HOTELLING TAX COMPETITION

Myrna Wooders
And
Ben Zissimos

No 668

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

THE UNIVERSITY OF WARWICK

Hotelling Tax Competition[1]

Myrna Wooders
University of Warwick

Ben Zissimos
University of Birmingham

Preliminary draft: June 2002

This draft March 2003

ABSTRACT: This paper shows how competition among governments for mobile firms can bring about excessive differentiation in levels of taxation and public good provision. Hotelling's Principle of Minimum Differentiation is applied in the context of tax competition and shown to be invalid. Instead, when an equilibrium exists, differentiation of public good provision is maximized. Non-existence of equilibrium, which can occur, is a metaphor for intense tax competition. The paper also shows that, to some extent, perfect tax discrimination presents a solution to the existence problem created by Hotelling tax competition, but that the efficiency problem of Hotelling tax competition is exacerbated.

KEYWORDS: amenity competition, Hotelling, limit tax, perfect tax discrimination, single peaked profit function, tax competition.

JEL CLASSIFICATION NUMBERS:

CORRESPONDENCE: m.wooders@warwick.ac.uk, b.c.zissimos@bham.ac.uk.

[1] We are especially grateful to Amrita Dhillon for useful comments and conversations about this paper. We also thank seminar participants at a European Science Foundation Workshop held at Paris 1 in June 2002. Financial support from the ESRC's Evolving Macroeconomy Programme is gratefully acknowledged.

1 Introduction

Under the conventional view of 'government as a Leviathan', interjurisdictional competition has come to be thought of as useful, in that it constrains governments' self-serving activities. The view has been expounded by Brennan and Buchanan (1980), among others, who say that " ... intergovernmental competition may be constitutionally 'efficient', regardless of the more familiar considerations of interunit spillovers examined in the orthodox theory" (p.185). This thinking applies conventional wisdom about the beneficial effects of competition between firms to the case where (Leviathan) governments behave in monopolistic fashion, using the policy variables under their control to maximize the rents to office. Yet the empirical literature remains unable to find conclusive support for this view (see, for example, Oates 1985). The problem may be that this conventional wisdom is based on a standard model, where the focus is on competition over the price of a single homogeneous good or public good. Just as firms may compete over product characteristics as well as price, governments may compete over amenities as well as taxes.

The present paper puts forward the idea that Hotelling's (1929) model can be adapted to understand why competition between Leviathan governments does not promote efficiency. In his classic article, Hotelling (1929) called into question the extent to which competition promotes efficiency when firms compete not just over prices but over product characteristics as well, and when consumers' preferences for product characteristics vary. We question, along parallel lines, the extent to which competition promotes efficiency when governments compete not just over taxes but over levels of amenity provision, and when firms' preferences for levels of amenity provision vary.[2] Thus, our argument provides an explanation of why the empirical literature has remained inconclusive. While a number of papers in the tax competition literature have aspects of Hotelling's model, our paper represents the first occasion on which, to our knowledge, Hotelling's model has been adapted to think about competition in amenities and taxation.[3]

[2]We use the term 'amenity' because the usual attributes of a 'public good,' namely non-excludability and non-rivalry, are not features of the goods that governments provide in our analysis. We refer to firms' 'preferences' rather than firms' technologies to emphasise that each firm has a clearly defined *preferred* or *ideal level of amenity provision* from which the actual level can vary.

[3]We are not the first to model interjurisdictional competition in tax and spending levels between Leviathan governments as a two stage game; this approach has been taken previously by Edwards and Keen (1996) among others. Devereux, Lockwood and Redoano's (2002) model is similar to ours in that

A key element of our analysis, new in the field of tax competition, is that firms have diverse technological requirements for levels of amenity provision. Suppose, for example, that the amenity in question is a legal system. It is generally agreed that some type of legal system will benefit a firm in its production activities and in bringing goods to market. But the ideal level of coverage differs across firms and certainly across industries. One firm's necessary legal protection is another's excessive red tape.

In broad terms, some firms within an industry operate with much less input of government provided public amenities than others. Take firms in the apparel and clothing industry as an example. Those that produce designs at the cutting edge of fashion rely more heavily on government provided amenities such as intellectual property protection, the availability of highly trained staff, and good communications networks to reach their rarefied clientele. At the other end of the spectrum are firms turning out clothing using already established patterns and brand images, for example firms producing counterfeit Levis jeans. For such firms, arguably, the more lax the levels of intellectual property protection the better. Moreover, they may have limited need of highly trained staff, and basic communications may be sufficient.

In the previous literature, where all firms tend to have the same technological requirements for amenities, the forces of competition tend to push all governments in the same direction.[4] With technological diversity among firms, it is not clear whether competitive forces will act similarly to push all governments in the same direction, or whether they will be pushed apart. Hotelling's Principle of Minimum Product Differentiation predicts that governments will provide amenities at the same (inefficient) level. However, research by

firms' preferences for public good provision are captured by their location on an interval of the real line. But in their model, location captures the cost of relocation to another country rather than a "preferred level" of amenity provision that we have in our model.

[4]Situations where competition tends to push all governments away from efficiency are studied by Gordon and Wilson (1986), Wildasin (1988), Wilson (1986), Wooders, Zissimos and Dhillon (2002) and Zodrow and Miezkowski (1986). In a broarder context, Gordon and Wilson (1999) examine how the benefits derived by government officials from the size of the tax base can affect the design of the tax system itself. Situations where competition tends to promote efficiency are studied by Boadway, Cuff and Marceau(2002), Boadway, Pestieau and Wildasin (1989) Wildasin (1989), Wooders (1985) and Wooders, Zissimos and Dhillon (2002) among others. Oates and Schwab (1988) show that majority rule can select the efficient outcome when there is interjurisdictional compeition for mobile resources. Besley and Smart (2001) argue that the issue of whether tax competition raises or lowers efficiency depends on whether policians are more likely to be benevolent or rent-seeking. Gordon and Wilson (2002) show that efficiency is promoted by competition when 'officials benefit by taking a smaller piece from a larger pie'. See Wilson (1999) for a comprehensive review of the earlier literature.

d'Aspremont, Gabszwicz and Thisse (1979) has called into question Hotelling's Differentiation result. Extending the intuition arising from their results on competition between firms to competition between governments suggests that competition might instead *maximize* the differentiation between governments' levels of amenity provision. Demonstrating this constitutes one of the main contributions of our paper.

Before considering our equilibrium analysis, we explain in a bit more detail how our model compares to Hotelling's original work. In the classic Hotelling model, consumers are located on a beach. Two ice-cream sellers chose their locations on the beach to maximize sales. Each consumer has inelastic unit demand for a *single unit* of ice cream and the only issues affecting utility are the price that the consumer has to pay for an ice-cream and the distance that he has to walk to buy it. Thus, each consumer maximizes utility by purchasing ice cream from the seller from whom the 'delivered price', including the cost of going to get the ice-cream, is the lowest.

In our model, amenity space corresponds to the beach. The further to the right that a firm is located on the interval, the higher is its preferred level of amenity provision. While Hotelling's ice cream sellers choose where to locate on the beach, in our model each government chooses a level of amenity provision in its jurisdiction. By locating within a jurisdiction, each firm is provided with the level of amenities provided by that jurisdiction. As in Hotelling's original paper, each firm is able to sell a single unit. So the only issues affecting profits in our model are the tax that the firm has to pay and the difference between the firm's ideal level of amenity provision and the level actually provided in the jurisdiction where it locates. We refer to this difference between the firm's ideal level of amenity provision and the level actually provided by the government as the *degree of amenity mismatch*. The firm maximizes profits by locating in the jurisdiction where the cost of obtaining the amenity is lowest, given taxes in each jurisdiction and the degrees of amenity mismatch.

Of course, it would not be satisfactory simply to re-label Hotelling's (1929) model using the governments' variables instead of firms' variables and so on. A government's location is associated with its cost of amenity provision. In the conventional Hotelling set up, by contrast, costs of sellers are exogenous and are not linked to their location. (Applying our model to Hotelling's beach setting, it would be as if the beach gets hotter towards one end than the other, increasing a seller's costs to keep the ice cream cool.) This apparently minor modification to the set-up of Hotelling's model leads to some quite far reaching changes in its analytical properties.

The stages of the game in our model correspond to standard Hotelling analysis as well. In the first stage governments simultaneously choose the levels of amenity provision. In the second stage, after having observed each others' levels of amenity provision, governments set taxes. Of course, this ordering of events is by no means the only possible, and alternatives may well affect the outcome.[5] As Kreps and Scheinkman (1983) argue in their study of firm behavior, the appropriateness of the set-up, or the game context as they call it, is essentially an empirical matter. Certainly, it seems reasonable to argue that governments first put in place the capacity for amenity provision in the same way that firms set up the capacity for production at the first stage. Then in the second stage they announce taxes in the same way that firms announce prices.[6]

Aspects of our equilibrium analysis of our model carry over from d'Aspremont et al (1979) and Kreps and Scheinkman (1983). First, when equilibrium exists then, as in d'Aspremont et al, differentiation between governments in the level of amenity provision is *maximized*, contrary to the suggested prediction of Hotelling's original analysis. Given the adaptations of our model to a policy setting, however, the interpretation is different to the outcome analyzed by d'Aspremont et al. When differentiation is maximized, this implies that one government supplies no amenities at all whilst the other government supplies amenities at a maximal level.

In equilibrium governments make positive rents, as under Cournot competition, as opposed to zero rents, as under Bertrand competition. The result is particularly striking for the jurisdiction that supplies no amenities at all even though it levies a positive tax. This arises as a result of the monopolistic power that each government has over location within its jurisdiction. Each firm must have a jurisdictional location in order to produce, and the government of that jurisdiction is able to exploit its resultant power when setting

[5] In principle taxes could be set before amenity levels, both could be set at the same time, one government could behave as a Stackelberg leader at each stage and so on.

[6] In a wider setting, beyond the context of our model, governments have the power to tax citizens first and then spend the revenue on public services. But multinational firms can be thought of as more like customers, choosing to locate in a jurisdiction only once the amenity is available for use there.

taxes.

Recent research has drawn attention to the persistent differences between what have come to be known as the core and the periphery of Europe. The core includes Benelux, France, Germany and Italy. The periphery includes Spain, Portugal, Ireland and Greece. For example, Baldwin and Krugman (2000) show how significant differences in taxes, and therefore amenity provision, have persisted over the last thirty five years or so, even as capital markets have become more integrated.[7] Stylistically, the core of Europe could be associated with the high tax high amenity providing government of our model and the periphery could be associated with the low tax low amenity providing government. Our equilibrium prediction that differentiation between levels of amenity provision is maximized provides a way of understanding why these observed differences between the European core and periphery have persisted.

To fix ideas, return to the example of the clothing and apparel industry. Our analysis may suggest that the forces of competition drive governments in the European core to over-provide amenities in order to attract (or retain) the companies of haute couture, that have a preference for a relatively high level of amenity provision. Given that a government in the European periphery provides amenities at a relatively low level (none at all in this stylized setting) and sets taxes relatively low, a government in the core cannot do any better by mimicking the periphery government. At the same time, the amenities offered by core governments are not sufficiently important to the production technologies of more standard clothing producers, and it is not worth paying the higher taxes of the core in order to be able to locate there.

It is a possibility in our framework, however, that an equilibrium does not exist. When firms are highly responsive to a government's efforts to attract them to its jurisdiction by changing its level of amenity provision then this situation arises. Firms are more responsive to change when a move away from their ideal level of amenity provision incurs a relatively high cost. Non-existence of equilibrium in this present setting is a formal metaphor for intense tax competition. No equilibrium level of taxation exists at which

[7] The theoretical model presented by Baldwin and Krugman (2000) motivates persistent differences in taxation and amenity provision between the core and periphery by allowing the core to move first in the policy setting game. First mover advantage gives them an incentive to act as Stackelberg leaders, setting high taxes and providing a high level of amenities.

governments stop undercutting each other in tax levels.[8]

In light of the equilibrium existence issue raised by the foregoing analysis, perfect tax discrimination is analyzed to examine the extent to which it provides a solution. As with perfect price discrimination, where firms can tailor prices to individual consumers, under perfect tax discrimination governments can tailor taxes to individual producers. One interpretation is that governments are able to offer tax breaks from a uniform schedule to firms in order to attract them to the jurisdiction.[9] Bhaskar and To (2002) show that the issue of equilibrium existence in the Hotelling model is completely resolved under perfect price discrimination. In our model we find that allowing governments to discriminate perfectly in setting taxes only partially resolves the equilibrium existence problem. There is a larger range of values for which the cost of amenity mismatch supports an equilibrium. But even under perfect tax discrimination, if the cost of amenity mismatch is relatively high then tax competition is so intense that the system does not settle down to an equilibrium.

Finally, under conditions where equilibrium exists, efficiency implications of the respective regimes are compared. The same inefficiency exists under Hotelling tax/amenity competition with uniform taxes as under the conventional Hotelling model analyzed by d'Aspremont, Gabszwicz and Thisse (1979). Product differentiation is maximal and therefore excessive. Research by Spence (1976) (in the context of firms) suggests that giving governments more power to discriminate between firms in terms of the taxes they are charged will increase and possibly maximize efficiency. Bhaskar and To (2002) show that this reasoning carries over to the original Hotelling framework of firm location and production. But we find that for our model efficiency loss is *worse* under perfect tax discrimination. In equilibrium, both governments offer no amenities at all. This exerts a high efficiency loss on firms that have a high public good requirement, and leads to a lower aggregate level of efficiency. There is a key difference in Bhaskar and To's analysis

[8] At first sight, this appears to imply that rents fall to or below zero. This is not the case. As shown by d'Aspremont et al for prices, no equilibrium exists when a small reduction in taxes is sufficient to attract all firms to the jurisdiction. Then governments keep responding to each other's tax plans with smaller and smaller but unending tax reductions.

[9] Earlier research by Bond and Samuelson (1986), Black and Hoyt (1989), Haaparanta (1996) and King, McAfee and Welling (1993) model situations where governments offer some firms more favourable treatment than others but they either model competition for a single firm or assume firms' technological requirements for amenities are identical.

of firms. In their setting, each firm has the same fixed level of cost. In our analysis, recall that governments' costs depend on their level of amenity provision. Under perfect tax discrimination, the higher-amenity-providing government looses out to the lower one because of the higher cost of provision. This creates a unilateral incentive to deviate from any relatively high level of amenity provision, bringing about a 'race to the bottom' of taxes and amenity provision.

The paper proceeds as follows. Section 2 sets out the basic model. Sections 3, and 4 examine Hotelling tax/amenity competition, looking for existence of subgame perfect equilibrium under uniform taxation and perfect tax discrimination respectively. Section 5 then compares the welfare implications of the regimes when equilibrium exists. Section 6 concludes.

2 The Model

We adapt Hotelling's model to the problem of tax competition. The governments of two countries, A and B, compete over taxes and the level of amenity provision in attempting to persuade firms to locate in their jurisdictions. These governments are assumed to be Leviathans, maximizing the rents to office through amenity provision. There is a continuum of firms on a (non-empty) interval $s \in [0, z]$.[10] The position (fixed in technology space) of each firm in the interval $s \in [0, z]$ reflects its ideal level of amenity provision to facilitate production.

The location on the interval $[0, z]$ of the two governments A and B is given by variables a and b respectively. The variable a measures the distance from 0 and b measures the distance from z; $a + b \leq z$, $a \geq 0$, $b \geq 0$. The location of the government determines the level of amenity provision to each firm in the jurisdiction; a to each firm in Jurisdiction A and $(z - b)$ to each firm in Jurisdiction B. The tax on the firm positioned at s is τ_{As} if the firm locates in Jurisdiction A and τ_{Bs} if it locates in Jurisdiction B.

In conventional Hotelling fashion, each firm is able to sell a single unit and to charge

[10] This could be generalised so that there are a (uniform) number of firms at each point on the interval, but this would not add insight.

price $p = d$. The cost function for the firm at $s \in [0, z]$ is given by

$$c_s = \begin{cases} c + \tau_{As} + k\,|s - a| & \text{if the firm locates in Jurisdiction } A \\ c + \tau_{Bs} + k\,|s - (z - b)| & \text{if the firm locates in Jurisdiction } B. \end{cases}$$

If the firm at s locates in A, for example, it must pay private cost c, and tax τ_{As}. The firm's position s indicates its *ideal level of amenity provision.* The *degree of amenity mismatch* of the firm positioned at s is given by the distance of the firm from the location of the government. For example, if the firm locates in A then the degree of amenity mismatch is given by $|s - a|$. The impact on costs of a divergence from this ideal level of amenity provision would then be captured by the term $k\,|s - a|$, where k parameterizes the impact of the degree of amenity mismatch on costs. We refer to k as the *cost of amenity mismatch* for short. Firm profits are given by $\pi_s \equiv p - c_s$. To focus the analysis on location decisions, it will be assumed throughout that p is high enough to ensure that all firms make positive profits.

The model described above is illustrated in Figure 1. The figure shows the set of firms $s \in [0, z]$. The locations of governments A and B at points a and b are also pictured. The point $\hat{s}$ shows the position of the marginal firm choosing to locate in Jurisdiction A. The firm at $\hat{s}$ is indifferent between Jurisdiction A and B because it makes the same profits in either.

To summarize, in terms of their technological requirements for amenity provision, firms' positions are fixed, but firms are able to pick their preferred jurisdiction to maximize profits. Each government, on the other hand, is able to pick its level of amenity provision but obviously its jurisdiction (A or B) is fixed.

3 Uniform Taxation

Under a *uniform tax game*, each government is able only to set a uniform tax on the firms that choose to locate in its jurisdiction. Government A sets a tax $\tau_{As} = \tau_A$ and makes rents of $\tau_A - a$ on each firm in its jurisdiction while Government B sets a tax $\tau_{Bs} = \tau_B$ and makes rents of $\tau_B - (z - b)$ on each firm in its jurisdiction. It is a condition of equilibrium that $\tau_A - a \geq 0$. The same condition applies to Government B; $\tau_B - (z - b) \geq 0$.

Given that a and b measure the distances of governments A and B from 0 and z

respectively, and that $a + b \leq z$, it must be the case that $a < \hat{s} < b$. Then

$$-\tau_A - k\,|s - a| = -\tau_B - k\,|s - (z - b)|$$

Hence

$$\hat{s}\,(\tau_A, \tau_B) = \frac{\tau_B - \tau_A}{2k} + \frac{(z - b + a)}{2}.$$

A firm may be closer to one government, say Government A, in terms of its degree of amenity mismatch; $|s - a| < |s - (z - b)|$. But if the net cost of public good procurement is sufficiently low, the firm may choose to locate in Jurisdiction B, accepting a higher degree of amenity mismatch; formally, this holds when $-\tau_B - k\,|s - (z - b)| < -\tau_A - k\,|s - a|$. Thus if it could set $\tau_B < \tau_A$ by a sufficiently wide margin, Government B could attract any firm $s \in [0, z]$.

The solution to the governments' problems, the levels of amenity provision and the taxes that they set, can now be determined in the outcome of a game. The two governments, A and B, play respective pure strategies $\tau_A \in \mathbb{R}_+$ and $\tau_B \in \mathbb{R}_+$.[11] Payoffs are given by the 'rents to office' which are defined by the following rent functions:

$$r_A\,(\tau_A, \tau_B) =$$
$$\begin{cases} z\,(\tau_A - a) & \text{if } \tau_A < \tau_B - k\,(z - a - b) \\ \frac{1}{2}\,(z + a - b)\,(\tau_A - a) - \frac{1}{2k}\,(\tau_A - a)\,\tau_A + \frac{1}{2k}\,(\tau_A - a)\,\tau_B & \text{if } |\tau_A - \tau_B| \leq k\,(z - a - b) \\ 0 & \text{if } \tau_A > \tau_B + k\,(z - a - b) \end{cases}.$$

$$r_B\,(\tau_A, \tau_B) =$$
$$\begin{cases} z\,(\tau_B - (z - b)) & \text{if } \tau_B < \tau_A - k\,(z - a - b) \\ \frac{1}{2}\,(z - a + b)\,(\tau_B - (z - b)) - \\ \frac{1}{2k}\,(\tau_B - (z - b)) + \frac{1}{2k}\,(\tau_B - (z - b))\,\tau_A & \text{if } |\tau_A - \tau_B| \leq k\,(z - a - b) \\ 0 & \text{if } \tau_B > \tau_A + k\,(z - a - b) \end{cases}.$$

If $\tau_A < \tau_B - k\,(z - a - b)$ then Government A attracts all firms to locate in Jurisdiction A and it makes overall rents of $z\,(\tau_A - a)$; see the first line on the right hand side of the rent function $r_A\,(\tau_A, \tau_B)$. If Government A sets $\tau_A > \tau_B + k\,(z - a - b)$ then no firm

[11] It will be assumed throughout that mixed strategies in tax rates are not available to governments. This is generally deemed to be an acceptable assumption in the applied literature on policy setting in a perfect information environment. Intuitively, it would not be regarded as reasonable for a government to announce a policy of randomising over tax rates. Admittedly, there may be more complex tax setting environments in which mixed strategies would make more sense. Developments in that direction are left for further research.

finds it profitable to locate in Jurisdiction A and there are no rents to be made from office there; see the last line on the right hand side of $r_A(\tau_A, \tau_B)$. Over the *firm sharing interval*, $|\tau_A - \tau_B| \leq k(z - a - b)$, some firms locate in each of the jurisdictions. Then rents for Government A are given by $r_A(\tau_A, \tau_B) = (\tau_A - a)\hat{s}$, the reduced form of which is given in the middle line on the right hand side of $r_A(\tau_A, \tau_B)$.

The 'rent function' of Government A is shown in Figure 2 for a fixed value $\overline{\tau}_B$. It shows two discontinuities, which occur at the taxes $\tau_A = \tau_B - k(z - a - b)$ and $\tau_A = \tau_B + k(z - a - b)$. At each discontinuity, all firms are indifferent between locating in either of the two jurisdictions. This property of the pay-off function, that it has two discontinuities, is familiar from the previous literature on stability in Hotelling's model (see d'Aspremont, Gabszwicz, and Thisse 1979, for example).

It is clear that $r_A(\tau_A, \tau_B)$ is linear in τ_A for $\tau_A < \tau_B - k(z - a - b)$ and equal to zero for $\tau_A > \tau_B + k(z - a - b)$. To see that $r_A(\tau_A, \tau_B)$ is strictly concave over the firm sharing interval, note that $\partial^2 r_A(\tau_A, \tau_B)/\partial\tau_A^2 = -1/k$ over the interval $|\tau_A - \tau_B| \leqslant k(z - a - b)$. The same holds for $r_B(\tau_A, \tau_B)$.

Amenity provision and tax setting is modelled as a two stage game. In the first stage, the governments A and B simultaneously determine their levels of amenity provision. In the second stage, they set taxes. Once the governments' decisions have been taken, firms take taxes and amenities as given and choose their geographical locations (ie, A or B) to maximize profits. Each of the two stages constitutes a subgame for which it is possible to determine whether or not there exists a Nash equilibrium. Then we say that there exists a *subgame-perfect Nash equilibrium* if the players' strategies constitute a Nash equilibrium in every subgame. It follows that if in either period there exists no Nash equilibrium in pure strategies then there is no subgame perfect Nash equilibrium (in pure strategies). We identify conditions on the existence of a subgame perfect Nash equilibrium of this game.

3.1 Stage 2: Taxes

The purpose of this section is to solve for Stage 2, where the location of the two governments is taken as fixed at distances a and b from the ends of the interval $[0, z]$ (ie at distances a from 0 and b from z respectively). As we shall see, when a and b are 'too close' an equilibrium fails to exist.

For given locations a and b, a strategy τ_A^* of Government A is a *best response tax* against a strategy τ_B when it maximizes $r_A(\tau_A, \tau_B)$ on the whole of $\mathbb{R}_+$. A *Nash equilibrium in taxes* is a pair (τ_A^*, τ_B^*) for which (i) τ_A^* is a best response to τ_B^* and vice-versa (ii) $\tau_A^* \geq a$ and $\tau_B^* \geq z - b$.

By standard results, if the rent functions were everywhere continuous and concave, then existence of a unique best response would be guaranteed. Because the rent function for each government is discontinuous, the usual first and second order conditions cannot be used to find best responses. However, it will be possible to show that when a Nash equilibrium does exist it is unique. Moreover, the tax choice of each jurisdiction maximizes its rents, and maximal rents are given by the maximum of the rent function on the firm sharing interval $|\tau_A - \tau_B| \leq k(z - a - b)$; see Figure 2.

The first step is to solve for the tax that maximizes rent on the firm sharing interval.

Lemma 1. *Assume governments play a uniform tax game. For given τ_B, the unique tax that maximizes $r_A(\tau_A, \tau_B)$ on the firm sharing interval is*

$$\tau_A(\tau_B; a, b, k, z) = k\left(\frac{a + \tau_B}{2k} + \frac{(z + a - b)}{2}\right).$$

For given τ_A, the unique tax τ_B that maximizes $r_B(\tau_A, \tau_B)$ on the firm sharing interval is

$$\tau_B(\tau_A; a, b, k, z) = k\left(\frac{(z - b) + \tau_A}{2k} + \frac{(z - a + b)}{2}\right).$$

If $\tau_A(\tau_B; a, b, k, z)$ and $\tau_B(\tau_A; a, b, k, z)$ are set simultaneously, then they can be solved for simultaneously to obtain:

$$\tau_A(a, b, k, z) = \frac{1}{3}(2a + (z - b) + (a - b)k + 3kz);$$

$$\tau_B(a, b, k, z) = \frac{1}{3}(2(z - b) + a + (b - a)k + 3kz).$$

As the rent function is strictly concave on the firm sharing interval, each government has a unique maximizing tax on that interval, taking the tax set by the other government as given. From the positive sign that the tax of the other government takes on the right hand side, it is clear that taxes are strategic complements.

The second part of the result says that when both governments set $\tau_A(\tau_B; a, b, k, z)$ and $\tau_B(\tau_A; a, b, k, z)$ simultaneously, each can be expressed strictly in terms of model parameters; $\tau_A(a, b, k, z)$ and $\tau_B(a, b, k, z)$. Of course, if this is the case then $\tau_A(\tau_B; a, b, k, z)$ and $\tau_B(\tau_A; a, b, k, z)$ are mutual best responses and constitute a Nash equilibrium point. This will only be the case, though, if, given the other government's tax, there is no tax outside the firm sharing interval that yields higher rent.

It is straightforward to check whether the highest payoff is yielded by the rent maximizing tax on the firm sharing interval or some other tax that attracts all firms to the jurisdiction. This check is performed in the next result.

Lemma 2. *Under a uniform tax game, the tax $\tau_A(\tau_B; a, b, k, z)$ that maximizes $r_A(\tau_A, \tau_B)$ on the firm sharing interval $|\tau_A - \tau_B| \leq k(z - a - b)$ is a best response to τ_B if and only if, for any $\tau_B, \varepsilon > 0$,*

$$r_A(\tau_A(\tau_B; a, b, k, z), \tau_B) \geq z(\tau_B - k(z - a - b) - a - \varepsilon).$$

Similarly, the tax $\tau_B(\tau_A; a, b, k, z)$ that maximizes $r_B(\tau_A, \tau_B)$ on the firm sharing interval $|\tau_A - \tau_B| \leq k(z - a - b)$ is a best response to τ_A if and only if, for any $\tau_A, \varepsilon > 0$,

$$r_B(\tau_A, \tau_B(\tau_A; a, b, k, z)) \geq z(\tau_A - k(z - a - b) - (z - b) - \varepsilon).$$

The only meaningful alternative to a best response tax in the firm sharing interval is a best response tax that attracts all firms to the jurisdiction.[12] In the first inequality, $r_A(\tau_A(\tau_B; a, b, k, z), \tau_B)$ gives the maximum rent for Jurisdiction A on the firm sharing interval, and $z(\tau_B - k(z - a - b) - a - \varepsilon)$ gives the rent from setting a tax low enough to attract all firms to A. In the case of Government A, for example, this tax is $\tau_A = \tau_B - k(z - a - b) - \varepsilon$. The second inequality gives a parallel expression for Jurisdiction B. Recall that a firm would accept a higher degree of amenity mismatch if the tax were low enough to make the net cost of pubic good procurement lower. At the tax implied

[12] From Lemma 1, $\tau_A(\tau_B; a, b, k, z)$ and $\tau_B(\tau_A; a, b, k, z)$ are both non-negative. So given that each country has a positive share of firms rents cannot be negative, and raising taxes to the point where no firms are attracted to the jurisdiction can be rejected as a possible best response.

by the right hand side of the inequality all firms, even those which have a smaller degree of amenity mismatch with Government B, would locate in Jurisdiction A because of the more favorable tax. Lemma 2 says that the $\tau_A(\tau_B; a, b, k, z)$ that maximizes rents on the firm sharing interval is a best response tax if and only if no tax $\tau_A = \tau_B - k(z - a - b) - \varepsilon$ exists that yields higher rents.

We are now ready to state conditions on the existence and uniqueness of a Nash equilibrium in the second stage, taking locations a and b, and parameters k and z as given. It will show that an equilibrium of this Stage 2 subgame exists if and only if each government has a best response tax that is on its firm sharing interval.

Proposition 1. *Assume governments play a uniform tax game, and that a and b are fixed on the interval $[0, z]$, with $a + b \leq z$, $a \geq 0$, $b \geq 0$. For $a + b = z$, both governments are at the same location and there exists an equilibrium in which $\tau_A^* = a$, $\tau_B^* = z - b$.*

For $a + b < z$ there exists an equilibrium point if and only if the two following conditions hold:

$$\text{(C1): } r_A\left(\tau_A^*\left(\tau_B^*; a, b, k, z\right), \tau_B^*\right) \geq z\left(\tau_B^* - k(z - a - b) - a - \varepsilon\right) \Leftrightarrow$$
$$\frac{\left((a - b)k + (z - a - b) + 3kz\right)^2}{18k} \geq \frac{z\left(2(a + 2b)k + 2(z - a - b) - 3\varepsilon\right)}{3}$$

$$\text{(C2): } r_B\left(\tau_B^*\left(\tau_B; a, b, k, z\right), \tau_A^*\right) \geq z\left(\tau_A^* - k(z - a - b) - (z - b) - \varepsilon\right) \Leftrightarrow$$
$$\frac{\left((b - a)k - (z - a - b) + 3kz\right)^2}{18k} \geq \frac{z\left(2(2a + b)k - 2(z - a - b) - 3\varepsilon\right)}{3}$$

Whenever it exists, an equilibrium point is determined uniquely by the taxes

$$\begin{aligned}\tau_A^*(a, b; k, z) &= \frac{1}{3}\left(2a + (z - b) + (a - b)k + 3kz\right); \\ \tau_B^*(a, b; k, z) &= \frac{1}{3}\left(2(z - b) + a + (b - a)k + 3kz\right).\end{aligned}$$

The first line of conditions C1 and C2 is familiar from Lemma 2. Here in Proposition 1, however, equilibrium values have been substituted. The Proposition establishes conditions under which the taxes that maximize rents in the firm sharing intervals of each government are mutual best responses. It also shows that if such taxes are not mutual best responses then equilibrium fails to exist.

The second line of C1 and C2 gives conditions for existence and uniqueness in terms of model parameters a, b, k and z. As stated, these reduced form conditions are not transparent. However, in the next section where stage 1 of the game is solved it will become clear that $a = 0$ and $b = 0$ are the only candidates for equilibrium. Checking that C1 and C2 hold having made these substitutions for a and b is straightforward.

The intuition behind Proposition 1 can be understood as follows. First, the situation where $a+b = z$ is directly analogous to a standard model of Bertrand competition, where each government offers the same amenity level. So there exists a Bertrand equilibrium, which is efficient in that neither government makes rents.

Second, in the situation where $a + b < z$, so that governments supply differing levels of amenities, Proposition 1 says that an equilibrium exists if and only if the tax set by each government is in the firm sharing interval. Suppose not. Suppose at the rent maximizing tax, where firms are shared, one government can do better by setting a tax sufficiently low to attract all firms to its jurisdiction. Then the other government has an incentive to undercut the first. The undercutting process continues ad infinitum and equilibrium is never reached. This does not mean that taxes become infinitely negative. The budget surplus condition always holds. As d'Aspremont, Gabszwicz and Thisse (1979) show for firms, only a small tax reduction is needed in such a situation to attract all firms to the local jurisdiction.

Although the basic insight of d'Aspremont et al (1979) carries over the present context of tax competition, the analysis in the present context is more complicated. The additional complications arise because our model allows governments to differ by offering different levels of amenities. The choice of amenity level affects the government's cost of provision. Recall that this is somewhat different from the conventional Hotelling set-up where firms offer a product that is homogeneous in all respects other than the location at which it is supplied. Varying location does not affect a firm's costs in Hotelling's conventional model. In our setting, by contrast, varying location does affect a government's cost of amenity provision. This adds an extra part to the process of solving for equilibrium. Lemma 1 shows that taxes become strategic complements in the firm sharing interval. That is, τ_B enters positively in $\tau_A(\tau_B; a, b, k, z)$ and τ_A enters positively in $\tau_B(\tau_A; a, b, k, z)$. This is different from the analysis of d'Aspremont et al, where there is no strategic substitution or complementarity at all.

Because taxes are strategic complements in the firm sharing interval, conditions C1 and C2 are somewhat less transparent than in d'Aspremont et al (1979). A nice feature of their formative analysis is that each condition is shown to depend in a clear way on the difference between a and b. When a and b are 'too close' equilibrium fails to exist. It is through this route that d'Aspremont et al (1979) introduce their main result; that Hotelling's Principle of Minimum Differentiation fails to hold. Contrastingly, the relationship between a and b in C1 and C2 cannot be discerned so clearly in the present analysis. However, a nice clear alternative demonstration of the present model's failure to exhibit the Principle of Minimum Differentiation will be given in the next section.

3.2 Stage 1: Level of public good provision

We now solve for Stage 1, defining an equilibrium in locations, which determines the level of public good provision by the respective governments. For Government A, the rent function is $r_A(\tau_A, \tau_B)$. Using the equilibrium values $\tau_A^* = \tau_A^*(a, b; k, z)$ and $\tau_B^* = \tau_B^*(a, b; k, z)$ that we derived for Stage 2, the rent function for Government A can be written as follows:

$$r_A(\tau_A^*(a, b; k, z), \tau_B^*(a, b; k, z)) = r_A(a, b; k, z).$$

Similarly, the rent function for Government B can be written as follows:

$$r_B(\tau_A^*(a, b; k, z), \tau_B^*(a, b; k, z)) = r_B(a, b; k, z).$$

A location a^* of Government A is a *best response* against a location b when it maximizes $r_A(a, b; k, z)$ on the whole of $\mathbb{R}_+$. A *Nash equilibrium in locations* is a pair (a^*, b^*) such that a^* is a best response against b^* and vice-versa.

Substituting $\tau_A^* = \frac{1}{3}(2a + (z - b) + (a - b)k + 3kz)$ and $\tau_B^* = \frac{1}{3}(2(z - b) + a + (b - a)k + 3kz)$ into $r_A(\tau_A^*, \tau_B^*) = (\tau_A^* - a)\hat{s}(\tau_A^*, \tau_B^*)$, Government A's problem in Stage 1 of the game can be written as follows:

$$\max_a r_A(a, b; k, z) = \frac{((a - b)k + (z - a - b) + 3kz)^2}{18k}.$$

Similarly, Government B's problem can be written

$$\max_{b} r_B(a, b; k, z) = \frac{((b-a)k - (z-a-b) + 3kz)^2}{18k}.$$

The game played between these two governments has an unconventional but nonetheless appealing form. To demonstrate that the Principle of Maximum Differentiation holds, we will first show that the second derivative of the rent function is everywhere *nonnegative.* This implies that, when the first derivative of the rent function is strictly negative, each government's rents will be maximized by moving as far from the location of the other government (in amenity provision space) as possible.

Lemma 3 shows how the second order condition of the government's problem in the first stage is non-negative.

Lemma 3. *Assume a uniform tax game.*

$$\frac{\partial^2 r_A(a, b; k, z)}{\partial a^2} = \frac{(k-1)^2}{9k}, \quad \frac{\partial^2 r_B(a, b; k, z)}{\partial b^2} = \frac{(k+1)^2}{9k}$$

Lemma 3, along with (C1) and (C2), are used to check that in equilibrium rents to office cannot be increased by changing location.

Proposition 2. *There exists a unique subgame perfect Nash equilibrium in pure strategies of a uniform tax game if and only if* $0 < k \leq \frac{1}{7}$. *If such an equilibrium exists then it is characterized (uniquely) by the point* $a^* = b^* = 0$.

This result shows that an equilibrium exists only if and only if the costs of amenity mismatch are relatively low ($k \leq 1/7$). If an equilibrium exists then differentiation in amenity provision is maximized. (Recall that a measures the distance from 0 and b measures the distance from z.) To see why it is the case, consider the incentives to deviate from the equilibrium $a^* = b^* = 0$. As governments move away from each other they increase the degree of differentiation of the amenity level that they offer. This in turn softens the degree of tax competition that they face, which increases the rents that can be made from any given level of amenity provision. If the costs of amenity mismatch are relatively high ($k > \frac{1}{7}$) then more firms switch to the government that is closer to the centre of the interval, producing a unilateral incentive to deviate from $a = b = 0$. However, if governments have an incentive to deviate from $a = b = 0$ then equilibrium fails to exist. The reason is that as the governments move closer to the centre of the interval, tax competition becomes more intense. That is, the incentive for one government to reduce taxes and in so doing attract all firms to its jurisdiction increases. No equilibrium level exists at which taxes stop falling. Thus, in non-existence of equilibrium we have a formal metaphor for intense tax competition.[13]

Comparing the results obtained here with those of d'Aspremont, Gabszwicz and Thisse (1979), in their earlier analysis, when mismatch costs were linear, a subgame perfect Nash equilibrium failed to exist for all parameter values. D'Aspremont et al were able to demonstrate existence of equilibrium only in an alternative model where mismatch costs were quadratic. In our present model with just a linear framework, we have been able to show that existence of equilibrium or otherwise depends on the cost parameter associated with mismatch k. Quadratic costs are not required to show existence. This difference of model properties arises out of the differences of our model to the standard Hotelling set-up. In our model location affects rents directly through costs. For example, for Jurisdiction A, $r_A(\tau_A, \tau_B) = (\tau_A - a)\hat{s}(\tau_A, \tau_B)$. The analogous expression in the conventional Hotelling set-up would be $r_A(\tau_A, \tau_B) = \tau_A \hat{s}(\tau_A, \tau_B)$. The differences in model behavior are driven by the feature that location affects rents directly through costs.

Given the adaptations of the Hotelling model to our policy context, the interpretation is different to that provided by d'Aspremont et al (1979) as well. In the conventional model, other than location there is no difference between the characteristics of the products being supplied by the two firms. When differentiation is maximized this simply means that the goods are supplied at different locations. Here in the context of this present paper, when differentiation is maximized this implies that one government supplies no amenities at all whilst the other government supplies amenities at a maximal level.

[13] As mentioned in the introduction, this does not mean that taxes become infinitely negative. The budget surplus condition always holds. As d'Aspremont et al (1979) show for firms, only a small tax reduction is needed in such a situation to attract all firms to the local jurisdiction.

4 Perfect Tax Discrimination

In a *perfect tax discrimination game*, each government is able to set an individualized tax for each firm $s \in [0, z]$. Each government is able to set an individual tax for the firm at s, in the same way as firms that perfectly price discriminate are able to set an individualized price for each consumer. Unlike in the previous section where each government set a single tax which all firms locating in that jurisdiction had to pay, now each government is able to set a different tax for each firm. The two governments A and B then engage in Bertrand competition separately for each firm. In this section, we consider the extent to which perfect tax discrimination resolves the problems of existence of equilibrium under uniform taxation.

Thinking more loosely, there is an alternative interpretation of the perfect tax discrimination game. If there existed a uniform tax schedule in each country then this model of perfect tax discrimination could be seen as capturing the incentive for governments to offer individualized tax breaks to firms in order to attract them to the jurisdiction.

For each firm $s \in [0, z]$, the two governments, A and B, play respective strategies $\tau_{As} \in \mathbb{R}_+$ and $\tau_{Bs} \in \mathbb{R}_+$. The rent functions to competition for this single firm are given as follows:

$$r_{As}(\tau_{As}, \tau_{Bs}) =$$

$$\begin{cases} (\tau_{As} - a) & \text{if } \tau_{As} < \tau_{Bs} + k\left(|(z-b) - s| - |s - a|\right) \\ 0 & \text{if } \tau_{As} > \tau_{Bs} + k\left(|(z-b) - s| - |s - a|\right) \end{cases}.$$

$$r_{Bs}(\tau_{As}, \tau_{Bs}) =$$

$$\begin{cases} (\tau_{Bs} - (z-b)) & \text{if } \tau_{Bs} < \tau_{As} + k\left(|s-a| - |(z-b) - s|\right) \\ 0 & \text{if } \tau_{Bs} > \tau_{As} + k\left(|s-a| - |(z-b) - s|\right) \end{cases}.$$

The rent received by each government when $\tau_{As} - \tau_{Bs} = k\left(|(z-b) - s| - |s - a|\right)$ will be specified presently.

Each of the rent functions has a single discontinuity. An example of $r_{As}(\tau_{As}, \tau_{Bs})$ is shown in Figure 3. For any $\tau_{As} < \tau_{Bs} + k\left(|(z-b) - s| - |s - a|\right)$, the firm finds it profitable to locate in Jurisdiction A. That is, the difference between the costs of amenity mismatch $k\left(|(z-b) - s| - |s - a|\right)$ across the two jurisdictions is more than offset by the difference in the taxes. The government makes rent $\tau_{As} - a$ on the firm at s. If $\tau_{As} > \tau_{Bs} + k\left(|(z-b) - s| - |s-a|\right)$, the difference in taxes more than offsets the difference between the costs of amenity mismatch across the jurisdictions, and the firm locates in Jurisdiction B. Then, obviously, the government makes rents of zero on the firm at s.

The firm is just indifferent between the two jurisdictions at the point $\tau_{As} = \tau_{Bs} + k\left(|(z-b) - s| - |s-a|\right)$. This is the point of discontinuity in $r_{As}\left(\tau_{As}, \tau_{Bs}\right)$ shown in Figure 3. The difference in the costs of amenity mismatch and the difference in the taxes across the two jurisdictions is exactly equal. We need to specify how firm s will decide its location when it is just indifferent between jurisdictions. The following assumption stipulates that either jurisdiction is chosen with probability one half.

A1: *If* $\tau_{As} - \tau_{Bs} = k\left(|(z-b) - s| - |s-a|\right)$ *for* $s \in [0, z]$ *then* s *is indifferent between* A *and* B *and chooses each jurisdiction with probability* $\frac{1}{2}$. *The expected rent for Government* A *is* $\frac{1}{2}\left(\tau_{As} - a\right)$ *and the expected rent for Government* B *is* $\frac{1}{2}\left(\tau_{Bs} - (z-b)\right)$.

Again, as in Section 3, the level of amenity provision and tax setting is modelled as a two stage game. As before, the governments A and B simultaneously determine their levels of amenity provision in Stage 1, and set taxes in Stage 2. Each of the two periods constitutes a subgame for which it is possible to determine whether there exists a Nash equilibrium. Then there exists a *subgame-perfect Nash equilibrium* if the governments' strategies constitute a Nash equilibrium in every subgame. As in the previous section, it follows that if in either period there exists no Nash equilibrium then there is no subgame perfect Nash equilibrium.

4.1 Stage 2: Taxes

As usual, Stage 2 is solved for first, where the location of the two governments is taken as fixed at distances a and b from the ends of the interval $[0, z]$. For given locations a and b and for a given firm $s \in [0, z]$, a strategy τ^*_{As} of Government A is a *best response* against a strategy τ_{Bs} when it maximizes $r_{As}\left(\tau_{As}, \tau_{Bs}\right)$ on $\mathbb{R}_+$. A *Nash equilibrium in taxes for firm* s is a pair $\left(\tau^*_{As}, \tau^*_{Bs}\right)$ for which (i) τ^*_{As} is a best response to τ^*_{Bs} and vice-versa. (ii) $\tau^*_A \geq a$ and $\tau^*_B \geq z - b$.

Let $T_A = \{\tau_{As}\}_{s\in[0,z]}$ be a *tax schedule* for Government A, consisting of one tax for each firm, and similarly let $T_B = \{\tau_{Bs}\}_{s\in[0,z]}$ be a tax schedule for Government B. A pair of tax schedules, T_A^* and T_B^* is a *Nash equilibrium in taxes* if for each $s \in [0, z]$ the pair $(\tau_{As}^*, \tau_{Bs}^*)$ is a Nash equilibrium in taxes for firm s.

The literature on entry deterrence through pricing strategy has had to broach the issue of what constitutes a best response when payoff functions defined by the game are discontinuous and do not have a well defined maximum (in the sense that first derivatives are not equal to zero). This issue carries over to the present context where the payoff function is increasing up to the discontinuity; see Figure 3. In a model of continuous strategy choices, such a payoff function does not have a well defined maximum because, for any strategy chosen by a player, there is always a strategy that yields a slightly higher payoff. Consider, for example, the present setting where any choice of ε implies a tax $\tau_{As} = \tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right) - \varepsilon > 0$, $(\varepsilon > 0)$ and rent $r_{As} = \tau_{As} - a$. Government A could choose a smaller value for ε (whilst still maintaining $\varepsilon > 0$) thereby setting a higher tax and earning higher rent.

Dasgupta and Maskin (1986) provide a way of resolving this issue by defining (discrete) strategy choices over a grid. In such a framework, ε has a smallest value defined by the distance between grid lines. Their approach has gained substantive support in the literature and, in the present setting, has intuitive appeal. Let $\varepsilon > 0$ be thought of as the smallest monetary unit; one cent in the Euro zone or the US and a penny in Canada or the UK, for example. With a smallest money unit, the minimum amount by which one government can undercut the other is well defined as ε. Then $r_{As}(\tau_{As}, \tau_{Bs})$ has a well defined maximum. Strategies can be made continuous by making the distance between grid lines arbitrarily small.[14]

For our purposes, we simply define a 'limit tax' for a firm s as a tax very close to but less than the tax that would make the firm indifferent between the two jurisdictions. To formalize a limit tax, let $\varepsilon > 0$ be given. For a particular firm s, a tax τ_{Bs}, and amenity

[14] A formal game theoretic treatment, along the lines of Dasgupta and Maskin (1986), could be developed for Hotelling Tax Competition. In such an approach, discrete taxes would be defined over a grid, with distance between grid lines equal to ε, and ε would then be allowed to become arbitrarily small. Inclusion of such a derivation would not contribute substatively to the results that we discuss in the present paper. Such a formal treatement of limit pricing by firms has been undertaken by Chowdhury (2002). The price that maximises the payoff as the grid size becomes small is defined as the limit price.

levels a and b satisfying $z - b > a$, the *limit tax for Government A*, $\tau_{As}^{\lim}$, is given by:

$$\tau_{As}^{\lim} = \tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right) - \varepsilon.$$

Analogously, for a particular firm s, a tax τ_{As}, and amenity levels a and b satisfying $z - b > a$, the *limit tax for Government B*, $\tau_{Bs}^{\lim}$, is given by:

$$\tau_{Bs}^{\lim} = \tau_{As} + k\left(|s-a| - |(z-b)-s|\right) - \varepsilon.$$

Notice that the limit tax is not relevant for the case $z - b = a$, where competition between governments is analogous to Bertrand competition in homogeneous products. When setting a limit tax in Stage 2, Government A effectively takes a, b, k, s, z and τ_{Bs}, as given, so we write the limit tax $\tau_{As}^{\lim}$ as a function of ε only; $\tau_{As}^{\lim}(\varepsilon)$. Analogously , for the limit tax of Government B we write $\tau_{Bs}^{\lim}(\varepsilon)$.

The notion of limit tax that we introduce here extends to a tax policy setting the idea of a limit price originally introduced by Bain (1956). Bain suggested that pricing strategies could be used to discourage entry.[15] Bhaskar and To (2002) show that pricing strategies can be used to discourage entry into a market that is defined geographically. A particular firm can supply its nearby market relatively cheaply because it can provide the good in question at relatively low delivery cost. Then the limit price is the highest price the firm can charge without making it possible for other more distant firms to profitably supply the market. For limit pricing to be a best response, profits must be maximized if the firm is the local market's sole supplier.

In the policy setting of this present paper, tax strategies can be used to discourage competition for a particular set of firms defined not in terms of their location but in terms of their degree of amenity mismatch. A particular government can provide an amenity to a firm with a relatively small degree of amenity mismatch at a tax that enables the firm to make relatively high profits; the closer is the level of amenity provision to the firm's ideal the higher are the profits that the firm makes, all else equal. From the point of view of one government, the limit tax is the highest tax that it can set for a firm while making

[15] Spence (1977) re-interprets limit pricing as competition in capacities, where an incumbent accumulates a large capacity and thus charges a low price, deterring entry. Milgrom and Roberts (1982) formulate a model based on informational asymmetry, where an incumbent charges a low price to signal that profits in the market are low.

it impossible for the other government to profitably provide an amenity on more favorable terms. The limit tax then maximizes the rent that can be made.

Using the definitions of limit taxes, we can now characterize the best response for each government in Stage 2.

Lemma 4. *Consider a perfect tax discrimination game and assume A1 holds. Fix* a *and* b *so that* $z - b > a$.

If, for some firm $s \in [0, z]$, $a < \tau_{Bs} + k(|(z-b)-s| - |s-a|)$ *then for* $\varepsilon > 0$ *sufficiently small Government* A*'s unique best response is* $\tau^*_{As} = \tau^{\lim}_{As}(\varepsilon)$. *If* $a \geq \tau_{Bs} + k(|(z-b)-s| - |s-a|)$ *then* $\tau^*_{As} = a$ *is a best response for Government* A.

If, for some firm $s \in [0, z]$, $z - b < \tau_{As} + k(|s-a| - |(z-b)-s|)$ *then for* $\varepsilon > 0$ *sufficiently small Government* B*'s unique best response is* $\tau^*_{Bs} = \tau^{\lim}_{Bs}(\varepsilon)$. *If* $z - b \geq \tau_{As} + k(|s-a| - |(z-b)-s|)$ *then* $\tau^*_{Bs} = z - b$ *is a best response for Government* B.

The first part of the result says that if, from Government A's point of view, the degree of amenity mismatch with a firm at s is small relative to that firm's mismatch with Government B, then it is a best response for Government A to set a limit tax for that firm. Formally, if $a < \tau_{Bs} + k(|(z-b)-s| - |s-a|)$ then $\tau^*_{As} = \tau^{\lim}_{As}(\varepsilon)$. Notice that $\tau_{Bs} + k(|(z-b)-s| - |s-a|)$ is decreasing in the degree of amenity mismatch $|s-a|$, making the condition more likely to hold if s is close to a. For given tax and location of Government B, Government A limit taxes the firm so it just prefers to locate in A. If, on the other hand, $a \geq \tau_{Bs} + k(|(z-b)-s| - |s-a|)$ then Government A can do no better than to set $\tau^*_{As} = a$. Clearly, setting $\tau^*_{As} < a$ would make negative rents. And given that the firm is not attracted to A at $\tau^*_{As} = a$, then it certainly will not find $\tau^*_{As} > a$ more attractive. The second part of the result states that parallel arguments hold for the best response of Government B.

In Lamma 4 and in the following, we mean by '$\varepsilon > 0$ sufficiently small' that the smallest monetary unit is small enough to enable the government that has the smaller degree of amenity mismatch with a given firm to undercut the other government using taxes. That is, we rule out the possibility that one government is closer in amenity space to a firm than the other government but not able to undercut the other on taxes and still make positive rents because the smallest monetary unit is too large. The formal bound on the size of ε is established in the proof.

The best responses determined above are now used to define equilibrium in the next two propositions.

Proposition 3. *Consider Stage 2 of a perfect tax discrimination game, with a and b fixed on the interval $[0, z]$. Assume A1 holds and that $a + b \leq z$, $a \geq 0$, $b \geq 0$. If $k < 1$ then for $\varepsilon > 0$ sufficiently small there exists a unique Nash equilibrium in taxes for this stage of the perfect tax discrimination game. A unique Nash equilibrium in taxes for each firm $s \in [0, z]$ is determined by the following taxes:*
if $a + b = z$,

$$\tau_{As}^{*} = \tau_{Bs}^{*} = a = z - b;$$

if $a + b < z$,

$$\tau_{As}^{*} = \tau_{As}^{\lim}(\varepsilon), \ \tau_{Bs}^{*} = z - b.$$

Proposition 3 can be explained as follows. If $a + b = z$ then we have the standard Bertrand case. If $a + b < z$ then, with relatively low costs of amenity mismatch ($k < 1$), Government A is always able to undercut Government B by offering a lower tax to every firm $s \in [0, z]$.[16] Government A maximizes rents by setting a limit tax. Because the cost of amenity mismatch is relatively low (for $k < 1$), the (lower) limit tax set by Government A is always enough to more than compensate for the larger degree of amenity mismatch.[17]

In the next result we show that if $k \geq 1$ then it is not possible for Government A to undercut Government B for all firms. Even if Government A sets taxes as low as possible, at $\tau_{As} = a$, a set of firms will still be better off locating in B. Therefore, when analyzing the case where $k \geq 1$, it will be helpful to re-introduce the notion of the marginal firm, $\hat{s}$, that is just indifferent between locating in either country. In the perfect tax discrimination game, the definition must be altered to allow for the fact that firms face individualized

[16] Note that this possibility of undercutting depends on the existence of a sufficiently small monetary unit. As a gets arbitrarily close to $z - b$, the smallest monetary unit must become arbitrarily small. But for given a and b, such a smallest monetary unit (ε) can always be found.

[17] The value of ε must be small enough so that Government A can set a tax τ_{As} sufficiently low and still make positive rent $\tau_{As} - a$. An explicit upper bound for the smallest money unit $\varepsilon \in (0, \hat{\varepsilon})$, where $\hat{\varepsilon} = (1 - k)(z - a - b)/2$, is established in the proof.

taxes:

$$\hat{s}\left(\tau_{As}, \tau_{Bs}\right) = \frac{\tau_{Bs} - \tau_{As}}{2k} + \frac{(z - b + a)}{2}.$$

The outcome in Stage 2 of the perfect tax discrimination game with costs of amenity mismatch relatively high are characterized in the following proposition.

Proposition 4. *Consider Stage 2 of a perfect tax discrimination game, with a and b fixed on the interval $[0, z]$. Assume A1 holds and that $a + b \leq z$, $a \geq 0$, $b \geq 0$. If $k \geq 1$ then for $\varepsilon > 0$ sufficiently small there exists a unique Nash equilibrium in taxes for this stage of the perfect tax discrimination game. A unique Nash equilibrium in taxes for each firm $s \in [0, z]$ is determined by the following taxes:*
if $a + b = z$, then

$$\tau^*_{As} = \tau^*_{Bs} = a = z - b, \text{ for } a + b = z \text{ and } s \in [0, z];$$

if $a + b < z$, then

$$\begin{aligned}
\tau^*_{As} &= a,\ \tau^*_{Bs} = z - b \text{ for } s = \hat{s}, \\
\tau^*_{As} &= \tau^{\lim}_{As}(\varepsilon),\ \tau^*_{Bs} = z - b, for\ s \in [0, \hat{s}), \\
\tau^*_{As} &= a,\ \tau^*_{Bs} = \tau^{\lim}_{Bs}(\varepsilon) \text{ for } s \in (\hat{s}, z].
\end{aligned}$$

Proposition 4 works in exactly the same way as Proposition 3, except that Government B is able to limit tax the firms that are towards the upper end of $[0, z]$. Because the cost of mismatch is relatively high, firms towards the upper end of $[0, z]$ find it profitable to locate in B even when Government A sets its lowest possible tax $\tau^*_{As} = a$. Government B maximizes the rents that it extracts from them by setting a limit tax. In fact, Proposition 3 can be thought of as a special case of Proposition 4. In general, we should expect some firms to locate in each country. It is only when costs of amenity mismatch are below $k = 1$ that the government providing the amenity at a relatively low level can undercut the other government to such an extent that it attracts all firms.

Taking Propositions 3 and 4 together, we have seen that a Nash equilibrium exists for all possible values of k in Stage 2 of the perfect tax discrimination game. We close this subsection by making the observation formal.

Corollary 1. *Consider Stage 2 of a perfect tax discrimination game, with a and b fixed on the interval $[0, z]$. Assume A1 holds and that $a+b \leq z$, $a \geq 0$, $b \geq 0$ and $\varepsilon > 0$ sufficiently small. There exists a Nash equilibrium in taxes of the perfect tax discrimination game.*

4.2 Stage 1: Location

We now solve for Stage 1, defining an equilibrium in locations. Let T_A and T_B be tax schedules for Jurisdictions A and B respectively and let $r_A(T_A, T_B) = \int_{s\in[0,z]} r_{As}(\tau_{As}, \tau_{Bs})$ and $r_B(T_A, T_B) = \int_{s\in[0,z]} r_{Bs}(\tau_{As}, \tau_{Bs})$ be the corresponding *overall rent functions.* Using the equilibrium values $\tau^*_{As} = \tau^*_A(a, b, s, \varepsilon; k, z)$ and $\tau^*_B = \tau^*_B(a, b, s, \varepsilon; k, z)$ that we derived for Stage 2, the overall rent function for Government A can be written

$$r_A(T^*_A(a, b, s, \varepsilon; k, z), T^*_B(a, b, s, \varepsilon; k, z)) = r_A(a, b, s, \varepsilon; k, z).$$

Similarly, the overall rent function for Government B can be written

$$r_B(T^*_A(a, b, s, \varepsilon; k, z), T^*_B(a, b, s, \varepsilon; k, z)) = r_B(a, b, s, \varepsilon; k, z).$$

A location a^* of Government A is a *best reply* against a location b when it maximizes $r_A(a, b, s, \varepsilon; k, z)$ on the whole of $\mathbb{R}_+$. A location b^* of Government B is a *best reply* against a location a when it maximizes $r_B(a, b, s, \varepsilon; k, z)$ on the whole of $\mathbb{R}_+$. A *Nash equilibrium in locations* is a pair (a^*, b^*) such that a^* is a best reply to b^* and vice-versa.

First we characterize equilibrium when the cost of amenity mismatch is relatively low; that is, $k < 1$.

Proposition 5. *If $k < 1$ and $\varepsilon > 0$ sufficiently small then there exists a unique subgame perfect Nash equilibrium in pure strategies of the perfect tax discrimination game. Equilibrium is characterized by the point $a^* = 0$, $b^* = z$.*

In the unique equilibrium, neither government provides any amenities.[18] To see the significance of this result, first recall that in the more familiar setting of perfect price discrimination by (private goods producing) firms, costs are exogenously given and in

[18] Recall that b measures the distance from z, so when $b^* = z$ and $a^* = 0$ then both governments provide no amenities.

equilibrium, the price of the last unit sold is equal to its marginal cost (limit pricing) and so the outcome is efficient. A firm's profit is equivalent to its contribution to social welfare, so profit maximization is equivalent to social welfare maximization. But in our model, governments' costs are endogenously determined by their location. From any position where governments are providing a positive level of amenities, Government A makes positive rents by attracting all firms to its jurisdiction while Government B makes zero rents (Proposition 3). Therefore, no government wants to be in the position of Government B. Each government has a unilateral incentive to undercut the other by reducing the level of amenity provision, in turn reducing taxes and attracting all firms to its jurisdiction. Because costs of amenity mismatch are relatively low, any firm can be more than compensated for amenity mismatch through lower taxation. Hence we have a 'race to the bottom' in tax rates and public good provision.

We now move on to consider the situation where amenity mismatch has a 'large' impact on costs; that is, $k \geq 1$. From Proposition 4 we saw that if $k \geq 1$ then, given a and b, some firms locate in each jurisdiction in the equilibrium of Stage 2. We now use the equilibrium taxes from Proposition 4 to solve overall rent functions in locations a and b for Stage 1. The overall rent function $r_A(a, b, s, \varepsilon; k, z)$ is shown to be strictly concave in a and the overall rent function $r_B(a, b, s, \varepsilon; k, z)$ is shown to be strictly concave in b. So from these we obtain candidates for equilibrium points a^* and b^* of Stage 1 of the game in the usual way. But these candidate points are based on the assumption that $a < z - b$. As we shall see, Proposition 6 shows that although b^* maximizes overall rents given $a < z - b$, Government B can make higher rents by setting $z - b \leq a$, presenting an incentive to deviate and undermining existence of equilibrium.

Assume $z - b > a$. Let $a^* \in \arg\max_a r_A(a, b, s, \varepsilon; k, z)$ and $b^* \in \arg\max_b r_B(a, b, s, \varepsilon; k, z)$. Using τ^*_{As} and τ^*_{Bs} from Proposition 4, note that

$$r_A(a, b, s, \varepsilon; k, z) = \int_{s\in[0,z]} r_{As}(\tau^*_{As}, \tau^*_{Bs}) = (a + (\hat{s} - a)/2)(1 + k)(z - a - b).$$

Taking the first derivative and solving for a yields a candidate for a^*:

$$a(b, k, z) = \frac{(k-1)(z-b)}{3k-1}.$$

Observe that for $k \geq 1$ the second derivative is negative – $\partial r_A/\partial a^2 = \frac{1}{2}\left(\frac{1}{k} - 2 - 3k\right) < 0$.

So the objective function is concave. Again, from Proposition 4,

$$r_B(a,b,s,\varepsilon;k,z) = \int_{s\in[0,z]} r_{Bs}\left(\tau^*_{As},\tau^*_{Bs}\right) = \left(b + (z-b-\hat{s})/2\right)(k-1)(z-a-b).$$

Taking the first derivative and solving for b yields a candidate for b^*:

$$b(a,k,z) = \frac{(1+k)(z-a)}{3k+1}.$$

Taking the second derivative, $\partial r_B/\partial b^2 = \frac{1}{2}\left(2+\frac{1}{k}-3k\right) \leq 0$ for $k \geq 1$. So the objective function is concave (weakly for $k=1$). The functions $a(b,k,z) = (k-1)(z-b)/(3k-1)$ and $b(a,k,z) = (1+k)(z-a)/(3k+1)$ are reaction functions and can be solved for simultaneously to obtain a unique crossing point:

$$\begin{aligned} a(k,z) &= \frac{(k-1)z}{4k} \text{ and} \\ b(k,z) &= \frac{(k+1)z}{4k}. \end{aligned}$$

At the points $a(k,z) = (k-1)z/4k$, $b(k,z) = (k+1)z/4k$, each government maximizes its rent, taking as given the location of the other. But also notice that in solving this problem it has been assumed that $a < z-b$. Indeed, $a(k,z) = \frac{(k-1)z}{4k} < \frac{(3k-1)z}{4k} = z-b(k,z)$. But to establish that this is indeed an equilibrium, it must be checked that Government B does not have an incentive to adopt a level of amenity provision $(z-b) \leq a$. It is through the recognition of the possibility that Government B may have an incentive to deviate by setting $(z-b) \leq a$ that we obtain the following surprising result:

Proposition 6. *If $k \geq 1$ then there exists no subgame perfect Nash equilibrium in pure strategies of the perfect tax discrimination game.*

The intuition behind the result is as follows. At $a(k,z) = \frac{(k-1)z}{4k}$, $z-b(k,z) = \frac{(k+1)z}{4k}$, Government A makes higher rents than Government B. The difference in rents when the Governments locate at these positions, and then adopt best response taxes in the second stage is $\frac{z^2}{4}$ in Government A's favour. But because A does so much better, Government B has an incentive to deviate from $b(k,z) = \frac{(k+1)z}{4k}$ by locating in the same position as Government A, $a(k,z) = \frac{(k-1)z}{4k}$, and setting taxes slightly lower than Government A. (Thus B gives some of the additional surplus $\frac{z^2}{4}$ back to firms in exchange for relocation to B.) Jurisdiction B does not need to worry about loosing the firms that, prior to the deviation, located in B because Government B makes more rents from the firms lured away from A. And prior to the deviation, B made zero rents from the firms that it now lures away from A. Thus, the rents that Government B makes under such a deviation are a net gain. This deviation contradicts equilibrium. Moreover, an equilibrium fails to exist because, from any position where $a \neq a(k,z)$, $b \neq b(k,z)$, there would be an incentive to move to these positions. And from these positions there is still an incentive to deviate, as just described. So no equilibrium can exist.

In the light of Corollary 1, the non-existence of equilibrium shown in Proposition 6 comes as a surprise. Corollary 1 shows that an equilibrium exists for all k. However, in Stage 2 of the game a and b are taken as fixed. In addition, it is assumed that $z - b \geq a$. The failure of equilibrium to exist comes about because a government positioned at $z - b$ on the interval has an incentive to deviate by setting a level of amenity provision equal to a and then undercut Government A on the tax. Then Government A has an incentive to deviate itself by changing its location. This possibility could not be accounted for in Stage 2 when locations were taken as fixed.

5 Efficiency

A standard social loss function is used to examine the efficiency implications of equilibrium (when it exists) under the respective regimes. The social loss function is of the form

$$L = \int_{s\in[0,\hat{s}]} k\,|s-a|\,ds + \int_{s\in(\hat{s},z]} k\,|z-b-s|\,ds.$$

This function aggregates the loss of potential profits that result from the divergence between amenity provision by each government and the ideal level of each firm.

Proposition 2 shows that a unique subgame perfect Nash equilibrium exists under the uniform tax game if and only if $0 < k \leq \frac{1}{7}$, and that the point $a^* = 0$, $b^* = 0$ is the equilibrium. Proposition 5 shows that a unique subgame perfect Nash equilibrium exists under the perfect tax discrimination game if $0 < k < 1$, and that the point $a^* = 0$, $b^* = 0$ is the equilibrium. To facilitate a comparison of efficiency across the two regimes, we assume that $0 < k \leq \frac{1}{7}$. Denote social loss under uniform taxation and perfect tax discrimination as L_u and L_p respectively. Then substituting equilibrium values and integrating it is immediate to see that

$$L_u = \left(\frac{1}{2}\right)^2 kz^2 < \frac{1}{2}kz^2 = L_p.$$

So under conditions where equilibrium would exist in both regimes, perfect tax discrimination brings about a lower level of social efficiency than uniform taxation under Hotelling amenity/tax competition. These solutions can be compared with the socially efficient outcome of $L^* = \frac{1}{8}kz^2$, which occurs when $a = b = \frac{z}{4}$.

6 Conclusions

This paper seeks an explanation of why competition between governments fails to promote efficiency. The explanation we propose builds on Hotelling's observation that when firms compete not just over prices but over product characteristics, and when consumers' preferences over product characteristics vary, then efficiency is not promoted by competition. In the policy setting of the present paper, competition between (Leviathan) governments fails to promote efficiency when governments compete over levels of amenity provision as well as taxes, and where firms' preferences for the level of amenity provision vary.

In the uniform tax game, when an equilibrium exists one government provides the amenity at a maximal level, which is inefficiently high, whilst the other government provides no amenity at all, which is inefficiently low. This result is driven by the variation in firms' ideal level of amenity provision. Then competition pushes governments 'too far' in opposite directions, rather than bringing about a universal race to the bottom or efficiency, the two outcomes on which most of the previous literature has focused.

The equilibrium that we demonstrate for uniform taxation appears to fit with recent empirical evidence, which shows persistent differences in levels of taxation and public good provision in areas where greater convergence had been expected. One example is in Europe, where a core and periphery has emerged despite significant efforts to avoid such an outcome. The core tends to be characterized by governments that tax and provide public amenities at a significantly higher level than in the periphery.

Interpreted more broadly, the equilibrium outcome may help to understand why aspects of economic development or legal reform may actually work against a government's (rent seeking) interests. A government in a country where public good provision is reckoned to be sub-optimally low may encounter resistance to reform. It has difficulties raising taxation because of resistance from both domestic and foreign firms whose original decision to locate or remain in that country was based on relatively low levels of amenity provision and taxation. An interesting thing about our analysis is that the usual presumption of downward pressure on developed country taxes and public good provision resulting from intergovernmental competition for firms does not follow. In this sense our theoretical predictions accord with the observation of a high-tax high-amenity providing core and low-tax low-amenity providing periphery of Europe. Our framework could similarly be used to help understand differences in amenity provision between the developed and developing worlds.

The failure of equilibrium to exist is taken as a metaphor for intense tax competition. When the level of amenity provision offered by governments is similar then the weight of competition falls on tax levels. In the limit, because there is very little to choose between the two governments in terms of amenity levels, each government can attract all firms to its jurisdiction by undercutting the other with a small reduction in the tax level. When the degree of amenity mismatch has a sufficiently large impact on firms' costs, making them relatively responsive to changes in levels of amenity provision, then the system never settles down to (subgame perfect Nash) equilibrium. The governments both have an incentive to offer similar levels of amenities in an effort not to loose firms to the other. From the view point of each government, there is no tax level at which the other government does not have an incentive to attract all firms by setting a tax that is slightly lower.

One way to circumvent the incentive for governments to undercut each other is for each to offer tailor made tax-amenity packages to firms. There is a widespread perception that tax breaks are used in a similar vein. We model this policy environment as a 'perfect tax discrimination game'. We show that under perfect tax discrimination the equilibrium existence issue is partially resolved but that efficiency is worse than under uniform tax discrimination. The price paid by governments for greater stability through 'head to head' competition for each firm is that, once again when equilibrium exits, each government can attract the firm in question by lowering taxes, resulting in a 'race to the bottom'. In equilibrium, no amenities are provided by either government. As with uniform taxation, though, when the degree of amenity mismatch has a sufficiently large impact on firms' costs, making them relatively responsive to changes in levels of amenity provision, then the system never settles down to equilibrium. When no amenities are being offered, one government has an incentive to deviate by offering a level of amenity provision at a relatively high level. But when one government offers a positive level of amenities, then the other government can always do better by setting amenities at a slightly lower level and undercutting the first using taxes.

An alternative way to prevent intensive tax competition might lie with tax harmonization. For taxes set in the second stage governments could agree to set the same tax. Then the only issue would be in setting the level of amenity provision in the first stage. In the model of this present paper it is clear that, given locations, under collusion the governments would have an incentive to raise taxes to the point where they had extracted all rents from firms. If perfect tax discrimination were possible then it is clear all rents would be extracted and the outcome would be efficient. Whilst economists might see such efficiency as an advantage, it is not clear that citizen-entrepreneurs would be happy to see all their profits transferred to politicians in the form of rents. Under uniform taxation the outcome is less obvious. Because of their differing requirements for amenity provision, firms make different profits. At a level of taxation where some firms could make positive profits and so a higher tax could extract further rents, other firms cannot make positive profits. The outcome would be dependent upon assumptions made about whether all firms must be profitable in equilibrium. The issue of tax harmonization within this framework is left to future research.

The framework of the present paper is similar to a Tiebout model in that all firms can 'vote with their feet' for the jurisdiction that makes them better off (see Oates and Schwab 1988, Wooders 1989). So the inefficiencies that arise in the present model may seem surprising given that such mobility promotes efficiency in a Tiebout setting. The difference in outcomes appears to lie in the fact that in our setting there are just two jurisdictions whilst in a Tiebout setting there are many, combined with the fact that governments in a Tiebout setting are not Leviathans.

One might conjecture that increasing the number of jurisdictions in the model of this present paper should bring about efficiency. On the face of it this appears to be true. To see why, assume that there are three governments in a uniform tax game. Introduce a third jurisdiction to the uniform tax game and assume that governments locate as far from each other as possible in amenity space, as in the equilibrium that we demonstrate for two jurisdictions. Computing social loss as in Section 5, we find that the same efficient level of social loss is obtained as if two governments had located at a quarter and three quarters of the way along the interval. But it is far from clear that the three jurisdiction outcome is in fact an equilibrium. The government in the middle attracts half of all firms whilst the other two share a half. Therefore the other two may well have an incentive to deviate from such a situation. Given the discontinuities in the reaction function, it is not clear whether existence of equilibrium can be established in the three jurisdiction game.[19]

It is worth considering the implications of the present analysis for the public choice literature on tax competition, of which Besley and Smart (2001) is an example. In that literature, citizens are able to use yardstick competition to evaluate the performance of policy makers who may or may not be self-interested. Yardstick competition is shown to be a relatively effective mechanism in an environment where preferences for public good provision are uniform. If the level of public good provision in the other jurisdiction is higher than at home then there is evidence of under-performance by domestic politicians. It remains to be investigated whether the same holds in an environment where preferences for public good provision varies. One possibility would be to allow citizens to choose between a benevolent dictator and a Leviathan in a framework like the one of the present paper, where agents' preferences for public good provision vary. It might then be possible to see whether Leviathan policy makers were induced to provide more efficient levels of public good provision or driven out of the policy arena all together. This seems like a promising area for further research.

[19] A larger number of agents has been introduced to a Hotelling framework by Salop (1979) where firms that compete for consumers are located on a circle. Note that such an approach would not be appropriate in our model because points on the interval denote levels of amenity provision rather than points in geographical space or time. So it does not make sense to join the two ends of the interval in order to form a circle.

References

[1] Baldwin, R. and P. Krugman (2000); "Agglomeration, Integration and Tax Harmonization," CEPR Discussion Paper, November 2000 no. 2630.

[2] Besley, T. and M. Smart (2001); "Does Tax Competition Raise Voter Welfare?" LSE Typescript.

[3] Black, D.A. and W. Hoyt (1989) "Bidding for Firms," *American Economic Review*, 79(5): 1249-1256.

[4] Boadway, R. P. Pestieau and D.E. Wildasin (1989); "Non-cooperative Behavior and Efficient Provision of Public Goods," *Public Finance*, 44(1): 1-7

[5] Boadway, R., K. Cuff and N. Marceau (2002); "Inter-jurisdictional Competition for Firms," *International Economic Review*, 43(5): 761-782.

[6] Bond, E. and L. Samuelson (1986); "Tax Holidays as Signals," *American Economic Review*, 76(4): 820-826.

[7] Bhaskar, V. and T. To (2002); "Is Perfect Price Discrimination Really Efficient? An Analysis of Free Entry," *Rand Journal of Economics* forthcoming.

[8] Brennan, G. and J.M. Buchanan, (1980); *The Power To Tax: Analytical Foundations of a Fiscal Constitution*, Cambridge University Press, Cambridge.

[9] d'Aspremont, C., J. Gabszewicz, and J. Thisse (1979); "On Hotelling's Stability in Competition," *Econometrica*, 17: 1145-1151.

[10] Dasgupta, P. and E. Maskin (1986); "The Existence of Equilibrium in Discontinuous Economic Games I. Theory." *Review of Economic Studies* 53, 1-26.

[11] Devereux, M., B. Lockwood and M. Redoano (2002); "Do Countries Compete over Corporate Tax Rates?" Warwick Economic Research Paper no. 642.

[12] Edwards, J. and M. Keen (1996); "Tax Competition and Leviathan." *European Economic Review* 40(1): 113-134.

[13] Gordon, R. and J.D. Wilson (1986); "An Examination of Mulitjurisdictional Corporate Income Taxation under Formula Apportionment," *Econometrica*, 54(6): 1357-1373.

[14] Gordon, R. and J.D. Wilson (1999); "Tax Structure and Government Behavior: Implications for Tax Policy," NBER Working Paper 7244.

[15] Gordon, R. and J.D. Wilson (2002); "Expenditure Competition," *Journal of Public Economic Theory* forthcoming.

[16] Haaparanta, P. (1996); "Competition for Foreign Direct Investment," *Journal of Public Economics*, 63(1): 141-53.

[17] Hotelling, H. (1929); "Stability in Competition," *Economic Journal*, 39: 41-57.

[18] King, I., R.P. McAfee and L. Welling (1993); "Industrial Blackmail: Dynamic Tax Competition and Public Investment," *Canadian Journal of Economics*, 26: 590-608.

[19] Kreps, D. and J.A. Scheinkman (1983); "Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes," *The Bell Journal of Economics*, 14(2): 326-337.

[20] Milgrom, P. and J. Roberts (1982); "Limit Pricing and Entry Under Incomplete Information," *Econometrica* 50: 443-460.

[21] Oates, W.E. (1985); "Searching for Leviathan: An Empirical Study," *American Economic Review*, 75: 748-757.

[22] Oates, W.E. and R.M. Schwab (1988); "Economic Competition Among Jurisdictions: Efficiency Enhancing or Distortion Inducing?" *Journal of Public Economics*, 35: 333-354

[23] Salop, S. (1979); "Monopolistic Competition with Outside Goods," *Bell Journal of Economics*, 10: 141-156.

[24] Spence, M. (1976); "Product Selection, Fixed Costs and Monopolistic Competition," *Review of Economic Studies* 43: 217-235.

[25] Spence, M. (1977); "Entry, Capacity, Investment and Oligopolistic Pricing," *Bell Journal of Economics*, 8: 534-544.

[26] Wildasin, D.E. (1988); "Nash Equilibria in Models of Fiscal Competition." *Journal of Public Economics*, (March) 35(2): 229-240.

[27] Wildasin, D.E. (1989); "Interjurisdictional Capital Mobility: Fiscal Externality and a Corrective Subsidy." *Journal of Urban Economics*, 25: 193-212.

[28] Wilson, J.D. (1986); "A Theory of Interregional Tax Competition." *Journal of Urban Economics*, 19: 296-315.

[29] Wilson, J.D. (1999); "Theories of Tax Competition." *National Tax Journal*, 52(2): 269-305.

[30] Wooders, M. (1989); "A Tiebout Theorem," *Mathematical Social Sciences*, 18: 33-55.

[31] Wooders, M., B. Zissimos, and A. Dhillon (2001) "Tax Competition Reconsidered." Warwick Economic Research Papers no. 622, (December) 52 pages. Available on-line at http://www.warwick.ac.uk/economics/research/twerps.html.

[32] Zodrow, G.R. and P. Mieszkowski (1986) "Pigou, Tiebout, Property Taxation and the Underprovision of Local Public Goods." *Journal of Urban Economics*, 19: 356-370.

A Appendix

Lemma 1. *Assume governments play a uniform tax game. For given τ_B, the unique tax that maximizes $r_A(\tau_A, \tau_B)$ on the firm sharing interval is*

$$\tau_A(\tau_B; a, b, k, z) = k\left(\frac{a+\tau_B}{2k} + \frac{(z+a-b)}{2}\right).$$

For given τ_A, the unique tax τ_B that maximizes $r_B(\tau_A, \tau_B)$ on the firm sharing interval is

$$\tau_B(\tau_A; a, b, k, z) = k\left(\frac{(z-b)+\tau_A}{2k} + \frac{(z-a+b)}{2}\right).$$

If $\tau_A(\tau_B; a, b, k, z)$ and $\tau_B(\tau_A; a, b, k, z)$ are set simultaneously, then they can be solved for simultaneously to obtain:

$$\tau_A(a, b, k, z) = \frac{1}{3}(2a + (z - b) + (a - b)k + 3kz);$$

$$\tau_B(a, b, k, z) = \frac{1}{3}(2(z - b) + a + (b - a)k + 3kz).$$

Proof. To maximize rents over the firm sharing interval, Government A solves the problem

$$\max_{\tau_A} r_A(\tau_A, \tau_B) = (\tau_A - a)\hat{s}(\tau_A, \tau_B).$$

Expanding the objecting function using $\hat{s}(\tau_A, \tau_B) = (\tau_B - \tau_A)/2k + (z - b + a)/2$, we obtain

$$(\tau_A - a)\hat{s} = \frac{1}{2}(z + a - b)(\tau_A - a) - \frac{1}{2k}(\tau_A - a)\tau_A + \frac{1}{2k}(\tau_A - a)\tau_B.$$

Setting the first order condition equal to zero and rearranging obtains $\tau_A(\tau_B; a, b, k, z)$. The second order condition is

$$\frac{\partial(r_A(\tau_A, \tau_B))}{\partial\tau_A} = -1/k,$$

so $r_A(\tau_A, \tau_B)$ must be strictly concave and $\tau_A(\tau_B; a, b, k, z)$ is the unique maximizer on the firm sharing interval.

Government B solves the analogous problem

$$\max_{\tau_B} r_B(\tau_A, \tau_B) = (\tau_B - (z - b))(z - \hat{s}(\tau_A, \tau_B)).$$

Expanding the objecting function, we obtain

$$(\tau_B - (z - b))(z - \hat{s}) =$$
$$\frac{1}{2}(z - a + b)(\tau_B - (z - b)) - \frac{1}{2k}(\tau_B - (z - b)) + \frac{1}{2k}(\tau_B - (z - b))\tau_A.$$

Setting the first order condition equal to zero and rearranging obtains $\tau_B(\tau_A; a, b, k, z)$. The second order condition once again is

$$\frac{\partial(r_B(\tau_A, \tau_B))}{\partial\tau_B} = -1/k,$$

so $r_B(\tau_A, \tau_B)$ must also be strictly concave and $\tau_B(\tau_A; a, b, k, z)$ is the unique maximizer on the firm sharing interval. □

Lemma 2. *Under a uniform tax game, the tax $\tau_A(\tau_B; a, b, k, z)$ that maximizes $r_A(\tau_A, \tau_B)$ on the firm sharing interval $|\tau_A - \tau_B| \leq k(z - a - b)$ is a best response to τ_B if and only if, for any $\tau_B, \varepsilon > 0$,*

$$r_A(\tau_A(\tau_B; a, b, k, z), \tau_B) \geq z(\tau_B - k(z - a - b) - a - \varepsilon).$$

Similarly, the tax $\tau_B(\tau_A; a, b, k, z)$ that maximizes $r_B(\tau_A, \tau_B)$ on the firm sharing interval $|\tau_A - \tau_B| \leq k(z - a - b)$ is a best response to τ_A if and only if, for any $\tau_A, \varepsilon > 0$,

$$r_B(\tau_A, \tau_B(\tau_A; a, b, k, z)) \geq z(\tau_A - k(z - a - b) - (z - b) - \varepsilon).$$

Proof. For Government A, it is only necessary to check whether the tax $\tau_A = \tau_B - k(z - a - b) - \varepsilon$ yields a higher rent than $\tau_A = \tau_A(\tau_B; a, b, k, z)$; the rent maximizing tax on the firm sharing interval. By Lemma 1, $\tau_A(\tau_B; a, b, k, z) > 0$ and by construction $\hat{s} > 0$, so $r_A(\tau_A(\tau_B; a, b, k, z), \tau_B) > 0$. Therefore, the alternative of setting $\tau_A = \tau_B + k(z - a - b)$, which yields zero rents, cannot yield higher rents than setting $\tau_A = \tau_A(\tau_B; a, b, k, z)$.

A parallel argument holds for Government B.

Having ruled out $\tau_A = \tau_B + k(z - a - b)$ as a strategy for Government A, sufficiency is immediate by definition of a best response. The tax $\tau_A = \tau_A(\tau_B; a, b, k, z)$ yields a rent $r_A(\tau_A(\tau_B; a, b, k, z), \tau_B)$, while the tax $\tau_A = \tau_B - k(z - a - b) - \varepsilon$ yields a rent $z(\tau_B - k(z - a - b) - a - \varepsilon)$. If $r_A(\tau_A(\tau_B; a, b, k, z), \tau_B) \geq z(\tau_B - k(z - a - b) - a - \varepsilon)$ then by definition $\tau_A = \tau_A(\tau_B; a, b, k, z)$ is a best response. Conversely, if to the contrary, $r_A(\tau_A(\tau_B; a, b, k, z), \tau_B) < z(\tau_B - k(z - a - b) - a - \varepsilon) = r_A(\tau_A, \tau_B)$ for some tax $\tau_A = \tau_B - k(z - a - b) - \varepsilon$, then by definition $\tau_A(\tau_B; a, b, k, z)$ cannot be a best response to τ_B. A parallel argument holds for Government B. □

Proposition 1. *Assume governments play a uniform tax game, and that a and b are fixed on the interval $[0, z]$, with $a + b \leq z$, $a \geq 0$, $b \geq 0$. For $a + b = z$, both governments are at the same location and there always exists an equilibrium in which $\tau_A^* = a$, $\tau_B^* = z - b$.*

For $a+b<z$ there exists an equilibrium point if and only if the two following conditions hold:

$$\text{(C1): } r_A\left(\tau_A^*\left(\tau_B^*;a,b,k,z\right),\tau_B^*\right) \geq z\left(\tau_B^* - k\left(z-a-b\right) - a - \varepsilon\right) \Leftrightarrow$$
$$\frac{\left(\left(a-b\right)k + \left(z-a-b\right) + 3kz\right)^2}{18k} \geq \frac{z\left(2\left(a+2b\right)k + 2\left(z-a-b\right) - 3\varepsilon\right)}{3}$$

$$\text{(C2): } r_B\left(\tau_B^*\left(\tau_B;a,b,k,z\right),\tau_A^*\right) \geq z\left(\tau_A^* - k\left(z-a-b\right) - \left(z-b\right) - \varepsilon\right) \Leftrightarrow$$
$$\frac{\left(\left(b-a\right)k - \left(z-a-b\right) + 3kz\right)^2}{18k} \geq \frac{z\left(2\left(2a+b\right)k - 2\left(z-a-b\right) - 3\varepsilon\right)}{3}$$

Whenever it exists, an equilibrium point is determined uniquely by the taxes

$$\begin{aligned}\tau_A^*\left(a,b;k,z\right) &= \frac{1}{3}\left(2a + \left(z-b\right) + \left(a-b\right)k + 3kz\right);\\ \tau_B^*\left(a,b;k,z\right) &= \frac{1}{3}\left(2\left(z-b\right) + a + \left(b-a\right)k + 3kz\right).\end{aligned}$$

Proof. For $a+b=z$ both governments are located in the same place and we effectively have a standard Bertrand equilibrium in homogeneous products.

Consider the case where $a+b<z$. Following d'Aspremont et al (1979), begin by showing that any equilibrium must satisfy the condition $\left|\tau_A^* - \tau_B^*\right| < k\left(z-a-b\right)$. Suppose that on the contrary, $\left|\tau_A^* - \tau_B^*\right| > k\left(z-a-b\right)$. Then the government that charges the strictly higher tax gets zero rents and gains by charging a tax equal to that of the other, contradicting the fact that $\left(\tau_A^*, \tau_B^*\right)$ is an equilibrium.

Suppose then that $\left|\tau_A^* - \tau_B^*\right| = k\left(z-a-b\right)$. Take, for example, the case where $\tau_A^* - \tau_B^* = k\left(z-a-b\right)$. If $\tau_B^* = 0$ then the rents of Government B are zero and it would make positive rents by charging $0 < \tau_B^* < \tau_A^* + k\left(z-a-b\right)$. If $\tau_B^* > 0$ then there are two cases to consider: (i) Either Government A gets all firms to locate in A, in which case Government B can obtain positive rents by reducing τ_B^*. So Government B has an incentive to deviate from τ_B^*; a contradiction; Or Government A has only a share of all firms and is able to capture all of them and make larger rents by charging a slightly lower tax. Let $\overline{s} < z$ be given by $\hat{s} = \left(\tau_B - \tau_A + \left(z-b+a\right)k\right)/2k$ for which $\tau_A^* = \tau_B^* + k\left(z-a-b\right)$, given τ_B^*. At τ_A^*, Government A makes rents $r_A\left(\tau_A^*, \tau_B^*\right) = \tau_A^*\overline{s}$.

For $\tau_A = \tau_A^* - \varepsilon$, the government makes rents $r_A(\tau_A, \tau_B^*) = \tau_A z$. For $\tau_A = \tau_A^* - \varepsilon$, where $\varepsilon = \varepsilon(z - \overline{s})\tau_A^*/z > 0$ the government makes rents $\tau_A z = \tau_A^* \overline{s}$. So for all $0 < \varepsilon < \varepsilon(z - \overline{s})\tau_A^*/z$, it is the case that $r_A(\tau_A^* - \varepsilon, \tau_B^*) = \tau_A z > \tau_A^* \overline{s}$; a contradiction. The only remaining possibility is that equilibrium must satisfy $|\tau_A^* - \tau_B^*| < k(z - a - b)$.

By definition of the 'rent to office' functions $r_A(\tau_A, \tau_B)$ and $r_B(\tau_A, \tau_B)$, for any equilibrium (τ_A^*, τ_B^*), τ_A^* must maximize $\frac{1}{2}(z + a - b)(\tau_A - a) - \frac{1}{2k}(\tau_A - a)\tau_A + \frac{1}{2k}(\tau_A - a)\tau_B$ in the firm sharing interval $(\tau_B - k(z - a - b), \tau_B + k(z - a - b))$. An equivalent condition must hold for τ_B^*.

By Lemma 1, the first order conditions of this problem yield

$$\begin{aligned}\tau_A^*(\tau_B; a, b, k, z) &= \frac{a + \tau_B}{2} + \frac{(z + a - b)k}{2} \\ \tau_B^*(\tau_A; a, b, k, z) &= \frac{(z - b) + \tau_A}{2} + \frac{(z - a + b)k}{2}\end{aligned}$$

As we have just proved that firm sharing is necessary for equilibrium, the simultaneous solutions $\tau_A^*(a, b, k, z)$ and $\tau_B^*(a, b, k, z)$ given in Lemma 1 provide the equilibrium taxes.

To establish conditions under which this pair (τ_A^*, τ_B^*) is indeed an equilibrium, it remains to check that τ_A^* maximizes $r_A(\tau_A, \tau_B)$ not just on the interval $(\tau_B - k(z - a - b), \tau_B + k(z - a - b))$ but on the whole of the domain $\mathbb{R}_+$, and similarly for τ_B^*. For fixed a and b, if τ_A^* is to be an equilibrium strategy given τ_B^*, by Lemma 2 we must have that for any $\varepsilon > 0$,

$$\begin{aligned}r_A(\tau_A^*, \tau_B^*) &= (\tau_A^* - a)\hat{s} = \frac{((a - b)k + (z - a - b) + 3kz)^2}{18k} \\ &\geq z(\tau_B^* - k(z - a - b) - a - \varepsilon).\end{aligned}$$

Substituting for $\hat{s}$ using $\hat{s}(\tau_A, \tau_B) = (\tau_B - \tau_A)/2k + (z - b + a)/2$ and simplifying, we obtain condition (C1). By symmetry, we get (C2).

To show that (C1) and (C2) are also sufficient for (τ_A^*, τ_B^*) to be an equilibrium ti remains only to check that they imply $|\tau_A^* - \tau_B^*| \leq k(z - a - b)$. This completes the proof of our proposition. □

Proposition 2. *There exists a unique subgame perfect Nash equilibrium in pure strategies of a uniform tax game if and only if* $0 < k \leq \frac{1}{7}$. *If such an equilibrium exists then it is characterized (uniquely) by the point* $a^* = b^* = 0$.

Proof. Write $r_A(a,b;k,z)$ as $r_A(a,b)$ and $r_B(a,b;k,z)$ as $r_B(a,b)$ because k and z are held constant throughout.

First assume $0 < k \leq \frac{1}{7}$.

Suppose that the pair (a^*,b^*) is a Nash equilibrium, where *either* a is interior or b is interior (or both); $a \in (0,z)$ or $b \in (0,z)$. Take b^* as given and let $a^* \in (0,z)$. But by Lemma 3, $\partial^2 r_A(a^*,b^*)/\partial a^2 = (k-1)^2/9k > 0$. If $\partial r_A(a^*,b^*)/\partial a > (<)\, 0$ then rents can be increased by increasing (decreasing) a, contradicting equilibrium. If $\partial r_A(a^*,b^*)/\partial a = 0$ then rents can be increased either by increasing or by decreasing a, again contradicting equilibrium. The same argument can be made for $b^* \in (0,z)$, holding a^* constant, as $\partial^2 r_B(a,b)/\partial b^2 = (k+1)^2/9k > 0$.

Therefore, the only candidates for an equilibrium pair are the corner solutions $(a^*,b^*) = (0,0)$, $(0,z)$ and $(z,0)$ (noting that (z,z) violates $a+b \leq z$). The three cases are taken in order. First we show why $(a^*,b^*) = (0,0)$ is an equilibrium. First observe that $\partial r_A(a,b)/\partial a = (1-k)((1-k)a + (1+k)b - (1+3k)z)/9k$. Using $b^* = 0$, $\partial r_A(a,b^*)/\partial a = (1-k)((1-k)a - (1+3k)z)/9k < 0$ for all $a \in [0,z]$. To see this, note that even when a takes its largest positive value at $a = z$, $\partial r_A(a,b^*)/\partial a = -4(1-k)/9z < 0$. Thus we have a corner solution. Rents could be increased were it possible to reduce a below the level $a = 0$. But this is not possible so $a^* = 0$ is a best response to $b^* = 0$.

Now take $a^* = 0$ as given and observe that $\partial r_B(a^*,b^*)/\partial b = (1+k)((1+k)b + z(3k-1)z)/9k$. If $b = 0$ then $\partial r_B(a^*,b)/\partial b = (1+k)(3k-1)/9k < 0$. But if $b = z$ then $\partial r_B(a^*,b)/\partial b = 4(1+k)z/9 > 0$. So both $b = 0$ and $b = z$ could in principle be stable corner solutions (see from above that the second order condition is satisfied). The matter of which is a best response depends upon which yields the higher rent; $r_B(0,0) = ((3k-1)z)^2/18k$ or $r_B(0,z) = 8kz^2/9$. Solving $r_B(0,0) = r_B(0,z)$ in terms of k we find that $k = \frac{1}{7}$. It is then easy to see that $r_B(0,0) \geq r_B(0,z)$ for $0 < k \leq \frac{1}{7}$, with $r_B(0,0) > r_B(0,z)$ for $0 < k < \frac{1}{7}$. So $b^* = 0$ is a best response to $a^* = 0$. Therefore, $(a^*,b^*) = (0,0)$ is a Nash equilibrium in locations.

Next suppose $(a^*,b^*) = (0,z)$ is a Nash equilibrium in locations. But then (C1) fails;

$$\frac{((a-b)k + (z-a-b) + 3kz)^2}{18k} - \frac{z(2(a+2b)k + 2(z-a-b))}{3} = -\frac{10kz^2}{9};$$

Next suppose $(a^*, b^*) = (z, 0)$ is a Nash equilibrium in locations. But then (C2) fails;

$$\frac{((b-a)k - (z-a-b) + 3kz)^2}{18k} - \frac{z(2(2a+b)k - 2(z-a-b))}{3} = -\frac{10kz^2}{9};$$

a contradiction.

Now assume $k > \frac{1}{7}$.

Suppose that $(a^*, b^*) = (0, 0)$ is a Nash equilibrium in locations. But $r_B(0, z) > r_B(0, 0)$. So there is a unilateral incentive for Government B to deviate from $b^* = 0$; a contradiction.

The solutions $(a^*, b^*) = (0, z)$ and $(a^*, b^*) = (z, 0)$ can be ruled out as candidates for a Nash equilibrium in locations for the same reason as when $0 < k \leq \frac{1}{7}$; Conditions (C1) and (C2) fail in the respective cases. □

Lemma 4. *Consider a perfect tax discrimination game and assume A1 holds. Fix a and b so that $z - b > a$.*

If, for some firm $s \in [0, z]$, $a < \tau_{Bs} + k(|(z-b) - s| - |s - a|)$ then for $\varepsilon > 0$ sufficiently small Government A's unique best response is $\tau^*_{As} = \tau^{\lim}_{As}(\varepsilon)$. If $a \geq \tau_{Bs} + k(|(z-b) - s| - |s - a|)$ then $\tau^*_{As} = a$ is a best response for Government A.

If, for some firm $s \in [0, z]$, $z - b < \tau_{As} + k(|s - a| - |(z-b) - s|)$ then for $\varepsilon > 0$ sufficiently small Government B's unique best response is $\tau^*_{Bs}(\varepsilon) = \tau^{\lim}_{Bs}$. If $z - b \geq \tau_{As} + k(|s - a| - |(z-b) - s|)$ then $\tau^*_{Bs} = z - b$ is a best response for Government B.

Proof. It is assumed that $\varepsilon > 0$ and arbitrarily small. The exact bound on ε is established below.

Consider Government A's best response first. Fix a, b and τ_{Bs} so that $a < \tau_{Bs} + k(|(z-b) - s| - |s - a|)$ and suppose to the contrary that $\tau^{\lim}_{As} = \tau_{Bs} + k(|(z-b) - s| - |s - a|) - \varepsilon$ is not the unique best response. Then by definition, there must be some other tax that yields a higher rent. First suppose that the best response tax is lower than $\tau^{\lim}_{As}$, obtained by setting $\varepsilon' > \varepsilon$. Write $r_{As}(\varepsilon)$ for the rent obtained from setting tax $\tau_{As} = \tau_{Bs} + k(|(z-b) - s| - |s - a|) - \varepsilon$. Taking the difference in rents we obtain $r_{As}(\varepsilon') - r_{As}(\varepsilon) = -\varepsilon' + \varepsilon < 0$. So rents are lower under a lower tax; contradiction.

Next suppose that the best response tax is higher than $\tau^{\lim}_{As}$. Suppose that Government A raises the tax by the smallest possible amount, to $\tau_{As} = \tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right)$. Write the rent associated with this tax rate as $r_{As}(0)$. At this tax, the firm s is indifferent between the two jurisdictions. By A1, the firm s locates in A with probability $\frac{1}{2}$. Taking the difference in rents we obtain

$$\begin{aligned} r_{As}(0) - r_{As}(\varepsilon) &= \frac{1}{2}\left(\tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right) - a\right) \\ &\quad - \left(\tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right) - \varepsilon - a\right) \\ &= -\frac{1}{2}\left(\tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right) - a\right) + \varepsilon. \end{aligned}$$

But it is always possible to pick ε sufficiently small to ensure that $r_{As}(0) - r_{As}(\varepsilon) < 0$; contradiction.

By definition of $r_{As}(\tau_{As}, \tau_{Bs})$, if Government A sets a tax $\tau_{As} > \tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right)$ then $r_{As}(\tau_{As}, \tau_{Bs}) = 0$, whilst $r_{As}\left(\tau_{As}^{\lim}(\varepsilon), \tau_{Bs}\right) > 0$. So rents are lower under a higher tax; contradiction. So we have established that if $a < \tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right)$ then the unique best response is $\tau_{As}^{*} = \tau_{As}^{\lim}(\varepsilon)$.

Now fix a, b and τ_{Bs} so that $a \geq \tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right)$ and suppose to the contrary that $\tau_{As}^{*} = a$ is not a best response. Then by definition there must be some other tax that yields a higher rent. First note that $r_{As}(\tau_{As}^{*}, \tau_{Bs}) = 0$. Clearly, $\tau_{As} < a$ would yield $r_{As}(\tau_{As}, \tau_{Bs}) < 0$; contradiction. Now suppose $\tau_{As} > a$. But then $\tau_{As} > \tau_{Bs} + k\left(|(z-b)-s| - |s-a|\right)$ and so, by definition of the rent function, $r_{As}(\tau_{As}, \tau_{Bs}) = 0$. So rents are not higher under a higher tax; contradiction.

An analogous set of arguments can be used to establish the corresponding results for the best response of Government B. □

Proposition 3. *Consider Stage 2 of a perfect tax discrimination game, with a and b fixed on the interval $[0, z]$. Assume A1 holds and that $a + b \leq z$, $a \geq 0$, $b \geq 0$. If $k < 1$ then for ε sufficiently small there exists a unique Nash equilibrium in taxes for this stage of the perfect tax discrimination game. A unique Nash equilibrium in taxes for each firm $s \in [0, z]$ is determined by the following taxes:*
if $a + b = z$,

$$\tau_{As}^{*} = \tau_{Bs}^{*} = a = z - b;$$

if $a + b < z$,

$$\tau_{As}^{*} = \tau_{As}^{\lim}(\varepsilon), \ \tau_{Bs}^{*} = z - b.$$

Proof.

For $a + b = z$, both governments are located in the same place and we effectively have a standard Bertrand equilibrium in homogeneous products.

Consider the case where $a + b < z$. It is assumed that $\varepsilon > 0$ and arbitrarily small. An explicit upper bound $\overline{\varepsilon} = (1 - k)(z - a - b)/2$ for ε will be established in the proof below.

We will show that for all $s \in [0, z]$ the following pair $(\tau_{As}^{*}, \tau_{Bs}^{*})$ is a Nash equilibrium.

$$\begin{aligned}
\tau_{As}^{*} &= \tau_{As}^{\lim}(\varepsilon) = z - b + k(|z - b - s| - |s - a|) - \varepsilon; \\
\tau_{Bs}^{*} &= z - b.
\end{aligned}$$

First check the firm's location decision. We take the difference between the cost to locating in B and locating in A:

$$\begin{aligned}
c_{Bs}(\tau_{Bs}^{*}) - c_{As}(\tau_{As}^{*}) &= \tau_{Bs}^{*} + k|z - b - s| - \tau_{As}^{*} - k|s - a| \\
&= z - b + k|z - b - s| - (z - b) \\
&\quad -(k(|z - b - s| - |s - a|) - \varepsilon) - k|s - a| \\
&= \varepsilon
\end{aligned}$$

For each firm $s \in [0, z]$, profits made in Jurisdiction A are higher by ε than profits made in Jurisdiction B. Therefore, each firm locates in A.

To check that the pair $(\tau_{As}^{*}, \tau_{Bs}^{*})$ does indeed represent a Nash equilibrium, suppose not. Then either $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$ is not a best response to $\tau_{Bs}^{*} = z - b$ or vice versa. First suppose that $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$ is not a best response to $\tau_{Bs}^{*} = z - b$. If $\tau_{Bs}^{*} = z - b$ then for all $k < 1$,

$$a < \tau_{Bs}^{*} + k(|(z - b) - s| - |s - a|).$$

To see this, note that for $s \geq z - b$, it is the case that $|(z - b) - s| - |s - a| = -|z - a - b|$ and for $s < z - b$ it is the case that $|(z - b) - s| - |s - a| > -|z - a - b|$. Using this and $\tau_{Bs}^{*} = z - b$ we have $0 < z - a - b + k(|(z - b) - s| - |s - a|)$. But by Lemma 4, if

$a < \tau_{Bs}^{*} + k\left(\left|(z-b)-s\right| - \left|s-a\right|\right)$ then $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$ is a best response to $\tau_{Bs}^{*} = z-b$; a contradiction.

We now establish the upper bound $\overline{\varepsilon} = (1-k)(z-a-b)/2$ on ε. Recall that Lemma 4 required ε to be sufficiently small as to ensure that $r_{As}(0) - r_{As}(\varepsilon) < 0$. Let $\overline{\varepsilon} = (1-k)(z-a-b)/2$. If $\varepsilon < \overline{\varepsilon}$ then $r_{As}(0) - r_{As}(\varepsilon) < 0$ for all $s \in [0,z]$. To see why, use the fact that $\left(\left|(z-b)-s\right| - \left|s-a\right|\right) \geq -\left|z-a-b\right|$ and $\tau_{Bs}^{*} = z-b$ in the expression for $r_{As}(0) - r_{As}(\overline{\varepsilon})$:

$$\begin{aligned} r_{As}(0) - r_{As}(\overline{\varepsilon}) &= -\frac{1}{2}\left(z-a-b+k\left(\left|(z-b)-s\right| - \left|s-a\right|\right)\right) \\ &\quad +\frac{1}{2}(1-k)(z-a-b) \\ &\leq -\frac{1}{2}\left(z-a-b-k\left|z-a-b\right|\right) \\ &\quad +\frac{1}{2}(1-k)(z-a-b) \\ &= 0 \text{ for all } s \in [0,z]. \end{aligned}$$

It follows directly that if $\varepsilon < \overline{\varepsilon} = (1-k)(z-a-b)/2$ then $r_{As}(0) - r_{As}(\overline{\varepsilon}) < 0$ for all $s \in [0,z]$. So there exists an $\overline{\varepsilon}$ such that for all $\varepsilon \in (0,\overline{\varepsilon})$, $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$ is a best response to $\tau_{Bs}^{*} = z-b$.

Now suppose that $\tau_{Bs} = z-b$ is not a best response to $\tau_{As}^{*} = \tau_{As}^{\lim}(\varepsilon)$. If $\tau_{As}^{*} = \tau_{As}^{\lim}(\varepsilon)$ then

$$\begin{aligned} & \tau_{As}^{*} + k\left(\left|s-a\right| - \left|(z-b)-s\right|\right) \\ = \; & z-b+k\left(\left|z-b-s\right| - \left|s-a\right|\right) - \varepsilon \\ & +k\left(\left|s-a\right| - \left|(z-b)-s\right|\right) \\ = \; & z-b-\varepsilon. \end{aligned}$$

But by Lemma 4, if $z-b \geq \tau_{As}^{*} + k\left(\left|s-a\right| - \left|(z-b)-s\right|\right)$ then $\tau_{Bs}^{*} = z-b$ is a best response to $\tau_{As}^{*} = \tau_{As}^{\lim}(\varepsilon)$; a contradiction. Note that this does not depend on the value of k and s.

We now demonstrate uniqueness of this Nash equilibrium. We already know from Lemma 4 that $\tau_{As}^{*} = \tau_{As}^{\lim}(\varepsilon)$ is the unique best response to $\tau_{Bs}^{*} = z-b$. On the other hand, $\tau_{Bs}^{*} = z-b$ is not a unique best response to $\tau_{As}^{*} = \tau_{As}^{\lim}(\varepsilon)$, as any $\tau_{Bs}^{*} \geq z-b$

earns $r_{Bs}(\tau^*_{As}, \tau^*_{Bs}) = 0$ for Government B. However, $\tau_{As} = \tau^{\lim}_{As}(\varepsilon)$, $\tau_{Bs} > z - b$ is not a Nash equilibrium. To see this, set some $\tau_{Bs} > z - b$ and $\tau_{As} = \tau^{\lim}_{As}(\varepsilon) = \tau_{Bs} + k(|z - b - s| - |s - a|) - \varepsilon$. As A is limit pricing the firm s, the firm locates in A and Government B makes rent $r_{Bs}(\tau_{As}, \tau_{Bs}) = 0$. As long as $\tau^{\lim}_{Bs}(\varepsilon) > z - b$, Government B has an incentive to deviate from τ_{Bs} by setting $\tau^{\lim}_{Bs}(\varepsilon)$, attracting the firm s to Jurisdiction B and making $r_{Bs}(\tau_{As}, \tau_{Bs}) > 0$. Only at $\tau^*_{As} = \tau^{\lim}_{As}(\varepsilon)$, $\tau^*_{Bs} = z - b$ does Government B not have a deviation that could make positive rents. In order to attract the firm s to B the government must set $\tau_{Bs} < z - b$ and this would violate condition (ii) of equilibrium.

As we have characterized a unique Nash equilibrium for all $s \in [0, z]$, we have demonstrated that there exists a unique Nash equilibrium in taxes. □

Proposition 4. *Consider Stage 2 of a perfect tax discrimination game, with a and b fixed on the interval $[0, z]$. Assume A1 holds and that $a + b \leq z$, $a \geq 0$, $b \geq 0$. If $k \geq 1$ then for ε sufficiently small there exists a unique Nash equilibrium in taxes for this stage of the perfect tax discrimination game. A unique Nash equilibrium in taxes for each firm $s \in [0, z]$ is determined by the following taxes:*
if $a + b = z$, then

$$\tau^*_{As} = \tau^*_{Bs} = a = z - b, \text{ for } a + b = z \text{ and } s \in [0, z];$$

if $a + b < z$, then

$$\begin{aligned}
\tau^*_{As} &= a,\ \tau^*_{Bs} = z - b \text{ for } s = \hat{s}, \\
\tau^*_{As} &= \tau^{\lim}_{As}(\varepsilon),\ \tau^*_{Bs} = z - b, for\ s \in [0, \hat{s}), \\
\tau^*_{As} &= a,\ \tau^*_{Bs} = \tau^{\lim}_{Bs}(\varepsilon) \text{ for } s \in (\hat{s}, z].
\end{aligned}$$

Proof. For $a + b = z$, both governments are located in the same place and we effectively have a standard Bertrand equilibrium in homogeneous products.

Consider the case where $a + b < z$. It is assumed that $\varepsilon > 0$ and arbitrarily small.

First take the firm $s = \hat{s}$. To solve for its location, use $\tau^*_{As} = a$ and $\tau^*_{Bs} = z - b$ in $\hat{s} = (\tau_{Bs} - \tau_{As})/2k + (z - b + a)/2$ to obtain

$$\hat{s} = \frac{(1 + k)(z - b) + (k - 1)a}{2k}.$$

It is straightforward to verify that $a < \hat{s} < z - b$ for $k > 1$, and that $\hat{s} \to z - b$ from below as $k \to 1$ (from above). By construction, $s = \hat{s}$ makes the same profits in Jurisdiction A as in Jurisdiction B. Therefore, by A1, the probability that it locates in each jurisdiction is $\frac{1}{2}$.

We will now show that the following pair $(\tau^*_{As}, \tau^*_{Bs})$ is a Nash equilibrium for $s = \hat{s}$;

$$\begin{aligned} \tau^*_{As} &= a; \\ \tau^*_{Bs} &= z - b. \end{aligned}$$

To check that the pair $(\tau^*_{As}, \tau^*_{Bs})$ does indeed represent a Nash equilibrium for $s = \hat{s}$, suppose not. Then either $\tau_{As} = a$ is not a best response to $\tau^*_{Bs} = z - b$ or vice versa.

First suppose that $\tau_{As} = a$ is not a best response to $\tau^*_{Bs} = z - b$. For $\tau_{As} = a$, rents are given by

$$\begin{aligned} r_{As}\left(\tau_{As}, \tau^*_{Bs}\right) &= \tau_{As} - a \\ &= 0. \end{aligned}$$

Setting $\tau_{As} < a$ contradicts condition (ii) of equilibrium. If Government A deviates by setting $\tau_{As} > a$ then the firm makes higher profits by locating in Jurisdiction B, as a result of which $r_{As}\left(\tau_{As}, \tau^*_{Bs}\right) = 0$. So there exists no profitable deviation from $\tau_{As} = a$; contradiction. An analogous argument holds for $\tau^*_{Bs} = z - b$.

Next take firms in the interval $s \in [0, \hat{s})$. We will show that for all such firms the following pair $(\tau^*_{As}, \tau^*_{Bs})$ is a Nash equilibrium.

$$\begin{aligned} \tau^*_{As} &= \tau^{\lim}_{As}(\varepsilon) = z - b + k\left(|z - b - s| - |s - a|\right) - \varepsilon; \\ \tau^*_{Bs} &= z - b; \end{aligned}$$

where it is assumed that $\varepsilon > 0$ and arbitrarily small.

First check the firm's location decision. We take the difference between the cost to locating in B and locating in A:

$$\begin{aligned} c_{Bs}\left(\tau^*_{Bs}\right) - c_{As}\left(\tau^*_{As}\right) &= \tau^*_{Bs} + k\,|z - b - s| - \tau^*_{As} - k\,|s - a| \\ &= z - b + k\,|z - b - s| \\ &\quad - (z - b) - \left(k\left(|z - b - s| - |s - a|\right) - \varepsilon\right) - k\,|s - a| \\ &= \varepsilon \end{aligned}$$

For each firm $s \in [0, \hat{s})$, profits made in Jurisdiction A are higher by ε than profits made in Jurisdiction B. Therefore, each firm locates in A.

To check that the pair $(\tau_{As}^*, \tau_{Bs}^*)$ does indeed represent a Nash equilibrium, suppose not. Then either $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$ is not a best response to $\tau_{Bs}^* = z - b$ or vice versa. First suppose that $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$ is not a best response to $\tau_{Bs}^* = z - b$. Check that Government A makes non-negative rents at $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$. Otherwise condition (ii) of equilibrium is violated. Note that we can represent any firm $s \in [0, \hat{s})$ as $s = \hat{s} - \delta > 0$, where $0 < \delta \leq \hat{s}$. Using this notation, we find that $r_{As} = 2k\delta - \varepsilon$. To see why, use $\hat{s} = ((1+k)(z-b) + (k-1)a)/2k$, $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$ and $\tau_{Bs}^* = z - b$ in $r_{As} = \tau_{As} - a$. As $k, \delta > 0$, it is always possible to pick an ε sufficiently small to ensure that $r_{As} = 2k\delta - \varepsilon > 0$.

If $\tau_{Bs}^* = z - b$ then for $s \in [0, \hat{s})$,

$$a < \tau_{Bs}^* + k(|(z-b) - s| - |s - a|).$$

To see this, now use $s = ((1+k)(z-b) + (k-1)a)/2k - \delta$ and $\tau_{Bs}^* = z - b$ in the above expression to show that

$$\tau_{Bs}^* - a + k(|(z-b) - s| - |s - a|) = 2k\delta > 0.$$

But by Lemma 4, if $a < \tau_{Bs}^* + k(|(z-b) - s| - |s-a|)$ then $\tau_{As} = \tau_{As}^{\lim}(\varepsilon)$ is a best response to $\tau_{Bs}^* = z - b$; a contradiction.

Suppose that $\tau_{Bs} = z - b$ is not a best response to $\tau_{As}^* = \tau_{As}^{\lim}(\varepsilon)$. Exactly the same argument as in Proposition 3 is used to establish a contradiction. (Recall that the argument used in Proposition 3 was independent of the value of k and s). We have that, for all $s \in [0, \hat{s})$, the pair $(\tau_{As}^*, \tau_{Bs}^*)$ is a Nash equilibrium.

We now demonstrate uniqueness of this Nash equilibrium. Once again, exactly the same argument as in Proposition 3 is used to establish that $\tau_{As}^* = \tau_{As}^{\lim}(\varepsilon)$, $\tau_{Bs}^* = z - b$ is unique.

Now consider all $s \in (\hat{s}, z]$. For such firms we will show that the following pair $(\tau_{As}^*, \tau_{Bs}^*)$ is a Nash equilibrium:

$$\begin{aligned}
\tau_{As}^* &= a; \\
\tau_{Bs}^* &= \tau_{Bs}^{\lim}(\varepsilon) = a - k(|z - b - s| - |s - a|) - \varepsilon.
\end{aligned}$$

First check the firm's location decision. We take the difference between the cost to locating in B and locating in A:

$$\begin{aligned} c_{Bs}\left(\tau_{Bs}^{*}\right)-c_{As}\left(\tau_{As}^{*}\right) &= \tau_{Bs}^{*}+k\left|z-b-s\right|-\tau_{As}^{*}-k\left|s-a\right| \\ &= a-k\left(\left|z-b-s\right|-\left|s-a\right|\right)-\varepsilon+k\left|z-b-s\right|-a-k\left|s-a\right| \\ &= -\varepsilon \end{aligned}$$

So costs are lower and therefore profits are higher for the firm if it locates in Country B.

To check that the pair $(\tau_{As}^{*},\tau_{Bs}^{*})$ does indeed represent a Nash equilibrium for $s\in(\hat{s},z]$, suppose not. Then either $\tau_{As}=a$ is not a best response to $\tau_{Bs}^{*}=\tau_{Bs}^{\lim}(\varepsilon)$ or vice versa.

Suppose that $\tau_{As}=a$ is not a best response to $\tau_{Bs}^{*}=\tau_{Bs}^{\lim}(\varepsilon)$. If $\tau_{Bs}^{*}=\tau_{Bs}^{\lim}(\varepsilon)$ then

$$\begin{aligned} & \tau_{Bs}^{*}+k\left(\left|(z-b)-s\right|-\left|s-a\right|\right) \\ = \; & a-k\left(\left|z-b-s\right|-\left|s-a\right|\right)-\varepsilon \\ & +k\left(\left|(z-b)-s\right|-\left|s-a\right|\right) \\ = \; & a-\varepsilon \end{aligned}$$

But by Lemma 4, if $a\geq\tau_{Bs}+k\left(\left|(z-b)-s\right|-\left|s-a\right|\right)$ then $\tau_{As}^{*}=a$ is a best response to $\tau_{Bs}^{*}=\tau_{Bs}^{\lim}(\varepsilon)$ for Government A; contradiction.

Now suppose that $\tau_{Bs}=\tau_{Bs}^{\lim}(\varepsilon)$ is not a best response to $\tau_{As}^{*}=a$. Check that Government B makes non-negative rents at $\tau_{Bs}=\tau_{Bs}^{\lim}(\varepsilon)$. Using the fact that $s=\hat{s}+\delta>0$, where $\delta>0$, we find that $r_{Bs}(\tau_{As}^{*},\tau_{Bs}^{*})=2k\delta-\varepsilon$. To see this, use $\hat{s}=\left((1+k)(z-b)+(k-1)a\right)/2k$ and $\tau_{Bs}=\tau_{Bs}^{\lim}(\varepsilon)$ in $r_{Bs}=\tau_{Bs}-z-b$. As $k,\delta>0$, it is always possible to pick an ε sufficiently small to ensure that $r_{Bs}=2k\delta-\varepsilon>0$.

If $\tau_{As}^{*}=a$ then for $s\in(\hat{s},z]$,

$$z-b<\tau_{As}^{*}+k\left(\left|s-a\right|-\left|(z-b)-s\right|\right)$$

To see this, use $s=\left((1+k)(z-b)+(k-1)a\right)/2k-\delta$ and $\tau_{As}^{*}=a$ in the above expression to show that

$$\tau_{As}^{*}-(z-b)+k\left(\left|s-a\right|-\left|(z-b)-s\right|\right)=2k\delta>0.$$

But by Lemma 4, if $z - b < \tau^*_{As} + k(|s-a| - |(z-b)-s|)$ then $\tau^*_{Bs} = z - b$ is a best response to $\tau^*_{As} = a$ for Government B; contradiction.

We now demonstrate uniqueness of this Nash equilibrium. We already know from Lemma 4 that $\tau^*_{Bs} = \tau^{\lim}_{Bs}$ is the unique best response to $\tau^*_{As} = a$. On the other hand, $\tau^*_{As} = a$ is not a unique best response to $\tau^*_{Bs} = \tau^{\lim}_{Bs}(\varepsilon)$, as any $\tau^*_{As} \geq a$ earns $r_{As}(\tau^*_{As}, \tau^*_{Bs}) = 0$ for Government A. However, $\tau_{As} > a$, $\tau_{Bs} = \tau^{\lim}_{Bs}(\varepsilon)$ is not a Nash equilibrium. To see this, set some $\tau_{As} > a$ and $\tau_{Bs} = \tau^{\lim}_{Bs}(\varepsilon) = \tau_{As} - k(|z-b-s| - |s-a|) - \varepsilon$. As B is limit pricing the firm s, the firm locates in B and Government A makes rent $r_{As}(\tau_{As}, \tau_{Bs}) = 0$. As long as $\tau^{\lim}_{As}(\varepsilon) > a$, Government A has an incentive to deviate from τ_{As} by setting $\tau^{\lim}_{As}(\varepsilon)$, attracting the firm s to Jurisdiction A and making $r_{As}(\tau_{As}, \tau_{Bs}) > 0$. Only at $\tau^*_{As} = a$, $\tau^*_{Bs} = \tau^{\lim}_{Bs}(\varepsilon)$ does Government A not have a deviation that could make positive rents. In order to attract the firm s to A the government must set $\tau_{As} < a$ and this would violate condition (ii) of equilibrium.

As we have characterized a unique Nash equilibrium for all $s \in [0, z]$, we have demonstrated that there exists a unique Nash equilibrium in taxes. □

Proposition 5. *If $k < 1$ and $\varepsilon > 0$ sufficiently small then there exists a unique subgame perfect Nash equilibrium in pure strategies of the perfect tax discrimination game. Equilibrium is characterized by the point $a^* = 0$, $b^* = z$.*

Proof. We assume that $\varepsilon > 0$ and arbitrarily small. First we show that, for $a^* = 0$ and $b^* = z$, a^* is a best response to b^* and vice-versa.

Look for B's incentive to deviate. At $b^* = z$, $(a^* = 0)$, $|(z-b)-s| - |s-a| = 0$ for all $s \in [0, z]$. So $\tau^*_{Bs} = z - b = 0$ and $r_{Bs}(\tau^*_{As}, \tau^*_{Bs}) = \tau^*_{Bs} - (z-b) = 0$ for all s. Government B cannot deviate by raising b, so the only option would be to deviate by lowering b. But by Proposition 3, it follows that if Government B sets $b < z$ so that $\tau_{Bs} = z - b > 0$, whilst Government A sets $\tau^*_{As} = a = 0$, then all firms make higher profits by locating in Jurisdiction A; and so $r_{Bs}(\tau^*_{As}, \tau_{Bs}) = 0$ for all s. So there is no profitable deviation for B. By symmetry, A has no incentive to deviate by raising a.

Now suppose that some other equilibrium exists where $a \in (0, z]$ and $b \in [0, z)$ and $z - b \geq a$. If $z - b = a$ then $\tau^*_{As} = \tau^*_{Bs} = z - b$ and $r_{As}(\tau^*_{As}, \tau^*_{Bs}) = r_{Bs}(\tau^*_{As}, \tau^*_{Bs}) = 0$ for all s. But by Proposition 3, Government A could attract all firms by lowering a and make positive rents. By symmetry, Government B has an incentive to raise b to a point where $z - b < a$ and attract all firms in order to make positive rents. Therefore, no values $a \in (0, z]$ and $b \in [0, z)$ can be an equilibrium. $\square$

Proposition 6. *If $k \geq 1$ then there exists no subgame perfect Nash equilibrium in pure strategies of the perfect tax discrimination game.*

Proof. First let $z - b > a$. Let $a^* \in \arg\max_a r_A(a, b, s, \varepsilon; k, z)$ and $b^* \in \arg\max_b r_B(a, b, s, \varepsilon; k, z)$. Using τ^*_{As} and τ^*_{Bs} from Proposition 4,

$$r_A(a, b, s, \varepsilon; k, z) = \int_{s \in [0,z]} r_{As}(\tau^*_{As}, \tau^*_{Bs}) = (a + (\hat{s} - a)/2)(1 + k)(z - a - b).$$

Taking the first derivative and solving for a yields a candidate for a^*:

$$a(b, k, z) = \frac{(k-1)(z-b)}{3k-1}.$$

Recall that, for $k \geq 1$, $\partial r_A / \partial a^2 = \frac{1}{2}\left(\frac{1}{k} - 2 - 3k\right) < 0$. So the objective function is concave.

Again, from Proposition 4,

$$r_B(a, b, s, \varepsilon; k, z) = \int_{s \in [0,z]} r_{Bs}(\tau^*_{As}, \tau^*_{Bs}) = (b + (z - b - \hat{s})/2)(k - 1)(z - a - b).$$

Taking the first derivative and solving for b yields a candidate for b^*:

$$b(a, k, z) = \frac{(1+k)(z-a)}{3k+1}.$$

Recall once again that $\partial r_B / \partial b^2 = \frac{1}{2}\left(2 + \frac{1}{k} - 3k\right) \leq 0$ for $k \geq 1$. So the objective function is concave (weakly for $k = 1$). Solving $a(b, k, z)$ and $b(a, k, z)$ simultaneously for a and b in terms of parameters k and z we have

$$\begin{aligned} a(k, z) &= \frac{(k-1)z}{4k} \text{ and} \\ b(k, z) &= \frac{(k+1)z}{4k}. \end{aligned}$$

At the points $a(k, z) = (k-1)z/4k$, $b(k, z) = (k+1)z/4k$, each government maximizes its rent.

Now suppose, contrary to the statement of the proposition, that there exists a subgame perfect equilibrium of this game. Then given the global concavity of the payoff functions $r_A(a, b, s, \varepsilon; k, z)$ and $r_B(a, b, s, \varepsilon; k, z)$, equilibrium must be characterized by the points following points:

$$\begin{aligned} a^* &= \frac{(k-1)z}{4k}; \\ b^* &= \frac{(k+1)z}{4k}. \end{aligned}$$

Now using these values calculate the difference in rents to governments a and b:

$$r_A(a^*, b^*, s, \varepsilon; k, z) - r_B(a^*, b^*, s, \varepsilon; k, z) = \frac{z^2}{4}.$$

Therefore, for any location $z - b > a$ chosen by Government B, it can profitably deviate by choosing $z - b = a$ and setting a tax $\tau_{Bs} = \tau^*_{As} - \varepsilon$, for all firms in the interval $[0, \hat{s})$. (Part of the additional surplus $z^2/4$ is transferred to the firms in this interval when government B sets $\tau_{Bs} = \tau^*_{As} - \varepsilon$, inducing them to move to B.) This deviation contradicts equilibrium. As there is always an incentive to deviate from $a \neq a^*$, $b \neq b^*$ no equilibrium can exist. □

Figure 1

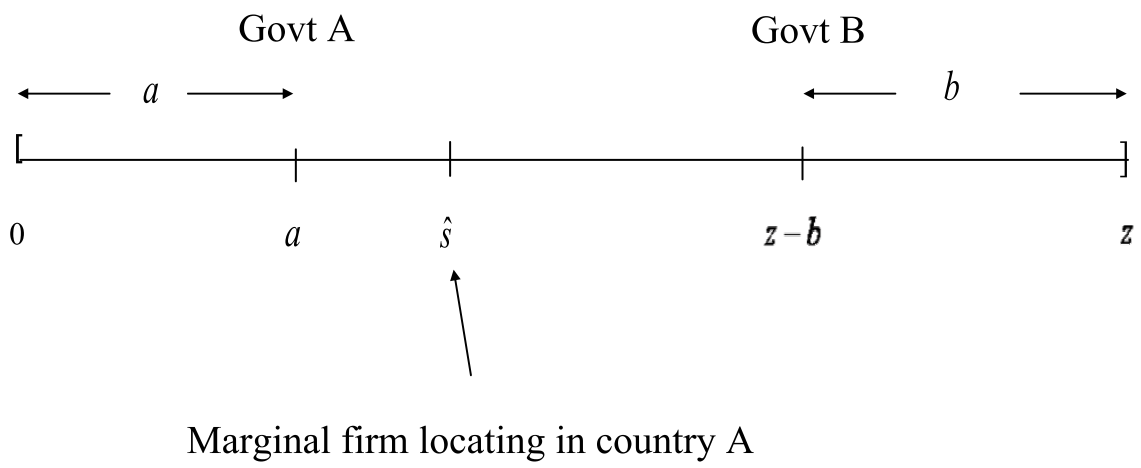
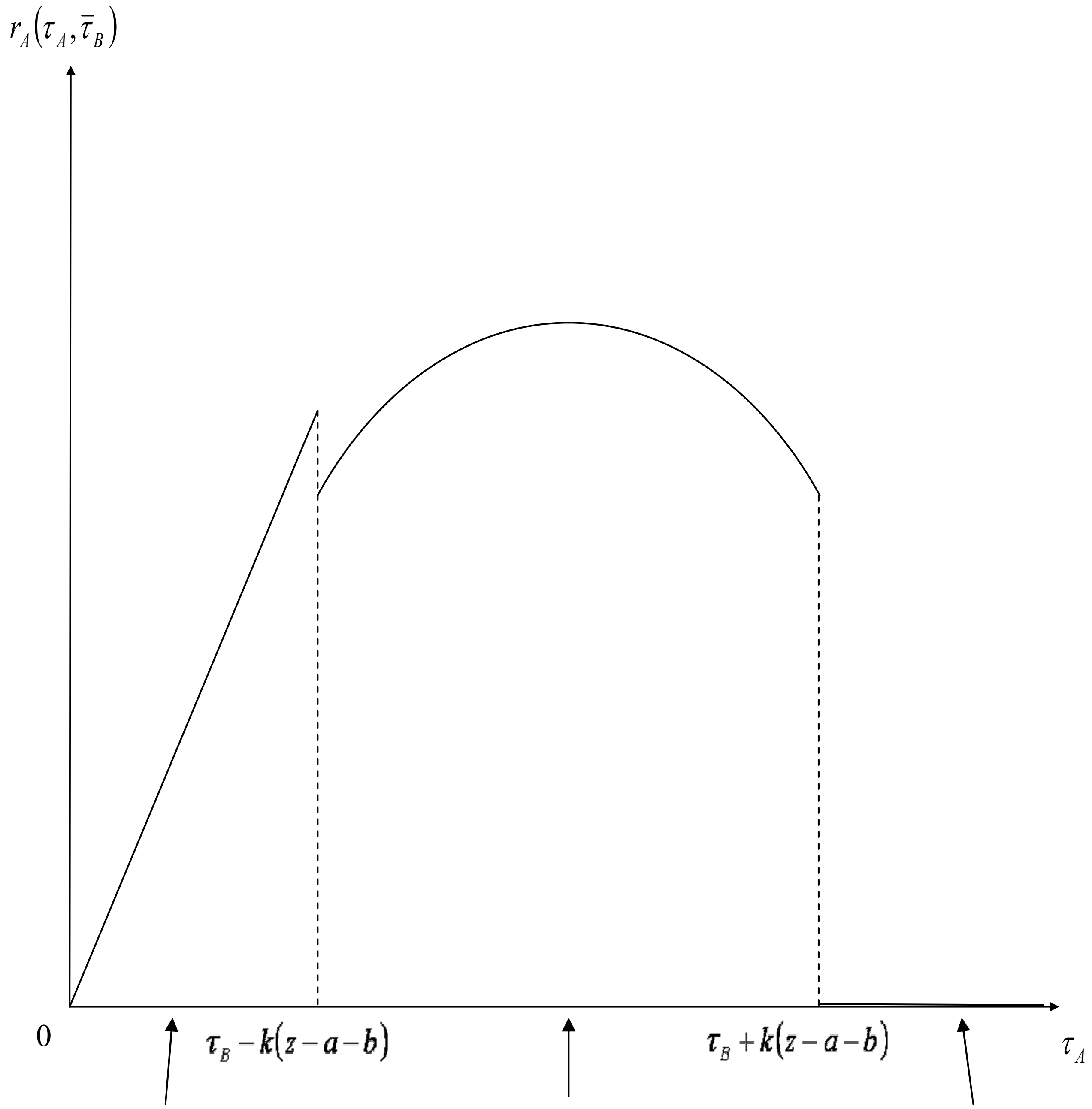


Figure 2



$r_A(\tau_A, \bar{\tau}_B)$
0
$\tau_B - k(z-a-b)$
$\tau_B + k(z-a-b)$
τ_A
All firms in A
Firm sharing interval
No firms in A

Figure 3