CANDIDATE STABILITY AND PROBABILISTIC VOTING PROCEDURES

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Candidate Stability and Probabilistic Voting Procedures

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Abstract

We extend the analysis of Dutta, Jackson and Le Breton (*Econometrica*, 2001) on strategic candidacy to probabilistic environments. For each configuration of the agenda and each profile of voters' preferences over running candidates, a *probabilistic voting procedure* selects a lottery on the set of candidates at stake. Assuming that candidates cannot vote, we show that *random dictatorships* are the only unanimous probabilistic voting procedures that never provide unilateral incentives for the candidates to leave the ballot independently of the composition of the agenda. However, more flexible rules can be devised if we focus on the stability of specific agendas

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1 Introduction

Social decision processes can be described as two-stage processes. In a first stage a set of possible candidates decide whether to enter the election or to leave the electoral fray. In a second stage, the society uses a voting rule in order to make a choice from the candidates at stake taking into account the preferences of the voters.

The analysis of voters' strategic incentives in the second stage of the public decision has received a considerable amount of attention in the theory of Social Choice. However, the study of candidates' strategic concerns is a relatively new issue in the literature and only a few works have addressed this topic. In fact, the traditional approach in the literature has usually assumed that the set of running candidates is exogenous and independent of any individual decision. Nevertheless, it is easy to find examples in which candidates may affect the outcome of the social choice simply by withdrawing their candidacy. For instance, we can think of an election with three candidates in which the winner is decided by plurality. A candidate with the highest number of first-preference votes may fail to win the election if another candidate drops out the fray to let the remaining candidate win. Hence, it becomes natural to investigate which voting rules will never provide such incentives for the candidates to leave the ballot since only for such voting rules the set of feasible alternatives can be considered exogenous.

In a recent paper, Dutta, Jackson and Le Breton [5][1] have inaugurated the normative analysis of candidates' incentives to quit the election. They introduce a condition on the voting rules called **candidate stability**. Given a set of initial candidates, a voting rule is candidate stable if a candidate never prefers to drop out the electoral race unilaterally rather than to stay in the fray. If the candidates are not allowed to vote, they prove that any deterministic voting rule satisfying the requirements of unanimity and candidate stability must be dictatorial (Theorem 1, DJL). If some candidates are also voters the analysis becomes problematic, but they obtain an impossibility result for a large class of deterministic voting rules (Theorems 2 and 3, DJL).

The purpose of this work is to investigate the strategic concerns of the candidates in a probabilistic framework. Thus, we generalize the analysis of DJL by modeling social choice

[1] Henceforth DJL.

processes as probabilistic voting procedures. A probabilistic voting procedure selects a lottery on the set of candidates at stake as a function of the set of entering candidates and voters' preferences over running candidates. Although probabilistic choices in a social context are sometimes criticized, the probabilistic framework provides many plausible voting rules which are interesting in their own right. These rules can incorporate certain notions of fairness and reasonable compromise that cannot be devised in a deterministic framework. In a situation of conflict between preferences of the voters (like, for example, the one resulting in the classical voting paradox) the refusal to entertain lotteries on candidates can lead to clearly unequitable social outcomes.

An alternative interpretation of probabilistic voting procedures fits nicely our analysis. Suppose that in the second stage of the public decision process voters play a voting mechanism that admits multiple equilibria at some profile of voters' preferences. Candidates know the possible equilibria for each profile of voters' preferences, but they do not have enough information about the strategies actually played by the voters and the equilibrium that eventually arises. Thus, candidates could not use any backward induction argument to focus on a specific equilibrium. However, they may assess a lottery assigning a probability to each possible equilibrium. In this situation, candidates consider that the outcome of the social choice process consists of a lottery on the set of candidates.

In order to analyze candidates' incentives to leave the ballot in probabilistic environments, we assume that the candidates are Expected Utility maximizers. We will see that when candidates are not allowed to vote, a probabilistic voting procedure is candidate stable if, whenever a candidate leaves the ballot, the probability that the society chooses any of the remaining candidates cannot decrease. As a first result, we show that, provided that candidates cannot vote, only probabilistic combinations of dictatorial probabilistic voting procedures, **random dictatorships**, satisfy unanimity and candidate stability at any possible configuration of the set of initial candidates. When we focus on a specific set of running candidates, the results depend crucially on the cardinality of is set. If there are at least four initial candidates we show that a unanimous and candidate stable probabilistic voting procedure must be a random dictatorship when a candidate drops her ballot but it may admit more flexible formulations when no candidate leaves the fray. Finally, if we consider an initial set of candidates containing only three candidates, then the decision power will still be concentrated in the hands of an arbitrary group of voters but the distribution of the veto power is not necessarily additive.

While random dictatorships play a crucial role in our characterizations, we do not view our results as negative or impossibility results. Random dictatorships have attractive features, which are connected with any intuitive notion of fairness. Hence, it is not clear that a random dictatorship should be undesirable in the same way in which a deterministic dictatorship can be considered undesirable.

Closely related to this article are the works of Ehlers and Weymark [6], Eraslan and McLennan [7] and Rodríguez-Álvarez [14].[2] They study the implications of candidate stability for voting correspondences, that is, rules that for each configuration of the ballot and each voters' preference profile select a set of candidates. In Ehlers *et al.* [6] and Eraslan and *et al.* [7] a strong version of candidate stability for correspondences is introduced, and it is shown that only dictatorial rules satisfy it together with unanimity. Moreover, in Eraslan *et al.* [7] voters are allowed to express weak preferences over candidates, and they show that only serially dictatorial rules are candidate stable and unanimous. On the other hand, in Rodríguez-Álvarez [14] the incentives of the candidates to quit an election are explicitly modeled by endowing them with different domains of preferences over sets of candidates. Each domain of preferences leads to different implications of candidate stability. Thus, negative results in the line of those of Ehlers *et al.* [6] and Eraslan *et al.* [7] are obtained when candidates' preferences are consistent with Expected Utility Theory and Bayesian Updating from some prior assessment.[3] On the other hand, more positive results are obtained when candidates compare sets consistently with extreme attitudes towards risk (like for instance according to the *leximin, maximin* or *maximax* criteria).

Before concluding this introduction, we want to refer to the interesting work by Pattanaik

[2]We address the interested reader to DJL and Rodríguez-Álvarez [14] for further references on the topic of endogenous agenda formation and strategic candidacy.

[3]Barberà, Dutta and Sen [2] propose this class of preferences over sets for the analysis of strategy-proof social choice correspondences.

and Peleg [13].[4] PP propose a probabilistic model of social choice and introduce probabilistic counterparts to the classical set of axioms of deterministic social choice. Their major concern is to analyze the degree of influence in the social choice that different coalitions of individuals may have.[5] However, they do not consider the analysis of candidates' incentives. Despite of the different motivations, their setting and results are closely related to ours. We will see that the set of axioms analyzed in PP is stronger than the one studied here. Hence, we can interpret our results theorem as generalizations of the main results in PP.

The remainder of the paper proceeds as follows. In Section 2, we introduce the set up and notation, while in Section 3 we study the implications of candidate stability and present the characterization theorems. We devote Section 4 to the proofs of the theorems. In Section 5 we conclude by discussing the case of voting candidates, the role of unanimity and a more general framework in which voters are also allowed to express their preferences over lotteries.

2 Definitions and Notation

2.1 Voters, Candidates and Preferences

Let $\mathcal{N}$ be a society formed by a finite set of voters $\mathcal{V}$, and an infinitely countable set of candidates $\mathcal{C}$, $\mathcal{N} = \mathcal{C} \cup \mathcal{V}$. We focus on the case in which there is no overlap between the sets of voters and candidates ($\mathcal{C} \cap \mathcal{V} = \{\varnothing\}$). In this scenario, we can isolate the incentives of the candidates to participate in an election, regardless of their concerns as voters.[6]

We assume that only finite agendas are feasible for this society. The whole set of potential candidates cannot run the election simultaneously. Let $2^{\mathcal{C}}$ denote the set of all finite subsets of $\mathcal{C}$. We call $A \in 2^{\mathcal{C}}$ an agenda. Our focus on finite agendas makes almost irrelevant our assumption on the infinite size of the set of potential candidates. It only implies that any possible agenda is a proper subset of another agenda containing exactly one additional candidate.[7]

[4]From now on PP.

[5]Nandeibam [12] extends the analysis of PP by permitting voters to express indifference between alternatives.

[6]When candidates can be voters, the analysis of stability becomes more complicated because candidates' preferences are assumed to favor their own election. We postpone the discussion of this interesting case to the concluding section.

[7]The reader will see that this assumption is not crucial for our results but it allows us to simplify the statement

Individuals are endowed with strict preferences on $\mathcal{C} \cup \{\varnothing\}$, where the empty set refers to the situation in which no candidate is elected. A strict preference is a complete, antisymmetric and transitive binary relation on $\mathcal{C} \cup \{\varnothing\}$. For any $i \in \mathcal{N}$, we denote by P_i a preference order of individual i, and by $\mathcal{P}^i$ the set of admissible preferences over candidates for individual i. We denote by $P \in \mathcal{P}^{\mathcal{V}}$ a voters preference profile. We assume that for all $i \in \mathcal{N}$ any candidate is preferred to the empty set. For any $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and P_i, $top\,(C, P_i)$ refers to the best element of C according to the preference order P_i. Preferences of voters over candidates are unrestricted, but each candidate considers herself as the best alternative, that is, for all $a \in \mathcal{C}$, and for all $P_a \in \mathcal{P}^a$, $a = top\,(\mathcal{C}, P_a)$. For each $A \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $P\,|_A$ denotes the restriction of P to the set A. Abusing notation, for any set $I \subseteq \mathcal{N}$, P^I refers to the restriction of the profile P to the members of I, $\mathcal{P}^I$ is defined as the set of admissible preferences profiles for I. For any set of voters I, we denote by $-I$ the set of voters $\mathcal{V} \backslash I$.

Candidates' Preferences over Lotteries.

Let $\mathcal{L}$ denote the set of lotteries on the set $\mathcal{C}$. For each $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ we can define:

$$\mathcal{L}_C = \left\{ \lambda = \{\lambda(c)\}_{c \in C} \in \mathbb{R}_+^{\#C} \text{ such that for all } c \in C, \lambda(c) \geq 0 \text{ and } \sum_{c \in C} \lambda(c) = 1 \right\}.$$

Then, $\mathcal{L} = \cup_{C \in 2^{\mathcal{C}} \backslash \{\varnothing\}} \mathcal{L}_C$.[8]

Candidates are endowed with complete, reflexive and transitive preferences over $\mathcal{L} \cup \{\varnothing\}$ and we assume that these preferences are consistent with the postulates of Expected Utility Theory. Moreover, candidates always prefer any $\lambda \in \mathcal{L}$ to the empty set. Formally, a utility function is a mapping $u_i : \mathcal{C} \to \mathbb{R}$. A utility function fits the preference ordering $P_i \in \mathcal{P}^i$ if for any $a, b \in \mathcal{C}$, $u_i\,(a) > u_i(b)$ if and only if aP_ib. Then, given two lotteries $\lambda, \lambda' \in \mathcal{L}$, a candidate $a \in \mathcal{C}$ with preferences over candidates P_a and consistent utility function u_a prefers the lottery λ to the lottery λ', if and only if:

$$\sum_{b \in \mathcal{C}} \lambda(b) u_a(b) > \sum_{b' \in \mathcal{C}} \lambda'(b') u_a\,(b').$$

of one of our theorems (Theorem 1).

[8]The notation $\#C$ stands for the cardinality of the set C.

Once defined the strict component of the candidates preferences over lotteries, the weak component is defined in the usual way. A candidate is indifferent between two lotteries λ, λ' if neither λ is preferred to λ' nor λ' is preferred to λ.

2.2 Probabilistic Voting Procedures

Definition 1. *A* ***probabilistic voting procedure*** *is a mapping* $p : 2^{\mathcal{C}} \times \mathcal{P}^{\mathcal{V}} \to \mathcal{L} \cup \{\varnothing\}$ *such that for all* $A \in 2^{\mathcal{C}}, a \in \mathcal{C}$ *and* $P \in \mathcal{P}^{\mathcal{V}}$*:*
i) $p(a, A, P) = 0$ *if* $a \notin A$ *and* $p(A, P) = \{\varnothing\}$ *if and only if* $A = \{\varnothing\}$,
ii) $p(A, P) = p(A, P')$ *for all* $P' \in \mathcal{P}^{\mathcal{V}}$ *such that* $P \mid_A = P' \mid_A$,
where $p(a, A, P)$ *denotes the probability assigned to candidate* a *according to the lottery* $p(A, P)$.

Item *i)* states that a candidate cannot be selected if she is not at stake. Moreover, whenever a candidate runs the election, the election always results in somebody selected.

Finally, *ii)* is in the spirit of Arrow's Independence of Irrelevant Alternatives. Namely, only voters' preferences over candidates who eventually run the election are relevant.

Our definition of probabilistic voting procedures is less general than the one proposed in PP since we embed Independence of Irrelevant Alternatives in the definition. A probabilistic voting procedure is a generalization of a single-valued voting procedure proposed in DJL. A single-valued voting procedure is restricted to select degenerate lotteries that assign probability one to a unique candidate. A probabilistic voting procedure is also more precise than a voting correspondence as defined in Ehlers *et al.* [6], Eraslan *et al.* [7], and Rodríguez-Álvarez [14] since it assigns a specific probability distribution to each preference profile and not only a set of possible outcomes. Thus, it is more flexible than a voting correspondence to meet voters' preferences. Notice that a probabilistic voting procedure may assign different probabilities to the candidates for different preference profiles even when the set of selected candidates do not change. Finally, a probabilistic voting procedure is a family of decision schemes as defined and analyzed in Gibbard [8], one for each configuration of the agenda.

2.3 Exit Stability and Candidate Stability

In this paper we are interested in designing probabilistic voting procedures for which the agenda can be considered exogenous and independent of the preferences of voters. In order to consider an agenda $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ as exogenous it will be necessary that:

- The candidates who are in the ballot ($c \in C$) have no incentives to leave the fray.

- The candidates who are not at stake ($b \in \mathcal{C} \backslash C$) cannot improve by entering the election.

Although it seems uncontroversial to introduce a stability condition for candidates who are in the ballot (they should not benefit by quitting), it is not clear which would be a correct statement of stability regarding the incentives of the non-running candidates. To ask for rules that never provide incentives to outsiders to enter would be too compelling and it would lead immediately to impossibility results. Hence, we only focus on the incentives of the running candidates to withdraw. By doing so, we accommodate circumstances in which the candidates are free to leave the election once they are at stake, but they cannot enter in the fray by themselves. Indeed, we will see that our non-exit conditions are rather powerful and they will reduce considerably the incentives of outsider candidates to run the fray.[9]

We present now two parallel stability conditions regarding strategic withdrawal of the candidates. Candidate stability is defined as in DJL. Given an initial agenda, a probabilistic voting procedure is candidate stable if the running candidates never have incentives to withdraw unilaterally their candidacy. On the other hand, exit stability condition captures the same idea, but applying it to any possible agenda. We say a probabilistic voting procedure p is exit stable if and only if a candidate never benefits by withdrawing her candidacy, independently of the remaining candidates who stay in the fray. Hence, for an exit stable probabilistic voting procedure any set of running candidates can be considered as stable, while candidate stability only refers to a specific set of candidates.

We provide now the formal definitions:

[9] See the discussion after Theorem 1 in Section 3.

Definition 2. *A probabilistic voting procedure p is **exit stable** if and only if for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $a \in C$, for all $P_a \in \mathcal{P}^a$, for all utility function u_a consistent with P_a and all $P \in \mathcal{P}^{\mathcal{V}}$:*

$$\sum_{b \in C} p\left(b, C, P\right) u_a\left(b\right) \geq \sum_{b' \in C \backslash \{a\}} p(b', C \backslash \{a\}, P) u_a(b').$$

Naturally, we do not include in the definition the empty set since no candidate can withdraw from an agenda in which no candidate runs the election. We finish this section with the definition of candidate stability.

Definition 3. *Given the agenda $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, a probabilistic voting procedure p is **candidate stable** at C if to stay in the fray when the remaining candidates in C also run the election is always a Nash equilibrium strategy for all the candidates in C. That is, for all $a \in C$, for all $P_a \in \mathcal{P}^a$, for all utility function u_a consistent with P_a and all $P \in \mathcal{P}^{\mathcal{V}}$, it holds that:*

$$\sum_{b \in C} p\left(b, C, P\right) u_a\left(b\right) \geq \sum_{b' \in C \backslash \{a\}} p(b', C \backslash \{a\}, P) u_a(b').$$

Clearly, a probabilistic voting procedure is exit stable if and only it is candidate stable for all agendas $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$.

We have to remark that our assumptions on candidates' preferences imply that both exit stability and candidate stability are empty of content for agendas with less than three candidates. At agendas with only one candidate at stake, this candidate is always elected with certainty and this candidate will never have incentives to leave the fray. On the same fashion, when a candidate quits from an agenda with only two candidates, the other candidate becomes elected which cannot represent an improvement for the withdrawing candidate.

2.4 Unanimity and Efficiency

In this work, we deal with probabilistic voting procedures that satisfy a minimal responsiveness condition to voters' preferences. Specifically, we require that when all voters agree on the best candidate, she is uniquely selected.

Definition 4. *A probabilistic voting procedure p is **unanimous** if for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, for all $c \in \mathcal{C}$ and all $P \in \mathcal{P}^{\mathcal{V}}$ such that $c = top\left(C, P_i\right)$ for all $i \in \mathcal{V}$, $p\left(c, C, P\right) = 1$.*

For all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, for all $I \subset \mathcal{V}$ and for all $P_I \in \mathcal{P}^I$, $Pareto\,(C, P_I) = \{c \in C$, such that there is no $b \in C$, bP_ic for all $i \in I\}$. We say that a candidate $b \in \mathcal{C}$ is Pareto dominated at profile $P \in \mathcal{P}^{\mathcal{V}}$ and agenda C, if there is another candidate $a \in A$ such that aP_ib for all $i \in \mathcal{V}$.

Definition 5. *A probabilistic voting procedure p is* ***(ex-post) Pareto efficient*** *if for all $c \in \mathcal{C}$, for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and all $P \in \mathcal{P}^{\mathcal{V}}$, $c \notin Pareto(C, P)$ implies $p(c, C, P) = 0$.*

Evidently, unanimity is less stringent than (*ex-post*) Pareto efficiency. Unanimity does not rule out the possibility that Pareto dominated candidates receive a positive probability.[10] We discuss the consequences of its relaxation in Subsection 5.2.

As candidate stability only applies to specific agendas, the previous definitions can be unnecessarily stringent when we focus on candidate stable probabilistic voting procedures. We provide now a restricted version of unanimity suitable for the study of specific agendas.

Definition 6. *A probabilistic voting procedure p is* ***unanimous at agenda*** *C if for all $a \in C$, for all agendas $A \in \{C, \{C \backslash \{c\}\}_{c \in C}\}$ and all $P \in \mathcal{P}^{\mathcal{V}}$ such that $a = top(A, P_i)$ for all $i \in \mathcal{V}$, $p(a, A, P) = 1$.*

3 Implications of Exit Stability and Candidate Stability in Probabilistic Environments

As it has been already mentioned, in this paper we assume that the candidates cannot vote. Hence, although exit stability and candidate stability are conditions regarding the preferences of the candidates, these preferences are not an input of a probabilistic voting procedure. Candidate stability and exit stability introduce restrictions on the social choice relating the outcome when all the candidates are at stake to the outcome when a unique candidate drops from the ballot.

We start this section by providing a crucial lemma that introduces the main implications of exit stability in our probabilistic environment. Namely, a probabilistic voting procedure is exit

[10] In probabilistic environments, it is possible to define another version of efficiency, *ex-ante* efficiency. *Ex-ante* efficiency requires that for any preference profiles there is no lottery that is unanimously preferred by all the voters to the outcome of the probabilistic voting procedure. It is easy to see that *ex-ante* efficiency is a stronger condition than *ex-post* efficiency and unanimity.

stable if and only if whenever a candidate withdraws from the poll, then no other candidate reduces her probability of being finally elected.

Lemma 1. *A probabilistic voting procedure p is exit stable if and only if for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, for all $a, b \in C$, and for all $P \in \mathcal{P}^{\mathcal{V}}$ it holds that $p(b, \mathcal{C} \backslash \{a\}, P) \geq p(b, \mathcal{C}, P)$.*

Proof. Assume that there exist $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $a, b \in C$, and $P \in \mathcal{P}^{\mathcal{V}}$ such that $p(b, C \backslash \{a\}, P) < p(b, C, P)$. Consider $P_a \in \mathcal{P}^a$, such that for any $c \in C \backslash \{b\}$, cP_ab. Given that C is finite and that by $i)$ in the definition of probabilistic voting procedure, for any $d \in (\mathcal{C} \backslash C)$, $p(d, C, P) = 0$, it is possible to find a utility function u_a fitting P_a with $u_a(b)$ small enough such that

$$\sum_{c \in C \backslash \{a\}} p(c, C \backslash \{a\}, P) u_a(c) > \sum_{c' \in C} p(c', C, P) u_a(c'),$$

which contradicts exit stability.

On the other hand, assume that for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, for all $a, b \in C$, and for all $P \in \mathcal{P}^{\mathcal{V}}$ it holds that $p(b, C \backslash \{a\}, P) \geq p(b, C, P)$. Consider first $P \in \mathcal{P}^{\mathcal{V}}$, $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and $a \in C$ such that for all $b \in C \backslash \{a\}$, $p(b, C \backslash \{a\}, P) = p(b, C, P)$, then $p(C, P) = p(C \backslash \{a\}, P)$, and trivially a cannot be better off withdrawing. Finally, consider $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, $P \in \mathcal{P}^{\mathcal{V}}$ and $a \in C$ such that there are some $b \in C \backslash \{a\}$ with $p(b, C \backslash \{a\}, P) > p(b, C, P)$ then it holds that,

$$\sum_{b \in C \backslash \{a\}} (p(b, C \backslash \{a\}, P) - p(b, C, P)) = p(a, C, P) > 0.$$

And as for all P_a, all u_a, and all $b \in \mathcal{C} \backslash \{a\}$, $u_a(a) > u_a(b)$,

$$p(a, C, P) u_a(a) > \sum_{b \in C \backslash \{a\}} (p(b, C \backslash \{a\}, P) - p(b, C, P)) u_a(b),$$

and then also,

$$\sum_{b \in C} p(b, C, P) u_a(b) > \sum_{b' \in C \backslash \{a\}} p(b', C \backslash \{a\}, P) u_a(b'),$$

and exit stability holds. ■

From Lemma 1 it is trivial to see that exit stability is equivalent to regularity as defined by PP.[11]. We state this fact in the following remark.

[11] See Section 4 and Definition 3.8 in PP.

Remark 1. *A probabilistic voting procedure p is exit stable if and only if for all $P \in \mathcal{P}^{\mathcal{V}}$ and $C, C' \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ with $C \subseteq C'$ and for all $c \in C$, $p(c, C, P) \geq p(c, C', P)$.*

By using the same arguments of Lemma 1 for a specific agenda C, we can prove and provide the following lemma characterizing the implications of candidate stability in our probabilistic environment.

Lemma 2. *Given the agenda $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, a probabilistic voting procedure p is candidate stable at C if and only if for all $a, b \in C$, and for all $P \in \mathcal{P}^{\mathcal{V}}$ it holds that $p\left(b, C \backslash \{a\}, P\right) \geq p\left(b, C, P\right)$.*

Remark 2. *If p is candidate stable at agenda C, then for all $P \in \mathcal{P}^{\mathcal{V}}$, and $b \in C$ such that $p(b, C, P) = 0$, $p(C, P) = p(C \backslash \{b\}, P)$.*

Proof. Assume the contrary, then there are $P \in \mathcal{P}^{\mathcal{V}}, b \in C$ such that $p(b, C, P) = 0$, but $p(C, P) \neq p(C \backslash b, P)$. In this case, it must be the case that there is $a \in C$ such that $p(a, C, P) > p(a, C \backslash \{b\}, P)$, which by Lemma 2 contradicts candidate stability at agenda C. ■

DJL have proved that only dictatorial rules are candidate stable and unanimous in deterministic environments. To check that dictatorial voting procedures are also exit stable is immediate. We can expect that probabilistic combinations of dictatorial rules, random dictatorships are also exit stable. According to a random dictatorship a group of voters (the "*vetoers*") have the possibility of becoming a dictator. The voters are asked for their best candidate, they introduce in a hat a given number of ballots with the name of their preferred candidate, and then a ballot is drawn at random. Notice that random dictatorships may assign different weights to different voters (that is, different numbers of ballots to be introduced in the hat), but the weights are invariant with respect to the agenda and the profile of voters preferences. Hence, a candidate has a positive probability of being elected if she is the best candidate for some voter and her probability of being elected is the proportion of ballots that have her name. We state this formally.

Definition 7 (*Dictatorship*). *A probabilistic voting procedure d_i is a* ***dictatorship*** *if there exists a voter $i \in \mathcal{V}$ such that for all $a \in \mathcal{C}$, for all $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$, it holds* *that:*

$$d_i(a, C, P) = \begin{cases} 1 & \textit{if } a = top(C, P_i), \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 8 (*Random Dictatorship*). *A probabilistic voting procedure p is a* ***random dictatorship*** *if it is a probabilistic convex combination of dictatorships. That is, if there is a group of voters $S \subseteq \mathcal{V}$, a set of weights $\{\alpha_i\}_{i\in S}$, ($\alpha_i > 0$ for all $i \in S$ and $\sum_{i\in S} \alpha_i = 1$) and a set of dictatorial probabilistic voting procedures $\{d_i\}_{i\in S}$ such that for all $C \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, and $P \in \mathcal{P}^{\mathcal{V}}$, $p(C, P) = \sum_{i\in S} \alpha_i d_i(C, P)$.*

It is easy to check that any random dictatorship is unanimous and exit stable. Notice that when a candidate withdraws the ballots that did not have her name remain with the same name, while those ballots that had her name are changed to another candidate. Hence, no other candidate's support is reduced, and no candidate can improve by leaving the fray.

In fact, just by rephrasing Theorem 4.14 in PP, we know that only probabilistic convex combinations of dictatorial rules are exit stable and efficient. Our Theorem 1 extends that result by relaxing efficiency to unanimity.

Theorem 1. *A probabilistic voting procedure is unanimous and exit stable if and only if it is a random dictatorship.*

The proof of Theorem 1 appears in the next section. The key point in the proof relies on the relation between candidate (exit) stable probabilistic voting procedures and candidate stable voting correspondences. By exploring this relation and applying a result in Rodríguez-Álvarez [14], we can show that any unanimous and exit stable probabilistic voting procedure only assigns positive probability to efficient candidates. Once efficiency is proved, the result follows immediately from Theorem 4.14 in PP, since exit stability is equivalent to regularity.[12]

With Theorem 1 at hand, we can evaluate the incentives of an outsider candidate to enter an election that uses a unanimous and exit stable probabilistic voting procedure. It is clear that only the candidates who are going to become the best candidate at stake for some "*vetoer*" can affect the social outcome. Moreover, the effect of their entry will be that they will become

[12] We present the results of PP and Rodríguez-Álvarez [14] in the following section as Propositions 1 and 2.

elected (with some positive probability). On the other hand, a candidate without any possibility of being elected does not have incentives to enter the fray since her entry has no effect in the social choice.

In order to analyze the implications of candidate stability, we cannot address to the results in PP. In fact, candidate stability and unanimity no longer compel the election to be driven by a random dictatorship. The following example shows the new possibilities that arise when we focus on the stability of fixed agendas.

Example 1. *Let $\mathcal{V} = \{1,2\}$, $C = \{a,b,c\} \subset \mathcal{C}$. Let d_1 and d_2 be the dictatorial probabilistic voting procedures associated to each voter. Now construct the probabilistic voting procedure $\bar{p}$ in the following fashion. For all $A \subseteq C$, and for all $P \in \mathcal{P}^{\mathcal{V}}$, $\bar{p}(A,P) = \frac{3}{4}d_1(A,P) + \frac{1}{4}d_2(A,P)$ if $Pareto(A,P) \neq C$; whereas $\bar{p}(x,A,P) = \frac{2}{3}$ if $x = top(C,P_1)$, $\bar{p}(x,C,P) = \frac{1}{6}$ otherwise, whenever $Pareto(A,P) = C$. Clearly, $\bar{p}$ is not a random dictatorship, since it allows for a more flexible outcome at preference profiles in which all the candidates are (ex-post) Pareto efficient. However, it is easy to see that $\bar{p}$ is unanimous and candidate stable at C. In Figure 1, we consider the preference profile $P \in \mathcal{P}^{\mathcal{V}}$ such that aP_1bP_1c , cP_2bP_2a. Candidate stability just implies that $\bar{p}(C,P)$ must be contained in inner triangle, as it is the case.*

Insert Figure 1.

The following Theorem 2 characterizes the family of probabilistic voting procedures satisfying candidate stability and unanimity at agendas containing at least four candidates. As we have seen in the previous example, candidate stability and unanimity allow for certain flexibility in the social choice when all the candidates are at stake. Nevertheless, it is only at the cost of distributing the decision power additively among the members of a group of voters when a candidate leaves the ballot.

Before following with the analysis of candidate stability and finite fixed agendas, we introduce now a bit of notation.

Let p be a probabilistic voting procedure and $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ a finite agenda. For any $P \in \mathcal{P}^{\mathcal{V}}$

and for any $a \in C$, define

$$L_p(a, C, P) = \{\lambda \in \mathcal{L}, \text{ such that for all } b \in C \backslash \{a\},\ \lambda(b) \leq p(b, C \backslash \{a\}, P)\}.$$

The set $L_p(a, C, P)$ contains all the lotteries that are admissible for $p(C, P)$ given the selection of p when the candidate a withdraws. At this point, we can present a rephrasal of Lemma 2 in terms of the sets of admissible lotteries. The probabilistic voting procedure p is candidate stable at agenda C if and only if $p(C, P) \in \bigcap_{b \in C} L_p(b, C, P)$.

Theorem 2. *Let $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ be a finite agenda containing at least four alternatives. A probabilistic voting procedure p is candidate stable and unanimous at C if and only if there is a group of vetoers $S \subseteq \mathcal{V}$, a set of weights $\{\alpha_i\}_{i \in S}$, ($\alpha_i > 0$ for all $i \in S$ and $\sum_{i \in S} \alpha_i = 1$), and a set of dictatorial probabilistic voting procedures $\{d_i\}_{i \in S}$, such that for all $P \in \mathcal{P}^{\mathcal{V}}$,*

$$p(C \backslash \{a\}, P) = \sum_{i \in S} \alpha_i d_i(C \backslash \{a\}, P) \text{ for all } a \in C \text{ and} \tag{1}$$

$$p(C, P) \in \bigcap_{b \in C} L_p(b, C, P). \tag{2}$$

We dub the class of probabilistic voting procedures satisfying (1) and (2) as ***modified random dictatorships****.*

Theorem 2 is parallel to the Theorem 4.11 in PP. Nevertheless, as we have already mentioned, it cannot be derived directly from it since candidate stability focuses on the stability of a particular agenda, while the results in PP apply regularity to all possible agendas.Evidently, as candidate stability is weaker than exit stability, Theorem 2 can be applied to provide an alternative prove of our Theorem 1. Moreover, Theorem 2 can be interpreted as a rationalizability result for probabilistic social choice in the same line of the results of Grether and Plott [9] and Denicolò [4] on non-binary choice. It is not necessary to apply the regularity argument to any possible agenda, it suffices to apply it to agendas in which one candidate is not in the ballot in order to get the modified random dictatorship characterization.

In the following lemma we specify the restrictions that $\bigcap_{b \in C} L_p(b, C, P)$ satisfies whenever a p is a modified random dictatorship.

Lemma 3. *Let $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ be a finite agenda and p be a modified random dictatorship with vetoers $S \subseteq \mathcal{V}$ and associated weights $\{\alpha_i\}_{i \in S}$. Then, for all $P \in \mathcal{P}^{\mathcal{V}}$:*

i) $\sum_{i \in S} \alpha_i d_i (C, P) \in \bigcap_{b \in C} L_p (b, C, P)$.

ii) *If $a, a' \in \bigcup_{i \in S} top (C, P_i)$ then there is no $\lambda \in \bigcap_{b \in C} L_p (b, C, P)$ such that $\lambda (a) = \lambda (a') = 0$.*

iii) $\sum_{i \in S} \alpha_i d_i (C, P) \neq \bigcap_{b \in C} L_p (b, C, P)$ *if and only if there is $a \in C$, such that for all $b \in C \backslash \{a\}$, there exists some $i \in S$ with $a = top (C \backslash \{b\}, P_i)$ and $a \neq top (C, P_i)$.*

Proof. i) It follows immediately from the fact that $\sum_{i \in S} \alpha_i d_i (C, P) \in \mathcal{L}$ and therefore for all $a, b \in C$, $\sum_{i \in S} \alpha_i d_i (a, C, P) \leq \sum_{i \in S} \alpha_i d_i (a, C \backslash \{b\}, P)$.

ii) Notice first that for any $P \in \mathcal{P}^{\mathcal{V}}$, $a \in \cup_{i \in S} top(C, P_i)$, $b \notin \cup_{i \in S} top(C \backslash \{a\}, P_i)$ and $\lambda \in L_p(a, C, P)$ by the definition of $L_p(a, C, P)$, $\lambda(b) = 0$. This point and $\lambda \in \mathcal{L}_C$ imply that for any $P \in \mathcal{P}^{\mathcal{V}}$, $a \in \cup_{i \in S} top(C, P_i)$ and $\lambda \in L_p(a, C, P)$ if $\lambda(a) = 0$ then $\lambda = \sum_{i \in S} \alpha_i d_i(C \backslash \{a\}, P)$. As $\cup_{i \in S} top(C, P_i) \subseteq \cup_{i \in S} top(C \backslash \{a\}, P_i)$ and $L_p(a, C, P) \subset \cap_{b \in C} L_p(b, C, P)$, item *ii)* follows immediately.

iii) Assume that for $P \in \mathcal{P}^{\mathcal{V}}$ there is not such candidate a, then for all $a \in C$ there is $b \in C \backslash \{a\}$ such that $\{i \in S,\ a = top(C \backslash \{b\}, P_i)\} = \{i \in S,\ a = top(C, P_i)\}$. Hence, for all $a \in C$ we get $\sum_{i \in S} \alpha_i d_i(a, C, P) = \min_{d \in C \backslash \{a\}} \sum_{i \in S} \alpha_i d_i(a, C \backslash \{d\}, P)$. Moreover, for any $\lambda \in \bigcap_{b \in C} L_p(b, C, P)$, we have that for all $a \in C$, $\lambda(a) \leq \sum_{i \in S} \alpha_i d_i(a, C, P)$. As $\sum_{i \in S} \alpha_i d_i(a, C, P) \in \mathcal{L}$, it must be the case that for all $a \in C$, $\lambda(a) = \sum_{i \in S} \alpha_i d_i(a, C, P)$, and then $\bigcap_{b \in C} L_p(b, C, P) = \sum_{i \in S} \alpha_i d_i(C, P)$.

Conversely, take $P \in \mathcal{P}^{\mathcal{V}}$ such that for some $a \in C$ for any $b \in (C \backslash \{a\})$, there is $i \in S$ with $a = top(C \backslash \{b\}, P_i)$, whereas $a \neq top(C, P_i)$. Notice first that

$$\sum_{i \in S} \alpha_i d_i(a, C, P_i) < \min_{d \in C} \left\{ \sum_{i \in S} \alpha_i d_i \left(a, C \backslash \{d\}, P\right) \right\}.$$

This implies there is $\lambda \in \mathcal{L}$ such that

$$\sum_{i \in S} \alpha_i d_i(a, C, P_i) < \lambda(a) \leq \min_{d \in C} \left\{ \sum_{i \in S} \alpha_i d_i \left(a, C \backslash \{d\}, P\right) \right\},$$

while for all $b \in C\backslash\{a\}$,

$$\lambda(b) \leq \sum_{i \in S} \alpha_i d_i \left(b, C, P\right) \leq \min_{d \in C} \left\{ \sum_{i \in S} \alpha_i d_i \left(b, C\backslash\{d\}, P\right) \right\},$$

and therefore $\lambda \in \bigcap_{b \in C} L_p(b, P)$ but $\lambda \neq \sum_{i \in S} \alpha_i d_i(C, P)$. ■

In item i) we show that the conditions of Theorem 2 define a non-empty class of probabilistic voting procedures since any random dictatorship always satisfies them.

Item ii) implies that at most one of the best preferred candidates according to the vetoers' preferences can receive null probability when no candidate leaves the agenda C.

Finally, item iii) characterizes the preference profiles in which a modified random dictatorship may be different than a random dictatorship. This situation may occur only at preference profiles in which there is a candidate that whenever another candidate leaves the fray, she becomes the best preferred candidate among those remaining in the fray for some voter with veto power. In those cases, this candidate will increase her possibility of being elected if any other candidate withdraws and then she can receive more probability than the assigned according by the random dictatorship lottery when all the candidates run the election.

Notice that whenever the number of candidates exceeds the number of vetoers in two, our characterization of modified random dictatorships collapses to random dictatorships, since there is no preference profile in which a candidate always increases her support when any other candidate leaves the ballot.[13] This case corresponds to situations in which the set of vetoers is relatively small with respect to the set of candidates and hence the power of decision is very concentrated in a few voters.

Just as a matter of clarification we present now an example of a modified random dictatorship. It resembles Example 5.6 in PP.

Example 2. *Let* $C = \{a, b, c, d\}$, *and* $\mathcal{V} = \{1, 2, 3, 4\}$; $S = \{1, 2, 3\}$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$. *Let* $\mathcal{P}^* \subset \mathcal{P}^{\mathcal{V}}$ *denote the set of preference profiles such that each vetoer has different top candidate but the remaining candidate is the second best for all the vetoers and the best for voter* 4. *Then,*

[13]This fact is also noted in PP for regular probabilistic voting procedures.

for all $A \in \{C, \{C \backslash \{a\}\}_{a \in C}\}$ *and all* $P \in \mathcal{P}^{\mathcal{V}}$, p^* *is defined in the following way:*

$$p'(A, P) = \begin{cases} \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) & \text{if } P \in \mathcal{P}^* \text{ and } A = C, \\ \sum_{i \in S} \alpha_i d_i(A, P) & \text{otherwise.} \end{cases}$$

It is easy to check that p' *satisfies the conditions of the Theorem 2. The set of admissible lotteries for* p' *is not a singleton if and only if all the vetoers report different top candidates and the remaining candidate is the second best for all the vetoers. We have tailored* p' *in such a way that the preferences of the voter without veto power are also relevant for the social choice.*

Theorem 2 does not cover the case of only three initial candidates. In fact, we need to consider agendas with at least four candidates at several steps of the proof to obtain the result. The following example shows that modified random dictatorships are not the only probabilistic voting procedures satisfying candidate stability and unanimity at agendas containing only three candidates.

Example 3. *Let* $C = \{a, b, c\}$. *Construct now a probabilistic voting procedure* p'' *such that for all* $a \in C$, *for all* $A \in 2^C \backslash \{\varnothing\}$, *and for all* $P \in \mathcal{P}^{\mathcal{V}}$,*:*

$$p''(a, A, P) = \begin{cases} \frac{1}{\#Pareto(A,P)} & \text{if } a \in Pareto(A, P), \\ 0 & \text{otherwise.} \end{cases}$$

p'' *is unanimous and candidate stable. Nevertheless, it is not candidate stable for any agenda with more than three candidates.*

Notice that p' is not a random dictatorship even when a candidate leaves the election, since the distribution of the veto power is not additive. Every group of voters has the same capacity to impose their most preferred candidate when only two candidates are at stake. In the next theorem, we see that any monotonic and sub-additive distribution of the veto power suffices to reconcile unanimity and candidate stability.

Theorem 3. *Assume the agenda* C *contains only three candidates. A probabilistic voting procedure is candidate stable and unanimous at* C *if and only if there is a group of voters* $S \subseteq \mathcal{V}$, *a set of weights* $\{\alpha_T\}_{T\subseteq S}$ *such that:*

$$p(a, C\backslash\{b\}, P) = \alpha_T \Leftrightarrow T = \{j \in S,\ a = top(C\backslash\{b\}, P_j)\} \text{ and,}$$
$$p(C, P) \in \bigcap_{b\in C} L_p(b, C, P),$$

for all $P \in \mathcal{P}^{\mathcal{V}}$ *and* $a, b \in C$. *Furthermore,*

i) $\alpha_{\{\varnothing\}} = 0$, *and for all* $T \in S$, $\alpha_T = 1 - \alpha_{S\backslash T}$,

ii) (monotonicity) *for all* $T, T' \subseteq S$ *implies* $T \subseteq T'$; $\alpha_T \leq \alpha_{T'}$,

iii) (sub-additivity) *for all disjoint* $T, T' \subset S$, $\alpha_T + \alpha_{T'} \geq \alpha_{(T\cup T')}$.

Although Theorem 2 does not cover the case of three candidates, as we will see in the following section, the main arguments in its proof can be applied to get the result at agendas with only three candidates. On the other hand, when there are only three initial candidates, a candidate stable probabilistic voting procedure is indeed regular and we can also address to Remark 4.15 in PP to prove that sub-additive distributions of the veto power among the vetoers are compatible with candidate stability and unanimity.

It may seem surprising that the implications of candidate stability depend so much on the number of candidates at stake. The differences between the results of Theorem 2 and Theorem 3 are due to the fact that when there are only three candidates, and one of them withdraws, the choice of the lottery on the remaining candidates is a binary choice. Thus, we can expect that the results with only three initial candidates are in the line of those in Barberà and Sonnenschein [3] and McLennan [10] on stochastic social preferences (or probabilistic binary choice.) As it is remarked in PP,[14] when there are three candidates, a candidate stable probabilistic voting procedure is rationalizable. That is, for every profile of individual orderings one can specify a probability distribution r over linear orderings of candidates, such that given a feasible agenda A, the probability of a candidate a being chosen by the voters from A, is the sum of the probabilities assigned by r to all those linear orderings $P' \in \mathcal{P}$ where $a = top(A, P')$. Nevertheless, when there are at least four candidates, regularity no longer implies rationalizability. But in this case, candidate stability forces the distribution of the veto power to be additive.

[14] See PP, Remarks 3.12 ; 4.15 and Lemma 3.13.

4 Proofs of the Theorems

As we have already mentioned the proofs of our Theorems rely on the results of PP and Rodríguez-Álvarez [14] that are stated as Proposition 1 and Proposition 2 below. In fact, our Theorem 2 cannot be proved from the results of PP, since candidate stability is weaker than regularity if there are at least four initial candidates. The key point in our results relies on the relation between exit (candidate) stable probabilistic voting procedures and candidate stable voting correspondences. But before we need the following definitions.

A *voting correspondence* v is a mapping $v : 2^{\mathcal{C}}\backslash\{\varnothing\} \times \mathcal{P}^{\mathcal{V}} \to 2^{\mathcal{C}}\backslash\{\varnothing\}$ such that:

i) For all $C \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$, $v(C, P) \subseteq C$.

ii) For all $C \in 2^{\mathcal{C}}\backslash\{\varnothing\}$, $v(C, P) = v(C, P')$ for all $P' \in \mathcal{P}^{\mathcal{V}}$ such that, $P\mid_C = P'\mid_C$.

The voting correspondence v satisfies unanimity at agenda C if for any $A \in \{C, \{C\backslash\{c\}_{c\in C}\}\}$, $a \in A$, and all $P \in \mathcal{P}^{\mathcal{V}}$, $a = top(A, P_i)$ for all $i \in \mathcal{V}$ implies $v(A, P) = a$. Finally, a voting correspondence v satisfies the *"no harm"* condition at agenda C if for all $P \in \mathcal{P}^{\mathcal{V}}$ and $a \in C$, $a \in v(C, P)$ implies $v(C, P) \subseteq v(C\backslash\{a\}, P) \cup \{a\}$. A voting correspondence v satisfies the *"insignificance"* condition at agenda C if for all $P \in \mathcal{P}^{\mathcal{V}}$ and $a \in C$ $a \notin v(C, P)$ implies $v(C, P) = v(C\backslash\{a\}, P)$.[15]

Proposition 1 (Theorem 4, Rodríguez-Álvarez [14]). *If a voting correspondence v is unanimous and satisfies "no harm" and "insignificance" at agenda C then there is a subset of voters $S \subseteq \mathcal{V}$ such that for all $A \in \{C, \{C\backslash\{a\}\}_{a\in C}\}$ and for all $P \in \mathcal{P}^{\mathcal{V}}$:*

a) $v(A, P) \subseteq Pareto(A, P_S)$,

b) for any $a \in C$, $v(A, P) \neq \{a\}$ if there are $i \in S$ and $b \in A$ with bP_ia,

c) for any $a \in C$, $\cup_{i\in S} top(C\backslash\{a\}, P_i) \subseteq v(C\backslash\{a\}, P)$.

[15]Lemma 4 in Rodríguez-Álvarez [14] proves that a voting correspondence satisfies no harm and insignificance conditions if and only if it is candidate stable when candidates compare sets of candidates according to the leximin criterion.

Following PP, we say that a probabilistic voting procedure satisfies *regularity* if for all $P \in \mathcal{P}^{\mathcal{V}}$, $C, C' \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and $c \in C$, $C \subseteq C'$ implies $p(c, C, P) \geq p(a, c', P)$.

Proposition 2 (Theorem 4.14, PP). *If there exists $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ with $\#C \geq \#\mathcal{V} + 2$ then a probabilistic voting procedure p satisfies regularity and efficiency if and only if p is a random dictatorship.*

The key point in the proofs of our results relies on the following observation. For any probabilistic voting procedure p we can define its auxiliary voting correspondence v_p in the following way, for any $c \in \mathcal{C}$, for any $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, and any $P \in \mathcal{P}^{\mathcal{V}}$, $c \in v_p(C, P)$ if and only if $p(c, C, P) > 0$. By *i)* and *ii)* of the definition of probabilistic voting procedures, v_p is well defined as a voting correspondence.

Before going to the proofs of the theorems we state a crucial lemma. It provides the link between candidate stable and unanimous probabilistic voting procedures and candidate stable voting correspondences.

Lemma 4. *Let $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$. If p is a candidate stable and unanimous at agenda C probabilistic voting procedure, then its associated voting correspondence v_p is unanimous and satisfies no harm and insignificance at C.*

Proof. It is immediate to see that the unanimity of p at C implies that v_p is unanimous. Now we check "no harm". Consider an arbitrary preference profile $P \in \mathcal{P}^{\mathcal{V}}$ and find $b \in C$ such that $p(b, C, P) > 0$. By candidate stability, we know that for all $a \in C \backslash \{b\}$ such that $p(a, C, P) > 0$, also $p(a, C \backslash \{b\}, P) > 0$ and by the definition of v_p, $a \in v_p(C, P)$ and $a \in v_p(C \backslash \{b\}, P)$. As the choice P was arbitrary, for all $b \in v_p(C, P)$, $v_p(C, P) \subseteq v_p(C \backslash \{b\}, P) \cup \{b\}$.

Finally, we prove "insignificance". Consider an arbitrary preference profile $P \in \mathcal{P}^{\mathcal{V}}$ and find $b' \in C$ such that $p(b', C, P) = 0$, by Remark 2 we know that $p(C, P) = p(C \backslash \{b'\}, P)$. Then for any $P \in \mathcal{P}^{\mathcal{V}}$ and any $b' \notin v_p(C, P)$, $v_p(C, P) = v_p(C \backslash \{b'\}, P)$, and as the choice of P we get the desired conclusion. ■

Proof of Theorem 1.

Sufficiency is clear, so we focus on necessity. Assume that the probabilistic voting procedure

p is exit stable and unanimous. As any exit stable probabilistic voting procedure is candidate stable at any agenda $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$, Lemma 4 applies. By Proposition 1, we know that for any $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ there is a group of voters $S \subseteq \mathcal{V}$ holding veto power for v_p, and therefore for all $P \in \mathcal{P}^{\mathcal{V}}$, $v_p(C,P) \subseteq Pareto(C, P_S)$, which implies that p is (*ex-post*) Pareto efficient.

Once we know that a unanimous and exit stable probabilistic voting procedure is indeed (*ex-post*) efficient, the result follows immediately. Notice that since C is an infinite set, we can always find an agenda C such that $\#C \geq \#\mathcal{V} + 2$. Moreover, we have seen in Remark 1 to Lemma 1 that exit stable probabilistic voting procedures satisfy regularity. Therefore, applying Proposition 2, we can see that any unanimous and exit stable probabilistic voting procedure is indeed a random dictatorship. ■

Proof of Theorem 2.

We start with sufficiency. By item $i)$ of Lemma 3, we know that a probabilistic voting procedure p satisfying the requirements of Theorem 2 exists. In order to check unanimity, notice that a random dictatorship is unanimous, and by $iii)$ of Lemma 3, p may differ from a random dictatorship only at non-unanimous profiles. On the other hand, it is not difficult to see that modified random dictatorships are candidate stable at C since for all $P \in \mathcal{P}^{\mathcal{V}}$, $p(C,P) \in \bigcap_{b \in C} L_p(b, C, P)$.

In order to prove necessity, we will assume that p is unanimous and candidate stable at the agenda C, with $\#C \geq 4$. Using the same arguments we have used in the proof of Theorem 1, Lemma 4 and Proposition 1 imply there is a set of voters S, such that for all $A \in \{C, \{C \backslash \{a\}\}_{a \in C}\}$, $a \in C$ and $P \in \mathcal{P}^{\mathcal{V}}$, $p(b, A, P) > 0$ only if $a \in Pareto(A, P_S)$.

The following claim presents a crucial neutrality result. It shows that the veto power of the vetoers does not depend on the names of the candidates.

Claim 1. *There is a set of weights $\{\alpha_T\}_{T \subseteq S}$ with $\alpha_T > 0$, and $\alpha_T + \alpha_{(S \backslash T)} = 1$ for all $T \subseteq S$ such that for all $a, b \in C$, $P \in \mathcal{P}^{\mathcal{V}}$ with $Pareto(C, P_S) = \{a, b\}$, it holds that:*

$$p(a, C, P) = \alpha_T \Leftrightarrow T = \{i \in S \text{ such that } aP_ib\}.$$

Proof. The proof of this claim proceeds by a series of steps. First, we are going to prove that if we fix the preferences of the voters not in S, whenever there exist two candidates that are the only efficient candidates according to the preferences of the vetoers, the probability of each candidate to be elected only depends on the group of vetoers who support them. In Step 2, we show that these probabilities do not depend on the names of the candidates. Finally, in Step 3, we see that the preferences of the voters without veto power are irrelevant.

Fix now a pair of candidates $a, b \in C$ and a preference profile $P \in \mathcal{P}^{\mathcal{V}}$, such that for some $T \subsetneq S$, $a = top(C, P_i)$, $b = top(C\backslash\{a\}, P_i)$ for all $i \in T$ while $b = top(C, P_j)$, $a = top(C\backslash\{b\}, P_j)$ for all $j \in (S\backslash T)$. Let $p(a, C, P) = \alpha$ and as $Pareto(C, P_S) = \{a, b\}$, $p(b, C, P) = (1 - \alpha)$.

Step 1. Consider now $P' \in \mathcal{P}^{\mathcal{V}}$, such that $P_S \mid_{\{a,b\}} = P'_S \mid_{\{a,b\}}$, $Pareto(C, P_S) = \{a, b\}$ while $P_{-S} = P'_{-S}$. We will prove that $p(C, P) = p(C, P')$.

Take now the profile of preferences $\tilde{P} \in \mathcal{P}^{\mathcal{V}}$ such that a vetoer $i \in S$ changes the order in which she compares two contiguous candidates with respect to her initial preferences. That is, there is $i \in S$, $d, e \in C$ ($\{d, e\} \neq \{a, b\}$) such that $P_{-i} = \tilde{P}_{-i}$; $P_i \mid_{C\backslash\{d\}} = \tilde{P}_i \mid_{C\backslash\{d\}}$, $P_i \mid_{C\backslash\{e\}} = \tilde{P}_i \mid_{C\backslash\{e\}}$ and dP_ie while $e\tilde{P}_id$. Notice that at least one of the candidates, say e, $e \notin \{a, b\}$. Moreover, $e \notin Pareto(C, P_S)$ and $e \notin Pareto\left(C, \tilde{P}_S\right)$. By Remark 2, $p(C, P) = p(C\backslash\{e\}, P)$, and also $p\left(C, \tilde{P}\right) = p\left(C\backslash\{e\}, \tilde{P}\right)$. Finally, applying $ii)$ of the definition of probabilistic voting correspondence we have that $p(C, P) = p\left(C, \tilde{P}\right)$. We can iterate this argument changing the preferences of only one vetoer each time, till we get the desired conclusion, that is $p(a, C, P) = p(a, C, P') = \alpha$ and $p(b, C, P) = p(b, C, P') = (1 - \alpha)$.

Step 2. Now, we prove that for any candidates $d, e \in C$ and any profile $P'' \in \mathcal{P}^{\mathcal{V}}$ such that $Pareto(C, P''_S) = \{d, e\}$ and $d = top(C, P_i)$ for all $i \in T$ while $e = top(C, P_j)$ for all $j \in (S\backslash T)$,, we have $p(d, C, P'') = \alpha$, $p(e, C, P'') = (1 - \alpha)$.

Consider a profile $\hat{P} \in \mathcal{P}^{\mathcal{V}}$ such that $a = top\left(C, \hat{P}_i\right)$, $b = top\left(C\backslash\{a\}, \hat{P}_i\right)$ for all $i \in T$ while $b = top\left(C, \hat{P}_j\right)$, $a = top\left(C\backslash\{b\}, \hat{P}_j\right)$ for all $j \in (S\backslash T)$ and $c = top\left(C\backslash\{a, b\}, \hat{P}_{i'}\right)$ for all $i' \in S$. By the arguments Step 1, we know that $p\left(a, C, \hat{P}\right) = \alpha$, $p\left(b, C, \hat{P}\right) = (1 - \alpha)$, since $Pareto\left(C, \hat{P}_S\right) = \{a, b\}$ and $P \mid_{\{a,b\}} = \hat{P} \mid_{\{a,b\}}$. Take now the preference profile $P^* \in \mathcal{P}^{\mathcal{V}}$ such that $\hat{P}_k = P^*_k$ for all $k \notin S$ while the voters in S the positions of the candidates b and

c change their positions, that is $aP_i^*cP_i^*b$ for all $i \in T$ and $cP_j^*bP_j^*a$ for all $j \in (S\backslash T)$, and $P_S^* \mid_{C\backslash\{b\}} = \hat{P}_S \mid_{C\backslash\{b\}}$ and also $P_S^* \mid_{C\backslash\{c\}} = \hat{P}_S \mid_{C\backslash\{c\}}$. As $c \notin Pareto\left(\mathcal{C}, \hat{P}_S\right)$, $p\left(c, \mathcal{C}, \hat{P}\right) = 0$. By Remark 2, we have that $p\left(C, \hat{P}\right) = p\left(C\backslash\{c\}, \hat{P}\right)$ and by $ii)$ in the definition of probabilistic voting procedure, $p\left(a, C\backslash\{c\}, \hat{P}\right) = p\left(a, C\backslash\{c\}, P^*\right) = \alpha$. Finally, candidate stability implies that $p\left(a, \mathcal{C}, P^*\right) \leq \alpha$. We now check that $p\left(a, \mathcal{C}, P'\right) = \alpha$. Given that $b \notin Pareto\left(C, P_S^*\right)$, from efficiency we obtain $p\left(b, C, P^*\right) = 0$ and by Remark 2, candidate stability implies $p\left(a, C, P^*\right) = p\left(a, C\backslash\{b\}, P^*\right)$. Notice now that by $ii)$ in the definition of probabilistic voting procedure $p\left(a, C\backslash\{b\}, P^*\right) = p\left(a, C\backslash\{b\}, \hat{P}\right) = \alpha$. Therefore, we have that $p\left(a, C, P^*\right) = \alpha$, $p\left(c, C, P^*\right) = (1-\alpha)$. Repeating the reasoning as many times as necessary we obtain the desired result.

Step 3. Finally, we have to prove that for all $P' \in \mathcal{P}^{\mathcal{V}}$ such that $P_S = P'_S$, $p\left(a, C, P'\right) = \alpha$ (and therefore also $p\left(b, C, P'\right) = (1-\alpha)$.)

Consider a profile of preferences $\bar{P} \in \mathcal{P}^{\mathcal{V}}$, such that there is a voter $k \notin S$ that switches the order in which she compares two continuous candidates (without any loss of generality dP_ke, $e\bar{P}_kd$, $P_k \mid_{\mathcal{C}\backslash\{d\}} = \bar{P}_k \mid_{\mathcal{C}\backslash\{d\}}$ and $P_k \mid_{\mathcal{C}\backslash\{e\}} = \bar{P}_k \mid_{\mathcal{C}\backslash\{e\}}$ and $P_{-k} = \bar{P}_{-k}$). Now, we have to analyze two possibilities: either $\{d, e\} \neq \{a, b\}$, or $\{d, e\} = \{a, b\}$. If $\{d, e\} \neq \{a, b\}$, by efficiency either d or e (or both) do not receive positive probability neither at profile P nor at profile P^*. We can assume without lost of generality that $d \notin Pareto\left(C, \bar{P}_S\right)$ and $p\left(d, C, P\right) = p\left(d, C, \bar{P}\right) = 0$. Remark 2 and $ii)$ in the definition of probabilistic voting procedure imply that $p\left(C, P\right) = p\left(C\backslash\{d\}, P\right) = p\left(C\backslash\{d\}, P^*\right) = p\left(C, P^*\right)$, which proves that the change of k's preferences has no effect on the choice. Finally, we check the case in which $\{d, e\} = \{a, b\}$. By the arguments in the previous paragraph and $ii)$ in the definition of probabilistic voting procedure, we know that $p\left(a, C\backslash\{b\}, \bar{P}\right) = \alpha$. Notice that also $P' \mid_{C\backslash\{b\}} = \left(P'_S, P^*_{-S}\right) \mid_{C\backslash\{b\}}$, and since $b \notin Pareto(C, P'_S)$, applying again $ii)$ of the definition of probabilistic voting procedure, we get $p\left(b, \mathcal{C}, \left(P'_S, P^*_{-S}\right)\right) = 0$. Remark 1 implies that $p\left(C, P^*\right) = p\left(C\backslash\{b\}, \left(P'_S, P^*_{-S}\right)\right)$, and $p\left(a, C, \left(P'_S, P^*_{-S}\right)\right) = \alpha$, $p\left(c, C, \left(P'_S, P^*_{-S}\right)\right) = (1-\alpha)$. Repeating the arguments with one such a switch in the preferences of one voter we obtain the result for arbitrary preference profiles of the voters without veto power.

It is immediate to see that the three steps suffice to conclude the proof of Claim 1.

Claim 2. *For all* $T, T' \subset S$ *with* $T \cap T' = \{\varnothing\}$, $\alpha_T + \alpha_{T'} = \alpha_{(T \cup T')}$.

Proof. Pick any two arbitrary disjoints subsets of the set of vetoers, $T, T' \subset S$, $T \cap T' = \{\varnothing\}$. Consider $P \in \mathcal{P}^{\mathcal{V}}$, $a, b, c, d \in C$ such that $aP_ibP_icP_idP_ie$ for all $i \in T$, $bP_jcP_jaP_jdP_je$ for all $j \in T'$, $cP_kaP_kbP_kdP_ke$ for all $k \in (S \backslash T)$, and for all $e \in (C \backslash \{a, b, c, d\})$. Construct now $P^1 \in \mathcal{P}^{\mathcal{V}}$ such that $P\mid_{C \backslash \{d\}} = P^1 \mid_{C \backslash \{d\}}$ and $d = top\left(C \backslash \{a\}, P_i^1\right)$ for all $i \in T$, $d = top\left(C, P_{i'}^1\right)$ for all $i' \in (S \backslash T)$. By the arguments of the previous claim, $p\left(a, C, P^1\right) = \alpha_T$. Analogously construct the profile $P^2 \in \mathcal{P}^{\mathcal{V}}$ such that $P\mid_{C \backslash \{d\}} = P^2 \mid_{C \backslash \{d\}}$ and $d = top\left(C \backslash \{b\}, P_j^2\right)$ for all $j \in T'$, $d = top\left(C, P_{j'}^2\right)$ for all $j' \in (S \backslash T')$, and we get, $p\left(b, C, P^2\right) = \alpha_{T'}$. In the same fashion take the profile $P^3 \in \mathcal{P}^{\mathcal{V}}$ such that $P\mid_{C \backslash \{d\}} = P^3 \mid_{C \backslash \{d\}}$ and $d = top\left(C \backslash \{c\}, P_k^3\right)$ for all $k \in S \backslash (T \cup T')$, $d = top\left(C, P_{k'}^3\right)$ for all $k' \in (T \cup T')$ and, $p\left(c, C, P^3\right) = \alpha_{S \backslash (T \cup T')}$.

Notice that $P\mid_{C \backslash \{d\}} = P^1 \mid_{C \backslash \{d\}} = P^2 \mid_{C \backslash \{d\}} = P^3 \mid_{C \backslash \{d\}}$. Then, candidate stability, *ii*) in the definition of probabilistic voting procedure and efficiency imply that

$$
\begin{aligned}
p(a, C, P) &= p\left(a, C \backslash \{d\}, P^1\right) \geq \alpha_T, \\
p(b, C, P) &= p\left(b, C \backslash \{d\}, P^2\right) \geq \alpha_{T'}, \\
p(c, C, P) &= p\left(c, C \backslash \{d\}, P^3\right) \geq \alpha_{S \backslash (T \cup T');}
\end{aligned}
$$

while, also by efficiency, $p(e, C, P) = 0$ for all $e \notin \{a, b, c\}$.

Now, we will show that the inequalities must be binding. Construct now the profile $P^* \in \mathcal{P}^{\mathcal{V}}$ such that $P\mid_{C \backslash \{b\}} = P^* \mid_{C \backslash \{b\}}$ and such that for all $e \in \{a, b\}$, $l \in S$, eP_l^*b, and therefore $Pareto(C, P^*) = \{a, c\}$. By the previous claim we have that $p(a, C, P^*) = \alpha_T$, and by Remark 2, $p(a, C \backslash \{b\}, P^*) = \alpha_T$. Then, using ii) in the definition of probabilistic voting procedure, we obtain $p(C \backslash \{b\}, P) = p(C \backslash \{b\}, P^*)$.

We can repeat the argument with candidates b and c to prove that $p(a, C, P) = \alpha_T$, $p(b, C, P) = \alpha_{T'}$ and $p(c, C, P) = \alpha_{S \backslash (T \cup T')}$. Moreover, by efficiency:

$$
\alpha_T + \alpha_{T'} + \alpha_{S \backslash (T \cup T')} = 1. \tag{3}
$$

Finally, note that as $\alpha_{S \backslash (T \cup T')} = \left(1 - \alpha_{(T \cup T')}\right)$, (3) implies $\alpha_T + \alpha_{T'} = \alpha_{(T \cup T')}$. This concludes the proof of Claim 2.

In order to complete the proof of the theorem, we only have to prove that for any $a, b \in C$, $P \in \mathcal{P}^{\mathcal{V}}$, $p(a, C\backslash\{b\}, P) > 0$ only if there is $i \in S$ with $a = top(C\backslash\{b\}, P_i)$, and moreover $p(a, C\backslash\{b\}, P) = \alpha_T$ where $T = \{i \in S,\ a = top(C\backslash\{b\}, P_i)\}$.

Consider $P \in \mathcal{P}^{\mathcal{V}}$, such that there are $a, b \in C$, $T \subset S$, $T = \{i \in S,\ a = top(C\backslash\{b\}, P_i)\}$. Construct now the profile $P' \in \mathcal{P}^{\mathcal{V}}$ such that $P\ |_{C\backslash\{b\}} = P'\ |_{C\backslash\{b\}}$ with $b = top(C\backslash\{a\}, P'_i)$ for all $i \in T$ for all $j \in (S\backslash T)$, while $b = top\left(C, P'_j\right)$. Then, we have that $p(a, C, P') = \alpha_T$, and therefore, by ii) in the definition of probabilistic voting procedure and by candidate stability $p(a, C\backslash\{b\}, P) = p(a, C\backslash\{b\}, P') > \alpha_T$. Repeating the argument with the remaining candidates $c \in \cup_{j\in S} top(C\backslash\{b\}, P_j)$, the additivity of the weights $\{\alpha_T\}_{T\subseteq S}$, we obtain that for any $P \in \mathcal{P}^{\mathcal{V}}$, $b \in C$:

$$p(C\backslash\{b\}, P) = \sum_{i\in S} \alpha_i d_i (C\backslash\{b\}, P).$$

Finally, candidate stability implies that $p(C, P) \in \bigcap_{b\in C} L_p(b, C, P)$. ■

Proof of Theorem 3.

In order to prove sufficiency we have to check that the conditions of Theorem 3 define a non-empty family of probabilistic voting procedures. Assume that p is a probabilistic voting procedure satisfying the conditions of the Theorem. Let $C = \{a, b, c\}$ and take any $P \in \mathcal{P}^{\mathcal{V}}$. Let S_a be the set of vetoers who prefer a to b and c, and define analogously S_b and S_c. Notice first that:

$$\begin{aligned} \alpha_{S_a} &\leq \min_{x\in\{b,c\}} \{p(a, C\backslash\{x\}, P)\}, \\ \alpha_{S_b} &\leq \min_{y\in\{a,c\}} \{p(b, C\backslash\{y\}, P)\}, \\ \alpha_{S_c} &\leq \min_{z\in\{a,b\}} \{p(c, C\backslash\{z\}, P)\}, \end{aligned} \tag{4}$$

On the other hand, as S_a, S_b and S_c form a partition of S, items i)and iii) imply that

$$\alpha_{S_a} + \alpha_{S_b} + \alpha_{S_c} \geq 1. \tag{5}$$

From (5) we know there is $\lambda \in \mathcal{L}$ such that $\lambda(a) \leq \alpha_{S_a}$, $\lambda(b) \leq \alpha_{S_b}$ and $\lambda(c) \leq \alpha_{S_c}$. Moreover, by (4), $\lambda \in \bigcap_{b\in C} L_p(b, C, P)$, and hence $\bigcap_{b\in C} L_p(b, C, P)$ is always non-empty which suffices to prove that there exists a probabilistic voting procedure satisfying the conditions of the theorem exists.

To check candidate stability is straight-forward since $p(C,P)$ always belongs to $\bigcap_{b\in C} L_p(b,P)$. Unanimity is also immediate from i).

The proof of necessity is parallel to the proof of Theorem 2. By an already familiar argument, from Lemma 4 and Proposition 1, we know there are a group of voters S holding veto power over p and only Pareto efficient candidates according to the preferences of the vetoers may receive positive probability. Moreover, a candidate receives probability 1 if and only if she is the top candidate for all the vetoers which implies item i).

The reader can check that proving Claim 1 at the proof of Theorem 2 we have only assumed the existence of three candidates. Hence this result can be applied when the initial set of candidates contains three candidates. Then from Claim 1, we know that whenever there are only two candidates at stake, say a and b, and a subset of the vetoers T prefer a to b, while the remaining vetoers prefer b to a then $p\left(a,\{a,b\},P\right) = \alpha_T$ while $p\left(b,\{a,b\},P\right) = \alpha_{(S\backslash T)} = (1-\alpha_T)$. Moreover, these weights are independent of the names of the candidates and of the preferences of the remaining voters.

Let us check monotonicity. It is trivially fulfilled when $\#S = 2$. So take $\#S \geq 3$, and without loss of generality, take the profile $P \in \mathcal{P}^{\mathcal{V}}$, such that aP_jbP_jc for all $j \in J$, bP_iaP_ic and bP_kcP_ka for all $k \in S\backslash\left(T\cup\{i\}\right)$. Then as candidate c is Pareto dominated according to the preferences of the vetoers, $p\left(C,P\right) = p\left(C\backslash\{c\},P\right)$, $p\left(a,C,P\right) = \alpha_T$ and $p\left(b,C,P\right) = (1-\alpha_T)$. Candidate stability implies that $p\left(a,C\backslash\{b\},P\right) = \alpha_{(T\cup\{i\})} \geq p(a,C,P) = \alpha_T$. Repeating the argument as many times as necessary, we obtain that $\alpha_T \leq \alpha_{T'}$ whenever $T \subseteq T' \subseteq S$.

Now, we complete the proof checking sub-additivity. Take two arbitrary sets of vetoers $T, T' \subset S$, such that $T \cap T'$ and consider a profile $P \in \mathcal{P}^{\mathcal{V}}$ such that $a = top(C,P_i)$, $b = top(C\backslash\{a\},P_i)$ for all $i \in T$, $b = top(C,P_j)$ $c = top(C\backslash\{b\},P_j)$ for all $j \in T'$, $c = top(C,P_k)$ and $a = top(C\backslash\{c\},P_k)$ for all $k \in S\backslash\left(T\cup T'\right)$. By candidate stability $p(a,C,P) \leq p(a,C\backslash\{b\},P) = \alpha_T$, $p(b,C,P) \leq p(b,C\backslash\{c\},P) = \alpha_{T'}$ and $p(c,C,P) \leq p(c,C\backslash\{a\},P) = \alpha_{S\backslash(T\cup T')}$. As $p(C,P) \in \mathcal{L}$, adding up the three inequalities we obtain:

$$\alpha_T + \alpha_{T'} + \alpha_{S\backslash(T\cup T')} \geq 1. \tag{6}$$

Finally, as $\alpha_{S\setminus(T\cup T')} = \left(1-\alpha_{(T\cup T')}\right)$, (6) implies $\alpha_T + \alpha_{T'} \geq \alpha_{(T\cup T')}$. As the choice of T and T' was arbitrary, this suffices to prove sub-additivity.

Just to conclude with necessity, notice that candidate stability implies that for any $P \in \mathcal{P}^{\mathcal{V}}$, $p(C,P) \in \bigcap_{b\in\mathcal{C}} L_p(b,C,P)$. ■

5 Concluding Remarks

A few comments on possible extensions to this work are in order.

5.1 Overlap between Voters and Candidates

This work only covers the case in which the sets of voters and candidates are disjoint. It is evident that in many real life public decision processes the candidates are also voters. Nevertheless, many problems arise when one tries to model the participation of the candidates also as voters, both in probabilistic as in deterministic environments.[16] Firstly, unanimity and efficiency lose their bite when candidates can vote, since they are always supposed to support their own election. Hence, it becomes necessary to introduce stronger versions of unanimity that do not take into account the self-preference of the candidates. But the main problem that we face is to attain clear implications of candidate stability. For instance, it is possible to construct degenerate candidate stable voting procedures by selecting the worst candidate of the candidate who decides to withdraw. A possibility is to impose "*ad hoc*" stability conditions (namely Lemma 1) in the same fashion of Ehlers *et al.* [6] and Eraslan *et al.* [7], but then the strategic interpretation of the framework would not be clear. If we impose Lemma 1 and allow the candidates to vote, we would obtain the following result, any probabilistic voting procedure satisfying Lemma 1 and a strong unanimity condition provided by DJL is a *random dictatorship* in which no candidate have veto power.

We want to remark that in the case in which candidates can vote, we cannot use the results in PP to get the characterization. When candidates are allowed to vote, voters' preferences are restricted by the self-preference of the candidates, while in PP only unrestricted strict preferences

[16] See Section 4 in DJL and Section 5 in Rodríguez-Álvarez [14].

are considered. Nevertheless, we can follow the reasonings in the proof of Theorem 2 and to extend them to variable agendas applying a result of Rodríguez-Álvarez [14], which does cover the case of voting candidates.

5.2 Relaxing Unanimity

Unanimity plays a crucial role in the proofs in DJL and Rodríguez-Álvarez [14], and also in our results. However it would be interesting to know what kind of candidate stable probabilistic voting procedures are ruled out by its requirements. As we have already noted, candidate stability has no bite when only two candidates can be elected. Probabilistic combinations of single-valued candidate stable voting procedures are candidate stable probabilistic voting procedures. Hence, we can construct non-unanimous candidate stable probabilistic voting procedures by mixing voting procedures with only two alternatives in their range. Many of these probabilistic voting procedures are by no means interesting. However, other attractive candidate stable but non-unanimous probabilistic voting procedures can be constructed in this fashion, as the next example shows.

Example 4. *Let $C \in 2^{\mathcal{C}} \backslash \{\varnothing\}$ and define for all $a, b \in \mathcal{C}$, for all $i \in \mathcal{V}$, and for all $P \in \mathcal{P}^{\mathcal{V}}$ the function $s_i^a : (C \backslash \{a\}) \times 2^{\mathcal{C}} \backslash \{\varnothing\} \times \mathcal{P} \to \{0,1\}$, in the following way,*

$$s_i^a(b, A, P) = \begin{cases} 1 & \text{if } aP_ib \text{ and } a \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Now let the probabilistic voting procedure $\tilde{p}$ be such that for all $A \in \{C, \{C \backslash \{b\}\}_{b \in C}\}$, for $a \in C$ and for $P \in \mathcal{P}^{\mathcal{V}}$, $\tilde{p}(a, A, P) = \frac{1}{\#\mathcal{V}(\#C-1)} \sum_{i \in \mathcal{V}} \sum_{b \in C \backslash \{a\}} s_i^a(b, A, P)$.

The voting procedure $\tilde{p}$ is clearly candidate stable at C since it is a combination of voting procedures with only two alternatives in their range. It is easy to see that $\tilde{p}$ is a probabilistic version of the Borda count in which the drop of a candidate is equivalent to being the last candidate for all the voters.[17]

[17] This example resembles the "*supporting size*" and "*positional voting*" methods proposed in Barberà [1] for the study of strategy-proof decision schemes.

5.3 Reporting preferences over lotteries

Our analysis admits an additional generalization. It would be natural to assume that voters (and not only candidates) are also expected utility maximizers. So society could use for the public choice the information contained in voters' preferences over lotteries.

Let $\mathcal{D}_{vNM}$ be the set of von Neuman-Morgenstern preferences over $\mathcal{L}$. For the spite of consistency with our previous work, we also assume that voters' preferences over degenerate lotteries that assign positive probability to only one candidate are strict. We denote by $\succsim \in \mathcal{D}_{vNM}^{\mathcal{V}}$ a profile of voters' preferences over lotteries.

We could define a *voting procedure over lotteries* π as a map that for each profile of voters' preferences over lotteries and each configuration of the agenda selects a lottery ($\pi : 2^C \backslash \{\varnothing\} \times \mathcal{D}_{vNM}^{\mathcal{V}} \rightarrow \mathcal{L}$), satisfying that i) non-running candidates cannot be elected ($\pi(A, \succsim) \in \mathcal{L}_A$) and ii) voters' preferences over lotteries containing non-running candidates are irrelevant ($\pi(A, \succsim) = \pi(A, \succsim')$ whenever $\succsim|_A = \succsim'|_A$).

Notice that a voting procedure over lotteries is a voting procedure as defined in DJL since voters express their preferences over the very outcome of the social choice. However, voting procedure over lotteries operate on a restricted domain of preferences. On the other hand a probabilistic voting procedure is equivalent to a voting procedure over lotteries satisfying the following invariance requirement, for any two profiles of voters' preference that coincide in their restriction to the set of degenerate lotteries, the result of the social choice must be the same.

It is clear that the definitions of exit (candidate) stability and unanimity can be straightforwardly translated to this general framework. Surprisingly, allowing for more flexibility in the voting procedure to meet voters' preferences does not create new possibilities. Hence, we can state a more general version of our previous Theorem 1.

Theorem 4. *A voting procedure over lotteries π is unanimous and exit stable if and only if π is a random dictatorship.*

We provide here a sketch of the proof, a complete proof is available from the author. It is easy to see that in this general framework Lemma 1 and Lemma 2 remain valid. However, we cannot address directly to the Theorems in PP and Rodríguez-Álvarez [14], since their results are obtained in the restricted framework in which voters report their preferences over candidates. Thus we need to employ more involving arguments.

In the same fashion we did for probabilistic voting procedures, for any exit stable voting procedure over lotteries π, we can define an auxiliary voting correspondence v_π in the natural way. For any $a \in C$, $A \in 2^C \backslash \{\varnothing\}$ and $\succsim \in \mathcal{D}^{\mathcal{V}}_{vNM}$, $a \in v_\pi (A, \succsim)$ if and only if $\pi (a, A, \succsim) > 0$. As exit stable voting procedures over lotteries are regular, the auxiliary voting correspondence v_π is rationalizable by a quasiorder.[18] Moreover, when only two candidates are at stake, *ii)* in the definition of voting procedure over lotteries implies that v_π only uses the information contained in the voters' preferences over single candidates.[19] This fact allows us to apply Mas-Colell and Sonnenschein [11] version of Arrow's Theorem and prove that there is a set of voters holding veto power over v_π, and therefore over π. Then, the arguments in the proof of our Theorem 2 imply that this distribution of the veto power over π is, in fact, additive and any exit stable and unanimous voting procedure over lotteries is indeed a random dictatorship.

In conclusion, the implications of exit stability and candidate stability do not change in the general framework. The results are the same and the proofs follow from similar (although more cumbersome) arguments. As the general framework does not provide new insights into the problem, we have preferred to highlight the less general one and follow more closely the original set up of DJL.

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[18] See the preliminaries of the proof of Theorem 4 in Rodríguez-Álvarez [14].

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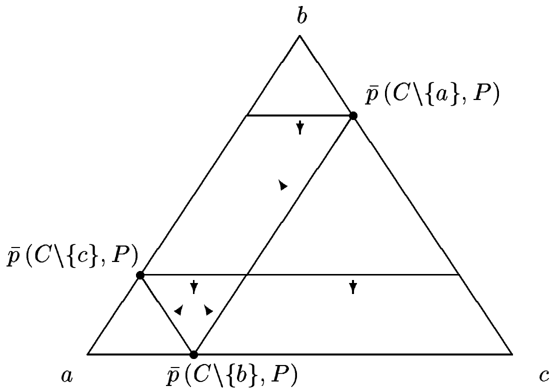

Figure 1: Example 1: $\bar{p}\,(C,P)$