NETWORKS AND FARSIGHTED STABILITY

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Abstract

The main contribution of this paper is to provide a framework in which the notion of farsighted stability for games, introduced by Chwe (1994), can be applied to *directed* networks. Then, using Chwe's basic result on the nonemptiness of farsightedly stable sets for games, we show that for any given collection of directed networks and any given collection of rules governing network formation, there exists a farsightedly stable directed network.

1 Introduction

We construct a framework in which the notion of farsighted stability for games, introduced by Chwe (1994), can be applied to collections of *directed* networks. Our construction proceeds in two steps. First, we extend the definition of a directed network found in the literature (e.g., see Rockafellar (1984)). Second, our extended definition allows us to introduce the notion a network formation network. We call such a network, a *supernetwork*. All networks are composed of nodes and arcs. In most economic applications, nodes represent economic agent, while arcs represent connections or interactions between agents. In a supernetwork, nodes represent the networks in a given collection, while arcs represent coalition moves and coalitional preferences over the networks in the collection. Thus, given any profile of agent preferences and any collection of directed networks, a supernetwork uniquely represents all the coalitional preferences and all the coalitional moves allowed by the rules governing network formation (i.e., the rules governing movement from one network to another) for the given collection of directed networks. By applying Chwe's basic

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result on the nonemptiness of farsightedly stable sets for games, we show that for any supernetwork corresponding to a given collection of directed networks, the set of farsightedly stable nodes is nonempty. Thus, we show that given any collection of directed networks, any collection of rules governing network formation, and any profile of agent preferences, there exists a farsightedly stable network.

In a directed network, each arc possesses an orientation or direction: arc j connecting nodes i and i' must either go from node i to node i' or must go from node i' to node i. In an undirected network, arc j would have no orientation and would simply indicate a connection or link between nodes i and i'. Under our extended definition of directed networks, nodes are allowed to be connected by multiple arcs. For example, nodes i and i' might be connected by arcs j and j', with arc j' running from node i to i' and arc j running in the opposite direction (i.e., from node i' to node i). Thus, if node i represents a buyer and node i' a seller, then arc j' might represent a flow of money (from buyer to seller) while arc j might represent a flow of goods or services (from seller to buyer). Also, under our extended definition arcs are allowed to be used multiple times in a given network. For example, arc j might be used to connect nodes i and i' as well as nodes i' and i''. However, we do not allow arc j to go from node i to node i' multiple times. By allowing arcs to possess direction and be used multiple times and by allowing nodes to be connected by multiple arcs, our extended definition makes possible the application of networks to a richer set of economic environments. Until now, most of the economic literature on networks has focused on linking networks (see for example, Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997)).

Given a particular directed network, an agent or a coalition of agents can change the network to another network by simply adding or subtracting arcs from the existing network, that is, by establishing or dissolving connections in accordance with certain rules. For example, if the nodes in a network represent agents, then the rule for adding an arc j from node i to node i' might require that both agents i and i' agree to add arc j. Whereas the rule for subtracting arc j, from node i to node i', might require that only agent i or agent i' agree to dissolve arc j. Given the rules governing network formation (i.e., the rules governing the addition or subtraction of arcs), a directed network is said to be farsightedly stable if no agent or coalition of agents is willing to alter the network (via the addition or subtraction of arcs) for fear that such an alteration might induce further network alterations by other agents or coalitions which in the end leave the initially deviating agent or coalition no better off - and possibly worse off.

A key step in our analysis of network formation and farsighted stability is the construction of a *supernetwork* representing agents' preferences and the rules governing network formation. Each node in a supernetwork represents a particular directed network, while arcs represent various types of connections between the networks. There are two broad categories of arcs in a supernetwork:(1) arcs representing coalition moves from one node to another (m-arcs) - and therefore, coalitional moves from one network to another; and (2) arcs representing pairwise coalitional preferences over nodes (p-arcs) - and therefore pairwise coalitional preferences over networks.

Arcs in each category (m-arcs and p-arcs) are indexed by the coalitions responsible for the arc. Thus, a supernetwork represents all the "coalitional moves" and all the "coalitional preferences" between networks in the given collection of networks. Using a result due to Chwe (1994), we show that the set of farsightedly stable nodes in any finite supernetwork is nonempty. Each farsightedly stable node in the supernetwork, in turn, represents a farsightedly stable network in the collection.

In current research, we are analyzing the efficiency properties of farsightedly stable networks. While here we focus on directed networks, the same methodology can be used to deduce the existence of farsightedly stable undirected networks (i.e., linking networks - such as the networks considered by Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997)). An excellent paper on stability *and* efficiency in undirected networks is Jackson (2001) (see also, Jackson and Watts (1998), Jackson and van den Nouweland (2000), Skyrms and Pemantle (2000), Watts (2001), and Slikker and van den Nouweland (2001)). In future research, we will focus on network formation dynamics - along the lines of the seminal paper by Konishi and Ray (2001) on coalition formation dynamics.

2 Directed Networks

We begin by giving a formal definition the class of directed networks we shall consider. Let N be a finite set of nodes with typical element denoted by i, and let A be a finite set of arcs, with typical element denoted by j. Arcs represent potential connections between nodes, and depending on the application, nodes can represent economic agents or objects such as positions in an organization or locations within a market area.[1]

Definition 1 *(Directed Networks)*
Given node set N and arc set A, a directed network, G, is a subset of $A \times (N \times N)$. We shall denote by $\mathbb{N}(N, A)$ the collection of all directed networks given N and A.

A directed network $G \in \mathbb{N}(N, A)$ specifies how the nodes in N are connected via the arcs in A. Note that in a directed network order matters. In particular, if $(j, (i, i')) \in G$, this means that arc j *goes from* node i *to* node i'. Also note that if the set

$$G(j) := \left\{(i, i') \in N \times N : (j, (i, i')) \in G\right\}$$

is empty, then arc $j \in A$ is not used in network G. If in our definition of a directed network, we also require that $G(j)$ be nonempty and single-valued, then our definition is the same as that given by Rockafellar (1984).

[1]Of course in a supernetwork, nodes represent networks.

Suppose that the node set N is given by $N = \{i_1, i_2, \ldots, i_5\}$, while the arc set A is given by $A = \{j_1, j_2, \ldots, j_5, j_6, j_7\}$. Consider the network, G, depicted in Figure 1.

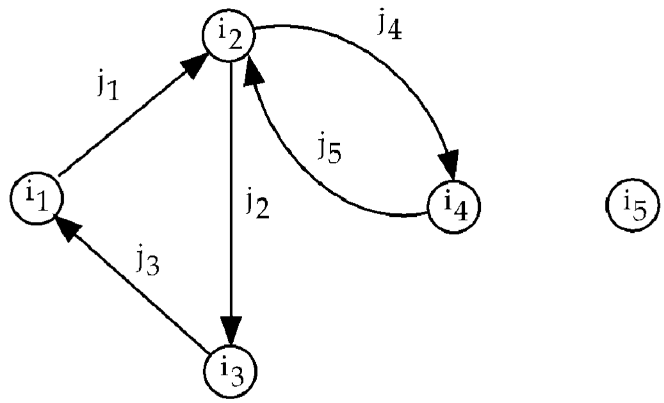

Figure 1: Network G

In network G, $G(j_5) = \{(i_4, i_2)\}$. Thus, $(j_5, (i_4, i_2)) \in G$. Also, in network G, arcs j_6 and j_7 are not used. Thus, $G(j_6) = \emptyset$ and $G(j_7) = \emptyset$. Finally, in network G, node i_5 is *isolated*, that is, node i_5 is not connected to any other nodes in the network by any arc going to or coming from node i_5.

Consider the new network, $G' \in \mathbb{N}(N, A)$ depicted in Figure 2.

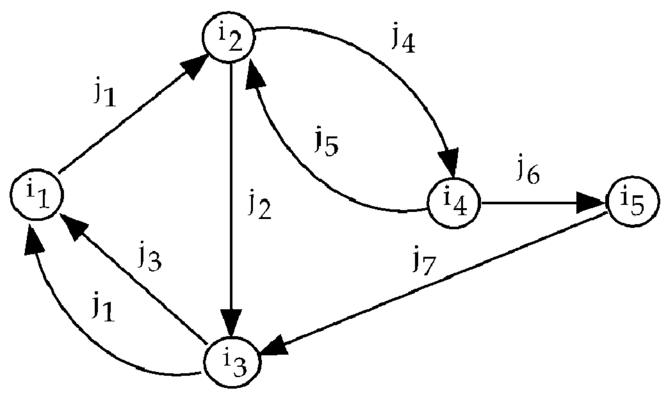

Figure 2: Network G'

In network G', $G'(j_1) = \{(i_1, i_2), (i_3, i_1)\}$. Thus, $(j_1, (i_1, i_2)) \in G'$ and $(j_1, (i_3, i_1)) \in G'$. Also, note that in network G', node i_5 is no longer isolated. In particular, arcs j_6 and j_7 are used to connect node i_5 to other nodes in the network - and in particular, $G'(j_6) = \{(i_4, i_5)\}$ and $G'(j_7) = \{(i_5, i_3)\}$. Finally, in Figure 2 note that nodes i_2 and i_4, as well as nodes i_1 and i_3 are connected by two arcs. Under our definition of a directed network it is possible to alter network G' by replacing arc j_5 from i_4 to i_2 with arc j_4 *from* i_4 *to* i_2. However, it is *not* possible under our definition to replace arc j_5 from i_4 to i_2 with arc j_4 from i_2 to i_4 - because our definition does not allow j_4 to go *from* i_2 *to* i_4 multiple times.

3 Supernetworks

Let $\mathbb{N} := \mathbb{N}(N, A)$ be a given collection of directed networks, with typical element denoted by G, and let 2^N denote the collection of all nonempty subsets (or coalitions) of N with typical element denoted by S. In this section, it is useful to think of N as representing a finite set of economic agents and each directed network $G \in \mathbb{N}$ as representing a particular configuration of connections between the agents in N via the arcs in A. For example, suppose that under the rules governing network formation, moving from network G (Figure 1) to network G' (Figure 2) involves four agents: agents i_1 and i_3 must agree to add arc j_1, agents i_3 and i_5 must agree add arc j_7, and agents i_5 and i_4 must agree to add arc j_6. Thus, in order to move from network G to network G', the members of coalition $\{i_1, i_3, i_4, i_5\}$ must agree to add connections (i.e., arcs) j_1, j_6 and j_7 to network G. Whether or not this happens depends upon the preferences of the individual agents, i_1, i_3, i_4, and i_5. We shall assume that each agent's preferences over networks are specified via each agent's network payoff function,

$$v_i(\cdot) : \mathbb{N} \to \mathbb{R}.$$

For each agent $i \in N$ and each directed network $G \in \mathbb{N}$, $v_i(G)$ is the payoff to agent i in network G. Agent i then prefers network G' in Figure 2 to network G in Figure 1 if and only if

$$v_i(G') > v_i(G).$$

Moreover, coalition $\{i_1, i_3, i_4, i_5\} \in 2^N$ prefers network G' to network G if and only if

$$v_{i_k}(G') > v_{i_k}(G) \text{ for all agents } i_k \text{ in } \{i_1, i_3, i_4, i_5\}.$$

Given a collection of directed networks $\mathbb{N}$ and agents' preferences over $\mathbb{N}$, we can give a very precise network representation of the rules governing network formation as well as agents' preferences. To begin, let

$$\mathbb{M} := \left\{m_S : S \in 2^N\right\} \text{ denote the set of m-arcs,}$$
$$\mathbb{P} := \left\{p_S : S \in 2^N\right\} \text{ denote the set of p-arcs,}$$
$$\text{and}$$
$$\mathbb{A} := \mathbb{M} \cup \mathbb{P}.$$

Given networks G and G' in $\mathbb{N}$, denote by

$$G \underset{m_{S'}}{\rightarrow} G'$$

(i.e., by an m-arc, belonging to coalition S', going from node G to node G') the fact that coalition $S' \in 2^N$ can change network G to network G' by adding or subtracting arcs to network G. Moreover, denote by

$$G \underset{p_{S'}}{\dashrightarrow} G'$$

(i.e., by a p-arc, belonging to coalition S', going from node G to node G') the fact that coalition $S' \in 2^N$ prefers network G' to network G.

Definition 2 *(Supernetworks)*
Given node set $\mathbb{N}$ *(a collection of directed networks,* $\mathbb{N}(N, A)$*) and arc set* $\mathbb{A} := \mathbb{M} \cup \mathbb{P}$*, a supernetwork corresponding to* $\mathbb{N}$ *is a subset* $\mathbf{G}$ *of* $\mathbb{A} \times (\mathbb{N} \times \mathbb{N})$.

Thus, for each $S' \in 2^N$, $m_{S'} \in \mathbb{M}$, and $p_{S'} \in \mathbb{P}$

$$(m_{S'}, (G, G')) \in \mathbf{G} \text{ if and only } G \underset{m_{S'}}{\rightarrow} G' \ ,$$
$$\text{and}$$
$$(p_{S'}, (G, G')) \in \mathbf{G} \text{ if and only if } G \underset{p_{S'}}{\dashrightarrow} G',$$

and thus, supernetwork $\mathbf{G}$ provides a network representation of all coalitional moves and all coalitional preferences.

Note that the set

$$\mathbf{G}(m_{S'}) := \left\{(G, G') \in \mathbb{N} \times \mathbb{N} : (m_{S'}, (G, G')) \in \mathbf{G}\right\},$$

contains all ordered pairs of networks, (G, G'), such that network G can be changed to network G' by the agents in coalition S', while the set

$$\mathbf{G}(p_{S'}) := \left\{(G, G') \in \mathbb{N} \times \mathbb{N} : (p_{S'}, (G, G')) \in \mathbf{G}\right\},$$

contains all ordered pairs of networks, (G, G'), such that network G' is preferred to network G by the agents in coalition S'. Also note that each agent's preferences over networks $\mathbb{N}$, defined via the network payoff function $v_i(\cdot)$, are automatically irreflexive. Thus, if $(p_{S'}, (G, G')) \in \mathbf{G}$, then $G \neq G'$.

4 Farsightedly Stable Networks

Given supernetwork $\mathbf{G}$, we say that network $G' \in \mathbb{N}$ farsightedly dominants network $G \in \mathbb{N}$ if there is a finite sequence of networks,

$$G_0, G_1, \ldots, G_h,$$

with $G = G_0$, $G' = G_h$, and $G_k \in \mathbb{N}$ for $k = 0, 1, \ldots, h$, and a corresponding sequence of coalitions,

$$S_1, S_2, \ldots, S_h,$$

such that for $k = 1, 2, \ldots, h$

$$(m_{S_k}, (G_{k-1}G_k)) \in \mathbf{G},$$
$$\text{and}$$
$$(p_{S_k}, (G_{k-1}G_h)) \in \mathbf{G}.$$

We shall denote by $G \lhd\lhd G'$ the fact that network $G' \in \mathbb{N}$ farsightedly dominates network $G \in \mathbb{N}$. Figure 3 below provides a network representation of farsighted dominance. In Figure 3, network G_3 farsightedly dominates network G_0.

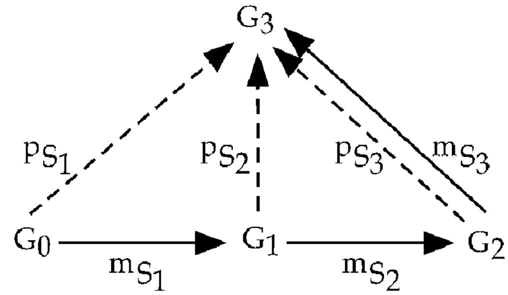

Figure 3: G_3 farsightedly dominates G_0

Definition 3 *(Farsightedly Stable Networks)*
Let $\mathbb{N} := \mathbb{N}(N, A)$ *be a collection of directed networks and let* $\mathbf{G} \subset \mathbb{A} \times (\mathbb{N} \times \mathbb{N})$ *be a corresponding supernetwork with node set* $\mathbb{N}$ *and arc set* $\mathbb{A} := \mathbb{M} \cup \mathbb{P}$*. A subset of directed networks* $\mathbb{F}_{\mathbf{G}}$ *is said to be farsightedly stable if*

$$\begin{gathered} \textit{for all } G_0 \in \mathbb{F}_{\mathbf{G}} \textit{ and } (m_{S_1}, (G_0, G_1)) \in \mathbf{G}, \\ \textit{there exists } G_2 \in \mathbb{F}_{\mathbf{G}} \\ \textit{with } G_2 = G_1 \textit{ or } G_2 \rhd\rhd G_1 \textit{ such that,} \\ (p_{S_1}, (G_0, G_2)) \notin \mathbf{G}. \end{gathered}$$

Thus, a subset of directed networks $\mathbb{F}_{\mathbf{G}}$ is farsightedly stable if given any network $G_0 \in \mathbb{F}_{\mathbf{G}}$ and any m_{S_1}- deviation to network $G_1 \in \mathbb{N}$ by coalition S_1 (via the adding or subtracting of arcs) there exists further deviations leading to some network $G_2 \in \mathbb{F}_{\mathbf{G}}$ where the initially deviating coalition S_1 is not better off - and possibly worse off.

There can be many farsightedly stable sets. We shall denote by $\mathbb{F}^{\star}_{\mathbf{G}}$ the largest farsightedly stable set. Thus, if $\mathbb{F}_{\mathbf{G}}$ is a farsightedly stable set, then $\mathbb{F}_{\mathbf{G}} \subset \mathbb{F}^{\star}_{\mathbf{G}}$.

We now have our main result on the uniqueness and nonemptiness of the largest farsightedly stable set.

Theorem 1 *(*$\mathbb{F}^{\star}_{\mathbf{G}} \neq \emptyset$*)*
Let $\mathbb{N} := \mathbb{N}(N, A)$ *be a collection of directed networks. Given any supernetwork* $\mathbf{G} \subset \mathbb{A} \times (\mathbb{N} \times \mathbb{N})$ *corresponding to* $\mathbb{N}$*, there exists a unique, nonempty, largest farsightedly stable set* $\mathbb{F}^{\star}_{\mathbf{G}}$*. Moreover,* $\mathbb{F}^{\star}_{\mathbf{G}}$ *is externally stable with respect to farsighted dominance, that is, if network* G *is contained in* $\mathbb{N} \backslash \mathbb{F}^{\star}_{\mathbf{G}}$*, then there exists a network* G' *contained in* $\mathbb{F}^{\star}_{\mathbf{G}}$ *that farsightedly dominants* G *(i.e.,* $G' \rhd\rhd G$*).*

Proof. The existence of a unique, largest farsightedly stable set, $\mathbb{F}^{\star}_{\mathbf{G}}$, follows from Proposition 1 in Chwe (1994). Moreover, since the set of networks $\mathbb{N} := \mathbb{N}(N, A)$ is finite and since the each agent's preferences over networks are irreflexive (i.e., $(p_{\{i\}}, (G, G')) \in \mathbf{G}$ implies $G \neq G'$), nonemptiness follows from the Corollary to Proposition 2 in Chwe (1994). ■

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