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Paul Beaudry and Charles Blackorby

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by

Paul Beaudry and Charles Blackorby

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Paul Beaudry: University of British Columbia, and NBER. (Email: Beaudry@econ.ubc.ca)
Charles Blackorby: University of Warwick and GREQAM (Email: c.blackorby@warwick.ac.uk)
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Send correspondence to Charles Blackorby at Economics Department, University of Warwick, Gibbet Hill Road, Coventry, UK, CV4 7AL; c.blackorby@warwick.ac.uk.

Abstract

This paper explores how to optimally set tax and transfers when taxation authorities: (1) are uninformed about individuals' value of time in both market and non-market activities and (2) can observe both market-income and time allocated to market employment. In contrast to much of the optimal income taxation literature, we show that optimal redistribution in this environment involves distorting market employment upwards for low net-income individuals through phased-out wage-contingent employment subsidies, and distorting employment downward for high net-income individuals through positive and increasing marginal income tax rate. We also show that workfare may also be used as part of an optimal redistribution program.

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1. Introduction

In most countries income redistribution is achieved through a variety of programs: these include direct income taxation, employment programs, welfare, unemployment insurance and pension schemes. Viewed as a whole, these programs create intricate incentives and complex redistribution patterns. Since the conditionality of these programs are quite varied, they generally result in a net tax-transfer system that depends not only on income but often depends on the extent of market participation as well. Reasoned economic policy should attempt to identify whether or not these programs are mutually consistent with the goal of redistribution.

The object of this paper is to explore the principals that should guide the evaluation of tax-transfer systems that depend on both market income and on quantity of time worked. In order to illustrate the type of issues we want to address, let us start with an example of an individual who pays taxes or receives transfers from a government depending on his or her interaction with three different systems: an income tax system, a social assistance system (welfare) and an unemployment insurance system.[1] The example is inspired by the Canadian social system, however, it has been purposely simplified to clarify issues and therefore the numerical values should be viewed as mainly illustrative.

Let y represent an individual's market income, let h represent the number of weeks (≤ 50) worked by an individual over a year and let T represent total taxes (net of transfers) paid by the individual over a year.

The income tax system:

If $y \leq \$6000$, there is no income tax; on income above \$6000, a marginal income tax of 20% is applied (i.e., total income tax equals $\max[.2(y-6000), 0]$).

The social assistance system (welfare):

If $y \leq \$6000$, the social assistance payment is $\$6000 - y$; if $y > \$6000$, there is no social assistance payment.

The unemployment insurance system:

Letting h be the number of weeks worked: if $h \leq 10$, the individual is not eligible for unemployment insurance; if $10 < h \leq 30$, then the individual is eligible for $h - 10$ weeks of unemployment insurance payments at 60% of weekly wages, up to a maximum

[1] For simplicity, we have not included in the example the interaction with the pension system. However, the issues we address are also potentially relevant for pension systems since these programs have pay-outs that depend both income earned and of amount worked.

payment of \$400 per week; if $30 < h < 50$, the individual is eligible for $50 - h$ weeks of unemployment insurance payments at 60% of weekly wages, up to a maximum payment of \$400 per week.

Consider the net tax implication of these three systems combined. The net amount of taxes paid (or transfer received) depends both on an individual's wage rate and on the number of weeks worked. Hence the pattern of tax rates rates faced by individuals varies with different market wage rates. In particular, consider the case where individual 1 earns \$600 per week worked, and individual 2 earns \$1000 per week worked. Then, the net taxes-transfers, T, paid by individuals 1 and 2 as a function of annual income is given below where, in calculating these tax rates, we assume that an individual receives an unemployment insurance payment for any eligible non-working weeks:

Tax function of individual 1:

If $y \leq 6000$, $T = y - 6000$ (marginal rate of 100%);
If $6000 < y \leq 18000$, $T = -.28(y - 6000)$ (marginal rate of -28%);
If $18000 < y$, $T = -3360 + .68(y - 18000)$ (marginal rate of 68%);

Tax function of individual 2:

If $y \leq 6000$, $T = y - 6000$ (marginal rate of 100%);
If $6000 < y \leq 10000$, $T = .2(y - 6000)$ (marginal rate of 20%);
If $10000 < y \leq 30000$, $T = 800 - .12(y - 10000)$ (marginal rate of -12%);
If $30000 < y$, $T = -1600 + .52(y - 30000)$ (marginal rate of 52%).

There are three aspects to notice about this tax-transfer system as also depicted in Figure 0. First, the tax rate depends not only on income but also depends on a worker's revealed market type, that is his or her wage rate. In particular, note that marginal tax rates are different at different income levels depending on a worker's wage rate. Second, the individuals face high marginal tax rates at both high and low income levels. Third, the individuals face negative marginal taxes rates for intermediate income segments. Let us emphasize that all these features stand in stark contrast to the prescriptions one would derive from a Mirrlees' type optimal tax problem. (There is a fourth property: the marginal taxes rates are neither monotonically increasing or decreasing; this does not contradict Mirrlees, but is of interest to us.) However, given that the above example allows taxes rates to be wage dependent, we immediately know that the Mirrlees' analysis does not directly apply and hence an alternative framework is needed.

In this paper, we examine an optimal income tax problem in hope of providing guidance on how to evaluate such a system. For example, we would like to know how to evaluate the efficiency of the properties noted above when the government can design a tax system that depends both on income and wage rates (or the amount of weeks worked). Moreover, since we believe than one of the concerns of governments is to avoid transferring substantial income to individuals that simply do not want to engage in market employment, our analysis recognizes that individuals may have different values for their non-market time.

Our approach to the problem follows the optimal non-linear income taxation literature as pioneered by Mirrlees (1971),[2] that is, we approach redistribution as a welfare maximization problem constrained by informational asymmetries. However, we depart in two directions from the Mirrlees' formulation. The first concerns the perceived need to target more effectively income transfers. For example, traditional welfare programs (or minimum revenue guarantees) are often criticized on the grounds that they transfer substantial income to individuals who value highly their non-market time, as opposed to transferring income only to the most needy. Although such a preoccupation is common, the literature is mostly mute on how to address this issue since the standard framework assumes that individuals value their non-market time identically. The second issue relates to the possibility of using work time requirements as a means of targeting transfers. Many social programs – such as most unemployment insurance programs or pension programs – employ information on time worked (either in years, weeks or hours) in order to determine eligibility; therefore it seems reasonable to allow for such a possibility when considering how best to redistribute income. Hence, the environment we examine is one where (1) taxation authorities are uninformed about individuals' potential value of time in market activities and about their potential value of time in non-market activities[3], and (2) income transfers can be contingent on both earned (market) income and on the allocation of time to market employment. Under the above assumptions, our redistribution problem formally becomes a multidimensional screening problem with two-dimensions of unobserved characteristics.[4]

Given the two-dimensional informational asymmetry, it is not surprising that the properties of the optimal redistribution program derived under our informational and observability assumptions are quite distinct from those found in the standard setup. More specifically, we show that optimal redistribution in our environment entails

[2] See also Mirrlees (1997).

[3] In our formulation, non-market activities can be interpreted as non-declared market activities.

[4] Screening problems with two-dimensions of unobserved characteristics are becoming more common in the literature. See Armstrong (1996) and Rochet & Chone (1998) for the state of the art in this literature and a discussion of some of the difficulties associated with solving such problems.

- distorting upwards the employment level of low net-income earners through negative marginal tax rates that are phased out, or equivalently, wage contingent phased-out employment subsidies;
- distorting downward the employment level of high net income earners through positive and increasing marginal tax rates;
- using public employment requirements (workfare) as a means to transfer income to individuals with very poor or non-existent market possibilities.

The above results provide a stark contrast with those of the non-linear taxation literature in large measure because in that literature the informational asymmetry is restricted to the value of market time.[5] Recall that the main prescriptions derived by Mirrlees are that

- marginal tax rates be everywhere non-negative, and
- there be a zero marginal tax rate on the most productive individual(s).[6]

In order to help understand these general results, we examine a number of special cases. We begin by considering a case that is, roughly, the dual of the Mirrlees' problem, that is, we examine a case where the market productivity of each individual is known but where only the distribution of the outside options is known. In this context, we show that that there exists a wage rate such that everyone above it faces non-negative marginal tax rates and everyone below it faces non-positive marginal tax rates. In other words, in this case, there is a simple dichotomy of treatment whereby individuals as either marginally taxed or marginally subsidized depending only on their wage rate. Moreover, for individuals that are taxed, they face increasing marginal tax rates. While for individuals that are subsidized, they face decreasing marginal subsidies.

We then show that this simple dichotomy disappears when neither market productivity nor non-market characteristics are observable. However, in this more general case, we are nevertheless able to show that the optimal tax structure maintains much the same flavor, that is, wage rates are extensively used to determine whether an individual is taxed or subsidized. In particular, we establish the optimality of phased-out wage subsidies as a means of redistributing income, that is, subsidies which are directed towards specific wage groups which decrease in intensity as a worker increases his work time. Moreover we show that, when market productivity is unobservable, the optimal redistribution program may

[5] Preferences in standard treatments are restricted to the single-crossing property without which virtually nothing can be said. We use an assumption that can be viewed as a strong version of the single-crossing property.

[6] This assumes that such an individual exists (Mirrlees (1997)). It has also been shown that, if the least productive individual is employed under the optimal scheme, then he or she also faces a zero marginal tax rate. However, this case is generally not considered to be very relevant since the least productive individual generally does not work under the optimal scheme.

have to marginally subsidize types that are highly skilled but work little in order to permit incentive compatible transfers to the most needy.

In order to facilitate comparison of our results with the standard ones found in the literature we consider in Section 6 two special cases: perfect negative correlation between the two sets of characteristics and perfect positive correlation between them. The former leads to an undistorted solution that is similar to that obtained by Dasgupta and Hammond (1980); the latter leaves the highly skilled undistorted but highly taxed while providing distorting wage subsidies for the low end of distribution.

Since his seminal contribution, Mirrlees' analysis has been extended in several direction.[7] In particular, Besley and Coate (1995) have shown that it is not optimal to complement the optimal non-linear taxation schedule with workfare, that is, it is never efficient to make income transfers contingent on public employment.[8] One surprising aspect of these properties is how they appear to conflict with many of the current policies debates which, *de facto*, tend to favor active employment programs such as employment subsidies (negative marginal taxation) and workfare.[9] Hopefully, this paper sheds new light on such policy debates.

The paper is structured as follows. In Section 2 we present the information constrained redistribution problem and derive simple properties of the associated optimal direct revelation mechanism as well as the properties of the tax system that implements the allocation. In Section 3 we define the concepts used to solve the problem and in Section 4 we analyze the dual to the Mirrlees' formulation, that is, the case where only the value of non-market opportunities are unknown.[10] In Section 5 we present our main results for the case where both the value of an individual's time in market and non-market activities are unknown, and we explore in detail the case where there are only two possible market wage rates but many outside options. In Section 6 we consider the problem when the the two sets of characteristics are correlated. We then discuss the implications of our results for workfare and other related issues.

[7] Many of the extensions of Mirrlees original analysis involve giving more tools to the taxation authorities. For example, see Guesnerie and Roberts (1987) or Marceau and Boadway (1994).

[8] This is not the main emphasis of this Besley and Coate paper. In fact, their main result is to show that workfare can be part of an optimal program when income maintenance is the objective as opposed to welfare maximization. They view their result as mainly providing a positive theory of workfare as opposed to a normative prescription. See also Brett (1998), Cuff (2000), and Kanbur, Keen and Tuomola (1994) on this issue.

[9] It is precisely this observation which motives the work by Besley and Coate (1995), Brett(1998), Cuff (2000), and Kanbur, Keen and Tuomola (1994).

[10] All proofs are relegated to the Appendix.

2. The Environment and the Pattern of Second Best Distortions

The economy has two sectors—a formal market sector and an informal, non-market or household sector. An individual can work in the formal/market sector at a wage rate no greater than his or her intrinsic productivity. Income earned in the formal/market sector can be observed and hence taxed. The amount of time allocated to the formal/market sector can also be observed. Besides working in the formal/market sector, an individual can also allocate time to production in the informal/household sector. Production in this sector is unobservable.[11] Each agent is endowed with a fixed number of hours which we have normalized to one; if individual i works for $h_i \geq 0$ hours in the formal sector, he or she has $1 - h_i$ hours available for producing goods in the informal/household sector.[12] Individuals have identical utility functions that are known and which depend upon the consumption of goods from both sectors of the economy. Individuals differ in their abilities and the ability level can vary across sectors. For example, one may be very productive in the formal/market sector but have low productivity in the informal/household sector or conversely.

Before describing our problem further, it is worth discussing our assumption regarding the observability of time worked, which could represent hours, weeks or years. This is particularly relevant since the more common assumption in the literature is that hours worked is not observable[13] and that only income is observable. We believe it is justified to assume that time worked is observable; in practice hours or weeks worked are used in many countries to determine eligibility for social programs. For example, in Canada, one of the biggest social programs is unemployment insurance. Eligibility and payments from Canadian unemployment insurance system depend explicitly on income and amount of time worked (both in terms of weeks and hours per week). This is a clear example of a large program that exploits information on time worked to determine transfers. Problems with measuring time worked does not appear very important.[14] As another example, currently,

[11] Alternatively, production in the household sector can be viewed as leisure with individuals having different tastes for leisure.

[12] It is assumed throughout the paper that the agent can choose how to allocate his or her time endowment in a continuous fashion. However, the results of this paper can be generalized to the case where h can only take on a discrete set of values (for example 0 and 1). The results generalize quite easily because the proofs do not exploit the continuous nature of the labor supply decision.

[13] Dasgupta and Hammond (1980) and Maderner and Rochet (1995) also examine optimal redistribution in environments where taxation authorities can transfer income based on market-income and market allocation of time. However, in these papers there is only one dimension of unobserved characteristics. See also Kesselman (1973) and Bloomquist (1981) for a related literature.

[14] There are obviously some groups in society for which it is very difficult to measure the amount of time worked, for example the self-employed. Accordingly, these groups are often excluded from programs such as unemployment insurance.

in Canada, there is a large scale experiment aimed at encouraging welfare recipients to work; this program is called the self-sufficiency project (see Card and Robins (1996) for details). One particular aspect of this program is that it explicitly requires individuals to work 30 hours per week in order to be eligible for a transfer; recipients are required to mail in pay stubs showing their hours of work and earnings for the month. Again, this illustrates that social programs currently use information on time-worked and therefore it seems relevant to allow for such a possibility in our analysis.

In our setup, types are indexed by $i \in I = \{1, \ldots, NM\}$ where for each type there is a two-tuple (ω_i, θ_i); $\omega_i \in \{\omega_1, \ldots, \omega_N\}$ is the productivity index of an individual of type i in the formal/market sector and $\theta_i \in \{\theta_1, \ldots, \theta_M\}$ is the productivity index in the informal/household sector. We assume that $\theta_i > 0$ and, for simplicity, we assume that $\omega_i \neq \theta_i$ for all i. The percentage of the population that is type i is denoted p_i and we impose no restrictions on the distribution of types.[15] Consumption in the formal sector is denoted c_i and is referred to as an individual's net (after tax) income. An individual's pre-tax income is denoted y_i, where pre-tax income is $y_i = h_i w_i$ with w_i being the wage rate received by type i in the formal/market sector. Individuals evaluate their well-being by means of a utility function, $U : \mathbf{R}_+ \mapsto \mathbf{R}$, which is defined on total consumption and assumed to be concave and differentiable. Individual i's total consumption is given by $c_i + (1 - h_i)\theta_i$. Implicit in this formulation is the assumption that the goods from the two sectors are perfect substitutes and that the production technology is linear.[16]

An allocation in this economy is a mapping that associates with every type (ω_i, θ_i) a triplet composed of (1) a consumption level for the formal/market sector good, (2) the hours supplied in the formal/market sector and (3) the wage rate in formal sector employment (or alternatively the income in the formal sector). Therefore an allocation in this economy corresponds to a sequence of the form $\{c_i, h_i, w_i\}_{i=1}^{NM}$. A particular element of this sequence, say (c_j, h_j, w_j), is referred to as j's allocation. Since we will also be interested in the tax structure that implements the optimal allocation, it is useful to denote the net tax paid by individual i by T_i recognizing that by definition T_i is equal $w_i h_i - c_i$.

[15] See Diamond (1998) for results obtained in the Mirrlees' framework when the the distribution of types is constrained. In Section 6 we analyze two extreme restrictions on the distribution of types, perfect positive and perfect negative correlation.

[16] The more general case would be to assume that utility is of the form $U(c_i, (1 - h_i)\theta_i)$. In this case, our results would likely depend on the cross-derivative of the utility function of which we know very little. Hence, we have chosen the case of perfect substitutes as reasonable starting point for the analysis of a problem with two dimensions of heterogeneity.

The government's objective is to maximize a utilitarian social welfare function[17] but is unable to implement a first-best optimum due to the asymmetry of information. In particular, it is assumed that the government cannot observe skill levels of individuals in either sector, that is, the government cannot observe either ω_i or θ_i. Under the above assumption, the government's maximization problem can be stated as follows.[18]

An optimal allocation is a $3NM$-tuple $\{\overset{*}{c}_i, \overset{*}{h}_i, \overset{*}{w}_i\}_{i=1}^{NM}$ that maximizes

$$\sum_{i=1}^{NM} p_i U\big(c_i + (1-h_i)\theta_i\big) \tag{2.1}$$

subject to

$$\sum_{i=1}^{NM} p_i c_i \le \sum_{i=1}^{NM} p_i w_i h_i, \tag{2.2}$$

and for all i

$$U\big(c_i + (1-h_i)\theta_i\big) \ge U\big(c_j + (1-h_j)\theta_i\big), \quad \forall j \quad \text{s.t.} \quad w_j \le \omega_i \tag{2.3}$$

$$U\big(c_i + (1-h_i)\theta_i\big) \ge U\big(\theta_i\big), \tag{2.4}$$

$$0 \le h_i \le 1 \tag{2.5}$$

In the above problem, (2.2) represents the materials balance constraint, (2.3) represents the incentive compatibility constraints and (2.4) represents the individual participation constraints. Since the incentive compatibility constraints in this problem are not standard, some clarification is in order. The implicit assumption in the above formulation is that an individual can costlessly mimic any other individual who has a lower market productivity, that is, individual i can choose to be employed in any job paying a wage $w \le \omega_i$. In effect, the incentive compatibility constraint (2.3) insures that individual i finds his or her allocation at least as good as that of any agent employed at a wage no greater than his or her own market productivity ω_i.[19] The participation constraints, (2.4), are also non-standard but are important in this framework. In effect, the participation constraint reflects our assumption that the government cannot impose (or collect) a positive tax on an individual with no market income, that is, the fruits of non-market activity are not transferable to the government.

[17] The results of this paper do not depend on a strict utilitarian perspective, but do depend on welfarism and quasi-concavity.

[18] Our formulation of the problem, which is standard in the taxation literature, restricts the government to a direct mechanism where individual allocations depend only on their own announcements.

[19] This formulation of incentive compatibility constraints has been examined by Dasgupta and Hammond (1980) in a one-dimensional problem.

Under this assumption, note that any individual can guaranty himself a minimum level of utility by simply not participating in market activities.[20]

Other aspects of this problem are worth explaining before proceeding. First, in hours-consumption space, the indifference curves of type (ω_i, θ_i) are parallel straight lines whose slope is θ_i. Utility increases in the northwesterly direction. At the point where an indifference curve hits the $h = 1$ line at consumption level c_i, that particular indifference curve represents utility level $u_i = U^{-1}(c_i)$. Because everyone has the same utility function, if the indifference curve of type (ω_i, θ_i) hits the $h = 1$ line at a consumption level higher than that of type (ω_j, θ_j), then, type (ω_i, θ_i) has a higher utility level than type (ω_j, θ_j). At any allocation, we can rank utility levels simply by observing where their indifference curves cross the $h = 1$ line.

The government's maximization problem can be simplified by restricting attention to allocations where w_i is set equal to ω_i. This property is stated in the following proposition along with a set of preliminary restrictions on an optimal allocation.

Proposition 1: If $\{\overset{*}{c}_i, \overset{*}{h}_i, \overset{*}{w}_i\}_{i=1}^{NM}$ is an optimal allocation, then

(a) $\overset{*}{w}_i = \omega_i$ for all i such that $\overset{*}{h}_i > 0$;

(b) If $\overset{*}{w}_i = \overset{*}{w}_j$, then $\overset{*}{h}_i > \overset{*}{h}_j \iff \overset{*}{c}_i > \overset{*}{c}_j$;

(c) If $\overset{*}{w}_i = \overset{*}{w}_j = \overset{*}{w}_k$ and $\overset{*}{h}_i > \overset{*}{h}_j > \overset{*}{h}_k$,
then
$\overset{*}{c}_j \geq \lambda \overset{*}{c}_i + (1-\lambda)\overset{*}{c}_k$ for $0 \leq \lambda \leq 1$ such that $\overset{*}{h}_j = \lambda \overset{*}{h}_i + (1-\lambda)\overset{*}{h}_k$;

(d) If $\overset{*}{h}_i \geq \overset{*}{h}_j$ and $\overset{*}{w}_i > \overset{*}{w}_j$, then $\overset{*}{c}_i \geq \overset{*}{c}_j$;

(e) If $\overset{*}{w}_i = \overset{*}{w}_j < \overset{*}{w}_k = \overset{*}{w}_l$ and, for $\lambda_1 \geq 1$ or ≤ 0 and $0 \leq \lambda_2 \leq 1$,
$\lambda_1 \overset{*}{h}_k + (1-\lambda_1)\overset{*}{h}_l = \lambda_2 \overset{*}{h}_i + (1-\lambda_2)\overset{*}{h}_j$
then
$\lambda_1 \overset{*}{c}_k + (1-\lambda_1)\overset{*}{c}_l \geq \lambda_2 \overset{*}{c}_i + (1-\lambda_2)\overset{*}{c}_j$.

Element (a) of Proposition 1 implies that we can find an optimal allocation by solving a simpler program that focuses only on the sequence $\{c_i, h_i\}_{i=1}^{NM}$. This simpler program is given by OP below. Elements (b) and (c) of Proposition 1 indicates that net income, c, must increase with h in a concave fashion, for a given level of w; and elements (d) and (e)

[20] It should be noted that most of the results of this paper do not depend on inclusion of these participation constraints. However, if they were not included, the optimal solution would sometimes have the property that non-participants in market activity would have to pay a lump-sum tax (which is the only tax that can be imposed on them). Since such an outcome does not seem to be reasonably implementable, we conduct our analysis while taking these constraints into account.

indicates how individual allocations must compare across individuals paid different wages. The implications of Proposition 1 are easily seen on Figures 1A through 1C. In each of these figures we plot a set of individual allocations by projecting them on either the $c-h$ or $c-w$ plane. Figure 1A illustrates the content of elements (b) and (c) by plotting three individual allocations with the same wage rate. We have joined the three points by a line in order to help visualized the concavity property implied by element (c). We refer to the line that joins equal wage allocations in the $c-h$ space as a consumption-hours profile since it represents how an individual (of a given market ability) perceives his or her net return to supplying different amounts of labor. Figure 1B complements Figure 1A by illustrating that net income must be non-decreasing in the wage for a given level of hours, as implied by element (d). Finally, Figure 1C illustrates, as implied by elements (d) and (e), that consumption-hours profiles for a given wage level must essentially lie below the consumption-hours profiles for a higher wage (the statement of point (e) makes precise the reason for the qualifier "essentially").

From Proposition 1, we can immediately infer a convexity property of the tax system that implements the optimal allocation. In effect, point (c) of Proposition 1 has the following corollary:

Corollary 1: If $\overset{*}{w}_i = \overset{*}{w}_j = \overset{*}{w}_k$ and $\overset{*}{h}_i < \overset{*}{h}_j < \overset{*}{h}_k$, then $T_j \leq \lambda T_i + (1-\lambda)T_k$ for $0 \leq \lambda \leq 1$ such that $\overset{*}{h}_j = \lambda \overset{*}{h}_i + (1-\lambda)\overset{*}{h}_k$

Note that the above corollary does not restrict net taxes to be increasing or decreasing with hours worked. Instead Corollary 1 indicates that, conditional one's market wage, net taxes should be a convex function of hours worked (or equivalently a convex function of income).[21] To see that Corollary 1 is a strong restrictions, one can go back to the example introduced in the introduction and notice that it does not satisfy this corollary. Hence, the current framework would suggest that such a tax system is likely to be inefficient. In particular, Corollary 1 implies that it is never optimal to have a net tax-transfer system where a worker faces, as hours worked increase, first positive marginal tax rates followed by a negative marginal tax. Corollary 1 will later be used in conjunction with other propositions to provide a comprehensive description of the tax system that implements the optimal allocation.

[21] Since the problem we are addressing is discrete, the notion of convexity refers only to the points actually implemented by the optimal program.

We can now take advantage of the results of Proposition 1 and replace the original program by the following program OP:

$$
OP = \begin{cases}
\{\overset{*}{c}_i, \overset{*}{h}_i\}_{i=1}^{NM} \quad \text{maximizes} \\
\sum_{i=1}^{NM} p_i U\big(c_i + (1-h_i)\theta_i\big) \quad \text{subject to} \\
\sum_{i=1}^{NM} p_i c_i \le \sum_{i=1}^{NM} p_i \omega_i h_i, \\
U\big(c_i + (1-h_i)\theta_i\big) \ge U\big(c_j + (1-h_j)\theta_i\big) \quad \forall i, \forall j \quad \text{such that}, \omega_j \le \omega_i, \\
U\big(c_i + (1-h_i)\theta_i\big) \ge U\big(\theta_i\big), \quad \text{and} \\
0 \le h_i \le 1 \quad \text{for all} \quad i.
\end{cases} \tag{2.6}
$$

To analyze this problem, we proceed in steps. In the Section 4, we characterize the solution for the case where there is only one market wage rate (which is of course known) and many unobservable non-market opportunities (many θs). We then analyze the case where the are many potential market wages and many outside options, but where the market productivities are observable. In this case, we show the existence of a wage rate that divides individuals between those that face positive and increasing marginal tax rates (as they increase hours worked) and those that are subsidized and face negative marginal tax rates. In the following section we analyze the general case in which there are many potential market wages and many outside options, all of which are unobservable. In particular, we present a detailed characterization of the case where there are two unknown market productivities but m unknown outside options, which illustrates all of our results.

3. Definitions

First, let us define $\overset{*}{H}_i$ to be an individual's market labor supply in the absence of informational constraints, that is, $\overset{*}{H}_i = 1$ if $\omega_i > \theta_i$ and $\overset{*}{H}_i = 0$ if $\omega_i < \theta_i$. For a given individual i, if $\overset{*}{h}_i < \overset{*}{H}_i$ we say that the optimal solution has induced individual i to be downward distorted. Conversely, if $\overset{*}{h}_i > \overset{*}{H}_i$, we say that individual i is upward distorted. It is worth emphasizing immediately the close connection between the nature of distortions and the pattern of marginal tax rates that implement the allocations. In particular, it should be obvious that a downward distorted individual must face a positive marginal tax rate if he were to increase hours worked, otherwise he would do it. Similarly, an upward distorted individual must face a loss in subsidy (a negative marginal tax rate) if he decreases his hours worked, otherwise it would be optimal for him to decrease his hours worked.

A standard procedure for analyzing problems with incentive compatibility constraints begins by determining which constraints are binding at an optimum. For example, in many problems with one dimension of unobserved characteristics, only the adjacent incentive compatibility constraints are binding at the optimum.[22] However, when there two (or more) dimensions of unobserved characteristics, such simple characterizations are not available. Accordingly, it is helpful to begin by characterizing the (necessary) properties of allocations that are linked together by incentive compatibility constraints at the optimum. In order to do so, we first need to define a set of concepts.[23]

- **An Attractor:** Type j is an attractor of i if

$$U\big(\overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_i\big) = U\big(\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_i\big) \quad \text{and} \quad \omega_j \leq \omega_i. \tag{3.1}$$

We say that j is the attractor and that the link itself is an attraction. An attraction links types i and j by an incentive compatibility constraint, where (1) it is a type i who can mimic type j and (2) it is type i who is indifferent between its own allocation and that associated with type j. Graphically, we depict the relationship of attraction between a type i and a type j by an arrow which links the allocation of i with that of j In Figure 2A i is attracted to j who is in turn attracted to k. In Figure 2B, it is the reverse. Notice that an attraction is not equivalent to an indifference relationship; it also involves the requirement that one be capable of mimicking, that is, $\omega_j \leq \omega_i$. The concept of a chain of attractors expands this notion to the case where types are linked together by a series of incentive compatibility constraints.

- **A Chain of Attractors:** Type j is an extended-attractor of i if there exists a sequence $k_1 \ldots, k_J$ such that type k_{l+1} is an attractor of k_l, $k_1 = i$ and $k_J = j$.

- **A Distinct-Attractor:** Type j is a distinct-attractor of i if j is an attractor of i and $(\overset{*}{c}_j, \overset{*}{h}_j) \neq (\overset{*}{c}_i, \overset{*}{h}_i)$.

[22] A standard procedure for solving one-dimensional problems first creates a relaxed problem by replacing the set of incentive constraints with only adjacent ones (downward or upward) and then demonstrates that the solution to the relaxed problem is in fact the solution to the original one. See, for example, Guesnerie and Seade (1982), Weymark (1986a, 1986b, 1987), Matthews and Moore (1987) and Besley and Coate (1995).

[23] These concepts are very similar to those used by Guesnerie and Seade (1982). However, in Guesnerie and Seade, when two individuals are on an indifference curve this implies that an incentive constraint is binding. This is not the case in the present setup and hence we have resorted to a more general set of concepts. Note that the first half of our definition of a attraction is the same as being W-linked in the language of Guesnerie and Seade.

A chain of attractors is useful when perturbing an allocation that reduces the utility level of some agent. For example, suppose that a perturbation reduces the consumption of type i, keeping h_i constant. In such a case, incentive compatibility requires that the consumption of all types in the chain of attractors of i must also have their consumption reduced by the same amount. Hence, the chain of attractors brackets together types whose incentive compatibility constraints must be treated simultaneously. Figures 2A and 2B are examples of chains of attractors. Similarly, the notion of a source is helpful when considering perturbations that increase the utility of some agent.

- **A Source :** Type j is a source of i if i is an attractor of j and

$$U\big(\overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_i\big) > U\big(\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_i\big) \quad \text{and} \quad \overset{*}{h}_j > \overset{*}{h}_i. \tag{3.2}$$

- **A Fundamental Source:** Type j is a fundamental source of i if it is an extended source[24] of i and $h_j = 1$.

Note that in Figure 2A that i is the fundamental source of j and k whereas j is a source of k. Note that the utility of k is higher than that of j which is in turn higher than the utility i. The role played by fundamental sources is critical in the paper. We show eventually that every downward distorted individual must be linked by a chain of sources to a fundamental source thus creating a concave chain of attractors.

4. The Dual to the Standard Model

In this section we assume that the market productivity of the individuals are known whereas only the distribution of non-market skills is known. Although, this dual to the standard problem may be of interest in its own right, it is analyzed here since it gives substantial insight into the nature of the solution of the general two-dimensional problem.

We first consider the case where there is only one market wage rate and show that the informationally unconstrained solution is optimal. We then proceed to the case where there are many known market productivites and many unobservable outside options. This is essentially the same as the problem OP with the caveat that the inequality $\omega_j \leq \omega_i$ in the statement is replaced with an equality, that is, individuals can only imitate others with the same wage. In this case we show that there is a wage rate such that everyone above

[24] An extended source is defined analogously to a chain of attractors.

it is weakly downward distorted (non-negative marginal tax rates) and everyone below it is weakly upward distorted (non-positive marginal tax rates).

4.1. One Market Type

When there is only one type in the market sector, that is, $\omega_i = \omega$ for all i, the solution is simple; it corresponds to the informationally unconstrained outcome. In effect, everyone who is more skilled in the market sector than in the informal sector works full time with $\overset{*}{h}_i = 1$ and has the same after-tax income, $c = \omega$, whereas everyone else works full time in the non-market sector, $\overset{*}{h}_i = 0$, and has zero consumption in the formal sector. To see why the this outcome is optimal, note that everyone in the market ($h > 0$), receives the same income, while everyone out of the market is better off than those working since for these individual $\theta > \omega$. However, since the non-working individuals have zero taxable income, it is not possible to transfer any resources from them towards the working individuals. Hence there is no possibility of utility equalizing transfers.

4.2. Observable market productivity

Let us now consider the case where there are many observable market productivities and many unobservable outside options. Given the observability of market productivity, incentive compatibility constraints are operational only for individuals with the same ω. In this case, the issue of a welfare improving redistribution scheme arises because there is the possibility to tax individuals with high market productivity and low values of θ in order to transfer the revenues to low-ω low-θ individuals. The question then is how to best to set up such a tax-transfer system. In particular, we would like to know if the optimal scheme distorts individuals' incentives to work, and if so, who's decisions are distorted. Proposition 2 provides a characterization of the pattern of distortions that arise in the optimal scheme.

Proposition 2 If market productivity is observable, the optimal solution implies the existence of a wage $\overset{*}{w}$ such that types with market productivity above $\overset{*}{w}$ are weakly downward distorted and types with market productivity below $\overset{*}{w}$ are weakly upward distorted; that is, there exists a $\overset{*}{w}$ such that:

If $\omega_i < \overset{*}{w}$ then $\overset{*}{h}_i \geq \overset{*}{H}_i$,

and

if, $\omega_i > \overset{*}{w}$ then $\overset{*}{h}_i \leq \overset{*}{H}_i$

Proposition 2 implies that in $\omega - h$ space (or equivalently the wage-income space) there exists is a horizontal line at $\overset{*}{\omega}$ such that all allocations above the line correspond to weakly downward distorted allocations and all allocations below the line correspond to weakly upward distorted allocations. Of course, the dividing line is determined endogenously as it depends upon the distribution of types. Before discussing the content of Proposition 2 further, it is useful to look at its implications for the tax rates that support the allocation. This is done in Corollary 2.

Corollary 2: If market productivity is observable and non-market options are not, then there exists a $\overset{*}{w}$ such that the taxes which implement the optimal allocation have the following property:

if, $w_i = w_j > \overset{*}{w}$ and $\overset{*}{h}_i < \overset{*}{h}_j$, then $T_i \leq T_j$
and
if $w_i = w_j < \overset{*}{w}$ and $\overset{*}{h}_i < \overset{*}{h}_j$, then $T_j \leq T_i \leq 0$.

Corollary 2 stresses the strong role played by wages in determining tax patterns when the taxing authority does not observe non-market opportunities, but observes market productivities. In effect, Proposition 2 and Corollary 2 indicate that it is optimal to divide individuals in two classes based solely on their market productivity. In the first group the high wage earners face positive marginal tax rates as they increase their hours worked (and furthermore we know from Corollary 1 that these marginal tax rates must be weakly increasing in hours worked or income). In contrast, the second class – the low wage earners – face negative marginal tax rates (subsidies) as they increase their hours worked (and these marginal subsidies are weakly decreasing in hours worked or income, from Corollary 1).

One of the interesting aspects to note about Proposition 2 and Corollary 2 is that, when market productivity is observable, it is never optimal to have an individual first face negative marginal taxes rates for low number of hours worked and then face positive marginal rates once he works a sufficient amount of hours. In effect, the optimal system in this case has an individual with a low market productivity (lower than $\overset{*}{w}$) always receives an ever larger total subsidy as he increases his hours worked–albeit at a decreasing rate. The reason for this result is that an individual who chooses a greater amount of hours is always revealing himself to be more "deserving" of redistribution than an individual with the same wage but who supplies less work, since greater working time reveals a low θ. As we will see, when market wages are unobservable, this property will be partly lost but the flavor is kept.

In order to get a deeper sense for why is can be optimal to distort upwards the employment decision of low ω individuals and distort downward the decisions of high ω individuals, it is helpful consider the chains of incentive compatibility constraints that become binding at the optima (Imagine these chains in the $c - h$ space.). For example, consider a group of highly skilled individuals that have $\theta < \omega$. Assuming it is desirable is to extract taxes from this group to transfer it to low wage individuals, it would be most desirable to tax such individuals using lump sum taxes. However, at some point, when the lump sum tax is sufficiently high, the participation constraint of the highest θ individual in the group will start to bind. At this point, the only choice is either to stop taxing this group, or to start distorting downwards the employment decision of the highest θ individual in order to allow further taxation of the other high wage individuals. In effect, the individual that is being distorted downwards then becomes an distinct attractor for the second highest θ in this wage group (as well as all others in this wage group), that is, there emerges a tight incentive compatibility constraint between types with the same market wage. If we try to tax this group even more, the type with the second highest θ in this wage group will eventually need to be distorted downwards and similarly becomes an attractor of the third highest θ in this wage group. As we take more money from this group it may become optimal to distort downward the work effort of the third highest θ who now becomes an attractor for the fourth highest and so on. This creates a chain of incentive compatibility constraints that we call a downward distorted chain. It should also be noted that the individual who works the least hours in this chain has the highest utility in the group, the individual who works the second smallest number of hours the second highest utility level and so on. This is seen by noting where the indifference curves hit the $h = 1$ axis in the $c - h$ space, since this provides a utility ranking.

In the case of individuals with poor market skills, optimal redistribution is achieve by creating a chain of tight incentive compatibility constraints with upward distortions. In particular, the reason for upward distortions now results from a desire to transfer income to only a subset of low wage individuals with sufficiently low θs. It is not generally optimal to simply transfer income to low wage individuals unconditionally, since this would also benefits very high θ individuals who receive high utility by not working. Hence, it is optimal to make the transfer conditional on working in the market. In effect, it is optimal to select allocations that give rise to a set of incentive compatibility constraints that leave indifferent an individual with a given θ between his own allocation and the allocation of the next lowest θ individual.

In the table below, we present an example of an optimal allocation and the associated tax system calculated using log utility. The outcome is depicted in Figure 3. The first column gives the name of the type, the second, the number of that type, the third the that type's market and non-market skill levels, and the last the optimal after-tax income (consumption) and hours-worked in the market place.

Type	Number	(ω, θ)	c	h
1	1	(1.0, 1.05)	0.0	0.0
2	5	(1.0, 0.9)	.581998	.646513
3	70	(1.0, 0.2)	.653089	1.0
4	12	(0.1, 0.2)	.492125	0.0
5	5	(0.1, .15)	.492125	0.0
6	30	(0.1, .05)	.641342	1.0

Table 1

It is clear, as can be seen in Figure 3, that the downward distorted chain in this example is from type 1 to type 2 to type 3 with type 3 being the fundamental source of types 1 and 2. The upward distorted chain runs from types 4 and 5 to type 6, and in fact crosses the downward distorted chain. Note that the crossing of the two chains is possible only because the high ω types cannot imitate the low ω types.

The optimal allocation in this example can be implemented by the following piece-wise linear tax function which depends upon both the wage rate (market productivity) and market income:

$$T(w, y) = -.49211 - 0.5y \quad \text{if} \quad 0 \leq y \leq 1.0 \quad \text{for} \quad w \leq 0.1 \tag{4.1}$$

and

$$T(w, y) = \begin{cases} 0.1y & \text{if} \quad 0 \leq y \leq .6465 \\ .065 + .8(y - .6464) & \text{if} \quad .6465 \leq y \quad \text{for} \quad w > 0.1. \end{cases} \tag{4.2}$$

It is easy to verify that the allocation given in Table 1 is implemented by this tax schedule. In order to highlight the properties of this tax schedule, it is depicted in Figure 4. In this figure we can see that the tax schedule in monotonically increasing (and convex) for the high wage group, and is monotonically decreasing for the low wage group, that is, the low wage group faces a negative marginal tax. Moreover, the high wage group's schedule starts at zero taxes for zero hours worked, which reflects a tight participation constraint, while the low wage group starts with a positive transfer a zero hours. From this figure, we can immediately see that such a tax schedule would not be incentive compatible if market wage were not observed since when wages are not observable, net taxes at zero hours worked must be the same for all individuals. In the next section we examine what changes when market productivity is not observable. In particular, we examine whether or the tax schedule that implements the optimal allocation still exhibits the monotonicity implied by Corollary 2, and whether it still admits negative marginal taxes rates for some wage groups.

5. The General Case

In this section we return to our basic problem, OP. In characterizing the solution to this problem, we present results in a manner that parallel those derived for the case of observable market productivity. In particular, we begin by providing a characterization in the space of market skills and hours worked. This characterization reflects the intuition gained from the observable wages case examined in the previous section, namely that it is generally optimal to downward distort high market productivity individuals and distort upwards low skill individuals. We then characterize the implied tax rates. In the subsection that follows, we analyze a special case where there are only two market productivities and many non-market skill levels, all unobservable to the tax authority. We provide a detailed characterization of this case. In the course of this characterization, we also present a numerical example to help the reader understand the nature of the general solution.

Proposition 3: For any solution to OP, $\{\overset{*}{c}_i, \overset{*}{h}_i\}$, there exists a non-increasing function $g : [0, 1] \mapsto [\underline{\omega}, \bar{\omega}]$. such that if

$$\omega_i < g(\overset{*}{h}_i) \longrightarrow \overset{*}{h}_i \geq \overset{*}{H}_i \tag{5.1}$$

and if

$$\omega_i > g(\overset{*}{h}_i) \longrightarrow \overset{*}{h}_i \leq \overset{*}{H}_i. \tag{5.2}$$

Figure 5 is a graphical illustration of the implications of Proposition 3. When projecting an optimal allocation on the space of wages and hours, $w - h$, the proposition indicates that this space can be divided into two areas by a non-increasing function. In the area to the southwest of the dividing line, only individuals with $\omega_i < \theta_i$ are mapped into this space. Hence, for individuals mapped into this sub-space, the distortions on time allocated to the market can only be positive and are implemented with negative marginal tax rates. In contrast, in the area to the northeast of the dividing line, any distortions on the allocation of market time must be negative and are implemented with positive marginal tax rates. This pattern of predicted distortions is not vacuous since it is generally the case that an optimal allocation is characterized by employment distortions of both types. Loosely speaking, Proposition 3 indicates that an optimal allocation is characterized by low market performers (in terms of wages and net income) having their labor supply (weakly) distorted upwards, while strong market performers have their labor supply (weakly) distorted downward. The contrast with Proposition 2 is interesting. In particular, when market productivity is observable, it is optimal to use only market wages to dichotomize individuals into upward and downward distorted allocations, whereas, when market productivity is not observed, it is optimal to use both wage information and hours information to decide whether an individual is to be downward or upward distorted.

Once again, before discussing the content of Proposition 3 in detail, it is helpful to first examine its implication for the structure of taxes.

Corollary 3: If neither market productivity nor outside options are observable, then, there exists a non-increasing function $g : [0, 1] \mapsto [\underline{\omega}, \bar{\omega}]$ such that the taxes which implement the optimal allocation have the following property:

If $w_i = w_j$, $\overset{*}{h}_i < \overset{*}{h}_j$ and $\omega_j < g(h_j)$, then $T_j \leq T_i \leq 0$ and
if $w_i = w_j$, $\overset{*}{h}_i < \overset{*}{h}_j$ and $\omega_i > g(h_i)$, then $T_i \leq T_j$

Corollary 3 and Proposition 3 indicate that allocations above and to the right of $g(\cdot)$ are implemented with non-negative marginal tax rates whereas those to the left and below are implemented with non-positive marginal tax rates. Note in particular that upward distorted allocations are achieved by negative marginal (and average) taxes and that by Corollary 1 and 3, these should be phased-out as hours increase. In effect, this setup supports negative marginal tax rates as part of an optimum but suggests that an individual who initially faces a subsidy when increasing his hours worked, may start facing a positive tax rate if he decides to work a sufficient number of hours. This last characteristic, whereby marginal tax rates may be first negative then positive for a given wage group, shows that the non-observability of market productivity changes quite significantly the nature of the optimal tax structure.[25]

The implications of Proposition 3 stand in stark contrast with those derived in the Mirrlees' framework. Let us recall that in the Mirrlees framework, an allocation can also be represented in the $\omega - h$ space. The resulting locus of points (in the case of a discrete set of types) has the property that net income is increasing in the wage (ability). In this

[25] Corollary 3 indicates that quantity of time worked at which one switches from receiving a marginal subsidy to facing a positive marginal tax rate is negatively related to one's wage rate, that is, higher wage workers always face such a switch at a lower level of hours worked than a lower wage worker.

case, we know that types with low net income and low ability generally have their labor supply decision distorted downward since both the substitution effect (induced by a positive marginal tax rate) and the income effect (induced by a negative average tax rate) favor a reduction in labor supply. In contrast, the highest ability individual, who is also the highest net income earner, generally has his or her labor supply weakly distorted upwards (relative to the informationally unconstrained case) since he or she generally faces a zero marginal tax rate but a positive average tax rate, which, if there is an income effect, tends to favor more labor supply.[26] Note that this pattern of distortions is virtually the opposite to that implied by Proposition 3.

5.1. Two Market Skill Levels

In this subsection we examine the case of two market productivities and many θs. We do this case in detail as it illustrates the most salient aspects of the general solution. In effect, we examine this case by breaking the problem into three parts, and thereby offering a constructive approach determining the optimal allocation. First we ask how to raise a given sum of money efficiently from the high ω types with $\theta < \omega$. Second, given the tax revenue, how should this sum be distributed efficiently among the low ω types, and then, we show how to put these two pieces together. Finally we show how to deal with the high wage types with $\theta > \omega$. In order to help understand the somewhat complex argument that follows we carry the following example along with the discussion. It is computed using log utility.

[26] For more details see Guesnerie and Seade (1982) and Weymark (1986a, 1986b, 1987).

Type	Number	(ω, θ)	c	h
1	1	(1.0, 1.05)	.2141	0.0
2	1	(1.0, 1.02)	.3947	0.172
3	5	(1.0, 0.7)	.5542	.4
4	70	(1.0, 0.2)	.674	1.0
5	15	(0.1, 0.2)	.3947	0.172
6	5	(0.1, 0.15)	.433	.4273
7	30	(0.1, .05)	.5189	1.0

Table 2

Figure 6 depicts this solution in $c - h$ space. Let the two market skills be given by ω_1 and ω_2 where $\omega_1 > \omega_2$ (1.0 and 0.1 in the example). Let I^1 be the set of types with skill rating ω_1 such that the informal skill is less than the market skill, that is

$$I^1 = \{(\omega_1, \theta_i^1) | \omega_1 > \theta_i^1\}. \tag{5.3}$$

Further, number these θs in descending order so that $\theta_1^1 > \theta_2^1 > \ldots > \theta_{n_1}^1$. Thus type (w_1, θ_1^1) is the type in this group that is least attached to the market. In the example, types 3 and 4 are in I^1 so that $\theta_1^1 = 0.9$ and $\theta_2^1 = 0.2$. The types in this group are relatively highly paid, and, in general, the planner wants to tax this group, subject to the incentive compatibility and participation constraints, in order to subsidize those with low wages and poor non-market skills.

Next, we partition the low market productivity types into two groups, I^2 and I^3 where the former have non-market skills above ω_2 but below θ_1^1 and I^3 whose non-market skills are even worse than their market skills;

$$I^2 = \{(\omega_2, \theta_i^2) | \omega_2 < \theta_i^2 < \theta_1^1\} \quad \text{and} \quad I^3 = \{(\omega_2, \theta_i^3) | \omega_2 > \theta_i^3\}. \tag{5.4}$$

It is the individuals in I^3 that the redistribution program is trying to help for these individuals have both poor market skills and poor outside options. It must however do so in a fashion such that those in I^2, as well as in I^1, do not wish to imitate them. In the example, type 7 is the only member of I^3 and types 5 and 6 comprise I^2. As above, the θs are numbered in decreasing order so that in the example $\theta_1^2 = .2$, $\theta_2^2 = .15$ and $\theta_1^3 = .05$.

Finally, let I^0 be the set of types such that the non-market skill level is greater than θ_1^1, that is,

$$I^0 = \{(\omega_i, \theta_i^0) | \theta_i^0 > \omega_1\}, \tag{5.5}$$

where $\theta_1^0 > \theta_2^0 > \cdots$.Thus types in I^0 have little attachment to the market in spite of being highly skilled and in an informationally unconstrained world spend full time at outside activities. Types 1 and 2 are the elements of I^0 in the example and $\theta_1^0 = 1.05$ and $\theta_2^0 = 1.02$.

Let us begin by seeing how to collect a given amount of revenue from the group I^1 efficiently. Begin by taking a lump-sum from every member of I^1 and having them work full time. Now suppose, for the given amount of required tax revenue, that it is optimal to distort downwards one member of this group. This type is always (ω_1, θ_1^1) because it is the least attached to the market. This can happen either because type (w_1, θ_1^1) has hit its participation constraint or because it is attracted to a type with good outside options which has positive consumption in the market economy. As this type is distorted downwards on its indifference curve by means of a positive marginal tax rate we can levy additional lump-sum taxes on the rest of group I^1 making certain that we take account of the fact that (ω_1, θ_1^1) is an attractor of (ω_1, θ_2^1); everyone in $I^1 \setminus \{1\}$ is bunched at $\overset{*}{h}_i = 1$. Note that at this allocation type (ω_1, θ_1^1) has a higher level of utility than the types in $I^1 \setminus \{1\}$. The amount of revenue that is being collected at this allocation is easy to calculate. The question is whether or not this is an efficient way to collect this amount of money. In general that depends upon the relative proportions of the different types in this group. Suppose, for example, that there are relatively few of type (ω_1, θ_2^1) compared to type (ω_1, θ_3^1). Then by distorting downward (ω_1, θ_2^1) by means of a positive marginal tax rate and maintaining its attraction to type (ω_1, θ_1^1), we can now lump-sum tax everyone in $I^1 \setminus \{1, 2\}$ remembering that type (ω_1, θ_2^1) is an attractor of (ω_1, θ_3^1). Of course, it might also be optimal to distort downwards the types (ω_1, θ_3^1) in which case they would be bunched with (ω_1, θ_2^1) or distinct from (ω_1, θ_2^1) and attracted to them. In either case, (ω_1, θ_3^1) is now an attractor of (ω_1, θ_4^1). Continue in this fashion until the set I^1 is exhausted. Clearly type (ω_1, θ_1^1) may or may not be on its participation constraint. Note, in particular, that the net income curve is concave; that is, moving from (ω_1, θ_1^1) to (ω_1, θ_2^1) and so on types face ever increasing marginal tax rates. For example, the right marginal tax rate of (ω_1, θ_1^1) is $(\omega_1 - \theta_2^1)/\omega_1$ and that of (ω_1, θ_2^1) is $(\omega_1 - \theta_3^1)/\omega_1$. There may be bunching at any kink on the net income curve but such bunching is never robust to changes in the distribution of types. Of course the amount of money that needs to be collected from this group has yet to be determined. In the example, type 3 is downward distorted with a net income .55 but working only 0.4. and is an attractor of type

4. Type 4 is also the fundamental source of type 3. This generates the downward distorted chain, 4-3-2-1, Figure 6.

Now suppose that we have a certain amount of revenue to use for redistribution. The redistribution program wishes first of all to transfer money to members of group I^3 who have poor market opportunities and even poorer non-market skills. As these types are better in the market than in the non-market it makes sense to have them work full time and to begin by giving them equal lump-sum transfers. Thus we are moving all types in I^3 up the $h = 1$ axis. However, types in I^2 can imitate everyone in I^3; hence as we transfer consumption to group I^3 we must also transfer money to members of I^2 moving them up the $h = 0$ axis so that they do not imitate the types in I^3. It is obvious that all the types in I^3 must be attractors of even the lowest θ in group I^2; that is (ω_2, θ_1^3) is an attractor of $(\omega_2, \theta_{n_2}^2)$. Everyone in I^3 is bunched on the $h = 1$ axis and everyone in I^2 is bunched on the $h = 0$ axis. At this allocation all types in I^3 have the same level of utility as type $(\omega_2, \theta_{n_2}^2)$ while all types in $I^2 \setminus \{n_2\}$ have higher levels of utility. There is a limit to these transfers to the types in I^2 and I^3 because, at some point, type (ω_1, θ_1^1) will imitate the types in I^2.

Suppose for example that there are relatively few of type $(\omega_2, \theta_{n_2}^2)$ compared to the rest of the members of I^2. Then we could give less money to everyone in $I^2 \setminus \{n_2\}$, a little more to everyone in I^3, by distorting the hours worked of (ω_2, θ_{n_2}) upward by negative marginal taxes making certain that (ω_2, θ_1^3) is still an attractor of $(\omega_2, \theta_{n_2}^2)$. In so doing however we must note that $(\omega_2, \theta_{n_2}^2)$ must now become an attractor of $(\omega_2, \theta_{n_2-1}^2)$. Of course this bunching of all the types in $I^2 \setminus \{n_2, n_2 - 1\}$ may not be optimal either. In may be that we should repeat the above exercise so that $(\omega_2, \theta_{n_2-1}^2)$ becomes an attractor of $(\omega_2, \theta_{n_2-2}^2)$ and so on. In any case the net income curve is again concave; there may be bunching at any of the kinks but this is not robust to changes in the distribution of types. Of course the beginning of this upward distorted chain is type (w_2, θ_1^2) thus exhausting group I^2. This can be seen in Figure 6 where type 7 receives a lump-sum transfer and is an attractor of type 6 which is upward distorted and is, in turn, an attractor of type 5. This yields an upward distorted chain, 5-6-7.

To put the two chains together there are two possibilities. First, the highest θ in I^2 may not be strictly upward distorted and hence is at $h = 0$. In this case the upward distorted chain must start where the indifference curve of (ω_1, θ_1^1) hits the $h = 0$ axis, say $\hat{c}_1^1$. Type (ω_2, θ_1^2), the first element of the upward distorted chain, cannot have a consumption level higher than $\hat{c}_1^1$ as that would induce type (ω_1, θ_1^1) to imitate type (ω_2, θ_1^2) and hence violate an incentive compatibility constraint. Having a consumption level lower than $\hat{c}_1^1$ cannot be optimal because the optimization program is trying to equalize utilities and could do so be moving the two chains closer together. The second possibility is that the highest θ

in I^2 is strictly upward distorted. Type (ω_2, θ_1^2) must now lie on the indifference curve of type (ω_1, θ_1^1). It cannot lie above it because of incentive compatibility and cannot lie below because of the optimization. Thus in both cases we have optimally adjusted the two chains. In Figure 6 type 5 is on the indifference curve of type 3 and hence is an attractor of type 3.

Now we have one downward distorted chain and one upward distorted chain. Type (ω_1, θ_1^1) is however attracted to the upward distorted type (ω_2, θ_1^2) and it is this that ties the two chains together. This leaves only the members of group I^0 who are unattached to the market and in an informationally unconstrained world would not work in the market at all. If type (ω_1, θ_1^1) is on its participation constraint then setting $h_0 = 0$ for all types in group I^0 is optimal. However, if (ω_1, θ_1^1) is not on its participation constraint then, at the least, everyone in I^0 will collect a lump sum where the indifference curve of (ω_1, θ_1^1) hits the axis at $\hat{c}_1^1$. This cannot in general be optimal. Moving all members of $I^0 \setminus \{\theta_1^0\}$ up the indifference curve of (ω_1, θ_1^1) saves money by forcing them to work while moving θ_1^0 down the $h = 0$ axis so as to maintain indifference with the rest of its group. In the example, as type 2 is moved up the indifference curve of type 3, income goes up at one unit per hour whereas consumption only goes up by 0.7 leaving a net gain to be distributed. The types in $I^0 \setminus \{\theta_1^0\}$ are moved up the indifference curve of (ω_1, θ_1^1) until they imitate one of the types on this indifference curve. Type (ω_2, θ_1^2) is on the indifference curve of type (ω_1, θ_1^1) and the best that can be done is to allocation all types in $I^0 \setminus \{\theta_1^0\}$ the same allocation as (ω_2, θ_1^2). In the example type 2 has the same allocation as type 5 on the indifference curve of type 3. Type 1 receives a lump-sum transfer and is indifferent between that and the allocation of types 2 and 5

This exhausts the types, that is, everyone has been assigned an optimal allocation. In this we see all the elements of Proposition 3. The types in I^1 have a high market productivity and are downward distorted whereas those in I^2 have low market skills and are upward distorted. Those in I^3 receive lump-sum subsidies and surprisingly, some of the high θs in I^0 may also be forced into the market place. This later argument demonstrates why Proposition 2 cannot go through in the general case. That is, there is, in general, no wage rate such that everyone above is downward distorted and everyone below is upward distorted.

Now consider the example in more detail. There are seven types. Types 1 and 2 are in I^0 with the high wage but are even more productive at outside activities. Types 3 and 4 are in I^1 but 4 is much more attached to the market than 3. 5 and 6 are the types in I^2 and only type 7 in I^3. The redistribution program would like to tax types in groups I^0 and

I^1 and redistribute the revenue to groups I^2 and I^3 but is prevented from using lump-sum taxation by the incentive compatibility constraints and participation constraints.

Type 4 pays a lump-sum tax of .33 but type 3 is downward distorted and is working only .4. Its indifference curve hits the $h = 0$ axis at $\hat{c}_1^1 = .27$. Thus the right marginal tax rate facing it is .8. Type 4 is an attractor of type 3 and the 4-3 incentive compatibility constraint is binding. This completes the downward distorted chain. To construct the upward distorted chain first move type 7 to (.32,1.0) and types 5 and 6 to (.27,0.0). At this point type 7 becomes an attractor of types 5 and 6 while they are, in turn attractors of type 3. The redistribution program however aims to improve more the lot of type 7. The best that can be done is to distort upwards the employment of types 5 and 6, moving them along the indifference curve of type 3 to maintain that incentive compatibility constraint and moving type 7 up the $h = 1$ axis while maintaining the incentive compatibility constraint 6-7. At some point it is efficient to distort upward type 6 even further, thus moving type 7 up the $h = 1$ axis so that type 6 becomes an attractor of type 5. Finally, the best that can be done with type 2 is to allocate it the same bundle as type 5 and give a lump-sum transfer to type 1 so that it is indifferent between the lump-sum transfer and the allocation of types 2 and 5. In the final solution type 3 is attracted to type 4, type 3 is attracted to type 5 and type 5 is attracted to type 6 which is in turn attracted to type 7. Types 1 and 2 are attracted to type 5.

This allocation can be implemented by the following piece-wise linear tax function which depends upon both the wage rate (market skill level) and market income:

$$T(w,y) = \begin{cases} -.2141 - 9.5y & \text{if} \quad 0 \leq y \leq .0172; \\ -.3775 - (y - .0172) & \text{if} \quad .0172 \leq y \leq .0433; \\ -.4036 - .5(y - .0433) & \text{if} \quad .0433 \leq y \leq .1 \quad \text{for} \quad w \leq .1; \end{cases} \tag{5.6}$$

and

$$T(w,y) = \begin{cases} -.2141 - .05y & \text{if} \quad 0 \leq y \leq .0172 \\ -.2227 + .3(y - .0172) & \text{if} \quad .172 \leq y \leq .4; \\ -.154 + .8(y - .4) & \text{if} \quad .4 \leq y \leq 1.0 \quad \text{for} \quad w > .1. \end{cases} \tag{5.7}$$

It should be noted that this example illustrates the features of Corollary 3. To illustrate this, and to allow easy comparison with the schedules derived for the observed productivity case, in Figure 7 we plot this tax schedule as a function of income for each wage group. As can be seen in the figure, low market wage types face declining marginal tax rates, whereas the high ω types initially face negative marginal taxes followed by positive marginal tax rates (at relatively high incomes). Obviously, both the schedule for the high wage group and the low wage group now start at the same net-tax level when supplying zero hours. Note that the presence of negative marginal taxes for rates for the high wage group is due to the presence of high ω in this group which would mimic low wage types if they were not offered employment subsidies.

We view the main insight of Corollary 3, as illustrated by the above example, as prescribing the use of subsidies, i.e. negative marginal tax rates for particular wage groups, as a means of redistributing income. This is in marked contrast to most of the optimal tax literature which prescribes marginal tax rates to be everywhere non-negative. Moreover, a key aspect to note about wage subsidies in our environment is that they are generally phased-out as individuals supply more work (as implied by Proposition 1.(c)). This contrasts with many commonly proposed wage-subsidy programs which do not include a phase-out, and hence, are often considered too expensive to implement. It should be noted that subsidy phase-outs are optimal in our environment both because they stop some high market-value individuals from taking advantage of such programs as well as allowing some low-market value individuals to take advantage of the relatively higher value of their non-market time.

6. Correlated Skills

The results above are derived without placing any restrictions on the distributions of market skills or the non-market skills; nor are any assumptions made about the correlation of the two distributions. As an empirical matter, no particularly reasonable hypothesis about these distributions or their possible correlations comes to mind. Do individuals with high market skills also have a high value of their time in non-market activities? Or, do those individuals with low market skills have valuable outside options? It seems to us that neither of these scenarios is obvious. Most likely, among those with high market valuations, some have good outside opportunities and some do not. The same seems probable for those with low market valuations. Nevertheless, from a theoretical point of view, an examination of the case of purely positive or purely negative correlation between the distributions is of some interest; such an assumption renders our problem one-dimensional and permits further comparison with results in the extant literature. We pursue that line of argument in this section.

Suppose first that the distributions of characteristics are perfectly negatively correlated, that is,

$$\omega_i > \omega_j \longleftrightarrow \theta_i < \theta_j \quad \text{for all} \quad i, j, \tag{6.1}$$

and that not all market skills are greater than or less than all non-market skills,[27] that is,

$$\min\{\omega_1, \ldots, \omega_M\} < \max\{\theta_1, \ldots, \theta_N\} \tag{6.2}$$

and

$$\max\{\omega_1, \ldots, \omega_M\} > \min\{\theta_1, \ldots, \theta_N\}. \tag{6.3}$$

Hence the higher the market value of one's time the lower is the value of one's outside option.

Proposition 4: If market and non-market skills are perfectly negatively correlated, (6.1), and (6.2)—(6.3) hold, then

$$\overset{*}{h}_i = \overset{*}{H}_i \quad \text{for all} \quad i. \tag{6.4}$$

Proposition 4 states that if the two distribution are perfectly negatively correlated, then at the optimum no individual has his or her work effort distorted either up or down. This proposition is reminiscent of a result of Dasgupta and Hammond [1980] where, in a Mirrlees setup with observable hours, they showed that a first-best outcome could be achieved and utilities equalized.[28] Here, as in Dasgupta and Hammond, individuals with a high value of market time have a lower reservation price on generating market income than individuals with a low market value time because of the assumed negative correlation. In such a case, redistribution is made easy since high types are ready to accept high lump sum taxes in order to have the right to work at a high paying jobs. However, whenever this correlation is not perfect, such a scheme is not incentive compatible and hence a distortionary scheme is likely needed. In order to highlight this point, we now examine the case of perfect positive correlation.

Suppose that the distributions of characteristics are perfectly positively correlated, that is,

$$\omega_i > \omega_j \longleftrightarrow \theta_i > \theta_j, \quad \text{for all} \quad i, j \tag{6.5}$$

and that

$$\max\{\omega_1, \ldots, \omega_M\} > \max\{\theta_1, \ldots, \theta_N\} \tag{6.6}$$

and

$$\min\{\omega_1, \ldots, \omega_M\} < \min\{\theta_1, \ldots, \theta_N\}. \tag{6.7}$$

This again reduces our problem to one that has effectively one dimension of non-observability. In this special case we can show that

[27] If not, the problem is trivial.

[28] In our framework we cannot generally equalize utilities because of the participation constraints. That is, if there were no participation constraints, and the two distribution were perfectly negatively correlated, then, the Dasgupta-Hammond result would hold.

Proposition 5: If market and non-market skills are perfectly positively correlated, as defined by (6.5), and (6.6)–(6.7) hold, then

$$\overset{*}{h}_i = \overset{*}{H}_i \quad \text{for all} \quad i \quad \text{such that } \omega_i > \theta_i. \tag{6.8}$$

Proposition 5 indicates that, even in the special case of perfect positive correlation between characteristics, it is only the individuals whose market value is greater that his or her nonmarket value who have undistorted employment decisions at the optimum; they work full-time. In contrast, for an individual whose non-market value exceeds his or her market value the employment decisions can still be distorted upwards. In order to see this last possibility, it is useful to turn to a simple example where characteristics are positively correlated.

Example 3:[29]

Consider a situation where there are only two types of equally likely individuals. Individuals of type 1 are the high productivity individuals with market productivity of 1 and non-market productivity of .8. Individuals of type 2 are low productivity individuals with market productivity of .45 and a non-market productivity of .5. Assuming log utility, it can be verified that the optimal allocation in this case is for individual 1 to work full time and receive an after tax income equal to .87, while individual 2 should spend .17 of his time working and receive .21 in after tax (transfer) income.[30] There are two aspects to note about this example. First, it satisfies the statement of Proposition 5 and the employment decision of individual 2 is upward distorted. This illustrates that upward distortions can arise. Individual 2 is upward distorted so that additional income can be transferred to him or her without inciting individual 1 to mimic. The implicit wage subsidy received by individual 2 is of the order of 200%, that is, by working 17% of the time this individual is receiving net income close to three times the market value of this time. However, this huge subsidy is necessary limited to a set number of hours. If the subsidy were not phased out after .17 hours, both types of individuals would want to take advantage of it making the whole scheme infeasible. In effect, this example illustrates why phased-out wage subsidies – that is, wage subsidies which decrease with the amount of time worked– are key to understanding how optimal redistribution is achieved in our environment.

[29] Two simple cases of correlation that are interesting to consider are where $\omega + \theta =$ constant and where $\omega = \theta$. In both these case, the optimum is given by *laissez-faire*. In the first case, this corresponds to the first best, while in the second case this corresponds to the only individually rational allocation.

[30] The exact numbers for individual 2 are .171429 and .207143.

7. Workfare

The analysis to this point demonstrates that an optimal redistribution program (in our environment) generally involves subsidizing employment of low market performers, by means of negative marginal tax rates, and by taxing the employment income of high market performers. As noted above, these results contrast markedly with much of the optimal taxation literature in which it is never optimal to transfer income through employment subsidies (negative marginal tax rates). From a policy point of view, workfare—a public work requirements—is a related and frequently discussed means of achieving redistribution. However, it has been shown by Besley and Coate (1995), that it is generally not welfare improving within the Mirrlees framework to complement the optimal non-linear income tax with workfare.[31]

To pose questions about workfare, we first define what is meant by workfare and then indicate how it can be integrated into our analysis. To be distinct from the subsidization of private sector employment, we consider workfare to be a requirement to work in a public sector employment program; in addition, we assume that individuals of all types have the same productivity in such a program and that the government does not learn an individual's type by having time allocated to the workfare program. This latter assumption appears reasonable given the likely institutional structure of most workfare programs.

We denote marginal productivity in the workfare program by ω^f (which can be negative) and the time requirement in workfare as h_i^f. With the addition of a workfare option, the government's problem is to choose an allocation of the form $\{\overset{*}{c}_i, \overset{*}{h}_i, \overset{*}{w}_i, \overset{*}{h}{}_i^f\}_{i=1}^{NM}$ which solves the following problem.

$$\max \sum_{i=1}^{NM} p_i U\big(c_i + (1 - h_i - h_i^f)\theta_i\big) \tag{7.1}$$

subject to

$$\sum_{i=1}^{NM} p_i c_i \leq \sum_{i=1}^{NM} p_i (w_i h_i + \omega^f h_i^f), \tag{7.2}$$

and for all i

$$U\big(c_i + (1 - h_i - h_i^f)\theta_i\big) \geq U\big(c_j + (1 - h_j - h_j^f)\theta_i\big), \quad \forall j \quad \text{s.t.} \quad w_j \leq \omega_i, \tag{7.3}$$

[31] In Brett (1998) public work is treated as a separate input. In this case, workfare may be employed as a last resort in an optimal redistribution program.

$$U\big(c_i + (1 - h_i - h_i^f)\theta_i\big) \geq U\big(\theta_i\big), \tag{7.4}$$

$$0 \leq h_i + h_i^f \leq 1, \quad h_i \geq 0, \quad h_i^f \geq 0 \tag{7.5}$$

Individuals can concurrently work in the workfare program and in private sector employment as long as the total time in these activities is no greater than the individual's endowment.

The following proposition relates the redistribution-workfare program to our previous analysis. In particular, starting from a given distribution of types, the proposition exploits the construction of a modified economy in which all types j with $\omega_j < \omega^f$ are relabeled so that $\omega_j = \omega^f$.

Proposition 6: If $\{\tilde{c}_i, \tilde{h}_i\}_{i=1}^{NM}$ solves OP for the modified economy (where types j with $\omega_j < \omega_f$ are relabeled so that $\omega_j = \omega^f$), then $\{\hat{c}_i, \hat{h}_i, \hat{w}_i, \hat{h}_i^f\}_{i=1}^{NM}$ solves the redistribution-workfare problem if for all i,

(1) $\hat{c}_i = \tilde{c}_i$,
(2) $\hat{w}_i = \omega_i$ and
(3) If $\omega_i > w^f$, $\hat{h}_i = \tilde{h}_i$ and $\hat{h}_i^f = 0$
(4) If $\omega_i = w^f$, $\hat{h}_i^f = \tilde{h}_i$ and $\hat{h}_i = 0$

Proposition 6 implies that workfare is to be used only if an individual's social productivity is greater in the workfare program than in private sector employment; otherwise, it is preferable to use employment subsidies instead of workfare requirements as a means of redistributing income. In part, this statement supports Besley and Coate (1995) finding against the use of workfare (when the objective is welfare maximization) since it is generally assumed that public works programs are unlikely to have greater social product than that associated with private employment. However, to be realistic, we must consider the possibility of individuals without private sector employment possibilities. Formally, in our notation this case corresponds to types with $\omega = -\infty$. It is exactly in such a case that it can be optimal to use workfare in our setup even with $w^f < 0$, while it is not welfare improving to do so in the Mirrlees setup (which is the situation analyzed by Besley and Coate, Section VII). The following example illustrates why a workfare requirement can be used to improve welfare even when $w^f < 0$.

Example 4 There are two individuals with log utility. Individual of type one has a market productivity normalized to one and values his or her non-market time at .8, that is, $\omega_1 = 1$ and $\theta_1 = .8$. Individual 2 is unemployed and values his or her non-market time at

.1, that is, $\omega_2 = -\infty$ and $\theta_2 = .1$. In the absence of the possibility of workfare, the social optimal would be characterized by transferring .1 unit from individual 1 to individual 2, thereby providing a utility level for individual 2 of log(.2). In this case, individual 2 can be considered to be in a welfare program without any work requirements. Now let us assume that the government can also impose work requirements, but that the work itself is socially unproductive with $w^f = -.2$. In this case, the social optimum is characterized by requiring individual 2 to spend .2 of his or her time in the workfare program in order to receive .16 units of the good. The utility of the type 2 individual is increased to log(.24). In effect, by tying the income transfer to workfare it has become possible to transfer more income to the unemployed individual since the resulting package remains unattractive to the employed individual. Hence, this example demonstrates why a socially costly workfare program may potentially be a desirable way to tie income transfers for individuals with low or non-existent private sector employment possibilities.

8. Conclusion

The object of this paper is to explore the principles that govern the design of an optimal redistribution program in which taxation authorities have both reasons and tools to favor programs that target transfers more effectively than simple negative income tax schemes.[32] To this end we have analyzed a variant of the optimal taxation problem pioneered by Mirrlees. Our departure consists of allowing for a greater scope of unobserved heterogeneity in the population and allowing the government to transfer income based on both market income and market labor supply. Our main finding is that, in contrast to much of the optimal taxation literature, optimal redistribution in this environment is achieved using employment subsidies on low market performers, positive marginal tax rates on high market performers and, in last recourse, workfare as a mean of transferring income to individuals with very poor or non-existent market opportunities.

How should these results interpreted? In our view, these results are not a call for re-designing income tax system to include a dependence on annual hours worked. Instead we view these results as supporting the potential relevance of certain active labor market programs as a complement to income tax as a means of redistributing income.[33] For example, these results provide potential support for programs, such as the Canadian Self-Sufficiency

[32] Avenues of future research include examining the value of rendering some informal activities observable through monitoring, and rendering the acquisition of skill endogenous.

[33] We also view these results as providing minimal guidelines of how such programs should interact with the income tax system in terms of the implied pattern of effective marginal tax rates.

Project, which supplements income to low wage earners who choose to work. More generally, we view our results as suggesting the use of phased-out wage subsidies as a means of redistributing income to low earners, that is, wage subsidies that decrease in intensity as an individual chooses to supply more labor. Such phased-out subsidy programs, in effect, allow substantial transfers to the most needy in society without inciting either high market-value individuals or high non-market value individuals to take advantage of it.

APPENDIX

Proof of Proposition 1:

Point (a). By contradiction. Suppose that for some j, $\overset{*}{h}_j > 0$ and $\overset{*}{w}_j \neq \omega_j$. Then consider setting $\overset{*}{w}_i = \omega_i$ for all i. In this case, all the constraints of the problem remain satisfied and the material balance constraint is relaxed. It is then possible to take the additional resources and share them equally, thereby creating an Pareto improvement.

Point (b). From incentive compatibility we know that $\overset{*}{c}_i - \overset{*}{c}_j \geq (\overset{*}{h}_i - \overset{*}{h}_j)\theta_i$. Since $\theta > 0$, $\overset{*}{h}_i > \overset{*}{h}_j$ if and only if $\overset{*}{c}_i > \overset{*}{c}_j$.

Point (c). The two incentive compatibility constraints associated with type j not mimicking either type i or k imply that, for any $0 < \lambda < 1$,

$$\overset{*}{c}_j - \overset{*}{h}_j\theta_j \geq \lambda\overset{*}{c}_i + (1-\lambda)\overset{*}{c}_k - \lambda\overset{*}{h}_i\theta_j - (1-\lambda)\overset{*}{h}_k\theta_j$$

.

If λ is further chosen such that $\overset{*}{h}_j = \lambda\overset{*}{h}_i + (1-\lambda)\overset{*}{h}_k$, the above inequality implies $\overset{*}{c}_j \geq \lambda\overset{*}{c}_i + (1-\lambda)\overset{*}{c}_k$.

Point (d). Incentive compatibility implies that $\overset{*}{c}_i + (1-\overset{*}{h}_i)\theta_i \geq \overset{*}{c}_j + (1-\overset{*}{h}_j)\theta_i$, and since, $\overset{*}{h}_i \geq \overset{*}{h}_j$ by assumption, this implies $\overset{*}{c}_i \geq \overset{*}{c}_j$.

Point (e). Proof by contradiction. Suppose there exist a λ_1 and a λ_2 such that, λ_1 is either ≤ 0 or ≥ 1, $0 \leq \lambda_2 \leq 1$,

$$\lambda_1\overset{*}{h}_k + (1-\lambda_1)\overset{*}{h}_l = \lambda_2\overset{*}{h}_i + (1-\lambda_2)\overset{*}{h}_j \tag{A.1}$$

and

$$\lambda_1\overset{*}{c}_k + (1-\lambda_1)\overset{*}{c}_l < \lambda_2\overset{*}{c}_i + (1-\lambda_2)\overset{*}{c}_j \tag{A.2}$$

• Two cases. First case. If $\lambda_1 \geq 1$, then (A.2) minus θ_k times (A.1) implies:

$$\lambda_1(\overset{*}{c}_k + (1-\overset{*}{h}_k)\theta_k) + (1-\lambda_1)(\overset{*}{c}_l + (1-\overset{*}{h}_l)\theta_k) < \lambda_2(\overset{*}{c}_i + (1-\overset{*}{h}_i)\theta_k) + (1-\lambda_2)(\overset{*}{c}_j + (1-\overset{*}{h}_j)\theta_k) \tag{A.3}$$

Furthermore, incentive compatibility associated with type k not mimicking either i or j implies:

$$\lambda_2(\overset{*}{c}_i + (1-\overset{*}{h}_i)\theta_k) + (1-\lambda_2)(\overset{*}{c}_j + (1-\overset{*}{h}_j)\theta_k) \leq \overset{*}{c}_k + (1-\overset{*}{h}_k)\theta_k \tag{A.4}$$

Combining (A.3) and (A.4) implies

$$\overset{*}{c}_l + (1-\overset{*}{h}_l)\theta_k > \overset{*}{c}_k + (1-\overset{*}{h}_k)\theta_k$$

which violates the incentive compatibility constraint associate with type k not mimicking type l.

Second case. If $\lambda_2 \leq 1$, a similar contradiction can be shown by subtracting θ_l times (A.1) from (A.2). ■

Proof Corollary 1:

From point (c) of Proposition 1, we know that for types i, j, and k with the same market wage w, incentive compatibility implies that

$$w\overset{*}{h}_j - T_j \geq (w\overset{*}{h}_i - T_i)\lambda + (w\overset{*}{h}_k - T_k)(1-\lambda) \tag{A.5}$$

for $\overset{*}{h}_j = \lambda\overset{*}{h}_i + (1-\lambda)\overset{*}{h}_k$

This directly implies that

$$T_j \leq \lambda T_i + (1-\lambda)T_k \tag{A.6}$$

■

Lemma A1: *If i is distorted, there must exist a k such that i is an attractor of k.*

Proof: Without loss of generality, suppose that i is downward distorted and that there is no k of which it is an attractor. Now pick $dc_i > 0$ and $dh_i > 0$ such that

$$dc_i - \theta_i dh_i = 0; \tag{A.7}$$

this deviation is physically feasible because i is downward distorted, $\overset{*}{h}_i < \overset{*}{H}_i$. Moreover, this deviation is such that the resulting allocation is incentive compatible, since there is no k that, by assumption, wants to mimic i, and this deviation satisfies the participation constraint since it leaves i indifferent. The deviation also releases resources since $w_i > \theta_i$. The resulting resources can be divided equally among all types (lump sum) in order to create a Pareto improvement. This contradicts the optimality of the solution. ■

Lemma A: *If i is downward distorted, then, for all j that are distinct attractors of i,*

$$\overset{*}{h}_j < \overset{*}{h}_i \quad \textit{and} \quad \overset{*}{c}_j < \overset{*}{c}_i. \tag{A.8}$$

Proof: First we established by Lemma A1 that because i is downward distorted there must exist a k such that i is an attractor of k

Now suppose (A.8) is false, that is,

$$\overset{*}{h}_j \geq \overset{*}{h}_i \quad \text{or} \quad \overset{*}{c}_j \geq \overset{*}{c}_i. \tag{A.9}$$

That j is a distinct attractor of i implies that

$$\overset{*}{c}_i + (1-\overset{*}{h}_i)\theta_i = \overset{*}{c}_j + (1-\overset{*}{h}_j)\theta_i \quad \text{and} \quad w_i \geq w_j. \tag{A.10}$$

Therefore, (A.10) and (A.9) imply that

$$\overset{*}{h}_j > \overset{*}{h}_i \quad \text{and} \quad \overset{*}{c}_j > \overset{*}{c}_i \tag{A.11}$$

because j is a distinct attractor of i. That i is an attractor of k implies

$$\overset{*}{c}_k + (1-\overset{*}{h}_k)\theta_k = \overset{*}{c}_i + (1-\overset{*}{h}_i)\theta_k \quad \text{and} \quad w_k \geq w_i. \tag{A.12}$$

Case 1: Suppose $\theta_i > \theta_k$; in this case k prefers the allocation of j to that of i and can mimic j yielding a contradiction.
Case 2: Suppose that $\theta_k > \theta_i$ and consider the following change in allocation for type i. Set $c_i = c_j$ and $h_i = h_j$. The resulting allocation remains incentive compatible and satisfies individually rational since the utility level of i has not changed. There can't exist a type l that would now want to mimic i since l would have already mimicked j. Furthermore, such a change in allocation frees up resources since i is downward distorted. Therefore, this again leads to a contradiction since these resources could be equally divided among all types in order to induce a Pareto improvement. ■

Lemma B: *If i is upward distorted, then, for all j that are distinct attractors of i,*

$$\overset{*}{h}_j > \overset{*}{h}_i \quad \textit{and} \quad \overset{*}{c}_j > \overset{*}{c}_i. \tag{A.13}$$

Proof: The argument proceeds just as in the proof of Lemma A. ■

Lemma C1: *If i is upward distorted, then, for all j that are attractors of i, such that*

$$\overset{*}{c}_i = \overset{*}{c}_j \quad \textit{and} \quad \overset{*}{h}_i = \overset{*}{h}_j, \tag{A.14}$$

either j is upward distorted or the set of distinct attractors of j is empty.

Proof: Suppose, to the contrary, that j is both downward distorted[34] and has a distinct attractor, say k. Because j is downward distorted, $w_j > \theta_j$ and as it is an attractor of i, $w_i \geq w_j$. By assumption i is upward distorted so that $\theta_i > w_i$ and hence $\theta_i > \theta_j$. Given that j is downward distorted and has a distinct attractor, k, we know from Lemma A that

$$\overset{*}{h}_k < \overset{*}{h}_j \quad \text{and} \quad \overset{*}{c}_k < \overset{*}{c}_j. \tag{A.15}$$

Because k is a distinct attractor of j, $w_j \geq w_k$ and hence, $w_i \geq w_k$. Using this, (A.14), (A.15), and the fact that $\theta_i > \theta_j$, shows that i prefers $(\overset{*}{c}_k, \overset{*}{h}_k)$ to $(\overset{*}{c}_i, \overset{*}{h}_i)$ and that i could imitate k. This cannot be optimal and the above supposition must be false. ■

Lemma C2: *If i is upward distorted, then, for all j that are distinct attractors of i, j is either upward distorted or the set of distinct attractors of j is empty.*

Proof: Suppose, to the contrary, that j is both weakly downward distorted and the set of distinct attractors of j is not empty. Because i upward distorted, $\theta_i > w_i$; j is an attractor of i implies that $w_i \geq w_j$; and j being an attractor of i and weakly downward distorted implies that $w_j > \theta_j$. Hence, it follows that $\theta_i > \theta_j$. If k is a distinct attractor of j, $w_j \geq w_k$, then, by Lemma A, $\overset{*}{h}_k < \overset{*}{h}_j$ and $\overset{*}{c}_k < \overset{*}{c}_j$. Because $w_i \geq w_j$, this means that i prefers the allocation of k and can imitate k yielding a contradiction. ■

Lemma C: *If i is upward distorted, then, for all j that are extended attractors of i,*

$$\overset{*}{c}_j \geq \overset{*}{c}_i \quad \textit{and} \quad \overset{*}{h}_j \geq \overset{*}{h}_i. \tag{A.16}$$

Proof: Consider an extended attractor $(j_1, j_2, \ldots, j_K)$. If j_1 is is a distinct attractor, then by Lemma B, (A.16) is satisfied; if non-distinct, (A.16) is trivially satisfied. If j_1 is downward distorted, then from Lemma C1 and Lemma C2 we know that $j_1 = j_K$ and the proof is complete. If upward distorted, repeat the above argument and the proof is completed by induction. ■

Lemma D1: *If i is downward distorted, then, for all j that are distinct attractors of i, j is either downward distorted or the set of distinct attractors of j is empty.*

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[34] Given (A.14), $\overset{*}{h}_j = 1$ is not possible because i is downward distorted.

Proof: Suppose, to the contrary, that j is both weakly upward distorted and the set of distinct attractors of j is not empty. Because i downward distorted, $\theta_i < w_i$; j is an attractor of i implies that $w_i \geq w_j$; and j being an attractor of i and weakly upward distorted implies that $w_j < \theta_j$. Hence, it follows that $\theta_i < \theta_j$. If k is a distinct attractor of j, $w_j \geq w_k$, then, by Lemma B, $\overset{*}{h}_k > \overset{*}{h}_j$ and $\overset{*}{c}_k > \overset{*}{c}_j$. Because $w_i \geq w_j$, this means that i prefers the allocation of k and can imitate k yielding a contradiction. ■

Lemma D: *If i is strictly downward distorted, then the set of sources of i is non-empty and each member j of the set of sources must be strictly downward distorted or $\overset{*}{h}_j = 1$. Moreover, θ_i must be greater than θ_j for all j in the set of sources.*

Proof: First of all, suppose that the set of sources is empty. This means that either there is no j such that i is an attractor of j or that if i is an attractor of j, then either

$$\overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_i \leq \overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_i \tag{A.17}$$

or

$$\overset{*}{h}_j \leq \overset{*}{h}_i \tag{A.18}$$

or both. If there is no j such that i is an attractor of j, then it cannot be optimal for i to be downward distorted. Thus if i is downward distorted there exists some j such that i is an attractor of j, that is

$$\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_j = \overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_j \quad \text{and} \quad w_j \geq w_i. \tag{A.19}$$

Among all j such that i is an attractor of j, pick that one with the smallest θ_j.

Next suppose that $\overset{*}{h}_j \leq \overset{*}{h}_i$. We know that $w_j \geq w_i > \theta_i$. If j is downward distorted and i is a distinct attractor of j, then, Lemma A yields a contradiction. Thus, if j is downward distorted, it must be that

$$\overset{*}{h}_i = \overset{*}{h}_j \quad \text{and} \quad \overset{*}{c}_i = \overset{*}{c}_j. \tag{A.20}$$

If $\theta_i > \theta_j$, because θ_j is the smallest and because $\omega_j > \theta_j$, we can increase j's consumption and hours worked with indifference and satisfy the incentive compatibility constraints. This generates a surplus and i being an attractor of j cannot be part of the optimal solution. Hence, we have that $\theta_j > \theta_i$. But this in turn means that there is no reason for j to be downward distorted and because i is downward distorted $\overset{*}{h}_j = 1$ is not possible. Thus, j is not downward distorted and we have that $\theta_j > w_j$ and hence that $\theta_j > \theta_i$. If i is a distinct attractor of j, Lemma B yields a contradiction, and so (A.20) holds. But this implies that i being an attractor of j cannot be part of the optimal solution. Thus, $\overset{*}{h}_j > \overset{*}{h}_i$.

Suppose that (A.17) is true. Substituting (A.19) into (A.17) shows that $\theta_j > \theta_i$ and hence there is no reason for i to be downward distorted and an attractor of j. Thus, θ_j cannot have been the smallest among the set of those j such that i is an attractor of j and the set of sources of i is non-empty.

Now applying (A.19) and the definition of a source, (3.2), shows that $\theta_i > \theta_j$ and hence that j is not upward distorted; j is either downward distorted or $\overset{*}{h}_j = 1$. ■

Lemma E: *If i is downward distorted and if j is a fundamental source of i ($\overset{*}{h}_j = 1$), then there is no k such that j is a distinct attractor of k.*

Proof: Suppose that such a k exists. Then we have

$$\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_j = \overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_j \quad \text{and} \quad w_j \geq w_i, \tag{A.21}$$

$$\overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_i > \overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_i \quad \text{and} \quad \overset{*}{h}_i < 1, \tag{A.22}$$

and

$$\overset{*}{c}_k + (1 - \overset{*}{h}_k)\theta_k = \overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_k \quad \text{and} \quad w_k \geq w_j. \tag{A.23}$$

From (A.21)—(A.23) and the downward distortion of i we obtain

$$w_k \geq w_j \geq w_i > \theta_i > \theta_j \tag{A.24}$$

where the last inequality follows from Lemma D. If $\theta_k < \theta_j$ then k is downward distorted and Lemma A yields $\overset{*}{h}_j < \overset{*}{h}_k < 1$, a contradiction. If $\theta_k > \theta_j$, then from (A.21) and (A.23) we have

$$\overset{*}{c}_k + (1 - \overset{*}{h}_k)\theta_k < \overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_k; \tag{A.25}$$

the latter means that k prefers the allocation of i and, because $w_k \geq w_i$, could obtain it. This cannot be part of an optimal solution. ■

Lemma F: *There do not exist i and j such that*

$$\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_i > \overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_i \tag{A.26}$$

and

$$\overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_j > \overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_j. \tag{A.27}$$

Proof: Suppose there exist i and j such that (A.26) and (A.27) hold; it follows that

$$(\overset{*}{h}_j - \overset{*}{h}_i)(\theta_j - \theta_i) > 0. \tag{A.28}$$

If $\theta_i > \theta_j$ and hence that $\overset{*}{h}_i > \overset{*}{h}_j$, it must be that j cannot imitate i and hence $w_j < w_i$. In this case however i prefers the bundle of j and can imitate j; this is not incentive compatible. A similar argument works if $\theta_j > \theta_i$. ■

Lemma G: *If i is downward distorted, j is upward distorted, f is a fundamental source of i, k is an extended attractor of j, $\overset{*}{c}_j \geq \overset{*}{c}_i$, $\overset{*}{h}_j \neq \overset{*}{h}_i$, and $\omega_j \geq \omega_i$ then,*

$$U(\overset{*}{c}_k + (1 - \overset{*}{h}_k)\theta_k) > U(\overset{*}{c}_f). \tag{A.29}$$

Proof: In order to prove Lemma G it is helpful to define the following concepts. For any upward distorted j, let

$$\mathcal{J} = (j_0, j_1, \ldots, j_M) \tag{A.30}$$

be a set of types that are distinct extended attractors of j where $j_0 = j$, and for all j_m and $j_{m+1} \in \mathcal{J}$, $\overset{*}{h}_{j_m} < \overset{*}{h}_{j_{m+1}}$ and $\overset{*}{c}_{j_m} + (1 - \overset{*}{h}_{j_m})\theta_{j_m} = \overset{*}{c}_{j_{m+1}} + (1 - \overset{*}{h}_{j_{m+1}})\theta_{j_m}$.

Now define the function $J : [\overset{*}{h}_j, 1] \mapsto [\overset{*}{c}_j, \bar{c}]$ such that

$$J(h) = \overset{*}{c}_{j_m} + (h - \overset{*}{h}_{j_m})\theta_{j_m} \quad \text{for} \quad h \in [\overset{*}{h}_{j_m}, \overset{*}{h}_{j_{m+1}}] \tag{A.31}$$

and

$$J(h) = \overset{*}{c}_{j_M} + (h - \overset{*}{h}_{j_M})\theta_{j_M} \quad \text{for} \quad h \in [\overset{*}{h}_{j_M}, 1] \tag{A.32}$$

Graphed in the $c-h$ space, the function $J(h)$ corresponds to a continuous and positively slopped line that connects by linear segments a set of distinct attractors of j and, if $\overset{*}{h}_{j_M} < 1$, extends this line along j_M indifference curve up to the point where $h = 1$.

For any downward distorted i, let

$$\mathcal{I} = (i_0, i_1, \ldots, i_N) \tag{A.33}$$

be a set of types that are distinct extended sources of j where $i_0 = i$,$i_N = f$ (f being a fundamental source of i), and for all i_n and $i_{n+1} \in \mathcal{I}$, $\overset{*}{h}_{i_n} < \overset{*}{h}_{i_{n+1}}$ and $\overset{*}{c}_{i_n} + (1 - \overset{*}{h}_{i_n})\theta_{i_{n+1}} = \overset{*}{c}_{i_{n+1}} + (1 - \overset{*}{h}_{i_{n+1}})\theta_{i_{n+1}}$.

Now define the function $I : [\overset{*}{h}_i, 1] \mapsto [\overset{*}{c}_i, \bar{c}]$ such that

$$I(h) = \overset{*}{c}_{i_{n+1}} + (h - \overset{*}{h}_{i_{n+1}})\theta_{i_{n+1}} \quad \text{for} \quad h \in [\overset{*}{h}_{i_n}, \overset{*}{h}_{i_{n+1}}] \tag{A.34}$$

Graphed in the $c-h$ space, the function $I(h)$ corresponds to a continuous and positively slopped line that connects by linear segments a set of distinct sources of i that extend all the way to on of its fundamental sources.

The functions $J(h)$ and $I(h)$ are not uniquely defined since the sets $\mathcal{J}$ and $\mathcal{I}$ are not generally unique. All statements invoking these functions refer to any function satisfying the above definition.

Given these definitions, we now show that if i and j satisfy the statement of the Lemma, then

$$J(h) > I(h) \quad \text{for} \quad h \in [\max(\overset{*}{h}_i, \overset{*}{h}_j), 1]. \tag{A.35}$$

Once it is shown that the above strict inequality must hold, it is then trivial to show that (A.29) must hold since (A.29) is just a special case when $h = 1$.

Let us first show that $J(\max(\overset{*}{h}_i, \overset{*}{h}_j)) \leq I(\max(\overset{*}{h}_i, \overset{*}{h}_j))$ leads to a contradiction. If $\overset{*}{h}_i = \max(\overset{*}{h}_i, \overset{*}{h}_j)$, then we know (from the definition of $I(h)$) that $I(\overset{*}{h}_i) = \overset{*}{c}_i$ and by assumption that $\overset{*}{c}_j \geq \overset{*}{c}_i$; however, as $\theta_j > \theta_i$ this leads to a contradiction. If $\overset{*}{h}_j = \max(\overset{*}{h}_i, \overset{*}{h}_j)$, then we know (from the definition of $J(h)$) that $J(\overset{*}{h}_j) = \overset{*}{c}_j$. Moreover, from the last statement in Lemma D, we know that $I(h) < \overset{*}{c}_i + (h - \overset{*}{h}_i)\theta_i$ for $h > \overset{*}{h}_i$ and hence $I(\overset{*}{h}_j) < \overset{*}{c}_i + (\overset{*}{h}_j - \overset{*}{h}_i)\theta_i$. By the fact that i is downward distorted, j is upward distorted, and $w_j \geq w_i$, we know that $\theta_j > \theta_i$. Combining these elements we obtained that $c_j = J(\overset{*}{h}_j) \leq I(\overset{*}{h}_j) < \overset{*}{c}_i + (\overset{*}{h}_j - \overset{*}{h}_i)\theta_j$. Rewriting these inequalities we obtain that $\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_j < \overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_j$, which is inconsistent with the incentive compatibility constraint associated with type j not mimicking type i. Hence if $I(h) \geq J(h)$ for some $h \in [\max(\overset{*}{h}_i, \overset{*}{h}_j), 1]$, it must be that $J(h)$ and $I(h)$ cross (or touch).

So let us denote by $\bar{h}$ be the first point where $J(\bar{h}) = I(\bar{h})$ and let j'_m be the type with the largest value of h in $\mathcal{J}$ that is strictly less then $\bar{h}$ and let $i_{n'}$ be the point in $\mathcal{I}$ with the smallest value of h that is larger or equal to $\bar{h}$. From the definitions it follows that

$$\overset{*}{c}_{j_{m'}} + (1 - \overset{*}{h}_{j_{m'}})\theta_{j_{m'}} \leq \overset{*}{c}_{i_{n'}} + (1 - \overset{*}{h}_{i_{n'}})\theta_{j_{m'}}, \tag{A.36}$$

$$\overset{*}{c}_{i_{n'}} + (1 - \overset{*}{h}_{i_{n'}})\theta_{i_{n'}} = \overset{*}{c}_{i_{n'-1}} + (1 - \overset{*}{h}_{i_{n'-1}})\theta_{i_{n'}}, \tag{A.37}$$

$$\overset{*}{c}_{i_{n'}} + (1 - \overset{*}{h}_{i_{n'}})\theta_{i_{n'}} < \overset{*}{c}_{j_{m'}} + (1 - \overset{*}{h}_{j_{m'}})\theta_{i_{n'}}, \tag{A.38}$$

and

$$\overset{*}{c}_{j_{m'}} + (1 - \overset{*}{h}_{j_{m'}})\theta_{j_{m'}} > \overset{*}{c}_{i_{n'-1}} + (1 - \overset{*}{h}_{i_{n'-1}})\theta_{j_{m'}}. \tag{A.39}$$

Combining (A.36) and (A.39) yields

$$\overset{*}{c}_{i_{n'-1}} + (1 - \overset{*}{h}_{i_{n'-1}})\theta_{j_{m'}} < \overset{*}{c}_{i_{n'}} + (1 - \overset{*}{h}_{i_{n'}})\theta_{j_{m'}} \tag{A.40}$$

which in conjunction with (A.37) implies that

$$\theta_{i_{n'}} > \theta_{j_{m'}}. \tag{A.41}$$

From (A.38) it is then clear that

$$\omega_{j_{m'}} > \omega_{i_{n'}}, \tag{A.42}$$

which implies in (A.36) holds with equality.

Since (A.36) holds with equality and $\omega_{j_{m'}} > \omega_{i_{n'}}$, Lemma E implies that $i_{n'}$ is downward distorted (since it is not a fundamental source) and Lemma A implies that $j_{m'}$ is upward distorted (since $0 < \overset{*}{h}_{j_{m'}} < 1$). Hence we know that $\theta_{j_{m'}} > \omega_{j_{m'}}$ and $\omega_{i_{n'}} > \theta_{i_{n'}}$. In conjunction with (A.41) we find that $\omega_{j_{m'}} < \omega_{i_{n'}}$, which contradicts (A.42). This completes the demonstration that $J(h) > I(h)$ (on their shared domain) and therefore implies that (A.29) must hold by the very fact that $\mathcal{J}$ and $\mathcal{I}$ can be chosen such that $J(1) = \overset{*}{c}_k + (1 - \overset{*}{h}_k)\theta_k$ and $I(1) = \overset{*}{c}_f$.

■

Lemma H: *If j is an extended attractor of an upward distorted i then*

(1) If there exists a k with $h_k = 0$ and $\omega_k \leq \omega_i$, then

$$U(\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_j) > U(\overset{*}{c}_k + \theta_j); \tag{A.43}$$

(2) otherwise

$$U(\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_j) > U(\theta_j). \tag{A.44}$$

Proof: Incentive compatibility implies that the relationships (A.43) and (A.44) hold with a weak inequality, therefore all that must be shown is that these relationships cannot hold with strict equality. Suppose they do hold with equality then, by the same argument as that used in Lemma C2, we know that j must be upward distorted. But if j is upward distorted and one of the two relationships holds with equality, then change the individual allocation of j to $\{\overset{*}{c}_j = \overset{*}{c}_k, \overset{*}{h}_j = 0\}$ if there is equality in case (1) and change it to $\{\overset{*}{c}_j = 0, \overset{*}{h}_j = 0\}$ if there is equality in case (2). The above modification of the allocation is incentive compatible (The argument here is similar to that of Lemma A.), and generates a surplus. Since the surplus can always be divided equally among individuals and thereby improve welfare, this leads to a contradiction. ■

Proof of Proposition 2: By contradiction, suppose there exists a type i and j, such that $\omega_i < \omega_j$, $\overset{*}{h}_i < \overset{*}{H}_i = 1$ and $\overset{*}{h}_j > \overset{*}{H}_i = 0$, which implies that $\theta_i < \theta_j$. Without loss of generality, suppose that j is the type with the highest h (hours) among the upward distorted types with $\omega = \omega_j$, and suppose that i is the type with the lowest h among the downward distorted types with $\omega = \omega_i$.

The first part of the argument is to show that the utility level of i is equal to $U(\theta_i)$, that is, type i is on his participation constraint. Suppose not. This with only happen if i had an attractor that has a utility level less than the fundamental source of i. Otherwise it would be desirable and incentive compatible (by Lemma D1) to reduce the after tax income of i and his attractor (if it exists) and transfer the resources to the fundamental source of i. But incentive compatibility implies that the attractor of i has utility greater than that of i, and hence i must be on his participation constraint and have utility $U(\theta_i)$.

The second step of the argument is to show that the only potential attractor of j is one with $h = 1$ and hence with the same utility as j (which is necessarily greater that the utility if i since $\theta_i < \theta_j$). Suppose not. Then j would have (by Lemmas B and C2) a downward distorted attractor, say type k, with type k having no attractor. In this case, it would be desirable and incentive compatible to reduce the after-tax income of both j and k and redistribute the resources to the fundamental source of k. The only potential impediment to such a transfer of resources would be that j is on his participation constraint, but this is impossible by Lemma A1, and hence the only potential attractor of j is one with the same utility as j and with no attractor.

Given the two arguments above, we know that the utility of j and his attractor (if it exists) are greater than the utility of the fundamental source of i, hence it would be desirable and feasible to transfer resources (keeping the hours allocations fixed) from j and his attractor to the the fundamental source of i. Hence a contradiction. ■

Proof of Corollary 2:

I) From Proposition 2, we know that if $\omega_i = \omega_j > \overset{*}{w}$, then $\overset{*}{h}_j > 0$, and hence that $\omega_j > \theta_j$. By hypothesis $\overset{*}{h}_i < \overset{*}{h}_j$, and therefore, $\theta_i > \theta_j$ by Proposition 1. Incentive compatibility implies, given that $\omega_i = \omega_j$, that

$$\omega_j \overset{*}{h}_i - T_i + (1 - \overset{*}{h}_i)\theta_i \geq \omega_j \overset{*}{h}_j - T_j + (1 - \overset{*}{h}_j)\theta_i \tag{A.45}$$

which upon rewriting becomes

$$T_i - T_j \leq (\omega_j - \theta_i)(\overset{*}{h}_i - \overset{*}{h}_j). \tag{A.46}$$

If $\omega_i = \omega_j > \theta_i$ this implies that $T_i \leq T_j$. Suppose therefore that $\theta_i > \omega_i$. Incentive compatibility yields

$$\omega_i \overset{*}{h}_j - T_j + (1 - \overset{*}{h}_j)\theta_j \geq \omega_j \overset{*}{h}_i - T_i + (1 - \overset{*}{h}_i)\theta_j \tag{A.47}$$

which can be rewritten as

$$(\omega_j - \theta_j)(\overset{*}{h}_j - \overset{*}{h}_i) \geq T_j - T_i. \tag{A.48}$$

If (A.48) holds with equality then $T_j > T_i$. Therefore we need only consider the case where $\overset{*}{h}_i = 0$ and (A.47) is a strict inequality. In this case, i's utility is greater than that of j and all sources and extended sources of j. If $T_i < 0$, then we can reduce the subsidy to i and transfer this to chain from j to its fundamental source contradicting optimality in which case (A.48)must hold with equality. If i is on its participation constraint, and hence that $T_i = 0$, we must show that $T_j \geq 0$. Suppose not and that $T_j < 0$. Because (A.47) holds with a strict inequality, if $\overset{*}{h}_j < 1$, we can reduce the subsidy to j and transfer it to those in that chain of attractors. This would increase total utility as those in the chain who work more hours have lower utility levels than j. Hence this case is impossible as well. Now consider the only remaining problem case where $T_j < 0 = T_i$, $\overset{*}{h}_i = 0$ and $\overset{*}{h}_j = 1$. In this case, there must exist a downward distorted type k, with $\omega_i > \omega_k > \overset{*}{w}$, otherwise $\overset{*}{w}$ could be selected anew such that $\omega_i < \overset{*}{w}$. Now consider increasing T_j and transferring the revenue to the fundamental source of k, denoted type f. Since in this case, $\theta_f < \omega_k < \omega_i < \omega_j - T_j$, it is desirable and incentive compatible to make such a transfer. Hence $\overset{*}{h}_i = 0$ and $T_j < T_i = 0$ is not possible, which confirms that $T_j \geq T_i$.

II) From Proposition 2, we know that if $\omega_i = \omega_j < \overset{*}{w}$ and $\overset{*}{h}_i < \overset{*}{h}_j$, then either

(i) $1 > \overset{*}{h}_j \geq 0$ and $\theta_j > \omega_j$ or

(ii) $\overset{*}{h}_j = 1$ and $\theta_j < \omega_j$.

In the first case, incentive compatibility implies

$$\omega_j \overset{*}{h}_j - T_j + (1 - \overset{*}{h}_j)\theta_j \geq \omega_j \overset{*}{h}_i - T_i + (1 - \overset{*}{h}_i)\theta_j \tag{A.49}$$

which implies $T_j \leq T_i$.

In the second case, if j is an attractor of i, then,

$$\omega_i \overset{*}{h}_i - T_i + (1 - \overset{*}{h}_i)\theta_i = \omega_j - T_j \tag{A.50}$$

which upon rewriting becomes

$$T_i - T_j = (\omega_i - \theta_i)(\overset{*}{h}_i - 1) > 0 \tag{A.51}$$

which implies that $T_i > T_j$. If j is not an attractor of i, the only remaining possibility is that $h_i = 0$, $T_i = 0$ and $T_j > 0$; then, there must exist a type k, with $\omega_i < \omega_k < \overset{*}{w}$, and $\overset{*}{h}_k > \overset{*}{H}_k = 0$, otherwise $\overset{*}{w}$ could be chosen to be greater that ω_i. In this case, it is optimal and incentive compatible to reduce the after tax income of types with $\omega = \omega_k$ and $h > 0$, and transfer the resources to j. Hence, $h_i = 0$, $T_i = 0$ and $T_j > 0$ is not possible and therefore $T_j \leq T_i$. Finally, since $h = 0$ implies that T must be greater than or equal to zero, it follows that $T_j \leq T_i \leq 0$. ■

Proof of Proposition 3 A function $g(\cdot)$ satisfying (5.1) always exist if there cannot exit a downward distorted i and an an upward distorted j such that $w_j \geq w_i$, $\overset{*}{h}_j \geq \overset{*}{h}_i$, and $(w_j, \overset{*}{h}_j) \neq (w_i, \overset{*}{h}_i)$. Therefore, let us assume that such i and j exist, and show that it leads to a contradiction. Note that this implies that $\overset{*}{c}_j \geq \overset{*}{c}_i$ for otherwise j would imitate i. From the fact that i is downward distorted and j is upward distorted we can immediately infer that

$$\theta_j > w_j \geq w_i > \theta_i. \tag{A.52}$$

Because $w_j \geq w_i$, it must be the case that (A.27) is false for otherwise j would imitate i. Hence, either

$$\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_j = \overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_j \tag{A.53}$$

or

$$\overset{*}{c}_j + (1 - \overset{*}{h}_j)\theta_j > \overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_j. \tag{A.54}$$

is true. If, (A.53) holds, then, as $w_j \geq w_i$, i is an attractor of j. If $\overset{*}{c}_j > \overset{*}{c}_i$, then i is a distinct attractor of j and by Lemma B, $\overset{*}{c}_j < \overset{*}{c}_i$, a contradiction. Therefore, $\overset{*}{c}_j = \overset{*}{c}_i$ and $\overset{*}{h}_j = \overset{*}{h}_i$. Because i is downward distorted, Lemma C1 implies that the set of distinct attractors of i is empty. That is, there does not exist k such that

$$\overset{*}{c}_i + (1 - \overset{*}{h}_i)\theta_i = \overset{*}{c}_k + (1 - \overset{*}{h}_k)\theta_i \quad \text{and} \quad w_i \geq w_k. \tag{A.55}$$

In this case, consider decreasing $\overset{*}{c}_i$ by ϵ ($dc_i < 0$) and transferring the resulting savings to a fundamental source of i denoted f. From Lemma D, such a fundamental source always exits. By Lemma H and the fact that $\theta_i < \theta_j$, type i's participation constraint cannot be strictly binding nor can can i be indifferent between his or her allocation and that of some individual k with $\overset{*}{h}_k = 0$. Therefore, by Lemma E and the fact that i has no distinct attractors, for ϵ small enough, such a transfer does not interfere with any of the incentive compatibility constraints or participation constraints. Moreover, from the definition of a source, $U(\overset{*}{c}_i+(1-\overset{*}{h}_i)\theta_i) > U(\overset{*}{c}_f)$ and therefore such a deviation would be welfare improving, hence a contradiction.

Thus suppose that (A.54) is true and let f be a fundamental source of i. From Lemma G we know that for every member of the extended attractors of j, their utility level must be strictly greater than that of f. Reduce by ϵ the consumption of all types that are extended attractors of j (including j itself), and transfer the resulting savings to a fundamental source of i. This deviation is welfare improving, by construction it is incentive compatible, and by Lemma H it satisfies the participation constraints; therefore, we have a contradiction. Hence, if $w_j \geq w_i$, $\overset{*}{c}_j \geq \overset{*}{c}_i$, and $(w_j, \overset{*}{c}_j) \neq (w_i, \overset{*}{c}_i)$, then j is downward distorted. ■

Proof of Corollary 3: The proof of this corollary proceeds along the same lines as the proof of Corollary 2.

I) If $\omega_i = \omega_j$, $\overset{*}{h}_i < \overset{*}{h}_j$ and $\omega_i > g(h_i)$, then $\theta_i < \omega_i$ and incentive compatibility implies

$$\omega_i\overset{*}{h}_i - T_i + (1-\overset{*}{h}_i)\theta_i \geq \omega_i\overset{*}{h}_j - T_j + (1-\overset{*}{h}_j)\theta_i \tag{A.56}$$

which upon rewriting yields

$$0 \geq (\omega_i - \theta_i)(\overset{*}{h}_i - \overset{*}{h}_j) \geq T_i - T_j \tag{A.57}$$

which implies $T_j \geq T_i$.

II) If $\omega_i = \omega_j$, $\overset{*}{h}_i < \overset{*}{h}_j$ and $\omega_j < g(h_j)$, then either $\overset{*}{h}_j > 0$ and $\omega_j < \theta_j$ or $\overset{*}{h}_j = 1$ and $\omega_j > \theta_j$

In the first case with $\overset{*}{h}_j > 0$ and $\omega_j < \theta_j$, incentive compatibility implies

$$\omega_j\overset{*}{h}_j - T_j + (1-\overset{*}{h}_j)\theta_j \geq \omega_j\overset{*}{h}_i - T_i + (1-\overset{*}{h}_i)\theta_j \tag{A.58}$$

which implies $T_j \leq T_i$.

In the second case, using the same arguments as in the proof of Corollary 2, one can rule out all the configurations with $\overset{*}{h}_j = 1$ and $\omega_j > \theta_j$ except ones where $T_j \leq T_i$. Once again, since at $h = 0$ taxes must be smaller or equal to zero, it follows that $T_j \leq T_i \leq 0$

■

Proof of Proposition 4: First suppose that there exists i such that $\omega_i < \theta_i$ and that $\overset{*}{h}_i > 0$. By Lemma A1, there exists k such that i is an attractor of k. If $\theta_k \leq \theta_i$ then there is, as in the reasoning of Lemma A1 no reason for i to be upward distorted; hence it must be the case that $\theta_k > \theta_i$. But by the hypothesis of negative correlation this implies that

$\omega_k < \omega_i$ in which case i cannot be an attractor of k yielding a contradiction. If $\omega_i < \theta_i$ then $\overset{*}{h}_i = 0$.

Now suppose that there exists i such that $\omega_i > \theta_i$ and the $\overset{*}{h}_i < 1$. By Lemma D these exists a source of I, say, f, which by Lemma F can be moved upwards. If there are no attractors of i we can move the source chain from i to f upward raising the utilities of everyone in the chain and hence this could not have been part of an optimal solution. Suppose that there is an attractor of i, say k. Then, $\omega_k \leq \omega_i$ which implies by the hypothesis of negative correlation that $\theta_k \geq \theta_i$ and hence that the utility of k is greater than or equal to that of i and hence of f. This is true for all extended attractors of i and hence the entire source chain can be moved upwards and this cannot have been part of an optimal solution. ■

Proof of Proposition 5: Consider first the case where $\omega_i > \theta_i$ and suppose that $\overset{*}{h}_i < 1$. By Lemma D, i has a source, say, f, where $\omega_f > \omega_i$ and by the hypothesis of positive correlation that $\theta_f > \theta_i$. This contradicts the claim that f is a source of i and hence $\overset{*}{h}_i = 1$.

Now consider the case where $\omega_i < \theta_i$. Example 2 in the text is a case where $\overset{*}{h}_i > 0$ and hence proves that complete efficiency cannot be attained in this case.■

Proof of Proposition 6: By contradiction. Suppose there is a $\{c'_i, h'_i, w'_i, h'^f_i\}_{i=1}^{NM}$ which is superior to the one defined in the proposition. Then it must be the case that there exist at least one j such that $h_j \neq 0$ and $h^f_j \neq 0$; otherwise $\{\tilde{c}_i, \tilde{h}_i\}_{i=1}^{NM}$ would not solve OP for the modified economy. However, if there exist such a j, we can free up resources by considering the following perturbation

(1) If $\omega_j > \omega^f$, set $h_j = h'_j + h'^f_j$ and $h^f_j = 0$

(1) If $\omega_j = \omega^f$, set $h^f_j = h'_j + h'^f_j$ and $h_j = 0$

Since such a perturbation allows all incentive compatibility constraints to remain satisfied and allows an equal redistribution of positive resources to every type, it allows for a Pareto improvement and thereby leads to a contradiction. ■

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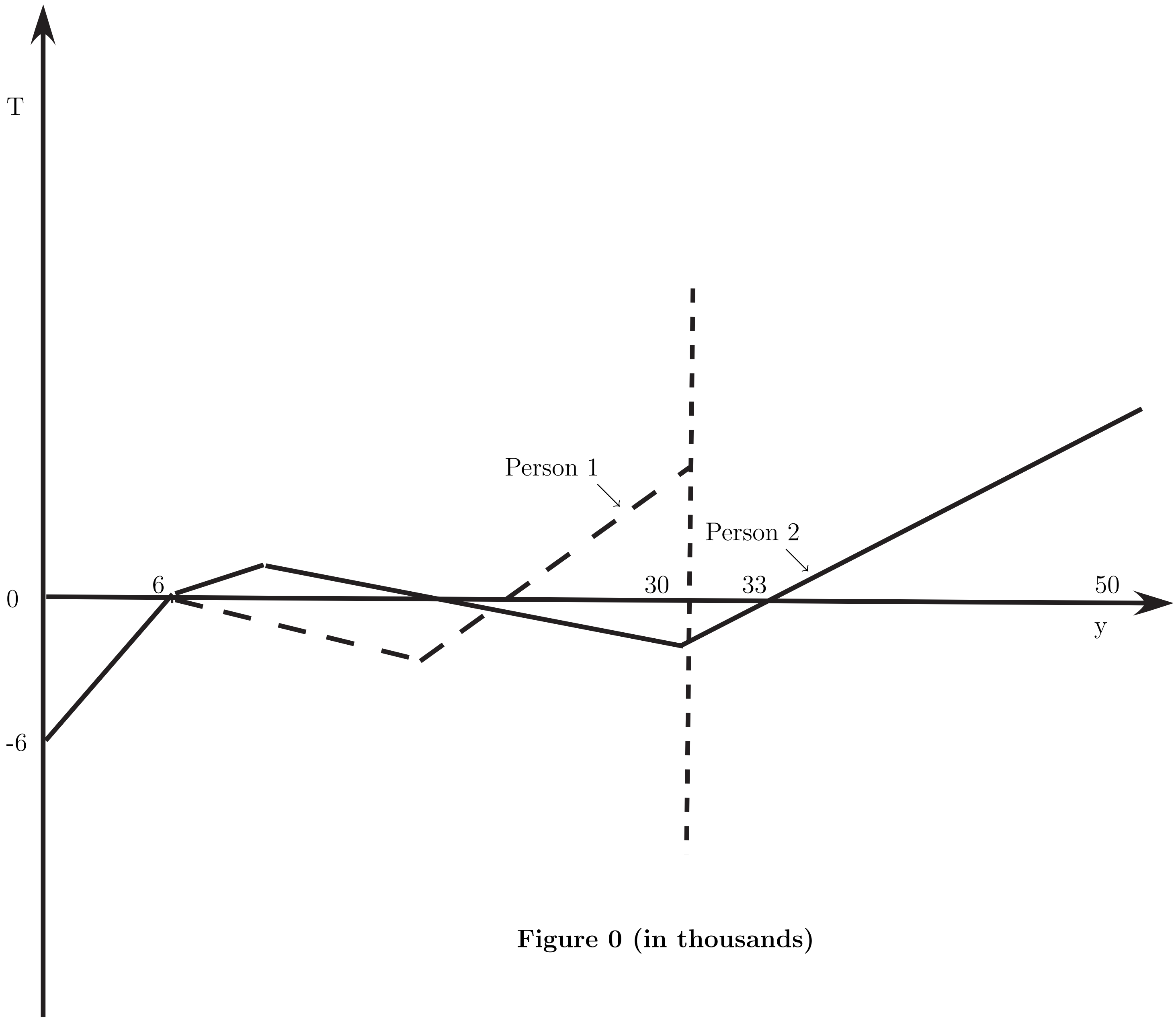
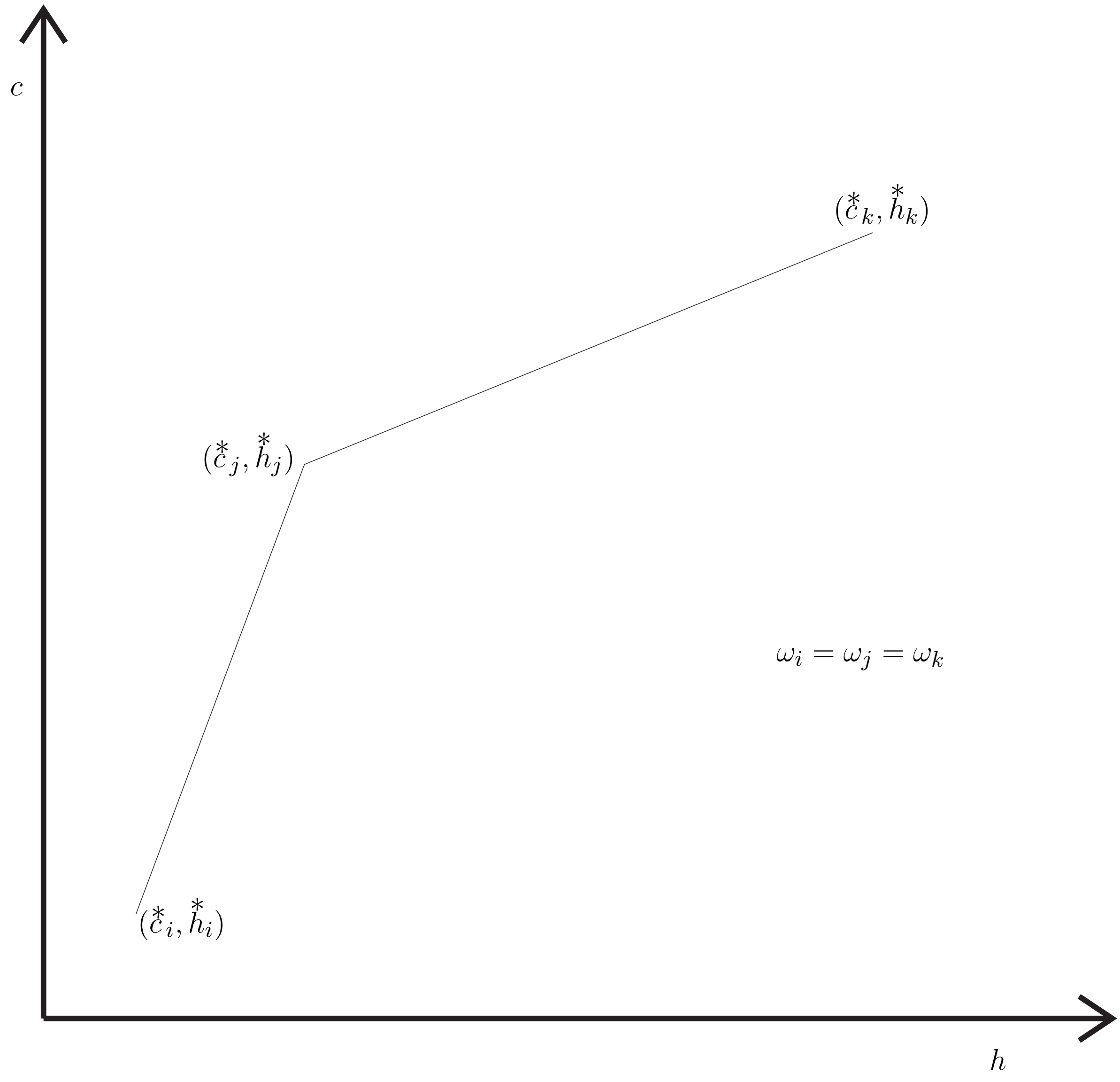
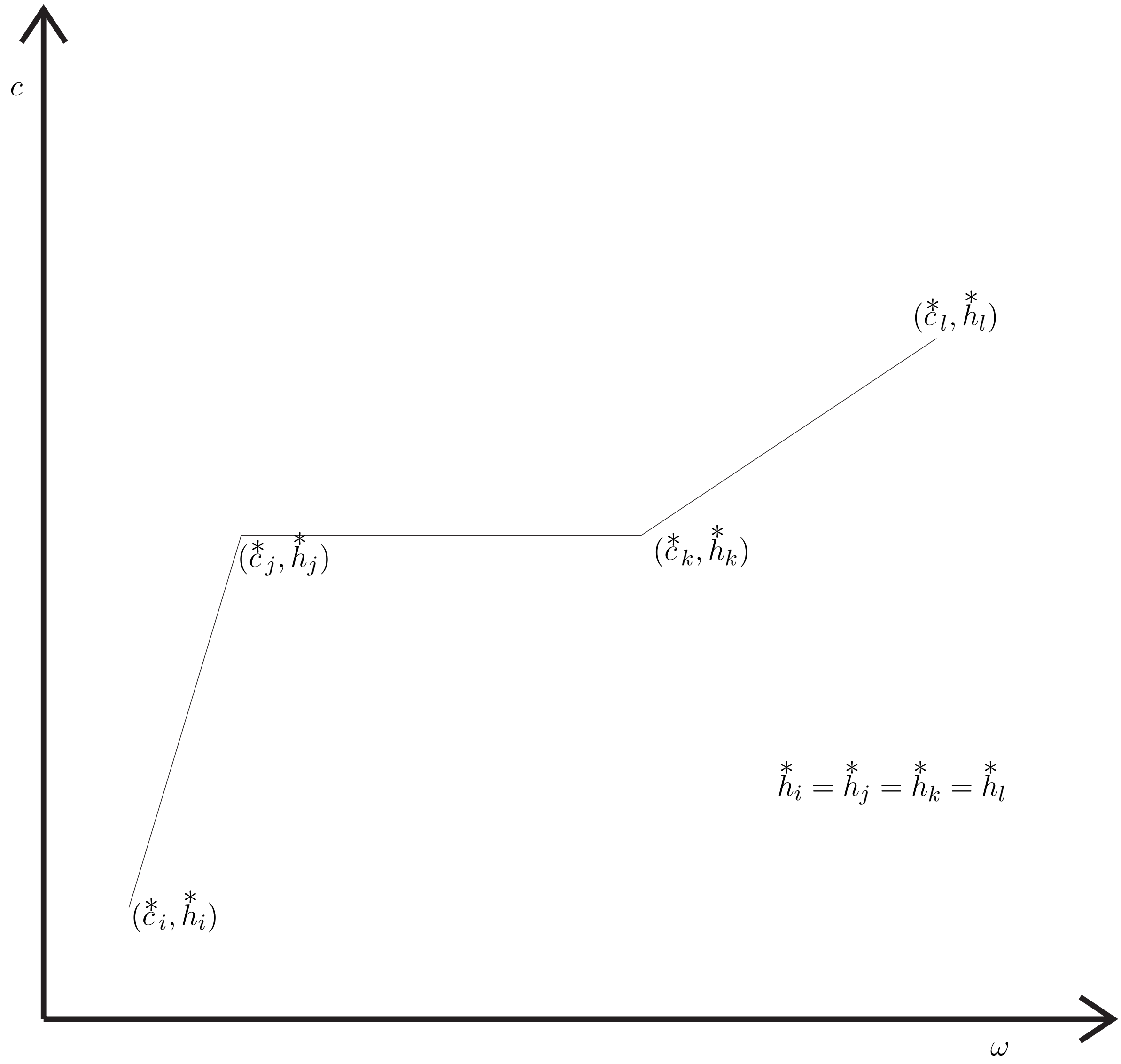


Figure 0 (in thousands)



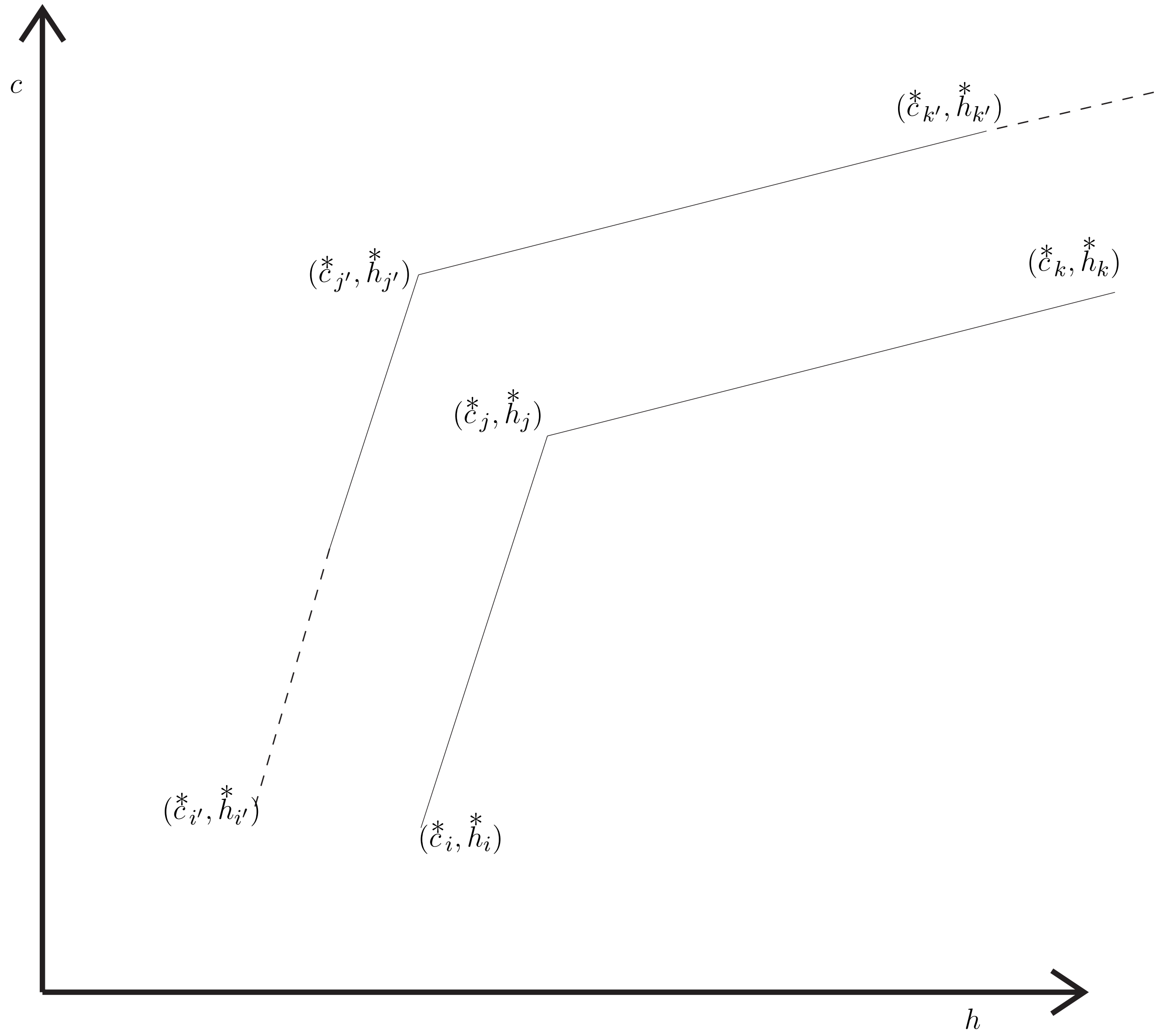
c
$(\overset{*}{c}_k, \overset{*}{h}_k)$
$(\overset{*}{c}_j, \overset{*}{h}_j)$
$\omega_i = \omega_j = \omega_k$
$(\overset{*}{c}_i, \overset{*}{h}_i)$
h

Figure 1A



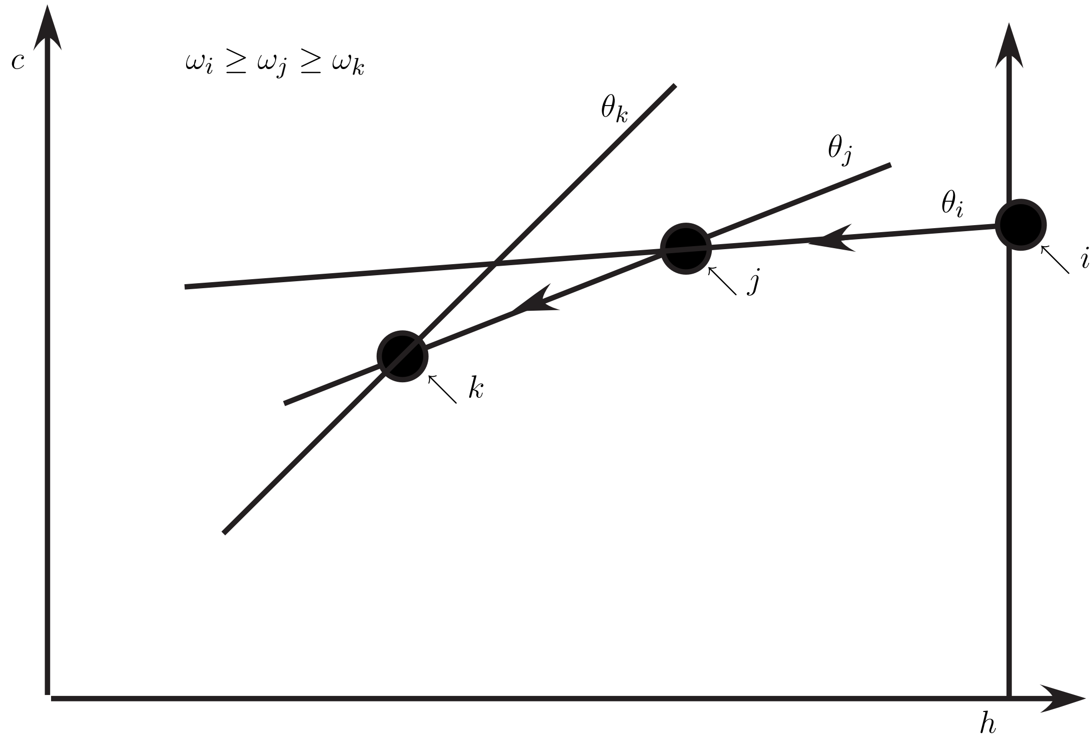
c
$(\overset{*}{c}_l, \overset{*}{h}_l)$
$(\overset{*}{c}_j, \overset{*}{h}_j)$
$(\overset{*}{c}_k, \overset{*}{h}_k)$
$\overset{*}{h}_i = \overset{*}{h}_j = \overset{*}{h}_k = \overset{*}{h}_l$
$(\overset{*}{c}_i, \overset{*}{h}_i)$
ω

Figure 1B



c
$(\overset{*}{c}_{k'}, \overset{*}{h}_{k'})$
$(\overset{*}{c}_{j'}, \overset{*}{h}_{j'})$
$(\overset{*}{c}_{k}, \overset{*}{h}_{k})$
$(\overset{*}{c}_{j}, \overset{*}{h}_{j})$
$(\overset{*}{c}_{i'}, \overset{*}{h}_{i'})$
$(\overset{*}{c}_{i}, \overset{*}{h}_{i})$
h

Figure 1C



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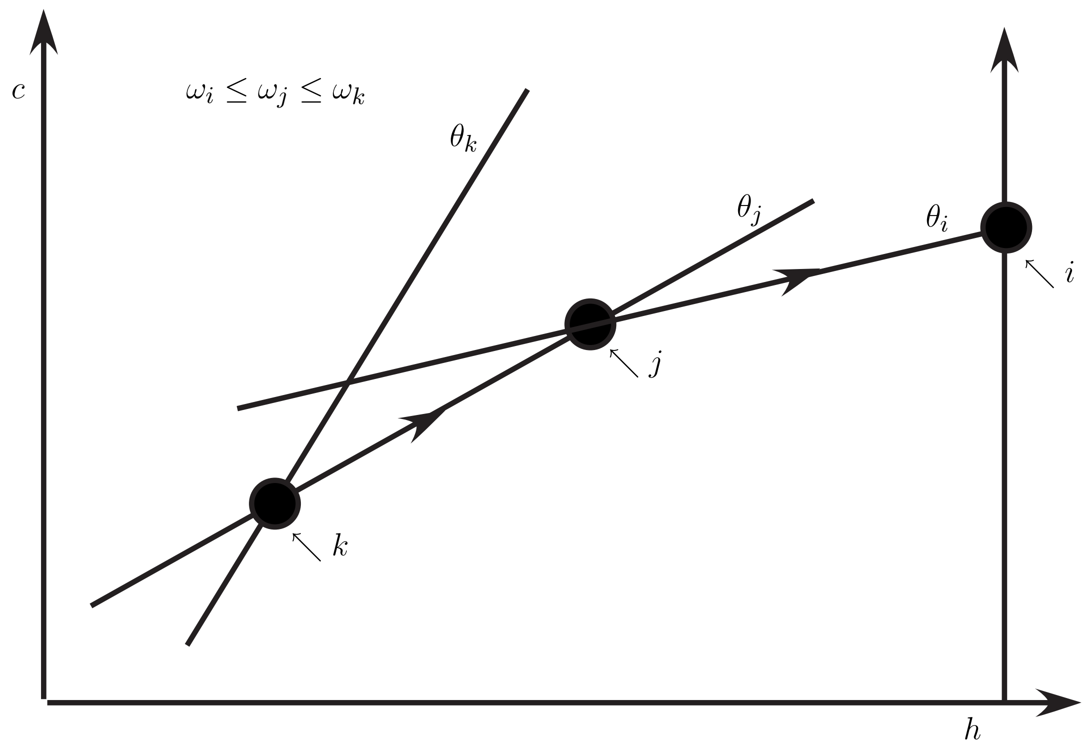

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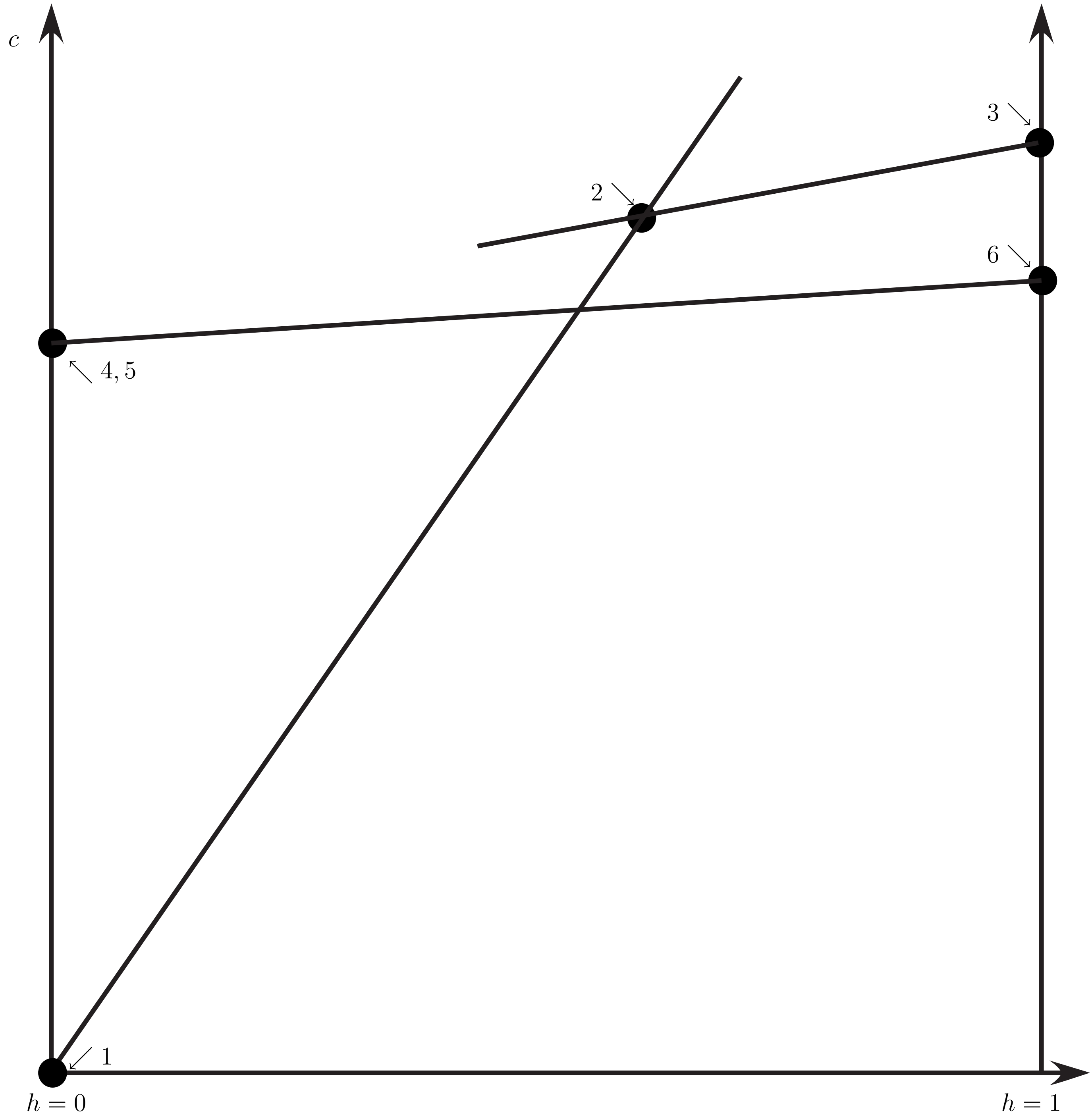

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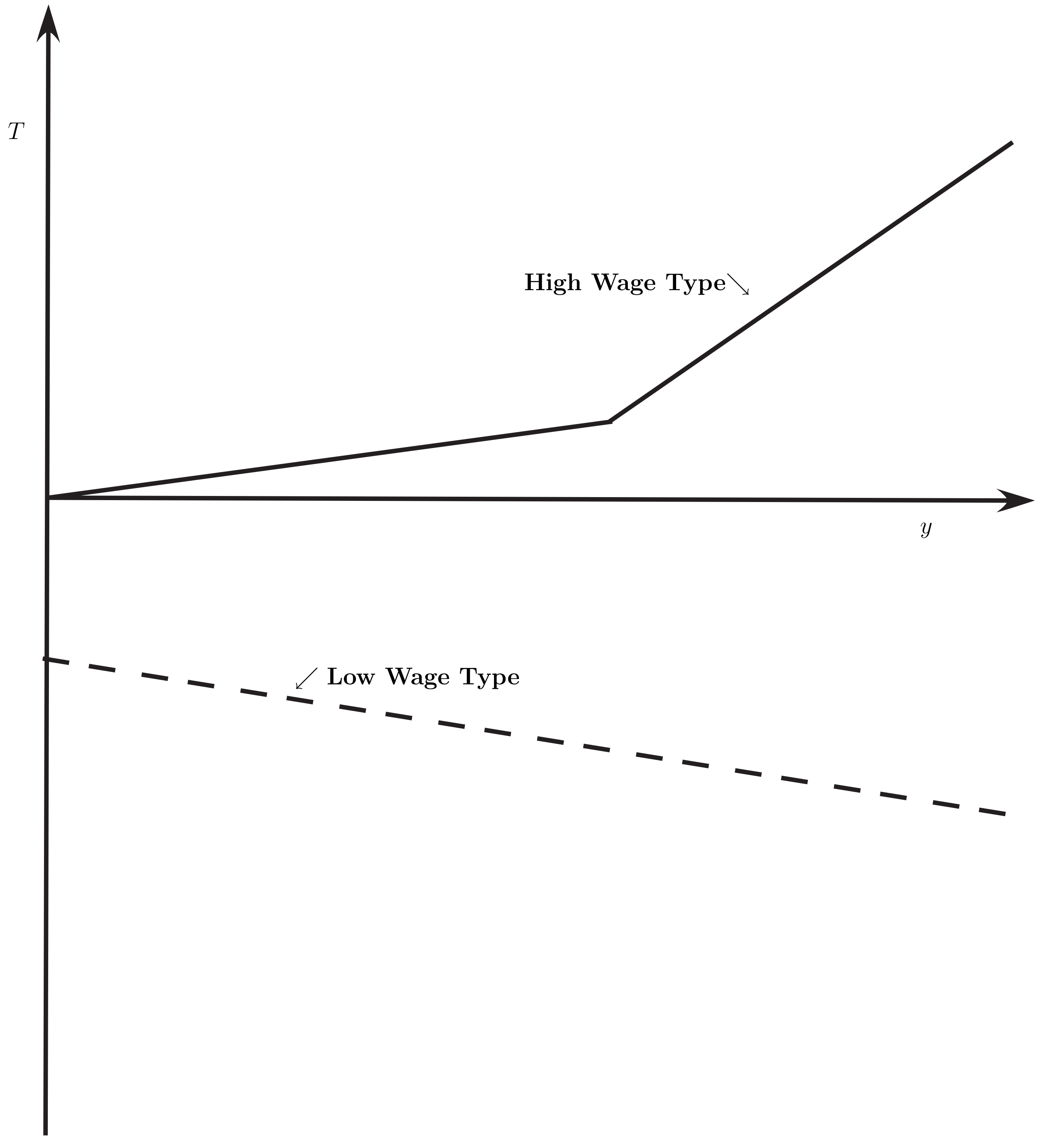

Figure 4

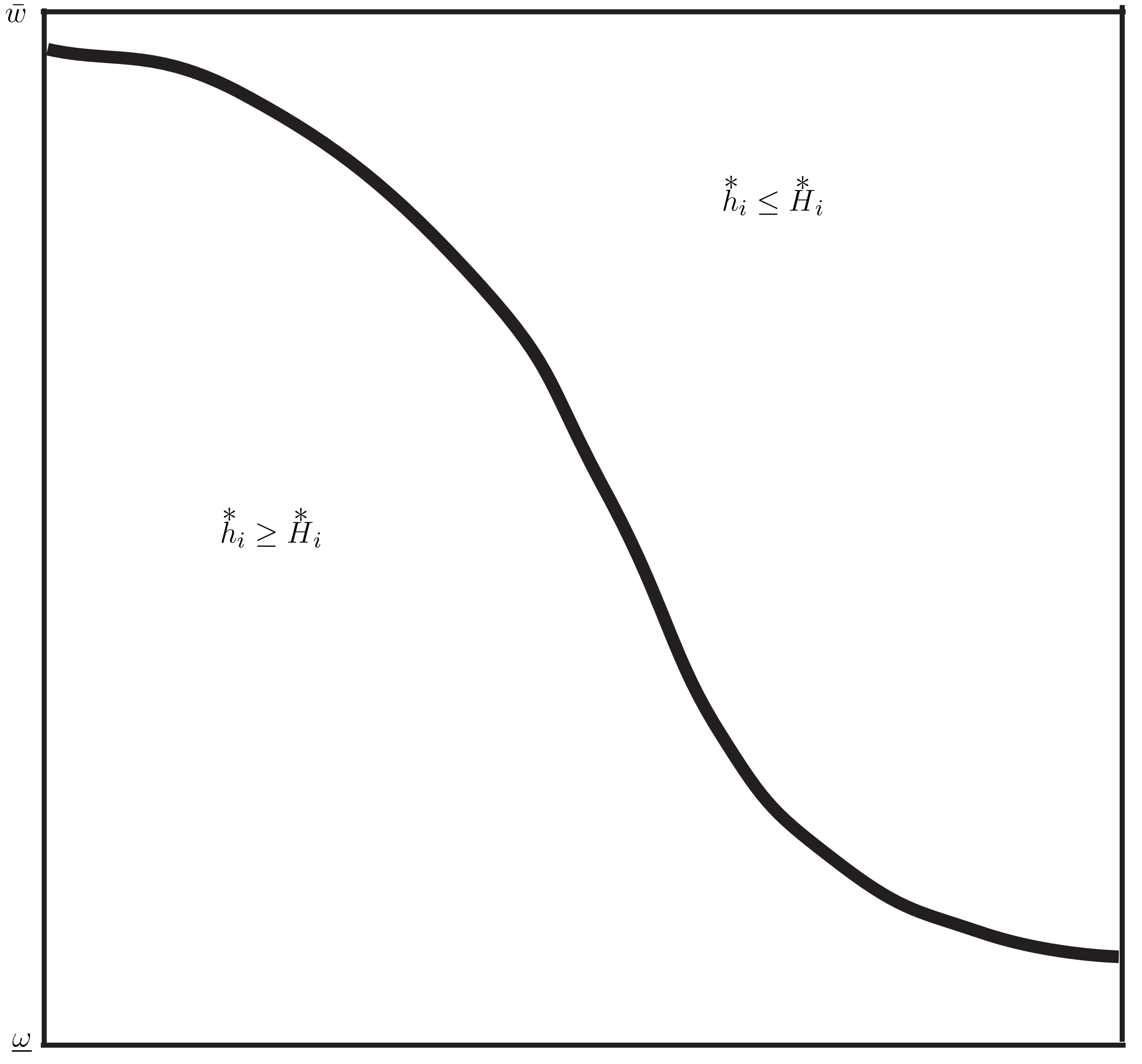
$\bar{w}$
$\overset{*}{h}_i \leq \overset{*}{H}_i$
$\overset{*}{h}_i \geq \overset{*}{H}_i$
$\underline{\omega}$
$h = 0$
$h = 1$

Figure 5

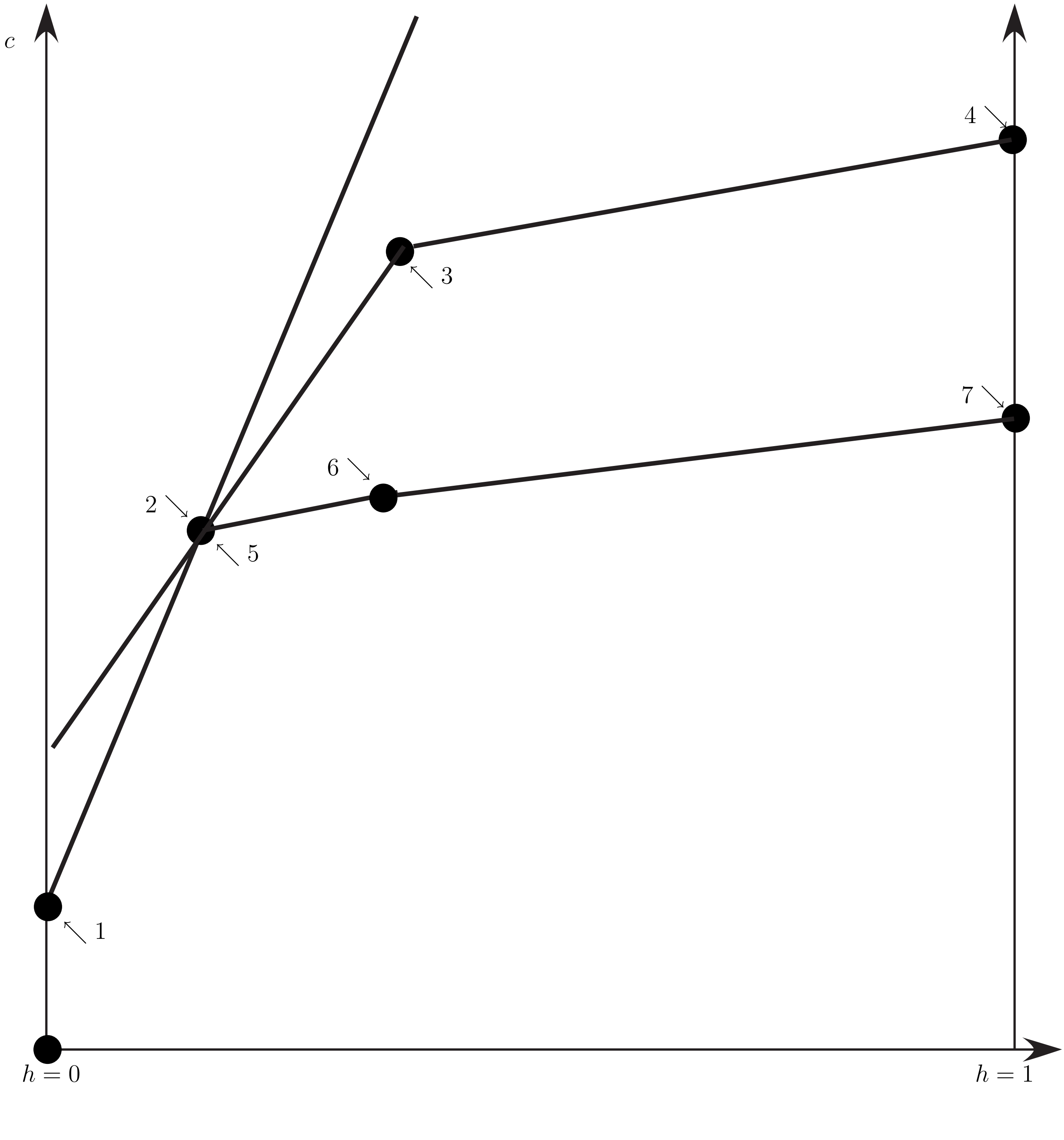

Figure 6

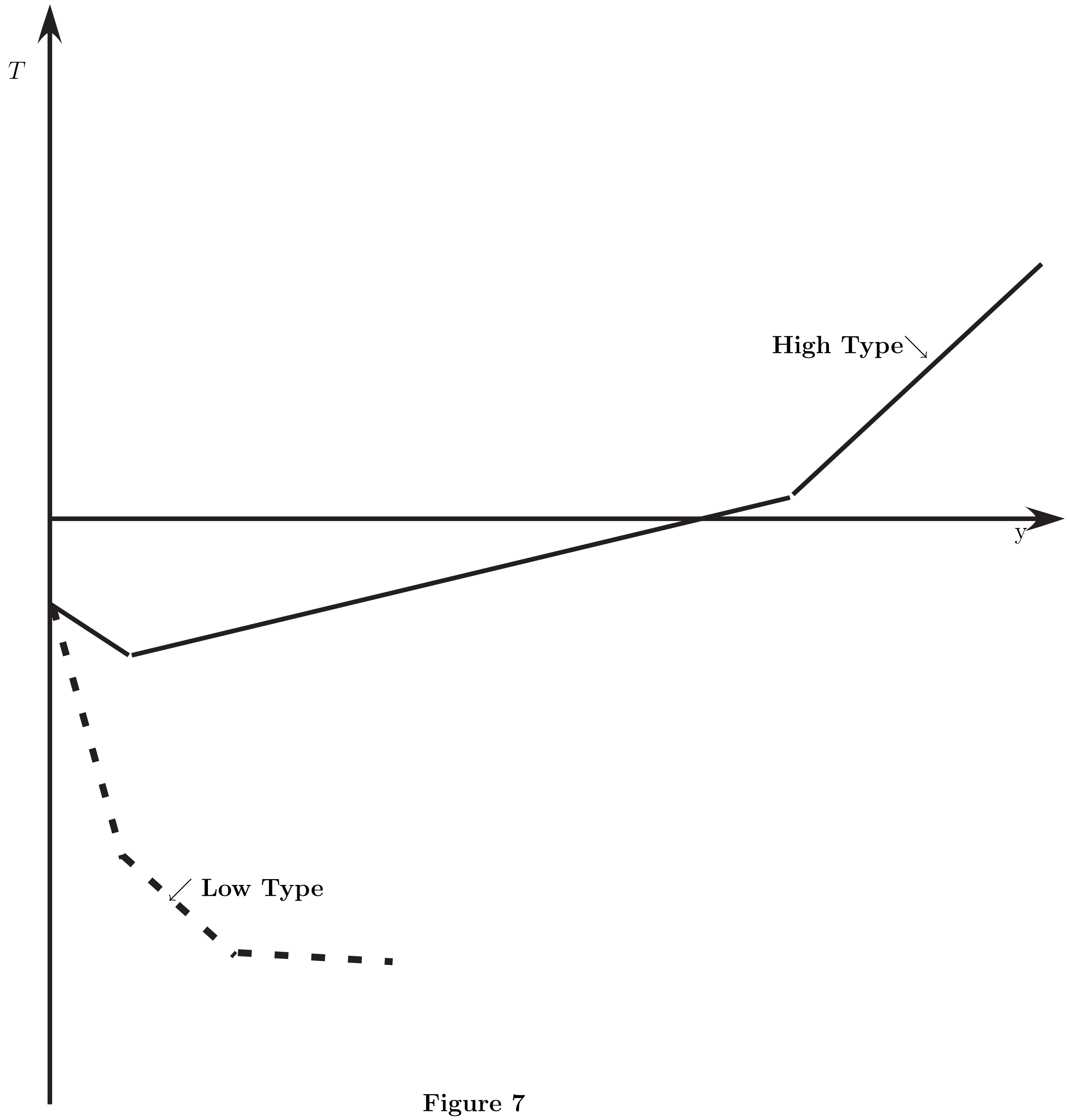

Figure 7