A COMPARISON OF THE PERFORMANCE OF FLEXIBLE FUNCTIONAL FORMS FOR USE IN APPLIED GENERAL EQUILIBRIUM ANALYSIS

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A COMPARISON OF THE PERFORMANCE OF FLEXIBLE FUNCTIONAL FORMS FOR USE IN APPLIED GENERAL EQUILIBRIUM ANALYSIS

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Abstract

This paper presents a procedure for testing the global properties of functional forms which recognizes their specific role in economic equilibrium modelling. This procedure is employed to investigate the regularity and the third-order curvature properties of three widely used flexible functional forms, the Translog, the Generalized Leontief and the Normalized Quadratic functional forms. We contrast the results from these flexible forms with a globally regular flexible form, the Non-separable Nested Constant-Elasticity-of-Substitution functional form. Our results indicate that inherently regular representations are best suited for equilibrium analysis.

1 Introduction

Flexible functional forms (FFFs) have been widely adopted for empirical econometric work, but there have been relatively few instances in which they have been employed to model production and consumption choices in applied general equilibrium models (Hudson and Jorgenson, 1984; Jorgenson and Slesnick, 1985; Reister and Edmonds, 1981). Few modellers have adopted FFFs for the reason that, in spite of their superior local approximations, they generally exhibit poor global properties.

Research on the behaviour of functional forms in equilibrium models has also been scant. Caves and Christensen (1980) built a number of examples to compare the regular domains of the Translog (TL; Christensen, Jorgenson and Lau, 1971) and Generalized Leontief (GL; Diewert, 1971) functional forms. Performance was found to depend on the initial specification of second order curvature conditions, with the TL being preferable when the cross Allen-Uzawa elasticities of substitution (AUES; Allen, 1938; Uzawa, 1962) are close to unity, and the GL being a good choice for cross AUESs close to zero. Reister and Edmonds (1981) analyzed the effects of replacing a Constant-Elasticity-of-Substitution (CES) specification (Uzawa, 1962) with a TL form in a simple equilibrium model and found substantial differences in their out-of-benchmark behaviour. Comparing the properties of the TL and GL functions, Despotakis (1986) concluded that the law of change of the AUES is different for these two functions, and noted that these differences can have important consequences for equilibrium analysis.

While in econometric modelling, functional forms are used to estimate the local characteristics of technologies or preference orderings from a given set of observations, in applied general equilibrium analysis functional forms are used as a global representation of technologies and preferences. Here, the information available to the modeller for the specification of technologies and preferences is typically local, i.e., limited to a small region of production or consumption sets. This local information is extrapolated to the full domain of the modelling exercise by specifying production or utility functions that are locally consistent with such information, an approach which is often referred to as "calibration" (Shoven and Whalley, 1992).

Thus, in applied general equilibrium applications the global properties of functional forms become important. Lack of global regularity, which may not be crucial for econometric estimation, may cause numerical solution methods to fail even when functions are well behaved at the equilibrium point. Furthermore, when analyzing discrete policy changes, third-order curvature properties, which are of little consequence for the purposes of econometric estimation, can crucially affect estimates of welfare impacts, both in total and at the margin.

This paper explores the global properties of four different flexible functional forms in order to assess their comparative performance and suitability for use in applied equilibrium modelling exercises. We develop a testing procedure which systematically investigates both the global regularity and the third-order curvature properties of functional forms, thus reflecting their specific role in applied general equilibrium modelling *vis a vis* econometric applications. Summary measures of global regularity are obtained by computing the area of the region in price space over which a cost function is well behaved (i.e., non-negative, monotonic, and concave in prices). We also obtain summary measures of global third-order curvature properties by computing the area of the region over which the function remains close, in a sense to be defined by the modeller, to a given local specification of curvature conditions.

The functional forms we examine are the Translog (Christensen, Jorgenson and Lau, 1971), the Generalized Leontief (Diewert, 1971), the Normalized Quadratic (NQ; Diewert and Wales, 1987), and the Nonseparable Nested CES (NNCES; Perroni and Rutherford, 1995). We choose to focus on the homogeneous, three-input case, both because it represents a good compromise between simplicity and generality and also because of its practical relevance (e.g., in the modelling of substitution possibilities among labour, capital and energy inputs).

Our tests uncover fundamental differences in the global behaviour of different functional forms and lead us to conclude that inherently regular representations are better suited for equilibrium modelling than traditional FFFs.

2 Evaluating the global properties of functional forms

Our discussion will be focused on production technologies with N inputs, one output and constant returns to scale. In most applications, a sufficient and convenient representation of such technologies is given by a continuous unit cost function, $C(p,1)$, where p denotes the input price vector and 1 is the production level. This will be hereafter referred to simply as $C(p)$ or C. The first derivatives of C, C_i, represent conditional input demands (Shephard's Lemma); these are homogeneous of degree zero in prices and, by Euler's Theorem, satisfy the adding-up condition $\Sigma_i\, p_i\, C_i = C$. The matrix of partial second derivatives $[C_{ij}]$ (the Hessian) is homogeneous of degree -1 in prices and satisfies the Cournot aggregation condition $\Sigma_j\, p_j\, C_{ij} = 0 \;\; (i = 1,..,N)$.

In the following, we will normalize prices so that price vectors lie in the unit simplex, $S^N \equiv \{p \mid p_i \geq 0;\ \Sigma_i\, p_i = 1\}$. We will also express conditional demands in terms of input value shares $\theta_i \equiv p_i\, C_i \,/\, C$, and the Hessian of the cost function in terms of Allen-Uzawa elasticities of substitution, which are defined as $\sigma^A_{ij} \equiv C_{ij}\, C \,/\, (C_i\, C_j)$. The AUES is a dimensionless index of curvature, and is thus scale-invariant. The Euler condition for the AUES matrix has the form $\Sigma_j\, \sigma^A_{ij}\, \theta_j = 0$.

Regularity

A cost function is regular (well-behaved) at a point p if its value $C(p)$ is non-negative, its first-derivatives $C_i(p)$ (which correspond to input demands) are non-negative (monotonicity), and if the Hessian $[C_{ij}(p)]$ is negative semidefinite (concavity, a sufficient condition for the choice of inputs $C_i(p)$ to minimize cost). Monotonicity implies $\theta_i(p) \geq 0$ and negative semidefiniteness of $[C_{ij}(p)]$ implies the same for $[\sigma^A_{ij}(p)]$.

We will refer to the range over which a function maintains monotonicity as the Monotonic Domain (MD). This can be characterized as

2.1 $\quad MD \equiv \{p \in S^N \mid C(p) \geq 0;\ \theta_i(p) \geq 0\}.$

The range over which a function maintains concavity will be referred to as its Concave Domain (CD):

2.2 $\quad CD \equiv \{p \in S^N \mid C(p) \geq 0;\ [\sigma^A_{ij}(p)]$ negative semidefinite$\}$.

The region of the price simplex over which a cost function is regular, which Despotakis (1986) termed the Outer Domain, is then simply

2.3 $\quad OD \equiv MD \cap CD.$

Third-order curvature properties

Despotakis (1986) defines the Inner Domain (ID) of a cost function as the region of the price simplex where the function provides a good approximation to the "true" technology. In applied modelling exercises, however, the information available to the modeller is typically limited to the calibration point p^0, and the "true" technology is unknown. In the absence of global information on the technology to be approximated, the modeller must adopt, explicitly or implicitly, certain assumptions concerning the out-of-benchmark characteristics of functions, on the basis of the local information available. For example, when choosing a CES representation, a modeller implicitly assumes that when moving away from the benchmark point the first and second derivatives of the cost function change in such a way as to ensure constancy of all cross AUESs. This corresponds to a specific set of conjectures about the third-order curvature properties of the cost function.

To make the notion of Inner Domain operational when only local information is available, we can employ a distance function Z:

2.5 $\quad Z(p) \equiv \| E(p),\ E(p^0) \|$

where $E(p)$ is a vector of curvature measures (e.g., elasticities) at p, and $E(p^0)$ represents the corresponding values at the benchmark point. The definition of E is left to the discretion of the modeller.

Once Z has been specified, the Inner Domain can be defined as the region of the unit simplex where the value of Z is less than or equal to a pre-specified tolerance level δ, i.e.,

$$2.4 \qquad \mathrm{ID}(\delta) \equiv \{p \in S^N \mid Z(p) \leq \delta\}.$$

The choice of a particular curvature index implies certain assumptions about the global characteristics of the cost function. For example, if the modeller believes that the "true" cost function exhibits constant value shares, then $E(p)$ will be chosen to represent a vector of value shares (in which case the best choice of functional form would naturally be a Cobb-Douglas cost function). In the following, we will restrict our discussion to a few second-order curvature indexes that have been proposed in the literature.

A well known dimensionless index of second-order curvature is the compensated price elasticity (CPE), which is defined as

$$2.5 \qquad \sigma^C_{ij} \equiv \partial \ln C_i \ / \ \partial \ln p_j = C_{ij} \ p_j \ / \ C_i .$$

A related measure of second-order curvature is the AUES, which has been already discussed. This can also be expressed as

$$2.6 \qquad \sigma^A_{ij} = \sigma^C_{ij} \ / \ \theta_j .$$

The AUES is a *one-input-one-price* elasticity of substitution (Mundlak, 1968), since, as (2.6) makes clear, it measures the responsiveness of the compensated demand for one input to a change in one input price. In contrast, the Morishima elasticity of substitution (MES; Morishima, 1967) constitutes a *two-input-one-price* elasticity measure, being defined as

$$2.7 \qquad \sigma^M_{ij} \equiv \partial \ln \ (C_i \ / \ C_j \ ) \ / \ \partial \ln p_j = \sigma^C_{ij} - \sigma^C_{jj} .$$

Note that, in general, the MES is not symmetric, i.e., $\sigma^M_{ij} \neq \sigma^M_{ji}$.

A third measure of curvature is the represented by the class of *two-input-two-price* elasticities of substitution, which take the form $\partial \ln \ (C_i \ / \ C_j \ ) \ / \ \partial \ln \ (p_j \ / \ p_i)$. One such index is the shadow elasticity of substitution (SES; Frenger, 1985), which is defined as

$$2.8 \qquad \sigma^S_{ij} \equiv (\theta_i \ \sigma^M_{ij} + \theta_j \ \sigma^M_{ji} \ ) \ / \ (\theta_i + \theta_j) .$$

When technologies are of the CES type, σ^A_{ij} , σ^M_{ij} , and σ^S_{ij} are all identical, but they are generally different otherwise.

In our tests, we define the distance function Z as a weighted sum of the square deviations from the benchmark elasticity of substitution matrix, where weights are chosen to be equal to the combined share of inputs i and j in total cost:

$$2.9 \qquad Z(p) = \frac{\Sigma_{i\neq j}\ [\theta_i(p^0) + \theta_j(p^0)]\ [\sigma_{ij}(p) - \sigma_{ij}(p^0)]^2}{\Sigma_{i\neq j}\ [\theta_i(p^0) + \theta_j(p^0)]\ [\sigma_{ij}(p^0)]^2}.$$

We employ four different versions of the above norm, respectively based on the CPE (σ^C), AUES (σ^A), MES (σ^M) and SES (σ^S).

3 Experimental design

We can obtain synthetic indexes, A_{MD} and A_{CD}, respectively for the Monotonic and Concave Domains, by measuring the volume of these regions as a proportion of the volume of the unit price simplex, i.e.,

$$3.1 \qquad A_{MD} \equiv \text{Area(MD)} / \text{Area}(S^N);$$

$$3.2 \qquad A_{CD} \equiv \text{Area(CD)} / \text{Area}(S^N).$$

For the Inner Domain we must also specify a tolerance level δ:

$$3.3 \qquad A_{ID}(\delta) \equiv \text{Area}[\text{ID}(\delta)] / \text{Area}(S^N).$$

Our testing procedure is as follows. The function under investigation is calibrated at a benchmark point p^0 to a given specification of derivatives up to the second order. The properties of the function (monotonicity, concavity, and $Z(\delta)$ for $\delta = 0.25$) are then systematically evaluated over a triangular grid on the price simplex. The resulting discrete mapping is then contoured to derive piecewise-linear approximations of the various domains. Finally, the approximated contour sets are used to compute the areas A_{MD}, A_{CD}, and $A_{ID}(\delta)$.

In the tests reported here we focus on the case $N = 3$ and use a grid containing 325 points. We choose p^0 to be the center of the unit simplex and consider two configurations of benchmark value shares: a symmetric configuration with θ = (1/3,1/3,1/3) and an asymmetric configuration with θ = (0.35,0.60,0.05). For each configuration of value shares, we examine a number of different benchmark configurations of second-order curvature conditions belonging to the *regular region*, i.e., the set of benchmark cross AUES configurations that are compatible with local concavity of the cost function.

Because of symmetry and homogeneity, only $H = N(N-1)/2$ elements of the matrix $[\sigma^A_{ij}]$ are independent, which implies that we can ignore the diagonal terms. Thus, the regular region Q is a subset of $\mathbf{R}^H$, bounded by $N - 1$ conditions for negative semidefiniteness of the AUES matrix. Q is a convex set, since a convex combination of two negative semidefinite matrices is also negative semidefinite. Moreover, Q is a cone, since multiplication of a negative semidefinite matrix by a positive scalar results in a negative semidefinite matrix. The latter property enables us to characterize the geometry of the regular region analyzing only the image of a projection of Q in $\mathbf{R}^{H-1}$. For this purpose we choose the following projection: the AUES matrix is divided by its largest positive off-diagonal element, so that the maximum off-diagonal element of the resulting matrix is always unity. Without loss of generality, we will assume that the element (1,2) is the largest cross AUES.

With $N = 3$ the regular region Q lies in $\mathbf{R}^3$, and the image of its projection lies in $\mathbf{R}^2$, bounded by the following constraints ($[\sigma^A_{ij}]$ denotes the normalized AUES matrix):

3.4 $\quad \sigma^A_{13} \leq 1$;

3.5 $\quad \sigma^A_{23} \leq 1$;

3.6 $\quad \sigma^A_{13} \geq -\theta_2 / \theta_3$;

3.7 $\quad \sigma^A_{23} \geq -\theta_1 / \theta_3$;

3.8 $\quad (\theta_3 / \theta_1)\, \sigma^A_{13} + (\theta_2 / \theta_1) \geq 0$;

3.9 $\quad \sigma^A_{13}\, \sigma^A_{23} + (\theta_1 / \theta_3)\, \sigma^A_{13} + (\theta_2 / \theta_3)\, \sigma^A_{23} \geq 0$.

The first two constraints follow from normalization. The remaining four are the sign constraints on the first and second principal minors for negative semidefiniteness.

We explore four "slices" through Q, those representing AUES matrices with maximum off-diagonal values respectively equal to 1/2, 1, 2 and 4. For each of these sections we examine a uniform grid of AUES configurations with between 47 and 52 points depending on the benchmark value shares. The resulting measures A_{MD}, A_{CD}, and $A_{ID}(\delta)$ are then averaged over all sample points.

Our testing procedure was applied to four different functional forms: TL, GL, NQ and NNCES. The TL and GL forms have been chosen because they are the best known among FFFs. The NQ form is included because it is globally concave (although it can lose monotonicity). The NNCES form is flexible and globally regular, and belongs to a family of functional forms that have been widely employed in the applied general equilibrium literature. All four functional forms, and the formulae used for parameter calibration, are described in Appendix A.

4. <u>Test results</u>

Results of Outer Domain calculations are summarized in Table 1, which reports average measures of MD, CD, and OD as a percentage of the price simplex for different specifications of maximum cross AUES values.

[Table 1 about here]

Our findings are consistent with earlier studies of the global properties of the TL and GL functional forms. The TL is prone to loss of concavity away from the benchmark point whenever the benchmark elasticities depart from unity. The GL tends to lose monotonicity as benchmark elasticities increase. The NQ remains concave over the entire domain, but tends to lose monotonicity at higher elasticities, much as the GL does. The Outer Domain for the NQ (as well as the TL and GL) falls below 50% of the price simplex when $\sigma^A_{12} = 4$ and benchmark value shares are equal. When benchmark shares are asymmetric and $\sigma^A_{12} = 4$, the TL, GL and NQ Outer Domain shrinks to roughly 1/4 of the price simplex. In contrast, the NNCES form is globally regular.

Table 2 reports corresponding results of Inner Domain calculations based respectively on the CPE, AUES, MES, and SES norms. The CPE based norm appears to be more volatile than the other measures, but the ranking of functional forms is consistent across all norms. The NQ performs poorly in all cases. In the symmetric case (upper panel), the TL form performs best for benchmark cross AUES values close to unity, and the GL form performs best for cross AUES values close to zero. In the asymmetric cases the TL performs rather poorly, particularly for low cross AUES values. The NNCES is the most consistent performer.

[Table 2 about here]

Comparison of Tables 1 and 2 suggests that the Outer Domain and the Inner Domain of a cost function might be correlated. We formally investigated this conjecture by computing the correlation coefficient between the areas of the Outer Domain and the AUES Inner Domain over the range of benchmark elasticity values (Table 3). These results confirm that "instability" in second-order curvature behaviour is closely associated with loss of regularity.

[Table 3 about here]

5 Summary and conclusion

This paper has presented a procedure for testing the global properties of functional forms which explicitly recognizes their role in equilibrium modelling. We have used this procedure to explore the regularity and third-order curvature properties of four flexible functional forms, and found that the Translog, Generalized Leontief and Normalized Quadratic all can lose regularity over large regions of the price simplex, particularly when the benchmark cross-elasticities are large. We also found that globally regular functional forms, like the NNCES, are better at preserving local calibration information over the domain of modelling exercises. For these reasons, we conclude that globally regular functional forms such as the NNCES are better suited for equilibrium analysis.

Further research should investigate the global properties of alternative specifications of the NNCES, in order to provide practitioners with some concrete guidance in the selection of nesting structure. A better understanding of the properties of the functional forms used in applied equilibrium exercises would improve transparency and ultimately contribute to users' understanding of model results.

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Table 1: Average measured Outer Domain dimensions

% of the price simplex

Symmetric Value Shares: θ=(0.33,0.33,0.33)

	Monotonic Domain				Concave Domain				Outer Domain			
σ^A_{12} (max)	.5	1	2	4	.5	1	2	4	.5	1	2	4
TL	71	82	81	32	27	62	73	60	25	61	65	27
GL	100	99	69	30	93	93	93	93	93	92	63	27
NQ	100	100	73	33	100	100	100	100	100	100	72	31
NNCES	100	100	100	100	100	100	100	100	100	100	100	100

Asymmetric Value Shares: θ=(0.35,0.60,0.05)

	Monotonic Domain				Concave Domain				Outer Domain			
σ^A_{12} (max)	.5	1	2	4	.5	1	2	4	.5	1	2	4
TL	69	86	72	28	46	71	76	64	41	70	61	25
GL	100	98	61	25	95	95	94	92	94	92	57	24
NQ	100	99	67	27	100	100	100	100	100	99	66	26
NNCES	100	100	100	100	100	100	100	100	100	100	100	100

Table 2: Average measured Inner Domain dimensions

% of the price simplex

Symmetric Value Shares: $\theta=(0.33,0.33,0.33)$

	Compensated				AUES				Morishima				Shadow			
σ^A_{12} (max)	.5	1	2	4	.5	1	2	4	.5	1	2	4	.5	1	2	4
TL	3	43	20	5	6	62	35	8	12	61	50	12	13	58	47	12
GL	25	40	17	4	67	59	22	6	81	75	30	8	83	76	29	8
NQ	4	4	3	1	11	9	6	3	21	12	6	2	20	14	6	2
NNCES	14	40	13	3	67	71	59	41	71	67	61	52	70	67	59	47

Asymmetric Value Shares: $\theta=(0.35,0.60,0.05)$

	Compensated				AUES				Morishima				Shadow			
σ^A_{12} (max)	.5	1	2	4	.5	1	2	4	.5	1	2	4	.5	1	2	4
TL	7	63	18	4	9	59	36	9	16	66	44	12	12	57	43	12
GL	42	63	17	4	84	61	27	7	93	70	33	9	91	68	32	9
NQ	2	2	2	1	1	1	1	1	6	5	4	2	21	15	9	3
NNCES	35	78	20	4	66	74	68	50	92	90	83	73	88	87	83	66

Table 3: Correlation between the AUES Inner and Outer Domains

	Symmetric Shares $\theta=(0.33,0.33,0.33)$				Asymmetric Shares $\theta=(0.35,0.60,0.05)$			
σ^A_{12} (max)	.5	1	2	4	.5	1	2	4
TL	0.92	0.92	0.84	0.90	0.96	0.97	0.93	0.77
GL	0.51	0.67	0.79	0.91	0.77	0.72	0.79	0.82
NQ	na	na	0.59	0.68	na	na	0.29	0.34

na: the NQ is globally regular for these shares and assumed σ^A values, and the correlation is therefore undefined.

APPENDIX A: Cost function definitions and calibration

In this appendix we summarize formulae for recovering parameters of the functional forms we tested. Here, the cost function is denoted as F, and the symbols C, θ and σ denote the given cost, input shares and Allen-Uzawa elasticities of substitution at the calibration point.

Translog

The TL form is defined as follows:

A.1 $\ln F(p) \equiv \ln b_0 + \Sigma_i b_i \ln p_i + 1/2 \Sigma_{ij} a_{ij} \ln p_i \ln p_j$

$\equiv \ln b_0 + L(p)$.

Restrictions:

A.2 $\Sigma_i b_i = 1$;

A.3 $a_{ij} = a_{ji}$, $i = 1,..,N$, $j = 1,..,N$;

A.4 $\Sigma_j a_{ij} = 0$, $i = 1,..,N$.

Calibration:

A.5 $a_{ij} = \theta_i \theta_j (\sigma_{ij} - 1)$, $i \neq j$;

A.6 $a_{ii} = -\Sigma_{j\neq i} a_{ij}$, $i = 1,..,N$;

A.7 $b_i = \theta_i - \Sigma_j a_{ij} \ln p_j$, $i = 1,..,N$;

A.8 $b_0 = C e^{-L(p)}$.

Generalized Leontief

The GL form is defined as follows:

A.9 $F(p) \equiv 1/2 \Sigma_{ij} a_{ij} (p_i p_j)^{1/2}$.

Restrictions:

A.10 $a_{ij} = a_{ji}$, $i = 1,..,N$, $j = 1,..,N$.

Calibration:

A.11 $a_{ij} = 4 \theta_i \theta_j C (p_i p_j)^{-1/2} \sigma_{ij}$, $i \neq j$;

A.12 $a_{ii} = [\theta_i C p_i - \Sigma_{j\neq i} a_{ij} (p_i p_j)^{1/2}] / p_i$, $i = 1,..,N$.

Normalized Quadratic

The NQ form is defined as follow:

A.13 $F(p) \equiv 1/2 (\Sigma_{ij} a_{ij} p_i p_j) / (\Sigma_i b_i p_i)$.

Restrictions:

A.14 $a_{ij} = a_{ji}$, $i = 1,..,N$, $j = 1,..,N$;

A.15 $b_i \geq 0$, $i = 1,..,N$;

A.16 $\Sigma_i b_i = 1$.

Calibration:

A.17 $a_{ij} = C [(\Sigma_k b_k p_k) \theta_i \theta_j \sigma_{ij} + b_i p_i \theta_j + b_j p_j \theta_i] / (p_i p_j)$.

We examined two alternative specifications, one in which $b_i = \theta_i$, and another in which $b_i = 1/N$. The first specification produces larger Inner Domain estimates and is used in the tests for Table 2.

Nonseparable Nested CES

We restrict our discussion to the case $N = 3$ (for the general N-input case see Perroni and Rutherford (1995)), and focus on a particular nesting structure, which we call "Lower Triangular Leontief" (LTL). Let us rearrange indices so that the maximum off-diagonal AUES element is σ_{12}. Then the three-input NNCES-LTL cost function can be described as

$$\text{A.18} \quad F(p) \equiv \varphi\ [(a_1\ p_1 + a_3\ p_3)^{(1-\gamma)} + (b_2\ p_2^{(1-\mu)} + b_3\ p_3^{(1-\mu)})^{(1-\gamma)/(1-\mu)}]^{1/(1-\gamma)} .$$

Restrictions:

$$\text{A.19} \quad \gamma \geq 0 ;$$

$$\text{A.20} \quad \mu \geq 0 ;$$

$$\text{A.21} \quad \varphi \geq 0 ;$$

$$\text{A.22} \quad a_i \geq 0 , \qquad i = 1,..,N ;$$

$$\text{A.23} \quad b_i \geq 0 , \qquad i = 1,..,N .$$

Calibration:

Let us denote with s_3 the fraction of the total input of commodity 3 which enters the first subnest of the structure described by (A.18) (with $1-s_3$ representing the fraction entering the second subnest). Let us also assume that all prices at the calibration point are unity. If we select

$$\text{A.24} \quad \gamma = \sigma_{12} ;$$

$$\text{A.25} \quad \mu = (\sigma_{12}\sigma_{13} - \sigma_{23}\sigma_{11}) / (\sigma_{13} - \sigma_{11}) ;$$

it can be shown that

$$\text{A.26} \quad s_3 = (\sigma_{12} - \sigma_{13}) / (\sigma_{12} - \sigma_{11}) .$$

The remaining parameters can then be recovered as follows:

$$\text{A.27} \quad \varphi = C ;$$

$$\text{A.28} \quad a_1 = \theta_1\ (\theta_1 + s_3\, \theta_3)^{\gamma/(1-\gamma)} ;$$

$$\text{A.29} \quad a_3 = s_3\, \theta_3\ (\theta_1 + s_3\, \theta_3)^{\gamma/(1-\gamma)} ;$$

$$\text{A.30} \quad b_2 = \theta_2\ [\theta_2 + (1 - s_3)\theta_3]^{(\gamma-\mu)/(1-\gamma)} ;$$

$$\text{A.31} \quad b_3 = (1 - s_3)\ \theta_3\ [\theta_2 + (1 - s_3)\ \theta_3]^{(\gamma-\mu)/(1-\gamma)} .$$