t,m_t) = \beta E_t( \frac{M_t}{M_{t+1}} m_{t+1} u_c(y_{t+1},m_{t+1}) ) \tag{3}$$

Now we posit the following monetary growth rule:

$$\frac{M_t}{M_{t+1}} = \mu\xi_{t+1}, \quad \text{where } \xi_{t+1} \text{ is i.i.d. with } E_t(\xi_{t+1}) \equiv 1 \tag{4a}$$

Upon substituting this into (3), note that we may factor out $E_t(\xi_{t+1})$ on the right-hand side. This is because, as (3) shows, $m_t$ is independent of $\xi_t$, so $\xi_{t+1}$ is likewise independent of $m_{t+1}$ (and, by assumption, of $y_{t+1}$). But $E_t(\xi_{t+1}) \equiv 1$, so we have:

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[4] Although precautionary demand has received attention in the cash-in-advance literature (e.g. Svensson (1985), Giovannini (1989), Bianconi (1992)), explicit discussion of it in the utility-of-money literature is notably absent.

[5] To obtain, set $c_t = y_t$, substitute $p_t/p_{t+1}$ out of (1) as $[m_{t+1}/m_t][M_t/M_{t+1}]$, and combine with (2a) to eliminate $i_t$.

$$m_t u_c(y_t,m_t) - m_t u_m(y_t,m_t) = \beta\mu E_t(\, m_{t+1} u_c(y_{t+1},m_{t+1})\,) \tag{3'}$$

All monetary shocks thus drop out: the solution of this equation for $\{m_t\}_1^\infty$ is unaffected by monetary uncertainty. However, if we instead use the rule:

$$\frac{M_{t+1}}{M_t} = \nu\zeta_{t+1}\,, \qquad \text{where } \zeta_{t+1} \text{ is i.i.d. with } E_t(\zeta_{t+1}) \equiv 1 \tag{4b}$$

(3) then becomes:

$$m_t u_c(y_t,m_t) - m_t u_m(y_t,m_t) = \frac{\beta}{\nu} E_t(\frac{1}{\zeta_{t+1}}) E_t(\, m_{t+1} u_c(y_{t+1},m_{t+1})\,) \tag{3'}$$

In this case a rise in $var_t(\zeta_{t+1})$ lowers $E_t(1/\zeta_{t+1})$, by Jensen's inequality. Since $E_t(m_{t+1}u_c(y_{t+1},m_{t+1}))$ is unaffected if the rise is temporary (the solution for $m_t$ is entirely "forward-looking"), $m_t$ now rises (if $u_{cm} \geq 0$). Thus, in the EOP model, whether future monetary uncertainty leaves unchanged or lowers the current price level depends, respectively, on whether expected *inverse* monetary growth is held constant - as in (4a) - or whether expected monetary growth itself is held constant - as in (4b).

Despite the fact that (4b) is the more obvious, we wish to argue that (4a) provides a better benchmark. The effect introduced by (4b) is a pure "convexity effect", unrelated to agents' preferences. To see its source, consider the real interest rate $r_{t+1} \equiv i_t p_t/p_{t+1}$, which is risky due to the risky $p_{t+1}$. It is intuitive that $p_{t+1}$ will vary proportionally with $M_{t+1}$; thus, since $r_{t+1}$ is a convex function of $p_{t+1}$, higher variance of $M_{t+1}$ about a given $E_t(M_{t+1})$ raises $E_t(r_{t+1})$. Hence if $E_t(r_{t+1})$ needs to remain constant for equilibrium (which is not quite true), either $p_t$ or $i_t$ (or both, as is the case here) must fall. On the other hand if $E_t(1/M_{t+1})$ is given - as under (4a) - $E_t(1/p_{t+1})$ and so $E_t(r_{t+1})$ are unaffected by higher variability of $M_{t+1}$. We shall refer to this as the "Fischer-Stulz" effect, since it was first noted by Stulz (1986), building on Fischer's (1975) observation that higher inflation variance raises the expected real interest rate.[6,7]

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6 Despite recognising the effect, Stulz does not suggest using the alternative growth rule in order to eliminate it. Nearly all comparable studies have used (4b). The first use of (4a) appears to be by Lucas and Stokey (1987), though they provide no explicit motivation for it; see also Sims (1992).

7 A further reason for wishing to set aside the effect is that welfare paradoxically *rises* as uncertainty increases: this is because utility is increasing in $m_t$.

Second, we turn to the BOP model. To exclude the "Fischer-Stulz" effect, we focus just on the growth rule (4a). The difference equation (counterpart of (3')) now becomes:

$$m_t u_c(y_t,m_t) \; = \; \beta\mu\xi_t \, E_t(\, m_{t+1}u_c(y_{t+1},m_{t+1}) + m_{t+1}u_m(y_{t+1},m_{t+1})\,) \tag{5}$$

Monetary shocks no longer drop out: since $m_t \equiv M_{t-1}/p_t$ here, and $M_{t-1}$ is predetermined, effects on $p_t$ of $\xi_t$ are bound now also to affect $m_t$. However, does $var_t(\xi_{t+1})$ matter for $m_t$? Since, by induction, $\xi_{t+1}$ affects $m_{t+1}$, higher variance of $\xi_{t+1}$ in general must affect the right-hand side of (5), and thereby $m_t$. Therefore in the BOP model "uncertainty irrelevance" breaks down. The intuition for this is that variability of $\xi_{t+1}$, by increasing the variability of $p_{t+1}$ and $\tau_{t+1}$, alters the precautionary demand for money.

## 3. Effects via Precautionary Demand

We here consider how effects via precautionary demand are sensitive to preferences. It is convenient to transform (5) into an equation in $x_t$,

$$x_t \; = \; \beta\mu\xi_t \, E_t(\, x_{t+1} + j(x_{t+1},y_{t+1})\,)$$

where

$$x_t \; \equiv \; m_t u_c(y_t,m_t) \qquad \text{or, inverted,} \qquad m_t \; \equiv \; h(x_t,y_t) \; ,$$

$$j(x_t,y_t) \; \equiv \; h(x_t,y_t)u_m(y_t,h(x_t,y_t)).$$

Substituting out $x_{t+1}$ by advancing the equation one period, we have:

$$x_t \; = \; [\beta\mu]^2\xi_t\overline{X} \; + \; \beta\mu\xi_t E_t(\, j(\beta\mu\xi_{t+1}\overline{X},\, y_{t+1})\,) \tag{6}$$

where

$$\overline{X} \; \equiv \; E_t(\, x_{t+2} + j(x_{t+2},y_{t+2})\,)$$

Note that $\overline{X}$ is unchanged by a temporary increase in uncertainty (i.e. a rise in $var_t(\xi_{t+1})$ with $var_t(\xi_{t+s})$, $s \geq 2$, unchanged). Then we can see from (6) that $x_t$ rises or falls depending on whether the function j(.) is convex or concave in x, i.e. on whether $j_{xx}$ is positive or negative. In the general case, we may compute $j_{xx}$ as:

$$j_{xx} = \frac{u_m}{mu_c^2}\,\frac{\left(\frac{-mu_{cm}}{u_c}\right)\left[1-\left(\frac{-mu_{mm}}{u_m}\right)\right]\left[2-\left(\frac{-mu_{cmm}}{u_{cm}}\right)\right] - \left(\frac{-mu_{mm}}{u_m}\right)\left[1-\left(\frac{-mu_{cm}}{u_c}\right)\right]\left[2-\left(\frac{-mu_{mmm}}{u_{mm}}\right)\right]}{\left[1-\left(\frac{-mu_{cm}}{u_c}\right)\right]^3}$$

Its sign thus depends on the elasticities of $u_c, u_m, u_{cm}$ and $u_{mm}$ with respect to m. To understand it further, we need to consider more restricted cases of u(c,m).

One special case is where utility is additively separable, so that $u_{cm} = 0$. Terms in $u_{cm}$, $u_{cmm}$ then drop out, which implies (noting $u_c, u_m > 0$, $u_{mm} < 0$) that $j_{xx}$ is positive or negative as $-mu_{mmm}/u_{mm}$ is greater or less than two, respectively. $-mu_{mmm}/u_{mm}$ has been termed the "coefficient of relative prudence". Kimball (1990) shows that such a measure, when applied to utility of consumption, is what determines the strength of precautionary saving; here we see that it also has an application as the determinant of precautionary money demand, and thereby of the effects of future monetary uncertainty. Intuitively, an increase in $var_t(\xi_{t+1})$ causes a "prudent" agent to try to increase its holdings of $M_t$. But the supply of $M_t$ is fixed: the higher demand has to be choked off by rises in the price of $M_t$ in terms both of goods ($1/p_t$) and of bonds ($i_t$). Viewed through (6), the convexity of j(.) causes $x_t$ to rise, which (since $u_{cm} = 0$) then raises $m_t$ or, equivalently, lowers $p_t$. A widely-used example[8] of an additively separable function is the logarithmic one:

$$u = [1-\alpha]\ln c + \alpha \ln m\,, \qquad 0 < \alpha < 1$$

Here $-mu_{mmm}/u_{mm} = 2$, its critical value: logarithmic utility yields "uncertainty irrelevance" even in the BOP model. This suggests that we should exercise care in drawing general conclusions from the many existing models which use this function.

Turning to non-separable preferences, the most plausible case would seem to be where $u_{cm} > 0$: money assists goods purchases and so raises the marginal utility of consumption. The following logarithmic/CES function provides an example of this:

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[8] See, amongst others, Gertler and Grinols (1982), Stulz (1986), Leeper (1991), Corsetti and Pesenti (1992), Grinols and Turnovsky (1993)

$$u = \ln( \{[1-\alpha]c^{\rho} + \alpha m^{\rho}\}^{1/\rho} )$$

$\rho < 0$ implies that c and m are gross complements: if we let $\rho \to -\infty$ they become perfect complements, and the function tends to $\ln( \min\{[1-\alpha]c, \alpha m\} )$, which is effectively a cash-in-advance model. $j_{xx}$ here turns out to be:

$$j_{xx} = \frac{u_m}{mu_c^2} \frac{\rho[\rho-1][1-\hat{\alpha}]}{[1-\rho\hat{\alpha}]} \quad \text{where} \quad \hat{\alpha} = \frac{\alpha}{\alpha + [1-\alpha][\alpha\beta/(1-\alpha)(1-\beta)]^{\rho/[\rho-1]}}$$

Since $\rho = 0$ reproduces the logarithmic function, we can see that $\rho < 0$, or complementarity, results in $j_{xx} > 0$ (provided $\alpha,\beta$ are not too large, ensuring $\hat{\alpha} < 1$), and thus in a similar negative effect of uncertainty on the price level to strong "relative prudence" above.

## 4. Conclusions

The choice of EOP versus BOP specification is important in discrete-time utility-of-money models. However, in continuous-time models it cannot arise, so we are naturally led to ask which specification fits this case. This can in principle be decided by comparing our results with Stulz's (1986), except that, since we know logarithmic utility is special, we must first generalise Stulz's utility function to avoid obtaining a misleading answer. When this is done the continuous-time model turns out to replicate the results of our EOP model: monetary uncertainty, appropriately measured, is irrelevant for current variables. This makes it possible to defend the EOP specification as being the better approximation to the continuous-time model. On the other hand, if we were to take the view that to model the precautionary motive for holding money is positively desirable, both the EOP specification and the continuous-time model are deficient in this light.

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