# DEBT RELIEF

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I develop a model of secondary market pricing of sovereign debt when the creditors can reduce the debt. The sovereign obtains a stochastic revenue flow from the external sector and have a constant debt flow obligation. Default is costly for both the sovereign and the creditors and the possibility to reduce debt creates a surplus. The creditors can capture this surplus only if they can continuously adjust the debt flow. With discrete adjustments, they have to share with the sovereign. I find that the more volatile the sovereign's revenue flow (ie, the underlying process is favorable for the sovereign), the bigger and later will be the debt relief. A country going through multiple debt reductions will get bigger reductions in later rounds, and the secondary market price of its debt will have bigger jumps at later reductions. Previous empirical work on sovereign debt provides empirical support for my model.

*/ I would like to thank my former thesis advisor Avinash Dixit for introducing me to the "art of smooth-pasting," and Marcus Miller and Jonathan Thomas for refreshing discussions. Discussions with Pierre Mella-Barral have been very useful in developing Section II.

## Introduction

The success of the Brady debt reduction plan for Mexico convinced the community of international organizations, commercial banks and debtor countries of the merits of debt reduction programs. By 1993, about half of all commercial bank debt to developing countries had been definitively restructured according to the Brady Plan approach.[1] As the commercial banks and international organizations perfect their techniques, we will more frequently observe debt reductions. There is however no theory about how the secondary market price of sovereign debt will change upon a debt relief, the timing of debt reductions, and the dynamics of prices as a country goes through a series of debt reductions. In this paper I present a theory of the dynamics of secondary market price of sovereign debt when creditors can offer debt relief and when such relief is anticipated by all market participants. I do not however touch on the problems of free-riding, coordination and heterogeneity among the creditors.

The basic assumptions of my model are as follows:

- Rather than the stock of debt and the interest payments on it, I consider the flow debt obligation of the sovereign. I assume that sovereign can smooth its debt obligations (thus I ignore debt reschedulings in response to temporary current account difficulties). I assume that the flow debt obligation remains constant unless creditors decide to reduce it.
- The sovereign cannot contract new debt.
- The sovereign raises revenue from the operations of the external sector. There is no problem of inability to service the debt. If this revenue falls short of its current flow debt, the sovereign can make up the shortfall from other sources of revenue. Since all transitory shocks to the cash flows can be smoothed out, I assume that the revenue from the external sector follows a geometric Brownian motion (ie, all shocks to the growth rate are permanent):

[1] Economist, September 1993. Other major cases include Venezuela, Uruguay, the Phillipines, Costa Rica, Nigeria, Brazil, and Argentina.

$$d X = \mu X dt + \sigma X d\omega \tag{1}$$

where $X$ is the instantaneous revenue, $\mu$ is the deterministic time trend, $d\omega$ is the increment of Wiener process (normally distributed with mean zero and variance $dt$), and $\sigma$ is the volatility parameter.

- When the sovereign defaults on its debt obligations or declare a moratorium, the country faces sanctions. I assume that upon default, all export-import operations cease, the country reverts to autarky, and, of course, the government can no longer raise revenue from the external sector. However, the creditors cannot seize and run the external sector. They can only seize whatever collateral they may have which I normalize to zero.
- Creditors can irreversibly reduce the flow debt obligation of the sovereign. The creditors, when need be, use this option to persuade the sovereign not to default.

The above assumptions satisfy the criteria Bulow and Rogoff (1989) set forth to distinguish sovereign lending from corporate lending: 1. ability to pay is never truly an issue; 2. collateral is of negligible value; 3. contracts are constantly subject to renegotiation. In my model 1. there is no limitation on the sovereign's ability to continue to make the debt payments; 2. there is no collateral; 3. flow debt obligation is reduced whenever both parties benefit from doing so.

In my model debt reductions prevent the sovereign from reverting to autarky. This is the same principle underlying the so called Debt Laffer curve. Krugman (1992) argues that reducing the debt provides incentives to the debtor country to adopt sound economic policies. Consequently, both the debtor country and the creditors may benefit from a debt reduction. If the total market value of a country's debt increases with a reduction in the face value of its debt, the country is said to be on the "right" side of its Debt Laffer curve. There is mounting evidence that the Debt Laffer curves do exist and debt reductions may indeed be warranted. Claessans (1990) found that most of the highly indebted and sub-Saharan countries are on the wrong side of the Debt Laffer curve. This is further confirmed by Lagae (1991) who found that 11 out of 15 most highly indebted countries are on the right side of their Debt Laffer curve. Yasayuki (1994) does not only confirm these findings for East and Southeast Asian countries, but he also finds that all countries on the right side of their Debt Laffer curve are also solvent. Thus debt reductions do not require the insolvency of the debtor.

My model prices the current debt obligations "on the debt-Laffer curve" by discounting for optimally timed future debt reductions. It can be enriched by considering policy options that may somehow improve the stochastic process governing the revenue from the external sector. Elsewhere I have made initial attempts to model real restructuring as a switching of stochastic processes (Hayri (1994)). Combining them awaits further work.

Three other models of sovereign debt use continuous-time stochastic processes to model the variability in the country's ability to pay: Claessens and van Wijnbergen (1993), Cohen (1993) and Bartolini and Dixit (1991). None of them deals with the debt reduction issue and they have different interpretations of $X$. They all assume that $X$ is the maximum the country can pay to the creditors. They differ in their assumptions about what happens when $X$ falls short of the debt obligations: Claessens and van Wijnbergen assume that any arrears are forgiven (so they are able to use straightforward option pricing formulae), Cohen assumes that $X$ is handed over to the creditors at all times, and Bartolini and Dixit assume that the arrears are rolled over. Thus in all three models, the debtors have no decisions to take. They all disregard the possibility that the country may refuse to pay its debt and face the consequences. In my model, $X$ is the net revenue the government gets from the external sector. The government has a choice between keeping up with the interest payments (no arrears are allowed) and default. When $X$ is below the contractual debt obligation, the government must resort to extraordinary measures to make up for the shortfall. The government can default when it is economically advantageous to do so. In the case of default, the country irreversibly loses its external sector and the creditors cannot run the external sector of the country. The creditors, expecting this bleak outcome, reduce the debt to prevent default. As a result, in my model default never occurs. Thus my model combines the previous three models with the models of strategic default by Bulow and Rogoff (1989) and Calvo and Kaminsky (1991).

I use techniques from so-called **real options** literature. A recent book by Dixit and Pindyck (1994) presents the main findings of this literature. The sovereign's problem here is similar to a firm's exit decision in McDonald and Siegel (1985) where output price is random and the firm can temporarily suspend its operations. Dixit (1989) calculates the value of a firm that can stop and restart production by incurring a fixed cost. His model is used to explain why firms may keep on operating while they are making losses. The sovereign in my model is akin to an owner injecting new cash into his firm to keep up with the debt payments. As my model is one of debt valuation, I do not allow temporary shutdowns. Instead I allow the creditors to reduce the debt flow, but assume that shutdown (default and subsequent return to autarky) is irreversible. A recent paper, Mella-Barral (1996), uses the same techniques to model the dynamics of corporate debt allowing for continuous debt reductions motivated by the avoidance of bankruptcy costs and adjusting the capital structure to exploit tax advantage of debt. Below I explain why the assumption of continuous debt reduction in the context of sovereign debt is not suitable and how such a model fails to deliver testable predictions.

*Outline of the paper*

Section I derives the value of non-renegotiable debt by using the techniques from Dixit (1992) and illustrates the existence of the Debt Laffer curve. Section II introduces continuous debt reduction and shows that the creditors can capture all ex-ante surplus with irreversible, and both ex-ante and ex-post surplus with temporary reductions. This section also shows the limitations of the assumption of continuous debt reduction. And I go on to develop a model with discrete reductions. Section III finds the optimal timing and Section IV the optimal size of a one-time debt reduction. The timing and size are then related to the properties of the underlying stochastic process. Section IV also discusses the issues of time-consistency and costs of debt reductions. Section V generalizes the model to n-reductions and presents the predictions of the model regarding the dynamics of the secondary market price of debt. Section VIII reviews the findings of the empirical studies on sovereign debt relief and finds support for the predictions of my model. I then recap my results and conclude.

## I. Valuation of non-renegotiable debt

Let me recap the setup discussed in the introduction: We will consider a sovereign who receives a revenue flow $X$ (which is assumed to follow a geometic Brownian motion with trend $\mu$ and volatility $\sigma$) from the external sector and has to service his external debt at a constant rate $D$. His creditors do not hold any collateral. If the sovereign stops servicing his debt, the country will lose its foreign trade privileges and the revenue flow from the external sector will stop.

In this section, I look at an extereme case where the debt cannot be renegotiated. In this case the sovereign faces an optimal stopping time problem: when to stop servicing the debt? He keeps up with the debt payments as long as the revenue flow is above a certain threshold S. Once the revenue goes below this threshold, the sovereign effectively abondons the external sector. The solution to this problem is well known and can be found in Dixit 1992. Below I restate the solution.

I assume that both the sovereign and the creditors discount the cash flows by $r$. Let $V(X; D)$ and $V_d(X; D)$ denote the value of sovereign's equity (ie, the expected present value of the revenue flow net of debt payments) and the value of debt as a function of the current revenue flow for a given level of constant debt flow $D$. The value of the equity has the following functional form (see Dixit 1992):

$$V(X;D) = \frac{X}{r-\mu} - \frac{D}{r} + A\,X^{-\alpha}\,. \tag{2}$$

where

$$\alpha = -\left(\tfrac{1}{2} - \mu/\sigma^2\right) + \sqrt{\left(\tfrac{1}{2} - \mu/\sigma^2\right)^2 + 2r/\sigma^2} \tag{3}$$

$r$ is the interest rate, $\mu$ and $\sigma$ are the trend and volatility of $X$, and $A$ is a constant. When $X$ is below $D$ the sovereign must either make up the shortfall from other sources or default. When to default (ie, abondon the external sector) is an optimal stopping time problem. Let $S$ be the default trigger (ie, the threshold for $X$ below which the sovereign defaults. The constant $A$ of the value function is determined by the boundary condition (ie, value matching condition) that upon default (at *S*) the value of equity is zero. :

$$V(S;D) \;=\; \frac{S}{r-\mu} \;-\; \frac{D}{r} \;+\; A\,S^{-\alpha} \;=\; 0\,. \qquad \text{(Value Matching)}$$

The sovereign chooses trigger $S$ to maximize the value of equity. Therefore the derivative of the value function must vanish at $S$. (This is the smooth-pasting condition. See Dixit 1992 for further details.)

$$V'(S;D) \;=\; \frac{1}{r-\mu} - \alpha A\,S^{-\alpha-1} \;=\; 0 \qquad \text{(Smooth Pasting)}$$

Solutions are

$$S \;=\; \frac{D}{r}\,\frac{(r-\mu)\,\alpha}{1+\alpha}\;, \qquad A \;=\; \frac{S^{1+\alpha}}{\alpha\,(r-\mu)} \;=\; a\,D^{1+\alpha} \quad \text{where} \quad a \;\equiv\; \frac{\big(\alpha\,(r-\mu)\big)^{\alpha}}{\big(r\,(1+\alpha)\big)^{1+\alpha}}. \qquad \textbf{(4)}$$

The value of debt (as a function of *X*) has a similar functional form:

$$V_d(X;D) \;=\; \frac{D}{r} \;-\; B\,X^{-\alpha}\;. \qquad \textbf{(5)}$$

Its constant, $B$, is determined by using the boundary condition that at the default trigger the value of debt is zero:

$$V_d(S;D) \;=\; \frac{D}{r} - B\,S^{-\alpha} \;=\; 0 \quad\Rightarrow\quad B \;=\; \frac{D\,S^{\alpha}}{r}\;. \qquad \textbf{(6)}$$

As the debt holders have nothing to do with the decision to default, the smooth pasting condition is not applicable here. Therefore, the value of debt at $X$ is $D/r(1 - S^{\alpha}\; X^{-\alpha})$.

As the external sector always generates positive net revenue, the abondonment is inefficient. Thus both parties would be better off the debt can be reduced and the sovereign postpones the abondonment (ie, uses a lower threshold). Adding up the value of debt[2] and equity, we obtain $X/(r\text{-}\mu)$ - $\alpha A\ X^{-\alpha}$. The second term is the net loss of both parties from their inability to renegotitate the debt.

The country is on the right side of the Debt Laffer curve, ie, reducing the debt flow increases its expected present value, only if[3]

$$\frac{X}{D} < \left[\frac{r-\mu}{r}\right] \frac{\alpha}{1+\alpha} (1+\alpha)^{\frac{1}{\alpha}} \tag{7}$$

The revenue to debt ratio ($X/D$) is similar to the export-debt service ration used to judge the relative liquidity position of the country. How low $X/D$ must be for a profiable debt reduction depends on how volatile the underlying process is. Considering moderately volatile processes ($\sigma^2 > 0.01$), we can show that the higher the volatility, the lower is the critical value of $X/D$ for a profitable debt reduction. This claim is incomplete because in calculating the value of debt we have not incorporated the expectation of a debt reduction. To do so, we should first model the process of debt reduction. The first alternative I consider in the next section is to assume continuous and costless debt reduction. However, as shown in the next section, that modelling strategy does not prove to be very successful and in the subsequent sections I first consider one-time discrete and then $n$-times discrete debt reductions.

## II. Continuous Debt Reduction

In this section I examine the value of debt (hence its secondary market price) under the assumption that the creditors can continuously and costlessly reduce the debt flow. Although this assumption

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2 In what follows we use $A_d = (1+\alpha)A$ which can be obatined from Equation 6.

3 To see where this condition comes from, note that

$$\frac{dV_d}{dD} = \frac{D}{r}(1 - S^{\alpha} X^{-\alpha}) - \frac{D}{r} X^{-\alpha} \alpha S^{\alpha-1} \frac{dS}{dD} = \frac{D}{r}(1 - (1+\alpha) S^{\alpha} X^{-\alpha})$$

allows us to obtain simple, closed-form solutions for the market price of debt, it does not provide any insights about the size and timing of debt reductions.

The creditors, when allowed to do so, may irreversibly reduce the debt service of the sovereign everytime its revenue flow hits a new low. The debt level will then be a function of $\check{X}$, where $\check{X}_t \equiv \inf\{X_s, 0 \leq s \leq t\}$. We modify the value functions to make the constants $A$ and $A_d$ functions of current debt flow $D_t$ ($D_0$ is the initial debt flow). Let $\check{X}(D)$ denote the level of $\check{X}$ at which debt is reduced to $D$. We call $\check{X}(D)$, for $D \in [0, D_0]$, debt reduction triggers. The following boundary conditions completely characterize the value functions:

- The value of equity must be zero at the time of each debt reduction (ie, for all $D$, $V(\check{X}(D), D) = 0$. If it were positive, the creditors could have further delayed the debt relief without risking default. If it were negative, the sovereign would have defaulted earlier.
- As such, the creditors' problem of maximizing the value of debt is equivalent to minimizing the value of equity. Thus the smooth-pasting condition must apply to the value of equity at every debt reduction trigger.
- The marginal value of debt is zero at every debt reduction trigger.

The first boundary condition is the main difference between the discrete and continuous approaches. I will discuss why it does not apply to the case of discrete reductions in the next section. With the possibility of continuous debt reduction, the creditors do not have to provide any incentives to the sovereign except when he is just about to default. And then, they only reduce the debt just enough to keep the sovereign indifferent between defaulting and keeping up with the payments, and thus prevent default. As explained in the next section, with discrete reductions, the creditors cannot do the same and they have to share the benefits of debt reduction with the sovereign.

Applying these boundary conditions to the now familiar value functions, we find the level of debt when $X$ hits a new low $\check{X}$, ie, the inverse of $\check{X}(D)$ (see Appendix for the derivation):

$$D(\check{X}) = \left[ \frac{r(1+\alpha)}{\alpha(r-\mu)} \right] \check{X} .$$

The value of debt at a debt reduction trigger is given by the following function (see Appendix):

$$V_d(\check{X}, D(\check{X})) = \left[1 - \frac{1}{(1-\alpha)}\right] \frac{D(\check{X})}{r} .$$

Debt is reduced to maintain a certain proportionality between debt and revenue flows. As at each debt reduction trigger the value of equity is zero, the value of debt equal to $\check{X}/(r - \mu)$. Market value of debt is also a constant proportion ($\alpha/(1+\alpha)$) of its face value.[4] Following a debt reduction, if the revenue goes up, so does the price of debt. If it goes down, there will be another debt reduction and the price of debt will be maintained. Hence $\alpha/(1-\alpha)$ is the lower bound of the secondary market price of sovereign debt. With continuous debt reduction, we do not observe jumps in the price at the time of debt reductions (because they take place continuously).

This model can easily be modified to accomodate reversible (ie, temporary) debt reductions: the creditors will keep the value of equity at zero while they are providing debt relief (ie, set the debt flow below its original level $D_0$.[5] As the debt relief does not contribute anything to the value of equity, the sovereign will be indifferent between default and carrying on at the default trigger $S$. The creditors will therefore provide debt relief only when $X < S$ and, to keep value of equity at zero, they will set the debt flow equal to $X$ (ie, confiscate the revenue flow). When $X$ exceeds $S$, they will demand the full amount (so there will be a jump in the debt obligation when $X = S$). So, there will be two states: when the revenue flow is above the default trigger, the debt flow will be fixed at $D$. When it falls below the default trigger, the creditors will confiscate the revenue flow.

This model is useless in assessing the impact of real life discrete debt reductions on price. These results do not provide us any insights about the size and timing of consecutive of debt reductions. As a first step to a general discrete debt reductions model, I start with a one-time, fixed size debt reduction, then allow variable size and, finally, generalize to multiple reductions.

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4 I take $D/r$ as the face value (ie, the value without the risk of default).

5 If the the value of equity is positive, they may raise the flow debt to boost the value of debt.

## III. Optimal timing of a fixed-size, one-time debt reduction

I now allow the creditors only one chance to reduce the debt. I start the derivation by assuming that the creditors will reduce debt by a fixed percentage $\varepsilon$. Let $D_0$ denote the initial debt flow and $D_1 = (1\text{-}\varepsilon)D_0$. Similarly $S_0$ and $S_1$ denote the associated default triggers as defined in Equation 4. In Section I, we have seen that the higher the debt obligation, the earlier the sovereign defaults and higher is the welfare loss due to non-renegotiability of debt. Consequently, there exists a Debt Laffer curve. When $X/D_0$ is below a certain level (given by Equation 7), the creditors may increase the value of debt by reducing the debt flow to $D_1$ (see Figure 1).

[ Figure 1 ]

As Figure 1 suggests for low enough values of $X$, the creditors are better off with a reduced debt flow. The reduction cannot simply take place at the point of intersection because the pre-reduction valuation of debt must account for the possibility of debt reduction. Let $V_0(X)$ and $V_1(X)$ denote the pre- and post-reduction value functions for equity (with associated constants $A_0$ and $A_1$). I will first derive the optimal timing of the debt reduction and then apply the value-matching condition to find the pre-reduction value function.

After the creditors exercise their option to reduce the debt by $\varepsilon$ percent, there cannot be any further reductions. Therefore the post-reduction value function is identical to the value function for non-renegotiable debt derived in Section I:

$$A_1 = A((1-\varepsilon)D_0) = a\,(1-\varepsilon)^{1+\alpha}\,D_0^{1+\alpha}\,. \tag{10}$$

The only benefit to the creditors of a debt reduction is to delay sovereign's default. Once the sovereign expects the debt reduction, he will not default at $S_0$. Knowing this, the creditors will reduce the debt as late as possible (ie, at the lowest possible $X$ value). But, of course, the sovereign will not wait for a debt relief that comes too late. Thus the creditors will reduce the debt at the latest time (at the lowest value of $X$) the sovereign may tolerate. Let $H$ be the debt reduction trigger so that the creditors will reduce debt when $X$ hits $H$. At the time of the debt reduction, the value of the equity will be $V_1(H) = D_1/r - A_1 H^{-\alpha}$. Without the debt reduction, the equity would be worthless at $H$. We want to know how long the sovereign will keep up with the debt payments to get this extra value $V_1(H)$. If he defaults prior to the debt reduction, he will forego $V_1(H)$. Therefore, the question is how long will he keep up with the debt payments when the value of equity is enhanced by $V_1(H)$? We can answer this question using the same setup as the Section I with simply adding $V_1(H)$ as a constant to the value function. The default trigger thus found, denoted by $\tilde{S}$, gives us the answer: the sovereign will wait for the debt reduction if it is expected to happen before $X$ hits $\tilde{S}$. If $\tilde{S}$ is less than $H$, the sovereign will not wait for the debt reduction scheduled to happen when $X$ hits $H$.[6] As the creditors will hold back the debt reduction as late as possible, the optimal debt reduction trigger is equal to $\tilde{S}$. The following equation gives us the optimal debt reduction trigger (see Appendix for the derivation):

$$A_1 H^{-\alpha} + c H - \frac{(2-\varepsilon) D_0}{r} = 0 . \tag{11}$$

where

$$c \equiv \frac{1 + 2\alpha}{\alpha (r - \mu)} . \tag{12}$$

One may argue that at $H$ the value of the equity without debt reduction must be negative and thus the actual increase in value is higher than $V_1(H)$. But, how could we determine the value of equity at $H$ without debt reduction? We know that it cannot be positive. However, it cannot be negative either: the value refers to the future. If it is negative, the sovereign have no incentive or obligation to continue with the debt payments. Thus the only correct answer is zero.

Is there a time-inconsistency problem here? To answer this question consider what would happen when the creditors do not reduce the debt at $H$. The most optimistic conjecture the sovereign may hold is that the debt reduction will take place when $X$ hits a lower trigger, say, $G < H$. Having waited till now, will the sovereign keep on waiting? The answer is simply no. The optimal stopping

[6] Note that $\tilde{S}$ is always greater than $S_1$ because the value function used in deriving $\tilde{S}$ is always greater than $V_1(X)$.

time problem does not depend on the current value of the process. The value of a debt reduction at $G$ is lower than $V_1(H)$ and thus at $H$, the sovereign defaults. Similarly, the creditors do not gain anything by refusing to undertake the debt reduction at $H$: refusing debt reduction will mean losing everyting.

Now let us discuss why the first boundary condition we used with continuous debt reduction does not apply here. That boundary condition states that the value of equity at the time of the debt reduction must be zero. If we apply it here, the post-reduction value of equity, by the value matching condition, must also be zero. As the post-reduction value is not affected by the timing of the debt reduction, it is zero only at $S_1$. Therefore, the first boundary condition implies that the debt reduction must be undertaken at the default trigger $S_1$. We have just shown that $H$ is greater than $S_1$ and the sovereign will not wait for a debt reduction offered at $S_1$. The first boundary condition applies only in the case of continuous debt reduction. In the case of discrete debt reductions, the creditors must share the surplus from the debt reduction with the sovereign.

By using $H$ we can calculate the original pre-reduction value functions for equity and debt. To find the pre-reduction value of equity (ie, the undetermined constant $A_0$), we use only the V-M condition at $H$. The S-P condition is not applicable here because the creditors, not the sovereign, makes the timing decision:

$$V_0(H) = V_1(H) \qquad \Rightarrow \qquad \frac{H}{r-\mu} - \frac{D}{r} + A_0 H^{-\alpha} = \frac{H}{r-\mu} - \frac{(1-\varepsilon)D}{r} + A_1 H^{-\alpha} . \tag{13}$$

Solving Equation 13 we get the following expression for $A_0$:

$$A_0 = A_1 + \frac{\varepsilon D}{r} H^{\alpha} \tag{14}$$

The total value (equity plus debt) depends on only the threshold where the external sector is abondoned (hence on $D_1$). The timing of the debt reduction affects only the relative values of debt and equity. Using the same technique in Section I[7] we can write the post-reduction value of debt as

$$V_{d,0}(X) \;=\; \frac{D_0}{r} \;-\; (\,A_0 + \alpha A_1\,)\,X^{-\alpha}\;. \tag{15}$$

$H$ is decreasing in the volatility and trend of the stochastic process. That is, if the process is more favorable to the sovereign, debt relief will come later. If the size of the reduction is smaller or the total debt is higher, the debt reduction will take place earlier (in the sense that the trigger will be lower). In this section we have shown that the optimal time of the debt reduction is the lates time the sovereign will tolerate and this timing decision is time-consistent. Unlike the case of continuous debt reduction, the creditors cannot appropriate all the surplus from the debt reduction. In the next section, we turn to the issue of optimal size of debt reductions.

## IV. Optimal size of a one-time debt reduction

I now look at the determination of the size of the debt reduction. In Section III, we have seen that given $\varepsilon$, the timing is free of time-consistency problem, but the determination of $\varepsilon$ itself is not. Take a debt reduction strategy of reducing the debt by $\varepsilon = ½$ so that $D_1 = ½D_0$. Let $H$ be the associated optimal reduction trigger. Note that $H > S_1$, where $S_1$ is the default trigger associated with debt flow of $½D_0$. Thus there exists a smaller debt reduction, say, $\varepsilon = 0.4$, such that when the debt is reduced to $0.6D_0$ at $H$, the sovereign may not immediately default. To allow for variable debt reductions, we introduce $\varepsilon$ to the value functions as an additional argument $V_0(X, \varepsilon)$ is the pre-reduction value of debt

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7 We can write the functional form of the post-reduction value of debt as $V_{d1}(X) = D_1/r - B_1 X^{-\alpha}$ and the pre-reduction value as $V_{d0}(X) = D_0/r - B_0 X^{-\alpha}$. From Section I we know that $B_1 = (1 + \alpha)A_1$. The constant of the pre-reduction value function, $B_0$, is determined by applying the value matching condition at the debt reduction trigger $H$:

$$\frac{D_1}{r} - (1+\alpha)A_1 H^{-\alpha} \;=\; \frac{D_0}{r} - B_0 H^{-\alpha} \qquad \Rightarrow \qquad B_0 \;=\; (1+\alpha)A_1 \;-\; \frac{\varepsilon D}{r} H^{\alpha}\;.$$

Substituting for the second term from Equation 14 we obtain $B_0 = A_0 + \alpha A_1$.

when the an $\varepsilon$-percent debt reduction is anticipated and $V_1(X, \varepsilon)$ is the value of debt after an $\varepsilon$-percent debt reduction.

Note that the sovereign would default at $H$ if debt reduction is delayed, but he will not do so if he does not receive the full reduction. This paradox is explained as follows: in the first case the question is whether to keep on paying $D_0$ in anticipation of the debt relief that has now been delayed. As $H$, by definition, gives us the longest the sovereign is prepared to wait for the reduction, he will not wait any longer than $H$. On the other hand, when he gets a smaller debt reduction, the question is when to default given the new debt flow of $D_1$. So the sovereign faces different questions depending on whether the creditors reneg on the timing or the size of the debt reduction. These questions have different answers and thus the timing of a debt reduction, given by $H$, is always time-consistent whereas its size may not be. If the creditors are better off by reducing the debt with a smaller amount, they will indeed do so. But the sovereign, anticipating this, will not wait till $H$ and default earlier. Thus a 50-percent reduction will be credible only if at its optimal debt reduction trigger, $V_{d,1}(H, \frac{1}{2}) \geq V_{d,1}(H, \varepsilon)$ for all $\varepsilon$. I will impose this time-consistency requirement in the determination of the optimal amount of debt reduction.

[ Figure 2 ]

First, examining the value of debt in Equation 5, we note that the pre-reduction value of debt depends on the amount of debt reduction only through its coefficient $B_0 = A_0 + \alpha A_1$ ($A_0$ and $A_1$ of course depend on $\varepsilon$). Therefore, for a given value of $X$, maximizing $V_{d,0}(X, \varepsilon)$ is equivalent to minimizing $B_0$. As debt reduction is bounded, $\varepsilon \in [0, 1]$, the following program always have solution (denoted by $\varepsilon^*$):

$$\underset{\varepsilon}{\text{Minimize}}\ B_0 = A_0 + \alpha A_1 = (1+\alpha)\, a\, (1-\varepsilon)^{1+\alpha} D_0^{1+\alpha} + \frac{\varepsilon D_0}{r} H(\varepsilon)^{\alpha} \qquad \textbf{Pre-RV}$$

$$\text{where } H(\varepsilon) \text{ satisfies } \quad a\,(1-\varepsilon)^{1+\alpha} D_0^{1+\alpha} H(\varepsilon)^{-\alpha} + cH(\varepsilon) - \frac{(2-\varepsilon) D_0}{r} = 0 .$$

Now refer to Figure 2 to see what may be a problem. Recall that $V_{d,0}(X;\ \varepsilon)$ is the value of debt with the anticipation an $\varepsilon$-percent debt reduction. For any $\varepsilon$, by definition, $V_{d,0}(X;\ \varepsilon^*) \geq V_{d,0}(X;\ \varepsilon)$ for all $X$. Note that this statement does not imply anything about post-reduction value of debt. As shown in Figure 2, there is a posibility that at the point of debt reduction, $H(\varepsilon^*)$, some other post-reduction value function, say $V_{d,1}(H(\varepsilon^*),\ \varepsilon)$ may dominate $V_{d,0}(H(\varepsilon^*),\ \varepsilon^*)$. That is, there may exists an $\varepsilon < \varepsilon^*$ such that $V_{d,1}(H(\varepsilon^*),\ \varepsilon) > V_{d,0}(H(\varepsilon^*),\ \varepsilon^*)$. Therefore, at $H(\varepsilon^*)$ the creditors would prefer to reduce debt by $\varepsilon$ rather than $\varepsilon^*$. Although $\varepsilon^*$ maximizes the pre-reduction value of debt, it would not be a credible debt reduction strategy. Also, note that the post-reduction value function associated with $\varepsilon^*$ dominates all other value functions associated with higher levels of debt reduction at their own triggers. Thus no $\varepsilon > \varepsilon^*$ can be a credible debt reduction strategy.

The reader may wonder whether the optimal timing of the debt reduction may be affected because the amount of reduction is now variable. As the owners will not tolerate losses beyond $H(\varepsilon^*)$, the debt reduction cannot happen later than $H$. On the other hand, reducing the debt earlier than $H(\varepsilon^*)$ cannot be time-consistent: will not solve the time consistency problem because the fact that $V_{d,1}(H(\varepsilon^*),\ \varepsilon) > V_{d,0}(H(\varepsilon^*),\ \varepsilon^*)$ and $\varepsilon < \varepsilon^*$ imply that at any earlier debt reduction at $X > H(\varepsilon^*)$, $V_{d,1}(X,\ \varepsilon) > V_{d,1}(X,\ \varepsilon^*)$.[8]

A credible (time-consistent) debt reduction strategy $\hat{\varepsilon}$ must have the property that at its trigger no other value function associated with some smaller debt reduction dominates it.[9] To define this property more carefully, for each $X$, we find the level of debt reduction that maximizes the post-reduction value of debt:

$$\underset{\varepsilon}{Max}\ V_{d,1}(X;\varepsilon) = \underset{\varepsilon}{Max}\ \frac{(1-\varepsilon)D_0}{r} - (1+\alpha)\ a\ (1-\varepsilon)^{1+\alpha}\ D_0^{-\alpha}\ X^{-\alpha}\ . \qquad \textbf{Post-RV}$$

In fact for each $\varepsilon$, there is some $X$ such that $\varepsilon$ solves (Post-RV). That gives us the function $X(\varepsilon)$:

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8 When the debt is to be reduced at $X > H$, the pre-reduction value must be recalculated, but the value matching condition at the point of debt reduction implies that the value of debt at $X$ will be equal to $V_{d,0}(X,\ \varepsilon^*)$.

9 Dixit 1992 uses the same principle to choose among alternative discrete investment projects.

$$X(\varepsilon) = b(1-\varepsilon)D, \qquad \text{where} \qquad b \equiv \left(a\,r(1+\alpha)^2\right)^{1/\alpha}$$

The undominated, thus credible, level of debt reduction is $\hat{\varepsilon}$ such that $X(\hat{\varepsilon}) = H(\hat{\varepsilon})$. To substantiate the result, we must prove that

1. $\hat{\varepsilon}$ is unique;

   Substitute $X(\varepsilon)$ in the equation for $H(\varepsilon)$:

$$a(1-\varepsilon)^{1+\alpha} D_0^{1+\alpha}\left(b(1-\varepsilon)D_0\right)^{-\alpha} + c\left(b(1-\varepsilon)D_0\right) - \frac{(2-\varepsilon)}{r} D_0 = 0 .$$

and there is a unique solution for $\hat{\varepsilon}$:

$$\hat{\varepsilon} = \frac{z-2}{z-1} \qquad \text{where} \qquad z \equiv r\left(a b^{-\alpha} + c b\right) > 2 .$$

Therefore, the size of the optimal debt reduction is independent of the level of debt flow.

2. $\hat{\varepsilon} < \varepsilon^*$ (otherwise there does not exist a credible debt reduction).

   The deriative of the objective function in the program (Pre-RV) is

$$-\,a\,(1+\alpha)^2\,(1+\varepsilon)^{\alpha}\,D_0^{1+\alpha} + \frac{D_0}{r}\,H(\varepsilon)^{\alpha} + \alpha\,\frac{\varepsilon D_0}{r}\,H(\varepsilon)^{\alpha-1}$$

evaluate it at $\hat{\varepsilon}$ by substituting $X(\hat{\varepsilon}) = b(1\text{-}\hat{\varepsilon})D_0$ for $H(\hat{\varepsilon})$:

$$\text{Sign}\left[-a(1+\alpha)^2(1-\hat{\varepsilon})^{\alpha}D_0^{\alpha} + \frac{b^{\alpha}}{r}(1-\hat{\varepsilon})^{\alpha}D_0^{\alpha} + \frac{\alpha\,\hat{\varepsilon}\,b^{\alpha-1}}{r}(1-\hat{\varepsilon})^{\alpha-1}D_0^{\alpha-1}H'(\hat{\varepsilon})\right] = \text{Sign}\left(H'(\hat{\varepsilon})\right)$$

The first two terms cancel out (from definition of $b$) and the terms multiplying $H'(\varepsilon)$ are all positive. The derivative of $H$ with respect to $\varepsilon$ can be derived by totally differentiating the equation for $H$:

$$H'(\varepsilon) = \frac{\frac{D_0}{r} - a(1+\alpha)(1-\varepsilon)^{\alpha} D_0^{1+\alpha} H^{-\alpha}}{\alpha\, a\,(1-\varepsilon)^{1+\alpha} D_0^{1+\alpha} H^{-(1+\alpha)} - c} \quad \text{and} \quad H'(\hat{\varepsilon}) = D_0 \frac{\frac{1}{r} - a(1+\alpha)\, b^{-\alpha}}{\alpha\, a\, b^{-(1+\alpha)} - c} < 0$$

The sign can be confirmed by substituting for the definitions of $a$ and $b$. As the derivative of the objective function evaluated at $\hat{\varepsilon}$ is negative, $\hat{\varepsilon}$ is always less than $\varepsilon^*$. If $\varepsilon^* = 0$, then there is no time-consistent debt reduction and the creditors never renegotiate the debt. In my exhaustive numeric calculations this never turned out to be the case.

First thing to note about the optimal amount of the debt reduction is that it is a fixed fraction of the outstanding debt flow. Thus the level of debt does not affect the percentage of debt to be forgiven. The following table shows how the optimal size and timing of debt reduction varies with the volatility and trend of the underlying stochastic process:

| | The sign of the derivative of | |
|---|---|---|
| | $\hat{\varepsilon}$ (amount of reduction) | $H$ (trigger) |
| with respect to trend: $\mu$ | - | - |
| with respect to volatility: $\sigma$ | + | - |

Note that although an increase in the trend diminishes the amount of debt reduction, the debt reduction still occurs later. Creditors reduce the debt of countries with good prospects (higher trend and smaller volatility) by a smaller amount and later. There is an asymmetry in how the size and timing of debt reduction are related to the stochastic process:

- amount of reduction depends on how favorable the process is to the creditors (ie, the higher the trend and the lower the volatility, the more favorable is the process for creditors and the smaller will be the debt reduction);
- timing is based on how favorable the process is to the sovereign (ie, the higher the trend and the volatility, the more favorable is the process for the sovereign and the later will be the debt reduction).

For example, a country with highly volatile revenue from its external sector may get a big reduction but the reduction will happen quite late. On the other hand a country with relatively stable revenue from its external sector will have an earlier debt relief, but the reduction will be smaller.

We may now consider what would happen when debt reduction is costly. In particular, when the creditors incur a fixed cost of debt reduction, they will compare the value of debt at the optimal debt reduction trigger with the value of debt with no-renegotiation. They will reduce the debt only if $V_{d,1}(H(\hat{\varepsilon}), \hat{\varepsilon}) - V_d(H(\hat{\varepsilon}))$ is greater than the cost of debt reduction. On the other hand, costs that are proportional to debt (such as costs due to delay and coordination among the creditors) reduction will only reduce the amount of debt reduction. As such they have no relevance for the decision on whether to undertake debt reduction or not. When the fixed costs are sufficiently high, there may not be any debt reduction. Or, although one debt reduction could profitable, two debt reductions may not be. Consequently, depending on the size of the fixed cost of renegotiating the sovereign debt, the creditors will be willing to reduce the debt only a finite number of times. To gain insights about the dynamics of size and timing of multiple debt reductions, I will generalize this model to *n*-reductions in the next section.

## V. Debt Valuation with n-reductions

I can generalize the model of the last section to *n*-reductions by using the same principles developed therein. The details of the derivations are in the Appendix. The *n*-reduction model allows us to develop predictions about the amount and timing of debt reductions, and by how much the price of debt will jump at consecutive debt reductions. In this analysis I assumed that the fixed cost of renegotiating sovereign debt is zero. The fixed costs are important in determining the optimal *n*. In practice they are relatively small (most costs are associated with delays and the length of the negotiations both of which are proportional to the amount of debt reduction) and often financed by the international organizations which support, if not organize, debt reduction negotiations. On one hand, even very small fixed costs are sufficient to make the number of debt reductions finite. On the other hand, small fixed costs will have neglible impact on the optimal size and timing of the debt reductions. In any case, I only report the results that are independent of the number of debt reduction rounds. The following results are valid for all parameter values. (The computations for all figures use $\sigma = 0.01$, $\mu = 0$, $r = 0.05$, and $n = 15$.)

[ Figure 3 ]

- Later debt reductions are bigger than the earlier debt reductions. Figure 3 shows the percentage debt reduction at each one of the 15 debt reduction triggers.

This confirms the view that desperate times require desperate measures. As such the optimal policy is to start with small reductions and, if need be, move on to bigger ones.

[ Figure 4 ]

- Price jump after the *m*th debt reduction is higher than the price after the *m*-1th debt reduction. Figure 4 shows secondary market price of debt right before and after each one of the 15 debt reductions. The triggers at which these reductions are carried out are shown on the x-axis.

Observing a series of debt reductions, we will see bigger and bigger jumps in the market price of debt.

[ Figure 5 ]

- Later debt reductions occur when the *X/D* ratio is lower and the *X/D* ratio recoveres to a higher level after the reduction (see Figure 5).

The very first debt reductions occur when the *X/D* ratio is around $\frac{\alpha(r-\mu)}{r(1+\alpha)}$, the level it maintains in continuous debt reductions. Later reudctions come when the sovereign's position is worse. These later reductions however shot up the *X/D* ratio.

These three results are linked together. As the creditors cannot commit to the size of future debt reductions, in the current debt reduction they consider future debt reductions. They can either provide bigger reductions upfront and, as they are big, they can wait longer before doing them or smaller and earlier reductions. This tradeoff may be resolved in either way, but providing bigger reductions is not a time-consistent strategy. Therefore, they cannot provide big reductions in the initial rounds. As all debt reductions are anticipated, price of debt sinks lower prior to larger reductions. With discrete and increasing debt reductions, the creditors keep the average revenue-debt service ratio at its desired level by allowing it to sink lower before bigger debt reductions.

On the other hand, whether debt reductions happen more or less quickly in the later rounds depends on the parameter values:

- with a large negative trend, or with a high volatility, or low interest rates, debt reductions speed up.

But except for this exteremes, the opposite is true: more time elapses between later rounds of debt reductions than the earlier ones.[10] However evidence provided by Backer 1992 suggests that having gone through a debt reorganization increases the probability of having another one. Thus it is possible that for most countries going through debt reorganizations have high volatility or high negative trend.

The following comparative statics are consistent with the one-time reduction model:

When the stochastic process becomes more favorable to the creditors (lower volatility and higher trend),

- debt will be reduced by smaller amounts;
- jumps in the market price of debt will be smaller.

When the stochastic process becomes more favorable to the debtors (higher volatility and trend),

- debt reductions will start later but be carried out faster (less time will elapse between consecutive reductions).

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[10] As the waiting time is independent of the current value of $X$, the higher the difference between the triggers of two consecutive debt reductions, the more time is expected to elapse between them. As we go through several rounds of debt reductions, they will be further and further apart.

## VI. Empirical findings

Below I gathered some evidence from various studies of debt relief/rescheduling. As there are no studies directly looking at debt reductions, I considered studies on debt reschedulings. As such I have implicitly assumed that the motives behind debt reschedulings are identical to those of debt reductions.

Lloyd-Ellis et al. (1989) find that "increases in the value of reschedulings and their grace periods lead to decreases in the probability of rescheduling." This supports my result that size and timing of reductions are negatively related. The larger the reduction, the later it happens. When creditors decide to reduce the debt by a larger amount, they will wait longer. Thus the net effect on a logit equation is a lower probability of rescheduling. The explanation advanced by the authors that "[B]anks have only allocated a certain amount of funds to rescheduling and once these have been committed the probability of further reschedulings during the period decline" can at best explain why an increase in the amount of funds may not necessarily increase the number of reschedulings, but it does not explain why that probability should decline.

In a later paper Lloyd-Ellis et al. (1990) have an apparently puzzling finding: the higher a country's ratio of reserves to its IMF quota, the bigger will be the next year's rescheduling -- if it happens. In the accompanying Tobit model, however, they find that the higher this ratio the less likely is a rescheduling in the next year. As the IMF quota is based on some measure of the country's GDP and volume of foreign trade, the ratio of reserves to the IMF quota is a measure of the cash flow of a the country -- a proxy for $X_t$. In terms of my model, the finding from the Tobit equation suggests that the countries with high $X_t$ are far away from their debt reduction triggers. The ones that have a rescheduling in the next year must be the ones with relatively higher volatility (or negative trend). My model predicts that high volatility (or negative trend) implies larger debt reductions happening relatively later. Therefore, my model can easily explain this puzzling empirical finding. In a separate study, Backer (1992) revises the results of Lloyd-Ellis et al.. In line with my model, he finds that higher levels of outstanding debt is associated with higher probability of rescheduling (that is, higher debt reduction trigger).

Calvo and Kaminsky (1991) is the only empirical study that explicitly deals with the relation between volatility and debt reduction. Three countries are included in their study. Consistent with the predictions of my model, they find that higher volatility implies a lower debt reduction trigger for Argentina and Mexico (coefficient of regression and their standard errors are -3.505 (0.61) and -11.186 (2.68)).[11] This finding contradicts their model where, in case of default, the government incurs a cost to verify the situation to its creditors and gets a state-contingent payment plan that smooths its consumption. When volatility is high, the government is inclined to use this insurance-cum-deafult mechanism more readily and hence their prediction that higher volatility implies an earlier debt relief.

Classens and van Wijnbergen (1993) investigate the debt reduction deal for Mexico. Their well-calibrated debt pricing model estimates that the total value of outstanding debt went up from $19.14 billion to only $19.26 billion (with market price increasing from $0.26 to $0.49). If it were not for the enhancements provided by the third parties, the total value of debt would have come down to $15.66 billion. The interesting conclusions of these numbers are that i.) total value of debt remained unchanged; ii.) enhamcement were needed because the debt reduction was carried out earlier than the creditors would have; iii) most of the transaction costs were absorbed by the third parties. This provides a justification for ignoring the fixed costs as they are either too small or likely to be financed by international organizations and other third parties.

## Conclusion

In this paper I present a model of sovereign debt dynamics. I assumed that the creditors can irreversibly reduce the debt flow to prevent default by the sovereign (in which case everything is lost). The fixed costs of debt reductions are mostly financed by international organizations or official creditors. As such the residual fixed costs to the creditors are too small to affect the optimal size and

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[11] Although the same coefficient for Brazil is positive (4.29 with a standard error of 2.342 with 9 observations), it is not significant (see their Table 7 on page 31).

timing of debt reductions. But, they are large enough to lead to a finite number of discrete debt reductions. Although I do not endogenize the number of debt reduction rounds in this verison, the results that I report do not depend on the number of debt renegotiation rounds.

My main results concerning the quantity and timing of debt relief and the dynamics of the market price of debt are as follows:

- Higher volatility implies that the size of the reductions will be larger but they will happen later;
- Later reductions will always be larger than the initial reductions.
- The jump in the secondary market price of debt will be higher at later rounds of debt reduction.
- In general the timing decisions depends on how favorable the process is to the debtor country (they happen later if it is favorable) and the size of the reduction depends on how favorable it is to the creditors (they are lower when the process is favorable).
- However not everything about timing obeys this principle. Whether the process speeds up as we go through multiple reductions or not depends on whether the process is unfavorable to the creditors or not.

My model characterizes how the secondary market price of debt will change at the time of debt reduction. The results caution against relying on the price increases as an indicator of the success of a debt relief plan: the jump in the price of debt will be much higher when the countries fortunes are really dwindling. Similarly, after each new debt reduction, the revenue-debt service ratio will be restored to a higher than the last reduction. If the jump in the price or the revenue-debt service ratio of a country is smaller than the previous debt reduction, the only conclusion we can draw is that the country got a debt reduction earlier than the creditors would ideally like to provide. I review the main results of the previous empirical works on sovereign debt. They are mostly consistent with my predictions and, furthermore, my model explains at least two of their puzzling findings.

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## Appendix to the Continuous Debt Reduction Model

The value of equity and debt are respectively given by:

$$V(X;D) = \frac{X}{r-\mu} - \frac{D}{r} + A(D)\,X^{-\alpha}$$

$$V_d(X;D) = \frac{D}{r} - B(D)\,X^{-\alpha}$$

Let *H*(*D*) denote the value of $\check{X}$ at which debt is reduced to $D$. The first boundary condition that the value of equity must be zero at the debt reduction triggers imply

$$V(H(D)\,;\,D) = \frac{H(D)}{r-\mu} - \frac{D}{r} + A(D)\,H(D)^{-\alpha} = 0$$

Smooth-pasting at the debt reduction triggers imply

$$V'(H(D)\,;\,D) = \frac{1}{\mu-r} - \alpha A(D)\,H(D)^{-\alpha-1} = 0$$

Solving these two equations, we get *A*(*D*) and *H*(*D*):

$$A(D) = a\,D^{1+\alpha} \qquad\qquad H(D) = \frac{\alpha(r-\mu)}{r(1+\alpha)}\,D = S(D)$$

Finally, the value of marginal debt is equal to 0, gives the following equation for derivative of *B*(*D*):

$$\frac{\partial\, V_d(H(D);D)}{\partial D} = \frac{1}{r} - B'(D)\,H(D)^{-\alpha} = 0$$

Integrating with respect to $D$ and using the definition of $a$, we can show that *B*(*D*) = *A*(*D*). Indicating that under continuous debt reduction, the expected total value of the revenue for any value of $X$ is $X/(r - \mu)$ and the value of debt reductions to the sovereign is exactly the risk premium on holding sovereign debt rather than riskless debt. Inverting $H(D) = \check{X}$, we get $D(\check{X})$. So the value of debt at $X = \check{X}$ is given by:

$$V_d(\check{X}, D(\check{X})) = \frac{D(\check{X})}{r}\left[1 - \frac{1}{1+\alpha}\right] .$$

After a debt reduction, a flow debt obligation with \$1 face value will be sold for $\alpha/(1+\alpha)$. Also note that there will never be default as the debt can be reduced infitely small amounts when $X$ is in the vicinity of zero. Thus the sum of the value of debt and equity always add up to $X/(r - \mu)$. Thus at any debt reduction trigger $X = \check{X}$, where the value of equity is zero, the value of debt will be $\check{X}/(r - \mu)$.

*The Case of Temporary Debt Reductions*

With temporary debt reductions, there are two different states: when $X > S$, the flow debt obligation is at its original level $D$; otherwise $D(\check{X}) = \check{X}$. The boundary conditions set forth in the text imply that in the latter state, the value of equity is always zero. The value of equity is subject to the value-matching and smooth-pasting conditions at the switching point between the two regimes. The equations are identical to the one used for the case of non-renegotiable debt: debt relief does not increase the value of equity. All the surplus generated through the renegotiation of debt is captured by the creditors. Thus value of debt in the first regime (no debt relief) is obtained by subtracting the value of equity from the expected discounted value of the revenue flow $(X/(r - \mu))$: $D/r - A(D)\, X^{-\alpha}$. When $X < S$, the value of equity is zero and flow debt obligation is equal to $X$. Therefore the value of debt is $X/(r - \mu)$.

**Appendix for the derivation of the optimal debt reduction trigger**

To answer the question in the setup of Section I, we look at the value of equity when debt flow is $D_0$. We add $V_1(H)$ to the value of sovereign's equity. The new default trigger will give us the rule that the sovereign will use in deciding to wait for the debt reduction. Note that this exercise is completely counterfactual and it is only a computational device to find out $H$. The sovereign's gain from debt reduction is $V_1(H)$ and we ask ourselves when will he default with this additional value. I use tildes to distinguish this new value function from the real one.

The value function has the same form as in Section 1:

$$\tilde{V}(X) = \frac{X}{r-\mu} - \frac{D}{r} + \tilde{A}\, X^{-\alpha} + V_1(H) \ .$$

At the default trigger, the value of equity is zero. The value-matching and smooth-pasting conditions at the default trigger $\tilde{S}$ are:

$$\tilde{V}(\tilde{S}) = \frac{\tilde{S}}{r-\mu} - \frac{D}{r} + \tilde{A}\, \tilde{S}^{-\alpha} + V_1(H) = 0 \ . \qquad \text{(Value Matching)}$$

$$\tilde{V}'(\tilde{S}) = \frac{1}{r-\mu} - \alpha \tilde{A}\, \tilde{S}^{-\alpha-1} = 0 \qquad \text{(Smooth Pasting)}$$

Solving these two equations we obtain the following equation for $\tilde{S}$:

$$\frac{1+\alpha}{\alpha(r-\mu)}\tilde{S} - \frac{D}{r} = -V_1(H) \ .$$

**Appendix for the model with n-reductions**

We now generalize the model of the previous section by allowing for $n$ debt reductions. Let us denote the debt level after the $i$th reduction by $D_i = (1 - \varepsilon_i)\, D_{i-1}$. Similarly, let $H_i$, $V_i$ and $A_i$ denote respectively the timing of, value of equity and the constant of the value function after the $i$th reduction. At every reduction point, the following value-matching condition for the value of equity must be satisfied (there is no smooth-pasting condition):

$$V_{n-i}\,(H_{n-i};\, D_{n-i}) \;=\; V_{n-i-1}\,(H_{n-i};\, D_{n-i-1}), \qquad \text{for } i \;=\; 0,..,n-1\ .$$

This condition may be expressed as a recursive formula for the constant of the value function:

$$A_{n-i-1} \;=\; A_{n-i} \;+\; \frac{D_{n-i-1}\,\varepsilon_{n-i}}{r}\; H_{n-i}^{\alpha}\ , \qquad \text{for} \quad i \;=\; 0,..,n-1 \qquad \textit{and} \qquad A_n \;=\; a\; D_n .$$

After the $n$-$i$-1 th reduction, we have to find how long (how low an $X$) will the sovereign tolerate in anticipation of the next debt reduction by $\varepsilon_{n-i}$ percent. This problem is identical to the problem of when does the sovereign default on non-renegotiable debt of $D_{n-i-1}$ when his equity is enhanced by the gain from the debt reduction (ie, $V_{n-i}(H_{n-i},\, D_{n-i})$). The answer to this question we replicate the smae analysis presented in the previous section of this appendix: We define a new value function, different than the original value function, denoted by tildes. We find the default trigger $\tilde{S}$ and the parameter of the new value function $\tilde{A}$, by imposing the value-matching and smooth-pasting conditions at $\tilde{S}$:

$$\tilde{V}(\tilde{S}) \;=\; \frac{\tilde{S}}{r-\mu} \;-\; \frac{D_{n-i-1}}{r} + \tilde{A}\; \tilde{S}^{-\alpha} \;=\; -\; V_{n-i}\left(H_{n-i},\; D_{n-i}\right). \qquad \text{(Value Matching)}$$

$$\tilde{V}'(\tilde{S}) \;=\; \frac{1}{r-\mu} \;-\; \alpha \tilde{A}\; \tilde{S}^{\,-\alpha-1} \;=\; 0 \qquad \text{(Smooth Pasting)}$$

The sovereign will keep up with the payments till $X$ hits $\tilde{S}$ and $\tilde{S}$ is given by the equation:

$$\frac{1+\alpha}{\alpha(r-\mu)}\,\tilde{S} - \frac{D}{r} = -V_{n-i}\left(H_{n-i}, D_{n-i}\right).$$

As the creditors will wait as long as the sovereign is willing to hold on, $H_{n-i} = \tilde{S}$. We thus obtain the optimal timing of the debt reduction:

$$A_{n-i}\,H_{n-i}^{-\alpha} + c\,H_{n-i} - \frac{2-\varepsilon_{n-i}}{r}\,D_{n-i} = 0 \qquad \text{for } i = 0,..,n-1\ . \qquad \text{(OT)}$$

For the optimal size of the reduction, we first find the expression for the value of debt after the $n$ - $i$th reduction. We start with the total value function. The revenue flow is terminated at the default trigger $S_n = [\alpha(r\text{-}\mu)/r(1+\alpha)]\, D_n$. The constant of the total value function is determined by the value-matching condition at $S_n$:

$$V_T(S_n);D) = \frac{S_n)}{r-\mu} + A_T\,S_n)^{-\alpha} = 0 \qquad \Rightarrow \qquad A_T = -\alpha\,a\,D_n^{1+\alpha} = -\alpha\,A_n\ .$$

Note that the total value is independent of how the debt reductions are timed and carried out. The value of debt can be found by subtracting the value of equity from the total value:

$$V_d(X;D_{n-i}) = \frac{D_{n-i}}{r} - \left(\alpha\,A_n + A_{n-i}\right)X^{-\alpha}\ . \qquad \text{(VoD)}$$

As before the time-consistency requirement for the size of the debt reduction implies that $\varepsilon_{n-i}$ must maximize the post-reduction value of debt at the debt reduction trigger $H_{n-i}$:

$$-\frac{D_{n-i-1}}{r} - \frac{\partial\,(\alpha A_n + A_{n-i})}{\partial\,\varepsilon_{n-i}}\,H_{n-i}^{-\alpha} = 0 \qquad \text{(T - C)}$$

In fact the solution of this equation has a nice form: $H_{n-i} = b_{n-i}\,D_{n-i}$. To show this we expand $\alpha A_n + A_{n-i}$ and show that it is equal to $b_{n-i}\,D_{n-i-1}^{1+\alpha}\,(1\text{-}\varepsilon_{n-i})^{1+\alpha}$. $b_{n-i}$ is a constant that depends on the <u>later</u>

debt reductions: $\varepsilon_n$, $\varepsilon_{n-1}$, $\varepsilon_{n-2}, \ldots, \varepsilon_{n-i+1}$; and the parameters of the stochastic process. The solution of the model depends on the recursive structure that allows us to solve for the optimal debt reductions starting from the last one and working backwards. This, of course, depends on the property that the percentage debt reduction in the last period is independent of the absolute level of debt. We can therefore find the whole sequence of debt reductions independent of the level of debt.

Now to expanding $A_{n-i}$:

$$A_{n-i} = A_{n-i+1} + \frac{1}{r} (1-\varepsilon_{n-i+1}) (1-\varepsilon_{n-i}) D_{n-i-1} H^{\alpha}_{n-i+1}$$

by repeatedly substituting for $A_j$'s on the right hand side, we get

$$A_{n-i} = A_n + \frac{1}{r} D_{n-i-1} \sum_{j=n-i-1}^{n} \varepsilon_j H_j^{\alpha} M(n-i, j-1)$$

where

$$M_{l,j} = M(l, j) \equiv \prod_{k=l}^{j} (1-\varepsilon_k) \qquad and \qquad M(l, j) = 1 \ \ for \ \ l > j .$$

Note that

$$D_j = (1-\varepsilon_{n-i}) D_{n-i-1} M(n-i+1, j) \qquad for \qquad j > n-i .$$

For $n$ we know that $H_n = b_n D_n$. What we can do is move iteratively and show that $H_{n-i} = b_{n-i} D_{n-i}$ for all $i$. I show one step of this induction for any $i$. Given that all $H_j$, $n \geq j \geq n-i$, are linear in $D_j$, we can write

$$H_j = b_j (1-\varepsilon_{n-i}) M(n-i+1, j) D_{n-i-1}$$

Substituting this in to the expression for $A_{n-i}$, we get

$$A_{n-i} = A_n + \frac{1}{r}\,(1-\varepsilon_{n-i})^{1+\alpha}\, D_{n-i-1}^{1+\alpha} \sum_{j=n-i-1}^{n} \varepsilon_j\,(1-\varepsilon_j)^{\alpha}\, b_j^{\alpha}\,\big(M(n-i\,,\,j-1)\big)^{1+\alpha)}$$

Let us expand $D_n$ to obtain:

$$A_n \;=\; a\,(1-\varepsilon_{n-i})^{1+\alpha}\, D_{n-i-1}^{1+\alpha}\, M(n-i-1\,,\,n)^{1+\alpha}$$

and

$$\alpha A_n + A_{n-i} \;=\; a\,(1+\alpha)\,(1-\varepsilon_{n-i})^{1+\alpha}\, D_{n-i-1}^{1+\alpha}\, M(n-i-1\,,\,n)^{1+\alpha}$$

$$+\; \frac{1}{r}\,(1-\varepsilon_{n-i})^{1+\alpha}\, D_{n-i-1}^{1+\alpha} \sum_{j=n-i-1}^{n} \varepsilon_j\,(1-\varepsilon_j)^{\alpha}\, b_j^{\alpha}\,\big(M(n-i\,,\,j-1)\big)^{1+\alpha)}$$

Taking the derivative with respect to $\varepsilon_{n-i}$, we get

$$\frac{\partial\, \alpha A_n + A_{n-i}}{\partial \varepsilon_{n-i}} \;=\; -(1+\alpha)(1-\varepsilon_{n-i})^{\alpha} D_{n-i-1}^{1+\alpha}\left[a\,(1+\alpha)\,M_{n-i-1\,,\,n}^{1+\alpha} + \frac{1}{r}\sum_{j=n-i-1}^{n} \varepsilon_j (1-\varepsilon_j)^{\alpha}\, b_j^{\alpha}\, M_{n-i\,,\,j-1}^{1+\alpha)}\right]$$

Substituting this back into T-C we get

$$b_{n-i}^{\alpha} \;=\; (1+\alpha)\left[r\,a\,(1+\alpha)\,M_{n-i-1\,,\,n}^{1+\alpha} + \sum_{j=n-i-1}^{n} \varepsilon_j (1-\varepsilon_j)^{\alpha}\, b_j^{\alpha}\, M_{n-i\,,\,j-1}^{1+\alpha}\right]$$

As proposed, $b_{n-i}$ does not depend on $\varepsilon_{n-i}$ or any previous reductions. Before we proceed, I would like to show that $b_{n-i}$ is decreasing in $i$. To see that consider $b_{n-i-1}$. Take out the first term of the summation in the second term, and divide by $(1 - \varepsilon_{n-i})^{1+\alpha}$ to obtain the following difference equation:

$$b_{n-i-1} \;=\; (1-\varepsilon_{n-i})\, b_{n-i} \;+\; (1+\alpha)\,\varepsilon_{n-i}\,(1-\varepsilon_{n-i})^{\alpha}\, b_{n-i}^{\alpha} \;=\; b_{n-i}\,(1-\varepsilon_{n-i})^{\alpha}\,(1+\alpha\,\varepsilon_{n-i}) \;<\; b_{n-i}.$$

Note that in the last inequality, the term that multiplies $b_{n-i}$ is decreasing in $\varepsilon_{n-i}$. Thus it reaches its maximum at $\varepsilon_{n-i} = 0$ and the theoretical maximum it could reach is 1. As $\varepsilon_{n-i} > 0$, $b_{n-i}$ is strictly decreasing in $i$. In fact $b_{n-i}$ is the revenue-debt flow ratio at the $n$-$i$th debt reduction trigger. Therefore, this ratio goes down in consecutive debt reductions.

Now we return to the optimal timing equation (OT) and substitute for $A_{n-i}$ and $H_{n-i}$:

$$(1-\varepsilon_{n-i})^{1+\alpha}\, D_{n-i-1}^{1+\alpha} \left[ a M_{n-i-1,\,n}^{1+\alpha} + \frac{1}{r} \sum_{j=n-i-1}^{n} \varepsilon_j\, (1-\varepsilon_j)^{\alpha}\, b_j^{\alpha}\, M_{n-i,\,j-1}^{1+\alpha} \right] b_{n-i}^{-\alpha}\, (1-\varepsilon_{n-i})^{-\alpha}\, D_{n-i-1}^{-\alpha}$$

$$+\; c\, b_{n-i}\, (1-\varepsilon_{n-i})\, D_{n-i-1} \;-\; \frac{1}{r}\, (2-\varepsilon_{n-i})\, D_{n-i-1} \;=\; 0\,.$$

$D_{n-i-1}$s drop out of the equation. We can then express $\varepsilon_{n-i}$ as function of the later debt reductions and the parameters of the stochastic process.

$$\varepsilon_{n-i} = \frac{z_{n-i}-2}{z_{n-i}-1}\,,\quad \text{where}\quad z_{n-i} \equiv \left[ r a M_{n-i-1,\,n}^{1+\alpha} + \sum_{j=n-i-1}^{n} \varepsilon_j\, (1-\varepsilon_j)^{\alpha}\, b_j^{\alpha}\, M_{n-i,\,j-1}^{1+\alpha} \right] b_{n-i}^{-\alpha} + r c b_{n-i}\,.$$

With some manipulations we get

$$z_{n-i} = \frac{1}{1+\alpha} \;-\; r\, a\, \alpha\, M_{n-i+1,\,n}^{1+\alpha}\, b_{n-i}^{-\alpha} \;+\; r\, c\, b_{n-i}\,.$$

As $b_{n-i}$ is decreasing in $i$ and $z_{n-i}$ is increasing in $b_{n-i}$, $z_{n-i}$ too is decreasing in $i$. Therefore, the size of optimal debt reduction is decreasing in $i$. In other words, the later the debt reduction the higher is the percentage of debt forgiven.

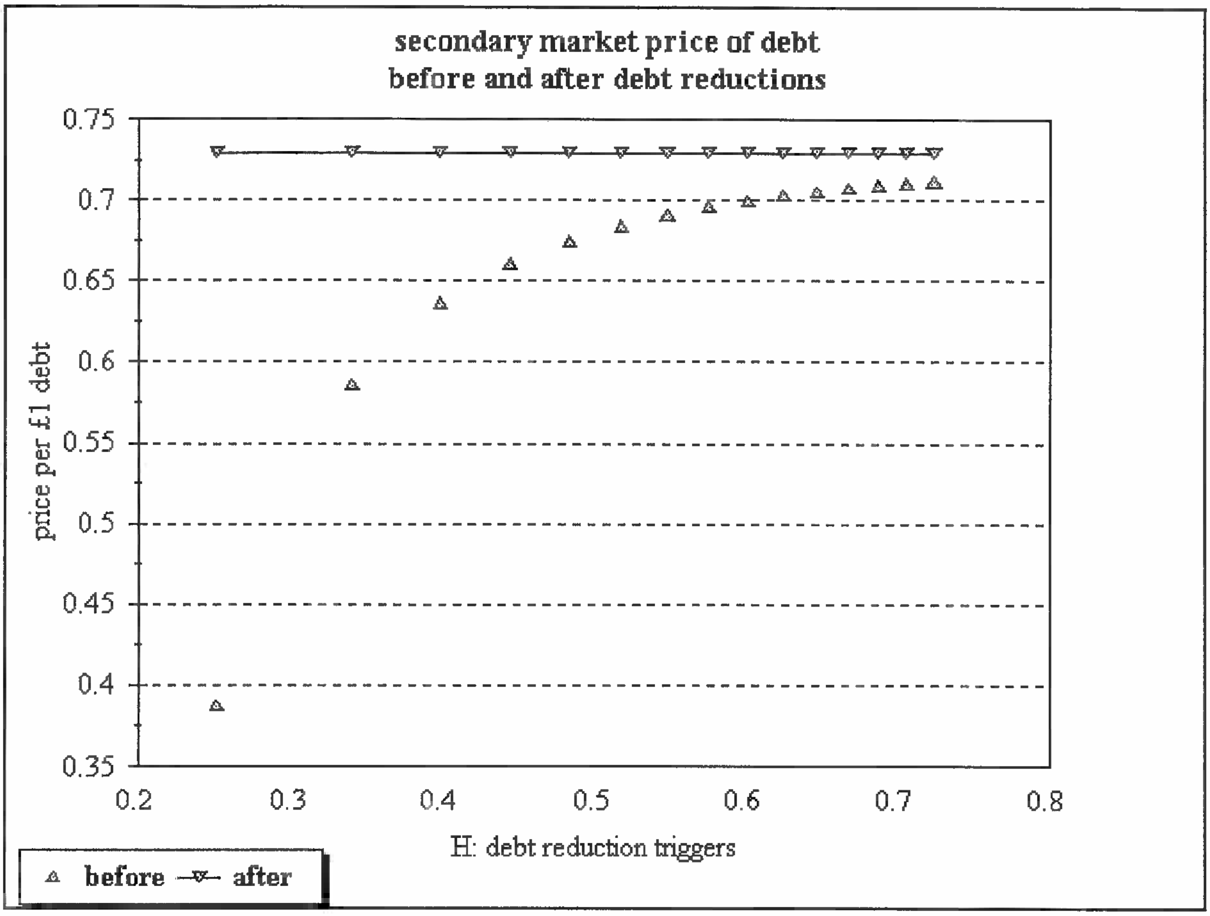
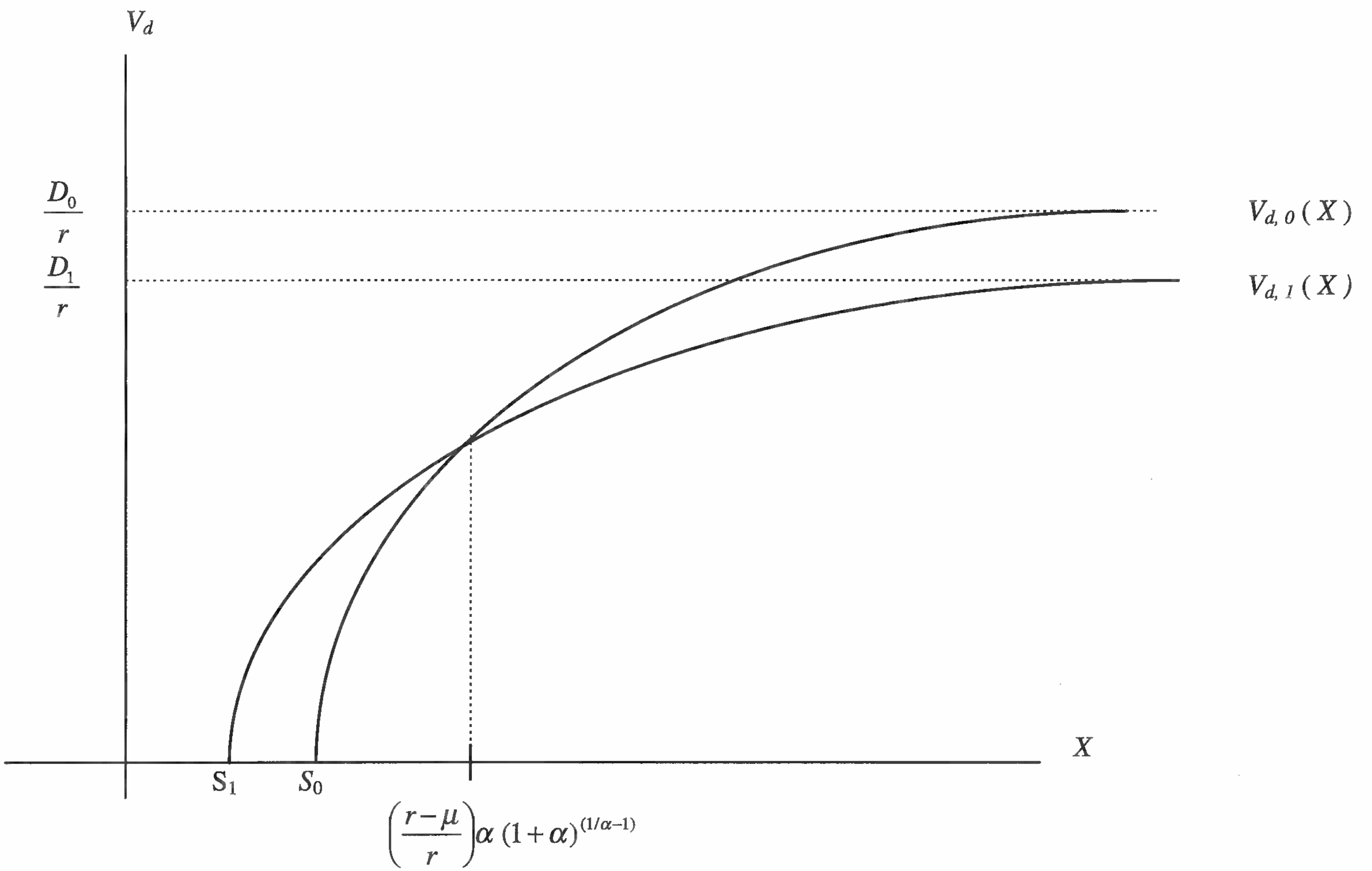



*Figure 3*

The upper line is the price of debt after the debt reduction. It is equal to the price obatined in the continuous debt reduction model. The lower curve shows the price of debt right before the debt reduction. The X-axis shows the value of *X* at which a debt reduction takes place. The first reduction is the farthest to the right.



$V_d$
$\frac{D_0}{r}$
$\frac{D_1}{r}$
$V_{d,0}(X)$
$V_{d,1}(X)$
$X$
$S_1$
$S_0$
$\left(\frac{r-\mu}{r}\right)\alpha\,(1+\alpha)^{(1/\alpha-1)}$


*Figure 1*

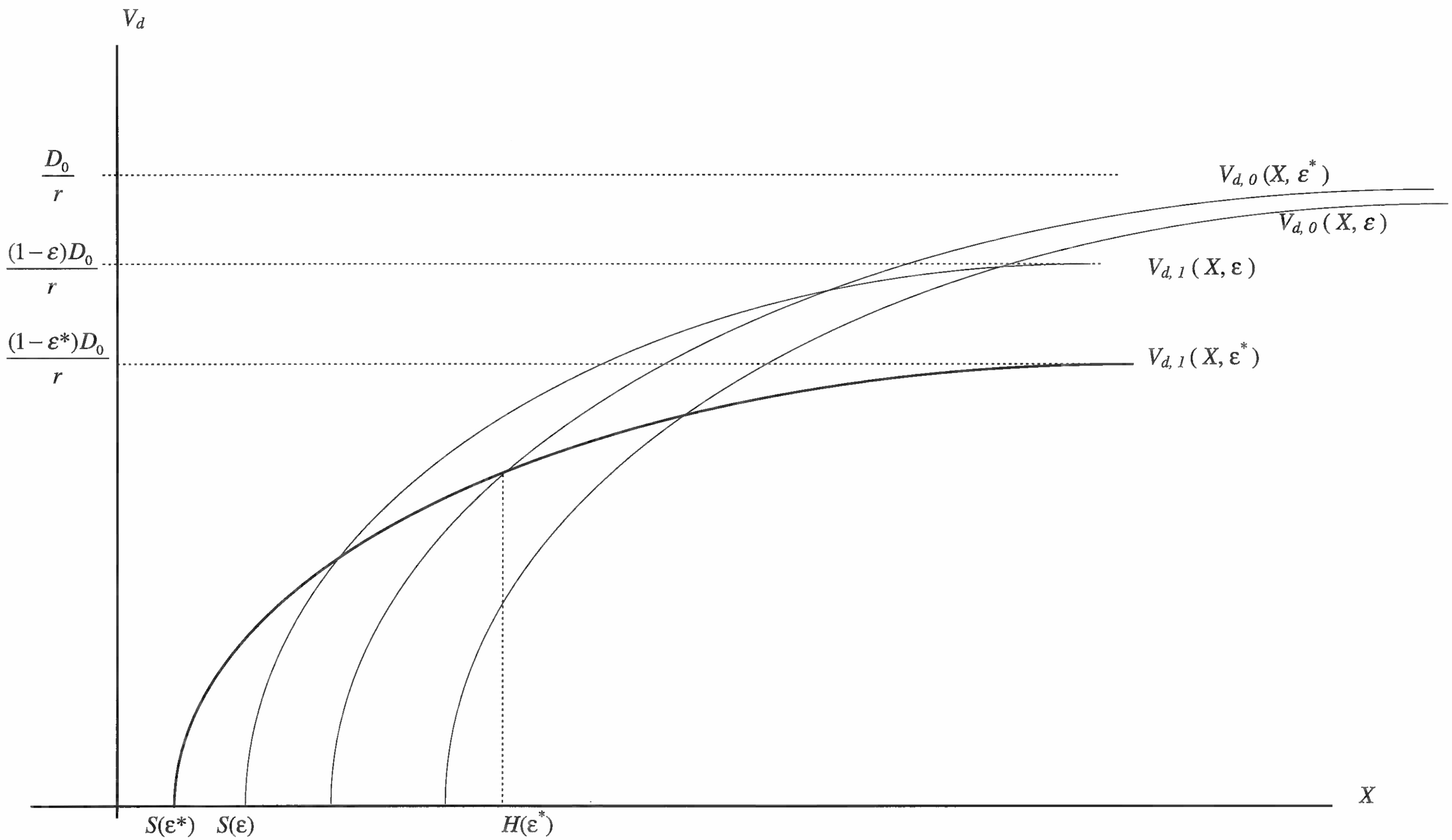

$V_d$
$\frac{D_0}{r}$
$\frac{(1-\varepsilon)D_0}{r}$
$\frac{(1-\varepsilon^*)D_0}{r}$
$V_{d,0}(X, \varepsilon^*)$
$V_{d,0}(X, \varepsilon)$
$V_{d,1}(X, \varepsilon)$
$V_{d,1}(X, \varepsilon^*)$
$X$
$S(\varepsilon^*)$
$S(\varepsilon)$
$H(\varepsilon^*)$

*Figure 2*

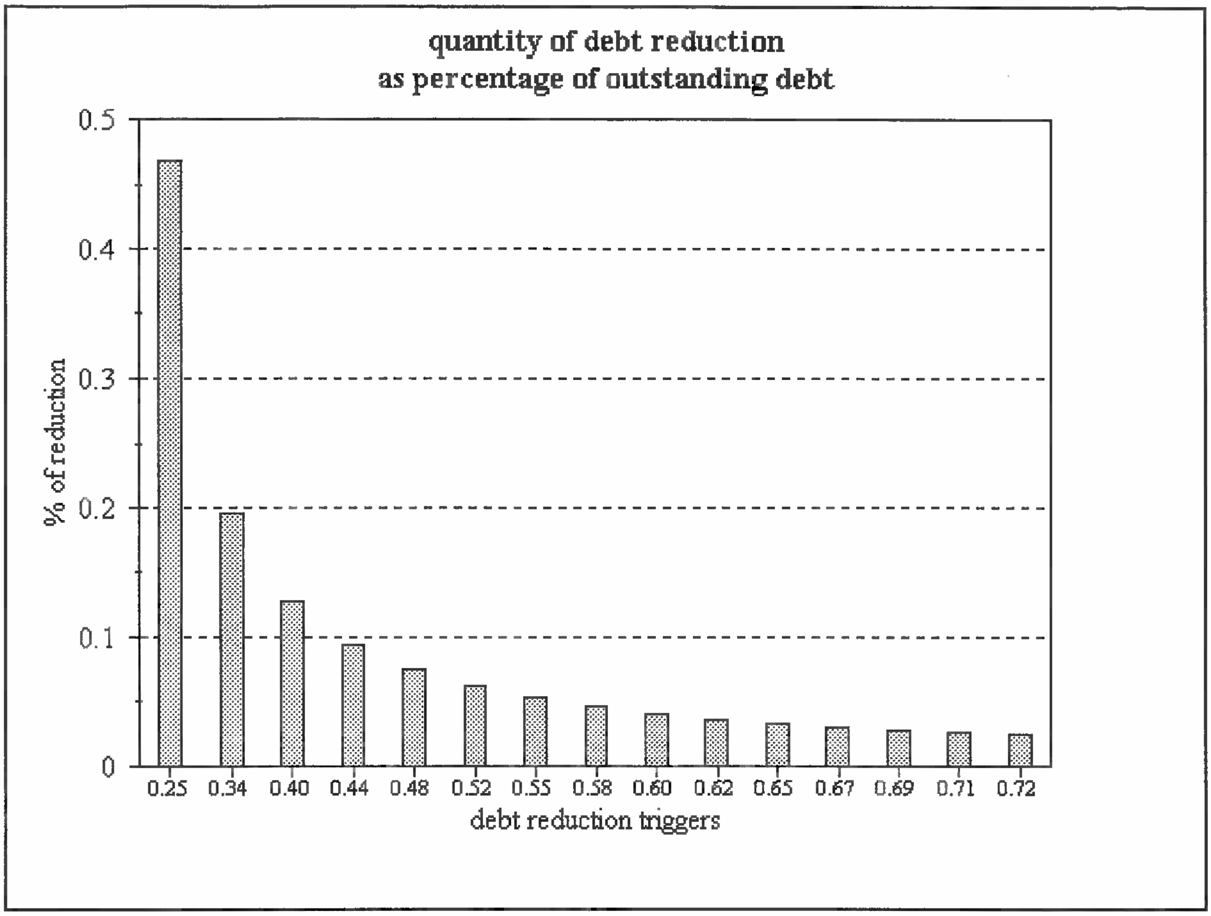



***Figure 4***

The values on the X-axis are the debt reduction triggers.

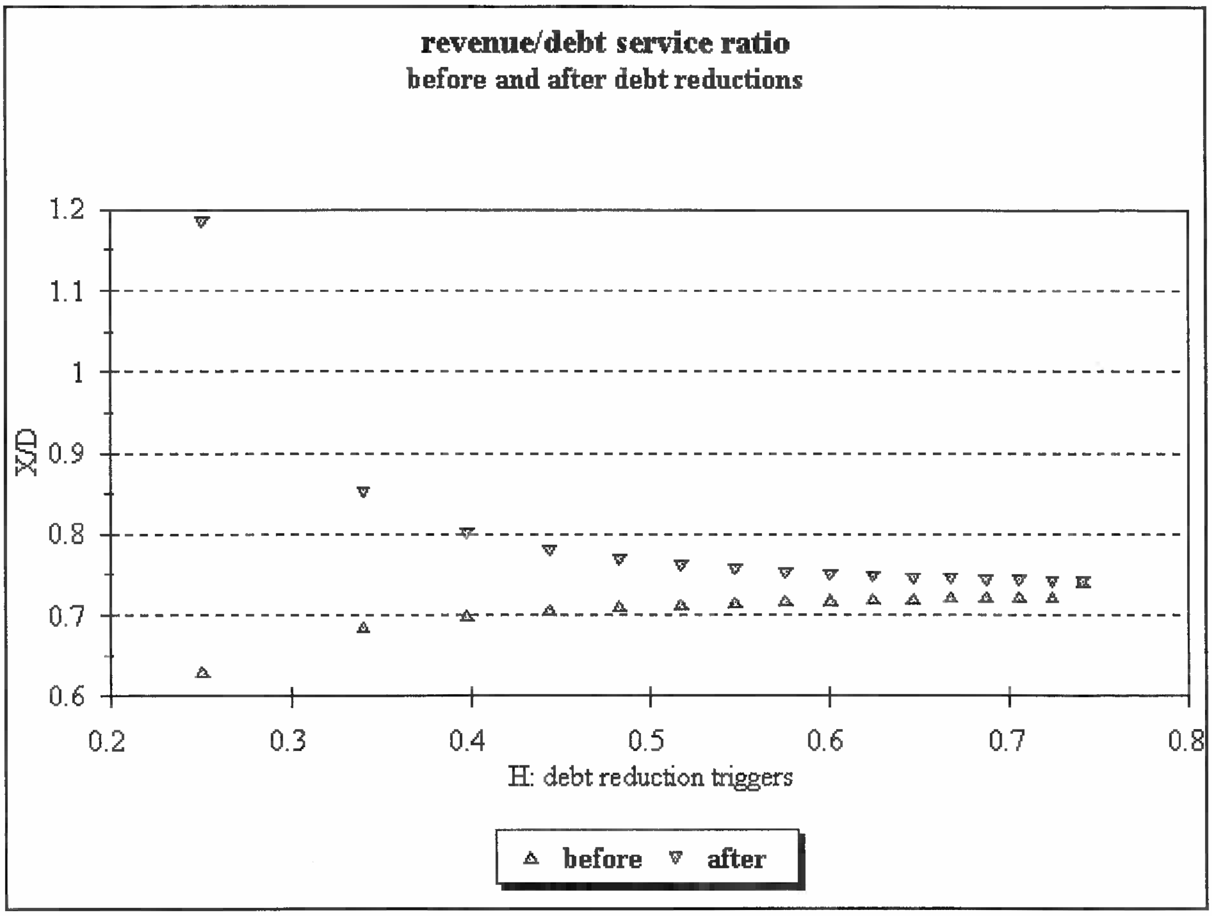



*Figure 5*

The upper curve is the *X/D* ratio after the debt reduction. The lower is the *X/D* ratio before the debt reduction. The X-axis shows the value of *X* at which a debt reduction takes place. The first reduction is the farthest to the right.

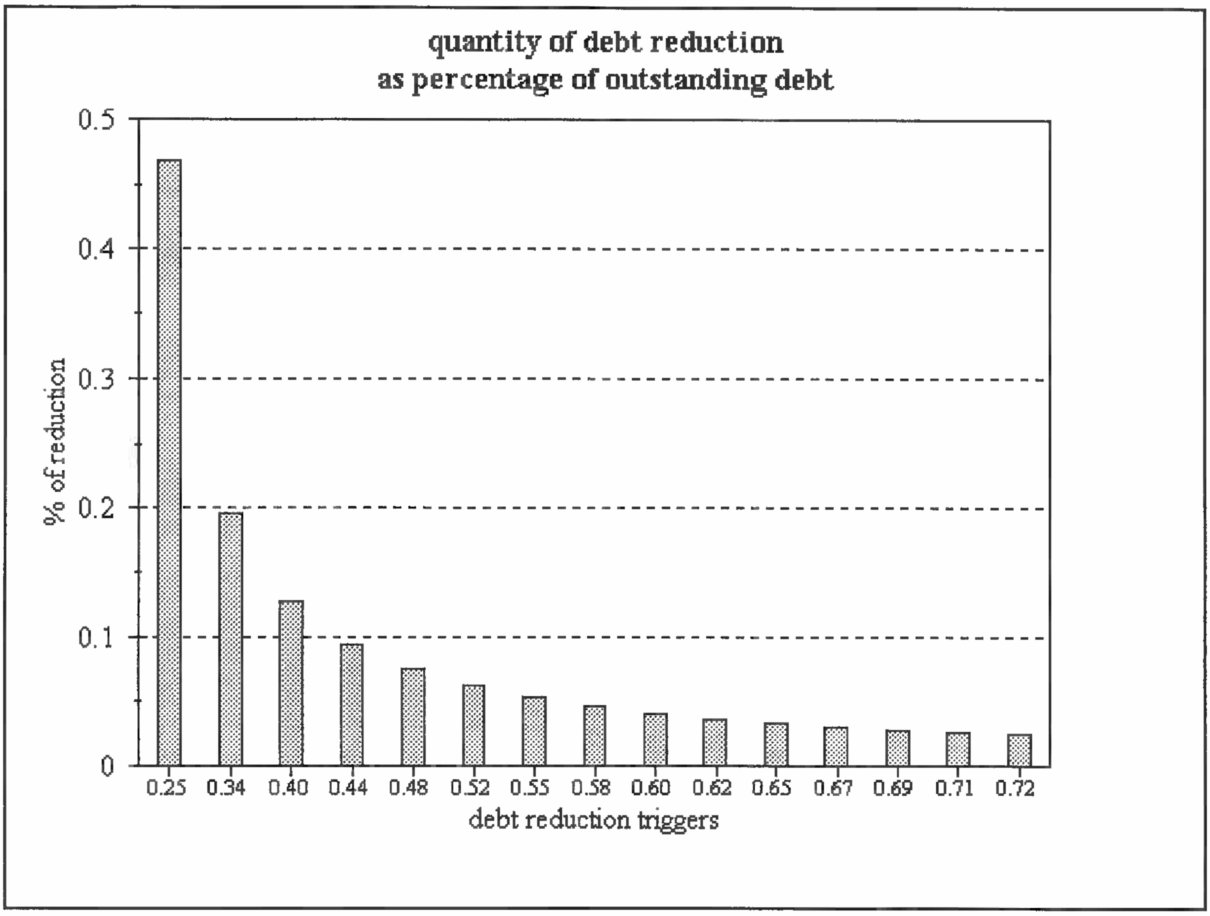

quantity of debt reduction
as percentage of outstanding debt
0.5
0.4
0.3
0.2
0.1
0
% of reduction
0.25 0.34 0.40 0.44 0.48 0.52 0.55 0.58 0.60 0.62 0.65 0.67 0.69 0.71 0.72
debt reduction triggers

*Figure 4*