COMPUTING POWER INDICES FOR LARGE VOTING GAMES: A NEW ALGORITHM

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This paper is circulated for discussion purposes only and its contents should be considered preliminary.

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Comments welcome.

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ABSTRACT

Voting Power Indices enable the analysis of the distribution of power in a legislature or voting body which uses weighted voting. Although the approach, based on co-operative game theory, has been known for a long time it has not been very widely applied, in part because of the difficulty of computing the indices when there are many players. This paper presents new algorithms for the classical power indices which have been shown to work well in real applications. We suggest that the availability of such accurate and efficient algorithms might stimulate further research in this under-researched field.

JEL Classification numbers: C63, C71, D71, D72

Key words: Voting; Weighted Voting; Power Index; Simple Game; Empirical Game Theory.

Computing Power Indices for Large Voting Games: A new algorithm

Power Indices are an important value concept for simple co-operative games with many applications to voting bodies which employ weighted voting. Although the classical power indices due to Shapley and Shubik (1954) and to Banzhaf (1963) have been applied to some well known cases, their wider application has been to some extent limited by the comparative difficulty encountered in computing them when the number of players is large. In this paper we propose new algorithms which use an approximation method but for which the error is very small and in practice appears to be negligible. We do not address the open question of the relative merits of the respective indices for measuring power but confine ourselves to computational aspects.

I. Power Indices: Notation

We assume a simple game of voting in a legislature with n members or players represented by a set $N = \{1, 2, \ldots, n\}$ whose voting weights are $w_1, w_2, \ldots, w_n$. The players are ordered by their weight representing their respective number of votes, so that $w_i \geq w_{i+1}$ for all i. We represent the combined voting weight of all members of a coalition represented by a subset T, $T \subseteq N$ by the function $w(T)$, $w(T) = \sum_{i \in T} w_i$. We will also have need of a sum of squares function: we define this as $h(T) = \sum_{i \in T} w_i^2$.

The voting decision rule is defined in terms of a quota, q. Thus a coalition of players represented by a subset T is winning if $w(T) \geq q$ and losing if

$w(T) < q$. It is customary to impose the restriction $q \geq w(N)/2$ to ensure the legislature is decisive and the voting game is a proper game.

A power index is a n-vector whose elements denote the respective ability of each player to swing a vote. In general a swing for a player, say player i, is a coalition which is losing until i joins it and it becomes winning. This we represent by a subset T_i such that:

$$q - w_i \leq w(T_i) < q \quad .$$

The power index for player i is defined as the relative number of swings for i with respect to a coalition model where, in some sense, each possible coalition is treated equally; if coalitions are regarded as being formed randomly then each is equally likely. Although the two indices employ the same general idea of a swing, they are mathematically distinct since they employ different coalition models which differ in what is meant by "equally likely". The Banzhaf index is based on considering coalitions as combinations of members in the sense of a list arranged in no particular order; they might be arranged alphabetically, or in any other arbitrary order: the ordering is irrelevant to the coalition. A member's power index is then the number of such coalitions it can swing from losing to winning, expressed as a proportion of either the total number of coalitions or the total number of such swings when all members are considered.

The Shapley-Shubik index, on the other hand, counts coalitions on the basis not only of swings, but also the order in which members are listed. Thus, given a particular swing for a member, the index takes into account the number of orderings of both the members of the winning coalition and the members not in the coalition: every reordering is counted as a different swing. The index is defined by expressing this number as a proportion of the number of orderings of all members.

For a given swing for player i, the number of orderings of the members of the subset T_i and its complement (apart from player i), $N\text{-}T_i\text{-}\{i\}$, is $t!(n\text{-}t\text{-}1)!$ where t is the number of members of T_i . (We can drop the subscript here for convenience; we use n and s for the numbers of members of sets N and S, etc.). The total number of swings for i defined in this way for this coalition model is $\sum_{T_i} t!(n-t-1)!$. The index, ϕ_i , is this number as a proportion of the number of orderings of all players in N,

$$\phi_i = \sum_{T_i} \frac{t!(n-t-1)!}{n!} \qquad (1)$$

The Banzhaf index assumes a coalition model in which orderings of players do not matter. The appropriate measure of the number of swings is then $\sum_{T_i} 1$. Two versions of the index are defined by expressing this number over different denominators. The Non-Normalized Banzhaf index (or Banzhaf Swing Probability), β_i', uses the number of coalitions which do not include i , the number of subsets of N -{i}, as denominator, 2^{n-1}:

$$\beta_i' = \sum_{T_i} 1/2^{n-1}. \qquad (2)$$

The Normalized Banzhaf Index, β_i, uses the total number of swings for all players as the denominator in order that it can be used to give a power distribution, in which the indices sum to unity over players:

$$\beta_i = \sum_{T_i} 1 / \sum_i \sum_{T_i} 1 \ . \qquad (3)$$

In the discussion of computation of the Banzhaf index we only need to consider the details of computing the swing probability version, (2), since $\beta_i = \beta_i' / \sum_i \beta_i'$.

To illustrate the effect of the different coalition models on the two indices, let us consider a voting body with 10 members. Suppose we wish to measure the power of member i. Let us compare the effect of the size of the coalition on the measurement of power according to the two indices. Consider two swings, coalitions T_i which are losing until i joins, one of 4 members and one of 7 members. The Banzhaf index treats these two swings equally: each counts as one swing. The Shapley-Shubik index, however, attaches different importance to them. For the coalition of size 4, the number of orderings of its members and the remaining 5 members is 4!5! = 2,880. The contribution of this swing to the index is then 2880/10! = 2880/3628800 = 0.000793. For the other coalition, however, its contribution to the index is 7!2!/10! = 10,080/3628800 = 0.002778. Thus the number of members in a swing coalition has a considerable importance to the computation of the Shapley-Shubik index, cases where the winning coalition and its complement are relatively equal being given much less weight.

Despite being so different in the way they count swings, the two indices have given results which have not been very different in some applications. However they have given results which have differed considerably in others. There is no clear guidance from the literature on the relative merits of the two indices. While political scientists and lawyers have tended to prefer the Banzhaf index and criticized the coalition model underlying the Shapley-Shubik index, mathematicians and game theorists have tended to the reverse preference ordering because of the uniqueness of the Shapley-Shubik index.

II. Computing the Indices by Direct Enumeration

Several methods are available to compute the Shapley-Shubik indices, with simple modifications for the Banzhaf indices: Direct Enumeration; Generating Functions (Mann and Shapley (1962)); Monte Carlo simulation (Mann and Shapley (1960); Multilinear Extensions (Owen (1972, 1975)); MLE Approximation (Owen (1972, 1975)).

The simplest method is Direct Enumeration, which is based on searching over all possible coalitions and applying the basic definitions of the indices by counting swings. We have implemented this method successfully by using an algorithm which finds each subset of players once only, then for each subset it finds all swings and updates expressions (1) and (2) repeatedly. That is, for a given subset $S \subseteq N$ the procedure finds all $T_i \subseteq S-\{i\}$ such that T_i is a swing. This is relatively straightforward but expensive in computer time. It is only feasible for small and medium values of n. Experience suggests it is practical for values of n up to about 25 but it very quickly becomes prohibitively slow beyond that because computer time is an exponential function of n. This method has the advantage that it can be applied not only to evaluating power indices for simple games but more generally it can be easily adapted to find Shapley values (and other value concepts for cooperative games which assign a characteristic function to each coalition of players).

The method of generating functions of Mann and Shapley (1962) and Owen's multilinear extensions (1972) are usually regarded as more suitable for small (or medium sized) games than for large games. Undoubtedly the direct use the latter very quickly runs into size constraints but we have not investigated the feasibility of the former as a general method for when n is large.

III. Owen's Approximation Algorithm

The method we propose here is a mixed algorithm which combines direct enumeration, described above, with the approximation method due to Owen (1972, 1975). This latter method has been used for large games in a number of studies. However, it is based on the central limit theorem for approximations to expressions (1) and (2) using the normal distribution function on the basis of assumptions of probabilistic voting. In order to develop our description of the algorithm proposed in this paper, it is first necessary to describe it.

Expression (1) for the Shapley-Shubik index can be rewritten by noting that the term inside the summation is a beta function:

$$\frac{t!(n-t-1)!}{n!} = \int_0^1 x^t(1-x)^{n-t-1}dx \tag{4}$$

The integrand on the RHS of (4), $x^t(1-x)^{n-t-1}$, can be regarded as the probability that the (random) subset T_i appears, when x is the probability that any member joins T_i, constant and independent for all players j, $j \in N - \{i\}$. Summing this expression over all swings, T_i, gives the probability of a swing for i. Let us call this probability $f_i(x)$:

$$f_i(x) = \sum_{T_i} x^t(1-x)^{n-t-1}. \tag{5}$$

Integrating x out of (5) gives the index, because, substituting (4) into (1) gives:

$$\phi_i = \sum_{T_i} \int_0^1 x^t(1-x)^{n-t-1}dx = \int_0^1 [\sum_{T_i} x^t(1-x)^{n-t-1}]\, dx$$

$$= \int_0^1 f_i(x)\, dx\,. \tag{6}$$

We can approximately evaluate ϕ_i using a suitable approximation for $f_i(x)$. In large games with many small weights, and no very large weights, this can be done with reasonable accuracy using suitable probabilistic voting assumptions and the normal distribution.

Assuming each player $j \neq i$ votes the same way as i with probability x, independently of the others, defines a random variable, v_j with the following dichotomous distribution:

$$\Pr(v_j=w_j) = x, \quad \Pr(v_j=0) = 1-x, \quad \Pr(v_j \neq w_i \text{ and } v_j \neq 0) = 0$$

Therefore its first two moments are:

$$E(v_j) = xw_j, \qquad Var(v_j) = x(1-x)w_j^2, \text{ all j.}$$

The total number of votes cast by players j in the same way as that of player i is a random variable $v_i(x) = \sum_{j \in N-\{i\}} v_j$. Then $v_i(x)$ has an approximate normal distribution with moments:

$$E(v_i(x)) = xw(N-\{i\}) = \mu_i(x), \text{ say, and}$$

$$Var(v_i(x)) = x(1-x)\, h(N-\{i\}) = \sigma_i(x)^2$$

Then the required probability,

$$f_i(x) = Pr[q - w_i \leq v_i(x) < q] \qquad (7)$$

can be obtained approximately using the normal distribution function, $\Phi(.)$ by evaluating the expression:

$$f_i(x) = \Phi\left(\frac{q-\mu_i(x)}{\sigma_i(x)}\right) - \Phi\left(\frac{q-\mu_i(x)-w_i}{\sigma_i(x)}\right). \qquad (8)$$

The Shapley-Shubik index in (6) is approximated by numerically integrating out x in (8). The Banzhaf index is obtained by setting $x = 0.5$ in (8).

This method has been used in a number of studies but its accuracy depends on the validity of the normal approximation. In many real world weighted voting bodies the approximation is not good and consequent computation errors are large because of the wide range of variation of the voting weights w_i.

IV. An Extended Owen Algorithm for Large Games

For games where n is too large for direct enumeration to be efficient (or feasible, and where some of the weights too large for Owen's algorithm to be accurate, an alternative can be used which combines the essential features of both. The procedure is as follows.

The players are divided into two groups: sets of major players $M = \{1, 2, \ldots, m\}$ and minor players $N - M$. The value of m is chosen for computational convenience, along a tradeoff between accuracy and efficiency. The algorithm finds all subsets of M in the same way as does the Direct Enumeration algorithm. Given a particular subset, $S \subseteq M$, it then evaluates the approximate conditional swing probability for each player making the standard assumptions about random voting by minor players only. This is used to find the joint probability of the coalition represented by the subset and the conditional swing. The index is obtained by summing these joint probabilities over all the subsets. There are two cases to consider: (1) player i is a major player, $i \in M$; (2) i is a minor player, $i \in N - M$.

(1) Major Players

It is necessary to search over all subsets of M which do not include player i and for each consider the probability of that subset and the probability of it being a swing for i. Suppose S is a subset of $M - \{i\}$. We let the swing probability be $f_i(x)$ as before. This can be written as:

$$f_i(x) = \Pr(\text{swing for } i) = \sum_S \Pr(S)\Pr(\text{swing for } i|S)$$

Defining the conditional probability of a swing given S as the function $g_i(S, x)$, and the probability of selecting S randomly by the function $p(s, x)$, we can write:

$$f_i(x) = \sum_S p(s, x)g_i(S, x).$$

The first factor inside the summation on the RHS is

$$p(s, x) = x^s(1-x)^{m-s-1}.$$

To find the second factor we define the random variable

$$v_i(x) = \sum_{j \in N-M} v_j ,$$

where v_j is as before, to represent the random number of votes cast by the minor players.

So, $\qquad E(v_i(x)) = xw(N\text{-}M) = \mu_i(x),$

and $\qquad Var(v_i(x)) = x(1\text{-}x)h(N\text{-}M) = \sigma_i(x)^2.$

Using these moments and the normal approximation to the distribution of $v_i(x)$, we can obtain the required probability as:

$$g_i(S, x) = \Pr[q - w(S) - w_i \leq v_i(x) < q - w(S)]$$

$$= \Phi\left(\frac{q - w(S) - \mu_i(x)}{\sigma_i(x)}\right) - \Phi\left(\frac{q - w(S) - w_i - \mu_i(x)}{\sigma_i(x)}\right). \qquad (9)$$

Therefore,

$$f_i(x) = \sum_{S \subseteq M-\{i\}} x^s(1\text{-}x)^{m-s-1} g_i(S, x). \qquad (10)$$

The required index is then:

$$\phi_i = \int_0^1 f_i(x)\, dx = \int_0^1 \Big[\sum_{S \subseteq M-\{i\}} x^s(1\text{-}x)^{m-s-1} g_i(S, x)\Big]dx$$

$$= \sum_{S \subseteq M-\{i\}} \int_0^1 x^s(1\text{-}x)^{m-s-1} g_i(S, x)dx$$

which can be found by searching over all subsets of M-{i}, integrating out x by numerical quadrature at each subset then summing. The Banzhaf index β'_i is obtained from (10) on setting x=0.5 instead of integrating it out, then summing to give $\beta'_i = f_i(0.5)$.

The search described here is over all subsets of M-{i}, but it is obvious that we can include M itself since $g_i(M, x) = 0$ for all $i \in M$.

(2) Minor Players

Now we describe the computation of the indices for the smaller players, $i \in N\text{-}M$. The subset S is now any subset of M. Since we are now treating the votes of all m major players as random (not just m-1 of them), we write the probability of the subset S:

$$\Pr(S) = p(S, x) = x^s(1-x)^{m-s}.$$

The behavior of the minor players other than i is described by a random variable $y_i(x) = \sum_{j \in N-M-\{i\}} v_j$ which has an approximate normal distribution with moments:

$$\mu_i(x) = xw(N\text{-}M\text{-}\{i\})$$

and $$\sigma_i(x)^2 = x(1-x)h(N\text{-}M\text{-}\{i\}).$$

Hence we can evaluate the conditional swing probability $g_i(S, x)$ which now is:

$$g_i(S, x) = \Pr[q - w(S) - w_i \le y_i(x) < q - w(S)]$$

approximately by the normal probability in expression (9) again after making the required notational substitutions.

Then, writing

$$f_i(x) = \sum_{S \subseteq M} p(s, x)\ g_i(S, x),$$

the Shapley-Shubik index is found again by quadrature, and the Banzhaf index by setting x=0.5, before summing, within the same subset search as before, $S \subseteq M$.

This algorithm has proved to be efficient and accurate in large games with $n > 100$ with a moderate and feasible choice of m. The obvious rule for choosing the value of m is that it should be large enough to ensure accuracy without being too large to permit all subsets of M to be enumerated feasibly in a reasonable computing time. Some examples of the application of the approach are given in the next section.

V. Some Examples

In this section we present the results of using the algorithm for practical computations. Our implementation uses a subroutine which finds every subset of a set exactly once, taken from Nijenhuis and Wilf (1983), a quadrature subroutine due to Patterson (1968) in NAG(1997) and double precision arithmetic throughout.

We present two examples: the United States Electoral College States game as reported in Owen (1995), and an artificial example with randomly generated weights. The application to the Electoral College States game is presented in order to establish the accuracy of the algorithm against previously published results. This example is a useful test case because we are in the fortunate position of having exact Shapley-Shubik indices. For this case however the Owen algorithm is reasonably accurate without the modifications proposed here. We therefore present a second example in which the weights have been generated randomly to be much more unequal and where accuracy is much improved by the use of the algorithm.

Example 1: The US Electoral College States Game, 1970 Census

Owen (1995) presents results for both indices obtained using his approximation method and also exact results for the Shapley-Shubik index obtained using the method of generating functions. This is therefore a suitable test bed to establish the utility of the algorithm. We have computed the power indices for the 51 states for all values of m from 0 to 20. The results are shown in Table 1 for the Shapley-Shubik index for m =0, 5, 10, 15 and 20. The algorithm is very accurate (to the reported accuracy of the indices given in Owen (1995)) for m=20; such slight differences as there are seem to be due to rounding error in the earlier results.

Figures 1 to 3 show the effect of changing m on the accuracy of the indices for certain states: those numbered i=1, 2, 3, 18 and 51. Figure 1 shows graphs of the effect of changing m on the Shapley-Shubik indices for these states in Figures 1(a) to 1(e) while Figure 1(e) shows the comparative computation errors for the five indices. Figures 2 and 3 show the comparable graphs for the Non-Normalized Banzhaf index and the Normalized Banzhaf index respectively. The general conclusion is that the algorithm performs well and that there is very little to be gained in improved accuracy by increasing m beyond 5.

Table 1: Shapley-Shubik Index for the Electoral College Game

Votes	States	m=0	m=5	m=10	m=15	m=20	Exact
45	1	0.08856	0.08838	0.08833	0.08832	0.08832	0.08831
41	1	0.07997	0.07979	0.07975	0.07974	0.07973	0.07973
27	1	0.05114	0.05100	0.05098	0.05097	0.05097	0.05096
26	2	0.04915	0.04901	0.04899	0.04898	0.04898	0.04898
25	1	0.04716	0.04702	0.04701	0.04701	0.04700	0.04700
21	1	0.03931	0.03919	0.03918	0.03917	0.03917	0.03917
17	2	0.03158	0.03148	0.03147	0.03147	0.03147	0.03147
14	1	0.02586	0.02578	0.02577	0.02577	0.02577	0.02577
13	2	0.02396	0.02390	0.02389	0.02389	0.02388	0.02388
12	3	0.02208	0.02202	0.02201	0.02201	0.02201	0.02200
11	1	0.02020	0.02015	0.02014	0.02014	0.02013	0.02013
10	4	0.01833	0.01828	0.01827	0.01827	0.01827	0.01827
9	4	0.01647	0.01642	0.01642	0.01641	0.01641	0.01641
8	2	0.01461	0.01457	0.01457	0.01456	0.01456	0.01456
7	4	0.01276	0.01273	0.01272	0.01272	0.01272	0.01272
6	4	0.01092	0.01089	0.01089	0.01088	0.01088	0.01088
5	1	0.009083	0.009059	0.009055	0.009054	0.009054	0.009053
4	9	0.007253	0.007234	0.007231	0.007230	0.007230	0.007230
3	7	0.005430	0.005416	0.005414	0.005413	0.005413	0.005412

q=269.5.

Figure 1 Electoral College: Shapley-Shubik Indices

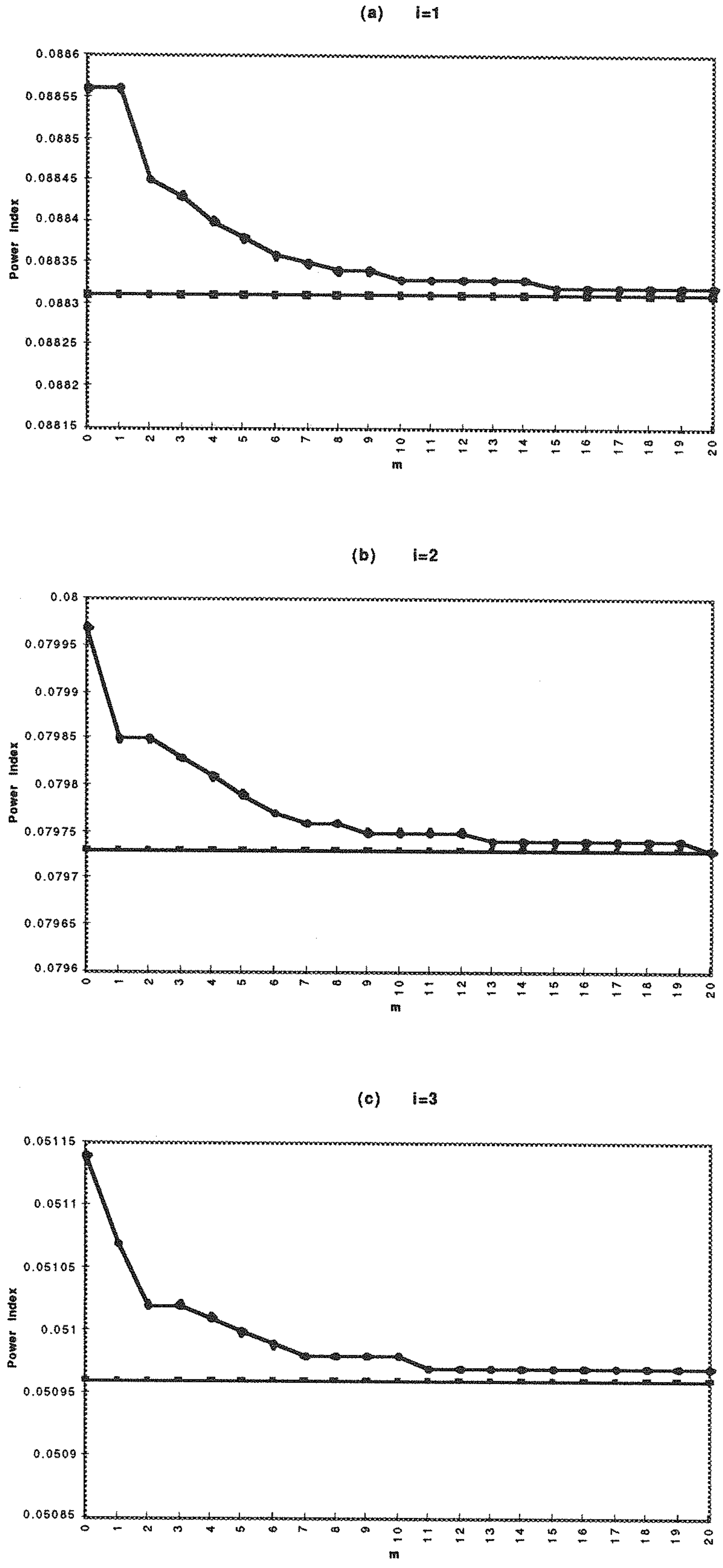
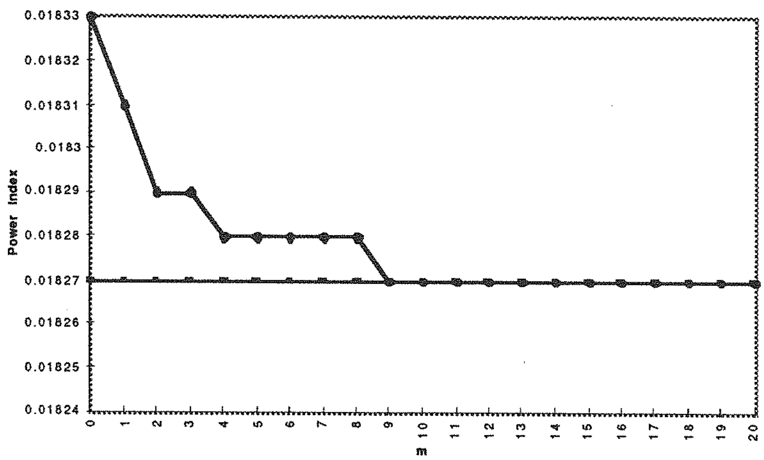
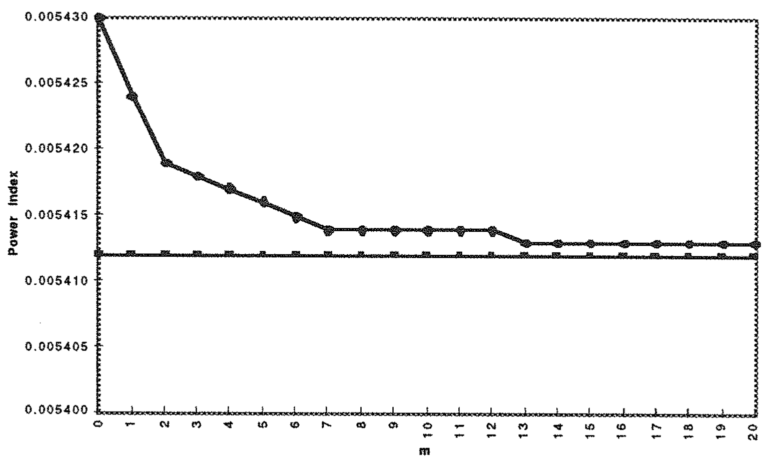


Figure 1 (Continued): Electoral College: Shapley-Shubik Indices

(d) i=18



(e) i=51



(f) Relative Errors, Shapley-Shubik Index

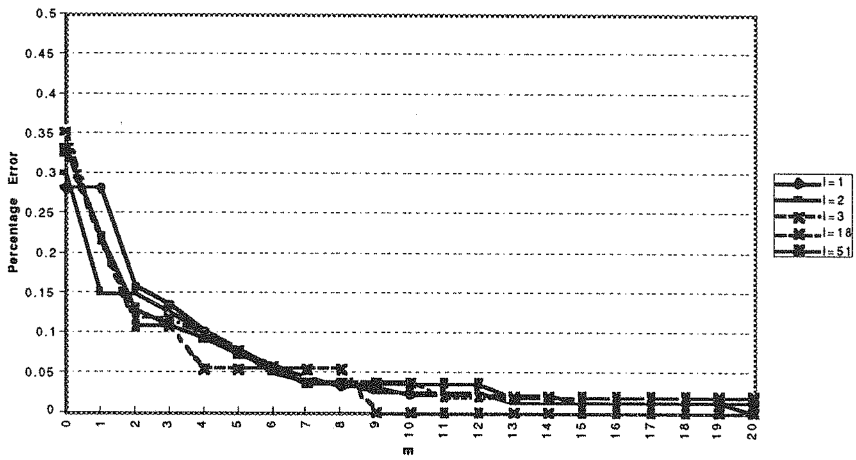
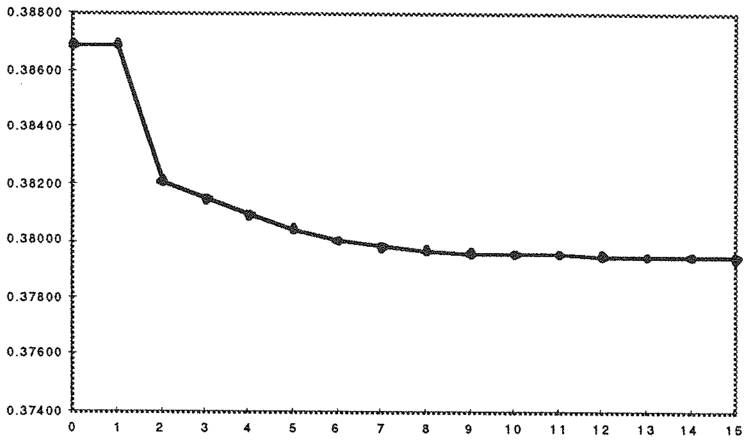
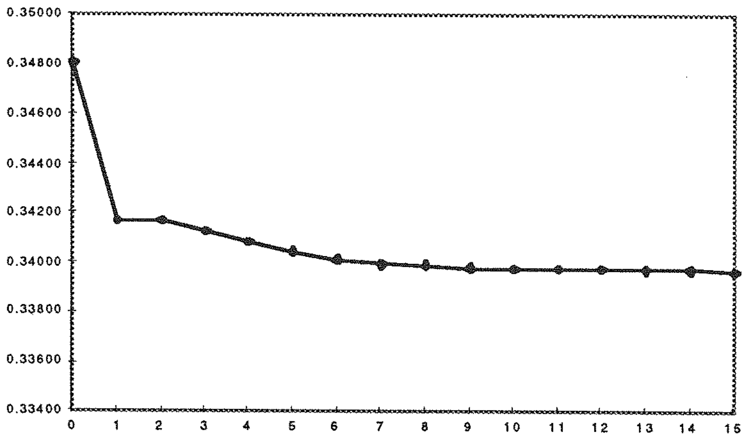


Figure 2 Electoral College: Non-Normalized Banzhaf Indices

(a) i=1



(b) i=2



(c) i=3

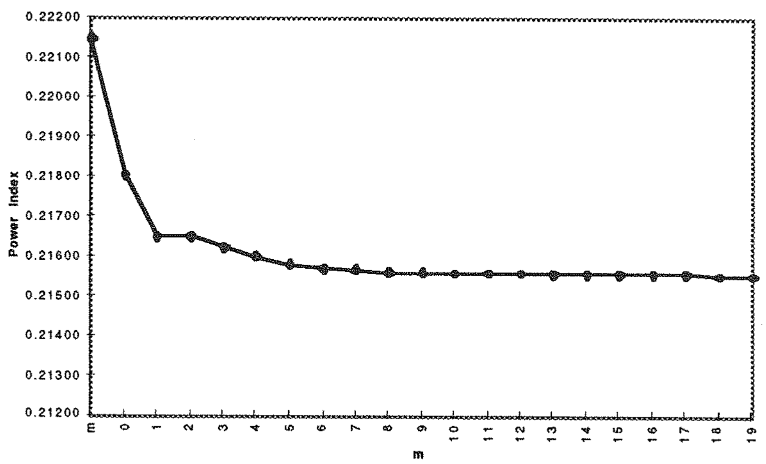
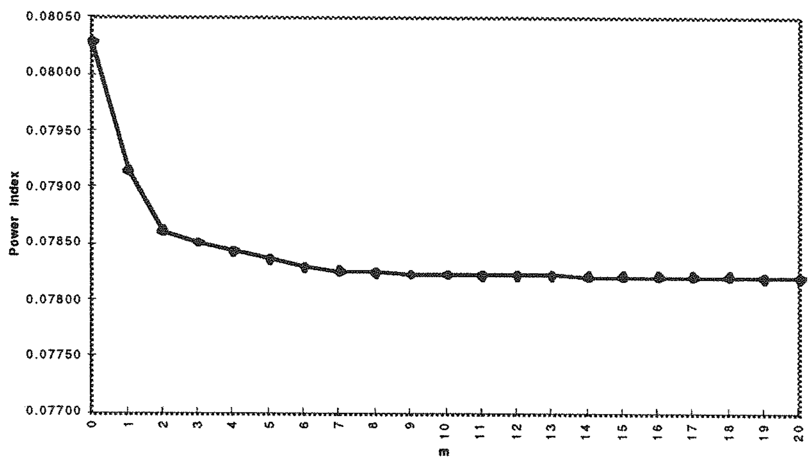
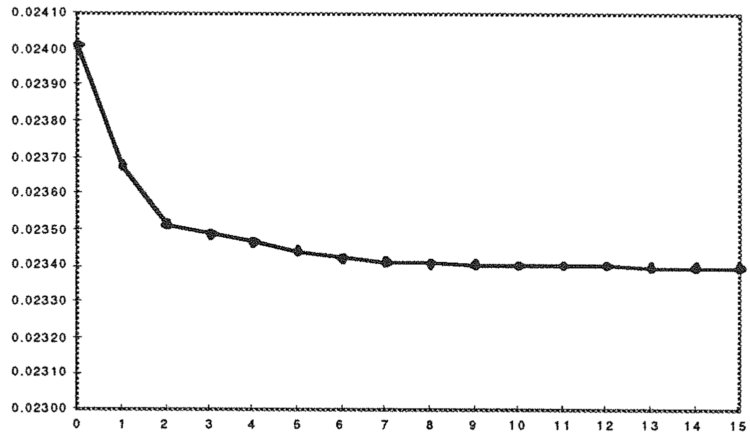


Figure 2 (Continued): Electoral College: Non-Normalized Banzhaf Indices

(d) i=18



(e) i=51



(f) Relative Errors, Non-Normalised Banzhaf Index

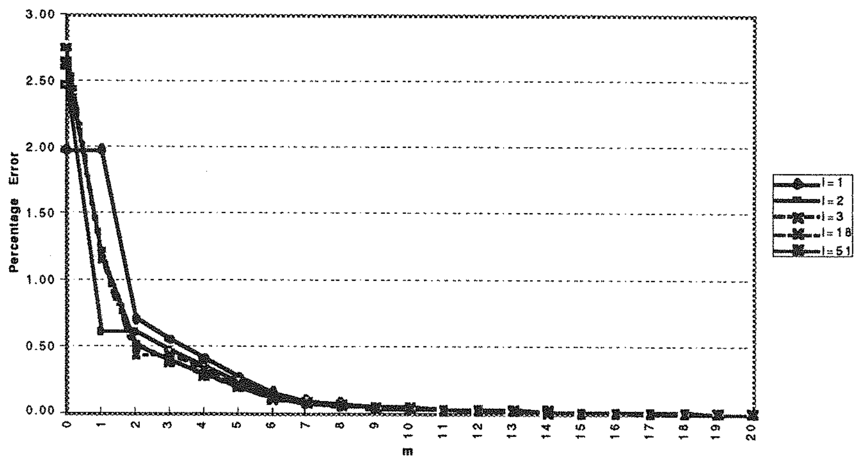
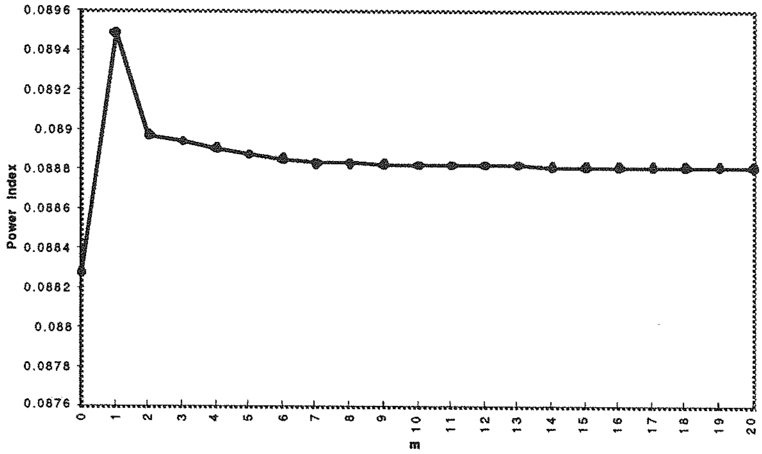
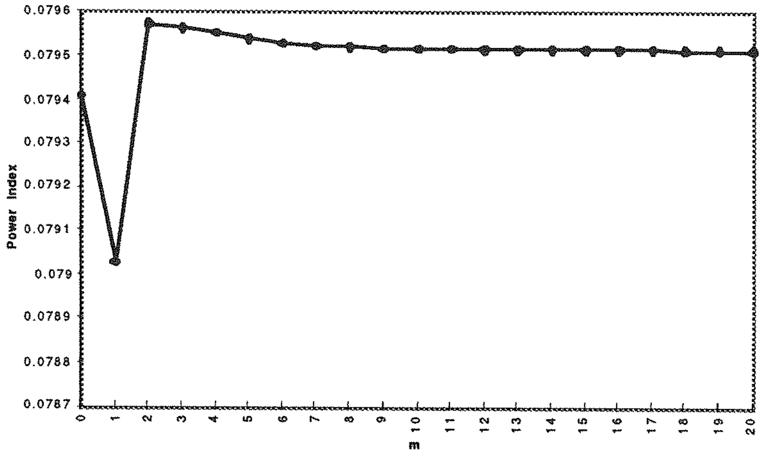


Figure 3: Electoral College: Normalized Banzhaf Indices

(a) i=1



(b) i=2



(c) i=3

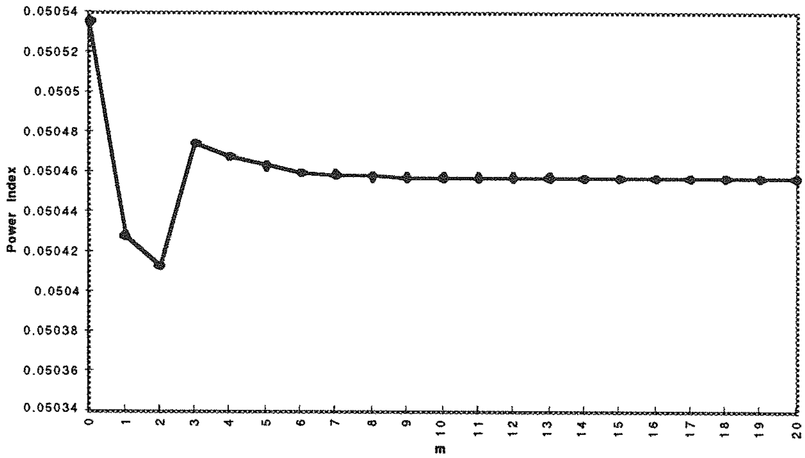
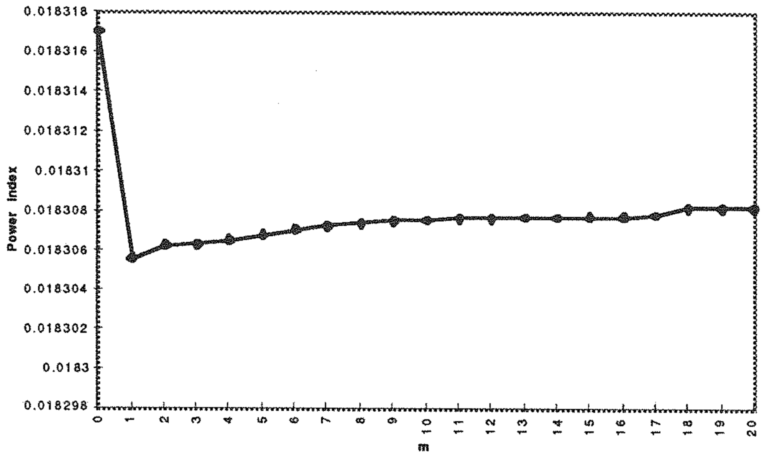
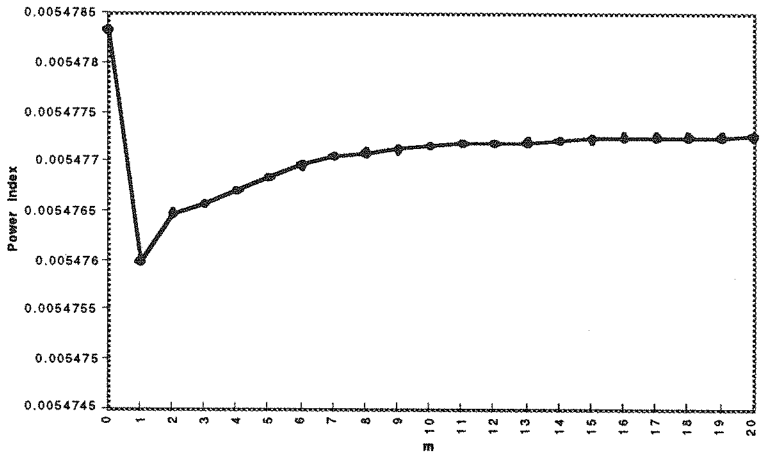


Figure 3 (Continued): Electoral College: Normalized Banzhaf Indices

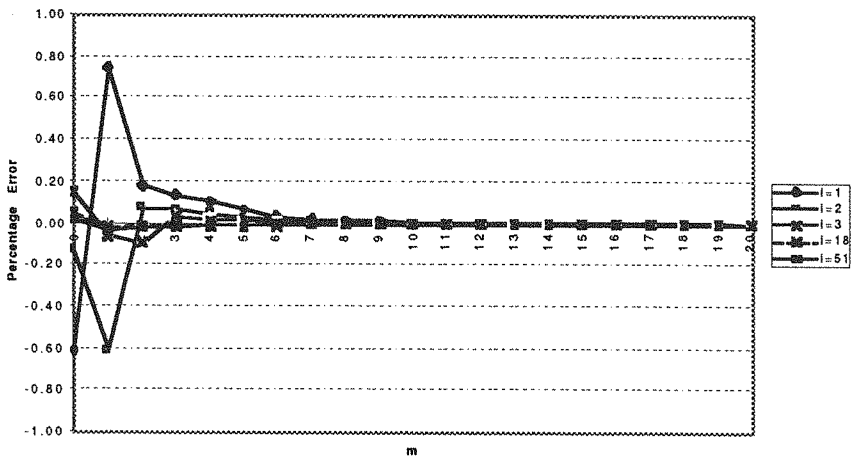
(d) i=18



(e) i=51



(f) Relative Errors, Normalised Banzhaf Index



Example 2

The second example is a game with artificially generated weights intended to capture the characteristic pattern often encountered in real voting bodies with a large number of members where there are a number of relatively large weights. The presence of these large players mean that the Owen algorithm is likely to be inaccurate. Very good results are obtained using our algorithm, however.

We have chosen n=100, q=55% and the weights have been generated at random (before being expressed as percentages) from a suitable distribution: w_1, w_2, . . . , w_{100} are a sample from the lognormal distribution $\Lambda(4.3, 3)$. The resulting distribution of votes is relatively concentrated. Although this concentration makes it necessary to choose a value of m greater than zero to get reasonable accuracy, for the same reason it is not necessary to use a large value of m. The results in Table 2 have been obtained using m=12. Table 2 shows the results for the players with the largest 10 weights and for i =30, 60, 100. It shows the two power normalized power indices, which sum to 1 over the players, and the power ratios, power index/ weight, which measure the discrepancy between voting weight and voting power.

The relative computation errors for various arbitrarily chosen players are shown in Figure 4. It is clear that while there are large errors for m=0 (the Owen MLE approximation) and for small m, they disappear quite quickly. For the large players the errors in all the indices are negligible for m≥5. For the small players the errors in the Shapley-Shubik indices and the Non-normalized Banzhaf indices become negligible for m≥5 and those in the Normalized Banzhaf indices for m≥7 in the case of the smallest player, i=100.

Table 2: Selected Voting Weights, Power Indices and Power Ratios

i	Voting Weights	Normalized Banzhaf Index	Banzhaf Power Ratio	Shapley-Shubik Index	Shapley-Shubik Power Ratio
1	0.2915	0.3522	1.2083	0.3506	1.2025
2	0.1960	0.1711	0.8733	0.1850	0.9440
3	0.1170	0.1147	0.9801	0.1142	0.9762
4	0.0970	0.0930	0.9588	0.0898	0.9262
5	0.0640	0.0708	1.1064	0.0566	0.8845
6	0.0248	0.0205	0.8276	0.0217	0.8742
7	0.0222	0.0185	0.8349	0.0194	0.8731
8	0.0200	0.0168	0.8394	0.0175	0.8717
9	0.0179	0.0151	0.8432	0.0156	0.8701
10	0.0168	0.0142	0.8449	0.0146	0.8692
...	...	...	...	...	...
30	0.0855	0.00073	0.8549	0.00073	0.8544
...	...	...	...	...	...
60	0.00636	0.000054	0.8550	0.000054	0.8535
...	...	...	...	...	...
100	0.0000001	0.00000009	0.9000	0.00000009	0.9000
Sum	1.0000	1.0000		1.0000	

q=0.55

Figure 4 Relative Approximation Errors

Figure 4 (a): Relative Errors: Shapley-Shubik Indices

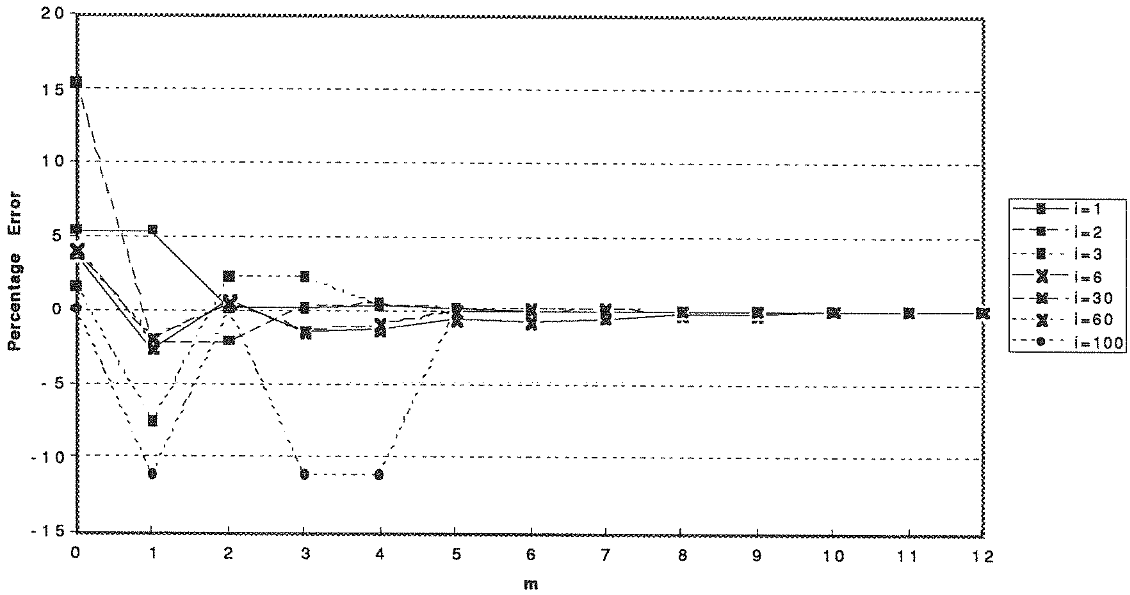


Figure 4 (b): Relative Errors: Non-Normalised Banzhaf Indices

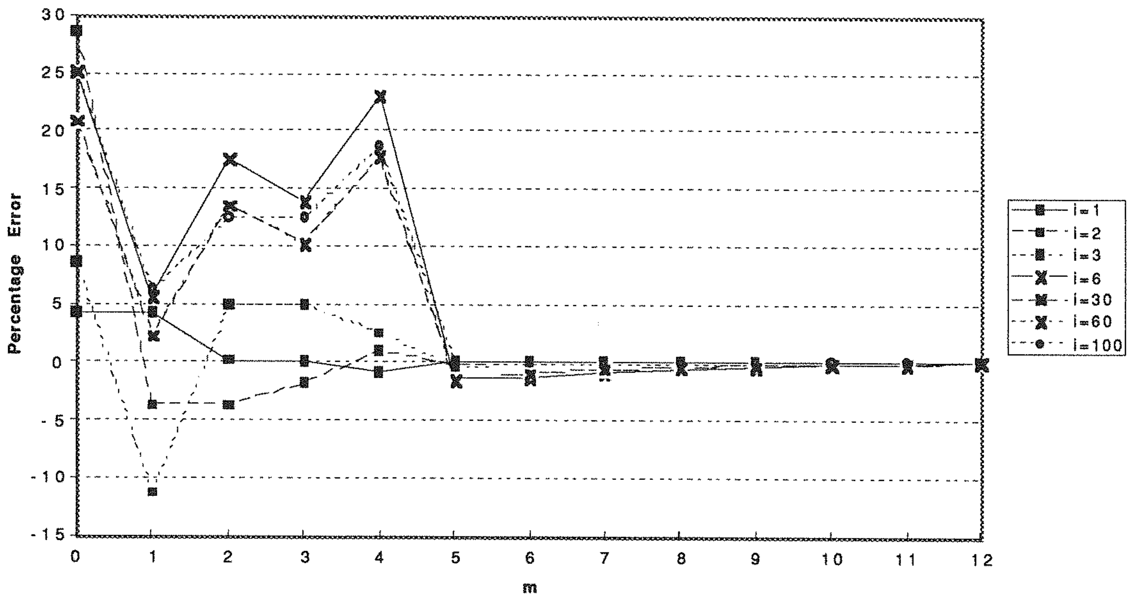
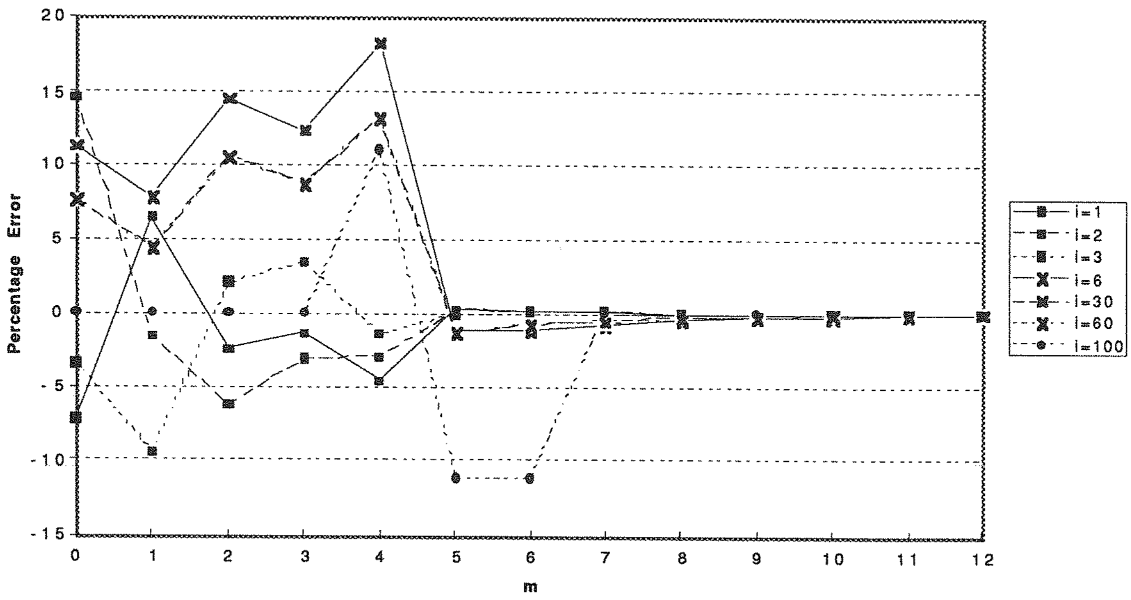


Figure 4 (c): Relative Errors: Normalised Banzhaf Indices



VI. Conclusion

We have described a new algorithm for computing values of voting games enabling the easier and better application of classical power indices to the kind of large weighted voting bodies which often occur in reality. An example is the Board of Governors of the International Monetary Fund, which has nearly 200 members, and employs weighted voting with weights varying over a wide range reflecting the financial quotas of each member country; an analysis of the resulting distribution of voting power has been made using the algorithm and is reported in Leech (1998). The method is capable of achieving a high degree of accuracy without excessive cost in terms of computing time in real applications. We suggest that advances in computing of this type might facilitate further analyses of power in weighted voting bodies and legislatures which could not only enhance our understanding of voting systems but also contribute to advances in the understanding of the relative properties and utility of the different indices.

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