ANALYTICAL RESULTS
FOR A MODEL OF PERIODIC CONSUMPTION

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Analytical results for a Model of Periodic Consumption

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Abstract:

This paper presents the partial analytical solution to a model of periodic consumption that incorporates imperfect capital markets and uncertainty. Our model assumes that consumption decisions occur more frequently than income receipt. We show that the week specific consumption functions can be ordered. At low levels of wealth these functions exhibit a "*u*-shaped" pattern between income receipts. We show analytically that changes in the level of the borrowing constraint affect only the level of consumption function and not the MPC, whilst mean preserving changes in uncertainty affect both.

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Key Words: Analytical consumption function, liquidity constraints; uncertainty.

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1. Introduction

Analytical solutions to models of consumption that allow for uncertainty and capital market imperfections are difficult to obtain when preferences exhibit precautionary motives. The literature (see for example Zeldes, (1989), Deaton (1991), Gourinchas and Parker (2002), Kelly and Lanot (2002)) has relied upon numerical methods to calculate solutions to these models. In some cases, the absence of analytical solutions makes it difficult to understand the response of consumption behaviour to some changes in the environment. In particular Carroll (2001) argues, on the basis of numerical calculations, that the effect of liquidity constraints and precautionary motives are indistinguishable in the context of a consumption model with income uncertainty.

A small number of context specific analytical results have been obtained. For example, assuming quadratic preferences, Carroll and Kimball (2001) provide results on the interaction of explicit borrowing limits and uninsurable income risk, showing that both factors induce precautionary saving (defined as the reduction in optimal consumption relative to the perfect capital and no uncertainty case). Mason and Wright (2001) assume that capital markets are perfect and give an analytical approximation to the linearised Euler equation, when both income and asset return uncertainties exist in small quantities. Rabault (2002) shows analytically that when borrowing is restricted to the present value of the minimum possible flow of future uncertain income, it is optimal for an individual to borrow the maximum occasionally even when precautionary motives exist. He also provides the analytical optimal consumption rule.

In this paper we model the short run consumption behaviour of individuals: i.e. consumption decisions are taken regularly (perhaps each week) while income receipt occurs with lower frequency (say monthly). In this context it is plausible to assume that the limit on unsecured borrowing is set exogenously and that there is a significant difference between the cost of that short term borrowing (on overdrafts or credit cards, the Annual Percentage Rate is close to 20%) and the return of short run liquid savings (usually close to 0%). In the short run we assume that uncertainty arises because of weekly variation in price and availability of goods in each week while income and all other characteristics are kept fixed. This framework shares some characteristics with the (conventional) life-cycle consumption problem, and disentangling the implications of imperfect capital markets and uncertainty on the optimal short-run behaviour of risk averse individuals will add to our understanding of life-cycle consumption behaviour.

We first show that is it is never optimal for an individual to set consumption equal in any two consumption periods (weeks) within the monthly payment cycle. We show that this is true over all levels of wealth for identical price draws. Assuming Constant Relative Risk Aversion (CRRA) preferences, we derive the analytical consumption function in each consumption period for sufficiently low wealth levels such that the optimal consumption plan through the month always leads to a binding borrowing constraint in the week before income receipt (the "fourth" week). We also derive the maximum value of wealth for which this analytical solution is defined. This allows us to examine the effect of changing the level of the borrowing constraint and uncertainty. We show that in general the Marginal Propensity to Consume (MPC) is independent of the level of the borrowing constraint, while the range of wealth for which the borrowing constraint always binds in the final week of the payment cycle, is decreasing in the level of the constraint. In contrast, the MPC is decreasing in the level of uncertainty.

Hence this paper contributes to the literature in two important ways. Firstly, we assume that capital market imperfections give rise to restrictions on the price of credit in addition to the bound on the level of borrowing previously obtained in the literature. In this framework, we then provide an analytical solution for the optimal consumption rule when there is uninsurable risk (to prices). Secondly, we use this analytical solution to calculate and compare the effects on the MPC of changing the parameters of the model. Given our modelling assumptions, our results challenge the conclusions of Carroll and Kimball (2001) and Carroll (2001) that liquidity constraints and uncertainty are "virtually indistinguishable". This arises because we focus on the effect on the MPC rather than on precautionary saving.

In Section 2 we briefly describe the model of short run consumption and present the stationary solution to that model.[2] Section 3 presents the analytical solution for wealth levels such that the liquidity constraint is binding. Section 4 presents the comparative statics and contrasts the effects of uncertainty and liquidity constraints on optimal consumption. Section 5 concludes.

2. A Model of Periodic Consumption

The model we present here is designed to explain the recurring consumption behaviour in the "short run" where income uncertainty is absent and all other characteristics (endogenous or not) are kept fixed. Evidence we report elsewhere (see Kelly and Lanot, 2002) indicates that the level of weekly consumption varies within a payment period, and we model the source of this variability by assuming that consumption price is random. This can be understood literally or, perhaps better, as a metaphor for variation in the availability or the quality of some goods which make the price of consumption a random variable. To make the analysis easier we assume that such variations occur independently and according to a known distribution.

We assume that there are four (weekly) consumption periods within each (monthly) payment cycle[3]. The maximum borrowing in each week, $\bar{d}_i$, $i = 1,.,4$, is defined as the exogenous limit on monthly borrowing $\bar{d}$, discounted by the weekly cost of borrowing, d; i.e. we have $\bar{d}_1 \equiv \bar{d}/(1+d)^3$, $\bar{d}_2 \equiv \bar{d}/(1+d)^2$, $\bar{d}_3 \equiv \bar{d}/(1+d)$ and $\bar{d}_4 \equiv \bar{d}$, while the minimum wealth possible at the start of each week is $\underline{w}_i \equiv -(1+d)\bar{d}_{i-1}$. We assume $(1+d)\bar{d} < y$ and hence total debt can always be repaid from monthly income. The budget constraint in week i is given by

[2] The empirical content of the model is investigated in Kelly and Lanot (2002).

[3] The analysis presented here does not rely on this specific division of the month into four weeks. In principle, the results can be extended to the case where a month is made of thirty days.

$$p_i c_i \le w_i + y_i + d_i \,, \tag{1}$$

and because income is received periodically, it enters the budget constraint only for the first period of the payment cycle. The evolution of wealth follows the process

$$w_{i+1} = \left(w_i + y_i - p_i c_i\right)\left(1+r\right) + d_i (r - d)\,, \tag{2}$$

for *i = 1,2,3,4* where *i+1 = 1* if *i=4* and where

p_i : price draw in the current period,

c_i : consumption in the current period

w_i : wealth at the start of the current period,

y_i : regular income such that $y_i = y$ if income is received in *i*, and $y_i = 0$ otherwise.

d_i : debt incurred in period *i*, subject to a positive upper limit $\overline{d}_i$,

r : return on savings,

d : cost of borrowing,

and we assume $d > r$.

Total expenditure in week *i* is given by $p_i c_i$ and following Deaton, (1991), we use cash in hand to refer to the sum of wealth and income in the first week and the level of wealth in subsequent weeks. We assume that prices are uncertain and thus uncertainty is "accumulated" over the payment cycle. As a result, the total level of uncertainty varies implicitly between weeks.[4] The distribution of prices is denoted F with support P which is assumed a closed interval $\left[\underline{p}, \overline{p}\right]$, with $\underline{p} > 0$ and $1 < \overline{p} < \infty$. We assume further that $\mathrm{E}_{\mathrm{P}}\left[p\right] = 1$ and that all bounded functions of price can be integrated over the price

[4] When income is received there four weeks remain before the next income and price uncertainty is at its relative peak, while in the week immediately before income is received, uncertainty is at its relative lowest since the receipt of income in the next period allows it to be dealt with afresh.

distribution. Price draws are assumed to be *iid* and the realisation of current and future price are denoted by p and p respectively. We assume that y_i, $\bar{d}$, r and d are exogenous and known with certainty.

Individuals are infinitely lived, and consumption is chosen to maximise the discounted sum of utility flows over all time periods t

$$\max E_0 \sum_{t=0}^{\infty} b^t u(c_t)$$

subject to the week i specific budget constraint (1), where $u(c)$ is continuous, increasing and concave. Dynamic programming methods allow us to find the optimal consumption in any t as a function of nominal wealth w and price p. The state space for this particular problem depends on the wealth level, the price draw and the specific week in the month. Thus we write four (week specific) Bellman equations; the argument of the maximum in each case gives the optimal consumption as a function of wealth and the price draw in that week. These equations are given by

$$V_i(w,p) = \max_{\substack{pc \le w+y+d \\ 0 \le d \le \bar{d}_i}} u(c) + b \begin{Bmatrix} E_P\left(V_{i+1}\left((1+r)(w+y_i-pc),p\right)\right) \; if \quad w+y_i-pc>0 \\ E_P\left(V_{i+1}\left((1+d)(w+y_i-pc),p\right)\right) \quad otherwise \end{Bmatrix} \qquad (3)$$

for $i = 1,2,3,4$ where $i+1 = 1$ if $i=4$.

We assume i) CRRA preferences with felicity function of the form

$$u(c) = \frac{1}{1-r} c^{1-r} \; for \; r > 1,$$

where r is the coefficient of relative risk aversion and ii) individuals are relatively impatient i.e. $b(1+d) < 1$.

Define the sets $W_i = [\underline{w}_i, \infty)$, and Ω_i the Cartesian product $[\underline{w}_i, \infty) \times \left[\underline{p}, \bar{p}\right]$. We assume that the optimal weekly expenditure functions $g_i(w,p)$, i=1…4, are bounded and continuous with respect to their arguments and that $\partial g_i(w,p)/\partial w \equiv g'_{i,w}(w,p) > 0$, for all $(w,p) \in \Omega_i$. The marginal utility of money $q_i(w,p)$ is given by $\frac{1}{p} l(c_i)$ where

$l\left(c_i\right) = c^{-r}$ and therefore $g_i\left(w, p\right) = p\left(pq\left(w, p\right)\right)^{-1/r}$. The solution to the model in terms of the marginal utility of money for each week i is given by

$$q_i\left(w,p\right) = \max\left\{\begin{array}{l} b\left(1+r\right)\int_P q_{i+1}\left(\left(1+r\right)\left(w + y_i - p\left(pq_i\left(w,p\right)\right)^{-\frac{1}{r}}\right)p\right)dF(p), \\ \min\left\{\frac{1}{p}\left(\frac{w+y_i}{p}\right)^{-r},\ b\left(1+d\right)\int_P q_{i+1}\left(\left(1+d\right)\left(w + y_i - p\left(pq_i\left(w,p\right)\right)^{-\frac{1}{r}}\right)p\right)dF(p)\right\} \\ \frac{1}{p}\left(\frac{w + y_i + \bar{d}_i}{p}\right)^{-r} \end{array}\right\}, \qquad (4)$$

from which the consumption function for week i is instantly recoverable[5]. This solution encompasses four regimes that can be described as follows:

Regime 1: $q_i\left(w,p\right) = b\left(1+r\right)\int_P q_{i+1}\left(\left(1+r\right)\left(w + y_i - p\left(pq_i\left(w,p\right)\right)^{-\frac{1}{r}}\right)p\right)dF(p)$.

This regime occurs when wealth is sufficiently high that the optimal consumption is below cash in hand. Positive wealth is carried over to the next period and the marginal utility of current wealth is given by the expected value of marginal utility of savings in the next period, discounted at r.

Regime 2: $q_i\left(w,p\right) = \frac{1}{p}\left(\frac{w+y_i}{p}\right)^{-r}$.

In this regime, consumption is exactly cash in hand and it is optimal to carry zero assets between weeks because the individual is neither sufficiently patient enough to save at r nor impatient enough to borrow at d. The marginal utility of wealth is given by the marginal utility of cash in hand.

Regime 3: $q_i\left(w,p\right) = b\left(1+d\right)\int_P q_{i+1}\left(\left(1+d\right)\left(w + y_i - p\left(pq_i\left(w,p\right)\right)^{-\frac{1}{r}}\right)p\right)dF(p)$.

This regime describes behaviour when consumption is greater than cash in hand. It is optimal for the individual to have some positive level of borrowing d^* and to begin the next period

[5] Proof of the existence of a stationary solution to (4) under the assumptions above, and the methodology for reaching its numerical solution are given in Kelly and Lanot (2002).

with negative wealth equal to $-(1+d)d^{*}$. The marginal utility of wealth is set equal to the expected marginal utility of borrowing in the next period discounted at d .

Regime 4: $q_i(w,p)=\frac{1}{p}\left(\frac{w+y_i+\bar{d}_i}{p}\right)^{-r}$ **.**

In this regime, the liquidity constraint is binding. Consumption is equal to the maximum amount available (cash in hand plus available borrowing $\bar{d}_i$). Minimum wealth $-(1+d)\bar{d}_i$ is held at the start of the next period. Hence the solution for marginal utility of wealth is equal to the marginal utility of the sum of all available cash in hand and maximum borrowing.

The minimum in (4) arises because capital market imperfections prevent an individual from receiving a return equal to d on saving. If this is optimal but saving with return r is not, the individual chooses a higher level of consumption (and a lower level of marginal utility) than if saving at d were possible. With CRRA preferences, zero consumption has an infinite marginal utility and therefore it is only optimal to exhaust all borrowing (Regime 4) in the week immediately before income receipt (week 4).

The first proposition shows that the week specific solutions to (4) at a given wealth level and for the same price draw are never equal.

Proposition 1

For any given $p \in P$, the four week-specific solutions to the functional equation (4) do not meet at any levels of wealth.

□

Proof: Shown in the appendix.

An important consequence of Proposition 1 is that for a given price draw any ordering of the week specific marginal utility functions which can be shown to hold over a given (possibly small) wealth interval, holds over the entire wealth range..

[Insert Figure 1 here]

The week specific consumption (or expenditure or marginal utility) functions can be solved numerically, following Deaton and Laroque (1995, 1996). Figures 1(a) and 1(b) describe the solution for assumed values of the parameters and illustrates the result in Proposition 1. Figure 1(a) shows the consumption function (recovered from (4)) for each week in the wealth-consumption space, for a given price draw. Holding wealth and price constant across weeks, the consumption functions exhibit a *u*-shape over the payment cycle with consumption highest in the week of income receipt, lowest the week after and then increasing until income is again received. Clearly this ranking is maintained for all wealth levels. Obviously along a sample path of price realisations and allowing wealth to follow the law of motion in (2) this pattern may not hold. Figure 1(b) shows the solution to (4) in the wealth - marginal utility space for weeks 2 to 4. The values of wealth where the solutions (dashed lines) follow the marginal utility of cash in hand (regime 2, thin continuous line) are clear. It is also evident that only the week 4 solution follows the marginal utility of cash in hand plus the maximum borrowing (regime 4, bold continuous line), because this is the only week in which borrowing the maximum is optimal.

3. Partial Analytical Solution

In week 4, if wealth is such that consumption is characterised by regime 4, then the liquidity constraint is binding, and the expenditure function $g_4(w,p)$ is a linear function of wealth. This linearity is transmitted to the consumption functions in weeks 2 and 3[6]. In this section we derive the analytical solution for these linear sections of $g_2(w,p)$ and $g_3(w,p)$, (and therefore $q_2(w,p)$ and $q_3(w,p)$) and calculate their range of definition (see Propositions 2 and 3 below) .

[6] This propagation of linearity is consistent with Carroll and Kimball (2001) who show that even without uncertainty, the concavity of a consumption function induced by a liquidity constraint in any future time period, propagates to the optimal consumption rule in all earlier time periods.

The calculation of the solution proceeds as follows: Firstly, assume that the monthly borrowing limit $\overline{d}$, is greater than or equal to some fraction g of income y (a lower bound for g is given explicitly in Proposition 4). This limit on debt is sufficiently large that borrowing is optimal in all weeks of the payment cycle and d is the interest rate applicable between weeks. Although the analysis can be generalised for smaller borrowing limits we argue that this case is the most relevant since in practice we can expect individual to be faced with such a constraint.

In addition, assume that in week 4, a price dependent threshold $\hat{w}_4(p)$ exists such that for all wealth levels w that satisfy $\underline{w}_4 \leq w \leq \hat{w}_4(p)$, the liquidity constraint binds, the marginal utility of money is given by $q_4(w,p) = p^{-1}\left(w+\overline{d}/p\right)^{-r}$ and the expenditure function is defined as $g_4(w,p) = p\left(pq_4(w,p)\right)^{-1/r} = w+\overline{d}$. Then the expenditure function in week 3 $g_3(w,p) = p\left(pq_3(w,p)\right)^{-1/r}$ such that $(1+d)\left(w-g_3(w,p)\right) \leq \min_p \hat{w}_4(p)$ can be calculated. This function describes optimal expenditure in week 3 for all values of wealth (below a price dependent threshold, $\hat{w}_3(p)$), such that wealth at the start of week 4 results in maximum borrowing being optimal in that week, for all prices. Similarly, we find the solution in week 2, $g_2(w,p) = p\left(pq_2(w,p)\right)^{-1/r}$, such that $(1+d)\left(w-g_2(w,p)\right) \leq \min_p \hat{w}_3(p)$. This function gives optimal expenditure in week 2 for wealth levels (below $\hat{w}_2(p)$), which lead to maximum borrowing in week 4 irrespective of the price outcomes in weeks 3 or 4. Proposition 2 provides the analytical solutions for $q_2(w,p)$, $g_2(w,p)$, $q_3(w,p)$ and $g_3(w,p)$; the analytical descriptions of $\hat{w}_i(p)$ and g are given in Propositions 3 and 4, respectively.

Proposition 2

Assume $\overline{d} \geq g\,y$, and for some subset of W_4, $\left[\underline{w}_4, \hat{w}_4(p)\right]$, for all $p \in P$, the solution for the marginal utility of money in week 4 is given by

$$q_4(w,p) = \frac{1}{p} l\left(\frac{w+\bar{d}}{p}\right) = \frac{1}{p}\left(\frac{w+\bar{d}}{p}\right)^{-r},$$

and the expenditure function is given by $g_4(w,p) = w + \bar{d}$. *For* $p \in P$ *and for a subset of* W_3, $[\underline{w}_3, \hat{w}_3(p)]$, *the marginal utility of money in week 3 is given by*

$$q_3(w,p) = \frac{1}{p}\left(\frac{a_3(p)w + b_3(p)}{p}\right)^{-r},$$

and the expenditure function by

$$g_3(w,p) = a_3(p)w + b_3(p),$$

where $a_3(p) = C_3(p)/(1+C_3(p))$,

$$b_3(p) = a_3(p)\bar{d}/(1+d),$$

$$C_3(p) = \left(b(1+d)^{1-r} p^{1-r} \int_P p^{r-1} dF(p)\right)^{-1/r}.$$

Similarly in week 2, for all $p \in P$ *and a subset of* W_2, $[\underline{w}_2, \hat{w}_2(p)]$, *the marginal utility of money is given by*

$$q_2(w,p) = \frac{1}{p}\left(\frac{a_2(p)w + b_2(p)}{p}\right)^{-r},$$

and the expenditure function by

$$g_2(w,p) = a_2(p)\, w + b_2(p)$$

where $a_2(p) = C_2(p)/(1+C_2(p))$,

$$b_2(p) = a_2(p)\bar{d}/(1+d)^2,$$

$$C_2(p) = \left(b(1+d)^{1-r} p^{1-r} \int_P p^{r-1} a_3(p)^{-r}\, dF(p)\right)^{-1/r}.$$

Clearly $a_2(p) \le a_3(p) \le 1$ *and* $b_2(p) \le b_3(p) \le \bar{d}$.[7] *By implication, the marginal utility functions are ordered as follows*

[7] Recalling that the minimum wealth decreases from week 2 to 4, we can conclude that the weekly consumption functions do not cross at low levels of wealth.

$$q_4(w,p) < q_3(w,p) < q_2(w,p)$$

given p, for wealth levels that satisfy $\left[\underline{w}_i, \hat{w}_1(p)\right]$.

In addition, $q_1\left(-(1+d)\bar{d}, p\right) < q_4\left(-(1+d)\bar{d}, p\right)$.

□

Proof: Shown in the Appendix.

Proposition 2 shows that the MPC is lowest in week 2 when uncertainty is relatively high and income receipt is two weeks hence, but highest in week 4 when uncertainty is resolved and income receipt occurs in the following week. The intercepts follow the same pattern. Expenditure between income receipts is financed from a fixed level of "resources" (i.e. from the cash in hand and available borrowing); the optimal expenditure behaviour is such that a lower proportion of any increase in wealth is consumed in week 2 relative to week 3, and in week 3 relative to week 4. Obviously, the optimal marginal utility functions are also ordered between weeks 2, 3 and 4. The following Corollary extends the ordering to the first week optimal marginal utility and confirms that the same ordering holds beyond the definition range of the functions above.

Corollary 1

Under the conditions given in Proposition 1 and 2, for any given $p \in P$ *, the optimal marginal utility functions are ordered as follows*

$$q_1(w,p) < q_4(w,p) < q_3(w,p) < q_2(w,p),$$

for any $w \in [\underline{w}_1, \infty)$. *By implication, for any given* $p \in P$ *, the ordering of the expenditure functions is*

$$g_2(w,p) < g_3(w,p) < g_4(w,p) < g_1(w,p),$$

for any $w \in [\underline{w}_1, \infty)$.

□

Proof: Shown in the Appendix.

These results substantiate analytically, the relative position of the consumption functions evident in the numerical solution shown in Figure 1(a) and 1(b) for any given level of wealth within the whole wealth range. Consumption is always largest in the first week, smallest in the second week and then increasing until the week of income receipt. The consumption functions never cross and therefore it is clear that the effects of changing uncertainty and proximity to income receipt persist even at high levels of wealth, (although the numerical results suggest the magnitude of these effects are decreasing in wealth). Table 1 provides numerical values of $a_i(1)$: the MPC in week 2 is close to one third and in week 3 is close to one half.

The analytical solutions for the MPC and expenditure functions given in Proposition 2 apply only to wealth levels such that maximum borrowing at the end of the payment cycle is optimal, irrespective of price. Proposition 3 defines exactly the upper bound (the price dependent threshold values) for which the analytical solutions apply in each week.

Proposition 3

Assume that $\bar{d} \geq g\,y$ and at the minimum wealth in week 1 $-(1+d)\bar{d}$, optimal consumption in each of the following weeks is given by the analytical functions in Proposition 2. Then the upper bound on the subset of W_4, $[\underline{w}_4, \hat{w}_4(p)]$ such that $q_4(w,p) = p^{-1}(w + \bar{d}/p)^{-r}$ is given by

$$\hat{w}_4(p) = p.(pD_1(r))^{-\frac{1}{r}} - \bar{d},$$

where

$$D_1(r) \equiv b(1+d)A_1(r)\left(-(1+d)\bar{d} + y + \bar{d}/(1+d)^3\right)^{-r} \int_P \left(1 + A_1(r)^{-\frac{1}{r}} p^{1-\frac{1}{r}}\right)^r dF(p),$$

and

$$A_1(r) \equiv b(1+d)^{1-r}\int_P p^{r-1} a_2(p)^{-r}\, dF(p).$$

The upper bound on the subset of W_3, $[\underline{w}_3, \hat{w}_3(p)]$, *such that for all* $p \in P$,

$$q_3(w,p) = \frac{1}{p}\left(\frac{a_3(p)w + b_3(p)}{p}\right)^{-r}$$

is given by

$$\hat{w}_3(p) = \frac{\min_p \hat{w}_4(p) + a_3(p)\bar{d}}{(1+d)(1-a_3(p))}.$$

Similarly, the upper bound on the subset of W_2, $[\underline{w}_2, \hat{w}_2(p)]$, *such that for all* $p \in P$,

$$q_2(w,p) = \frac{1}{p}\left(\frac{a_2(p)w + b_2(p)}{p}\right)^{-r}$$

is given by

$$\hat{w}_2(p) = \frac{\min_p \hat{w}_3(p)(1+d) + a_2(p)\bar{d}}{(1+d)^2(1-a_2(p))}.$$

□

Proof: Shown in the Appendix.

These threshold points are price dependent, and are determined by the level of uncertainty and the borrowing limit. If wealth is greater than the threshold point then the individual is not constrained in the final week of the payment cycle and the analytical solutions given in Proposition 2 do not apply.

The derivation of the analytical functions in Proposition 2, is conditional upon $\bar{d} \geq g\,y$ i.e. borrowing occurs throughout the payment cycle and d is the relevant interest rate for wealth carried between periods within the cycle. This assumption implies that the maximum value of the week 2 kink, $\max_p \hat{w}_2(p)$, is negative. Proposition 4 provides the expression for the minimum value of g in terms of the other parameters of the model, such that the condition is true.

Proposition 4

Assume wealth is at the minimum in week 1 $-(1+d)\overline{d}$. *If*

$$g > \left(\frac{1}{\left(1+C_2\left(\overline{p}\right)\right)\left(1+C_3\left(\underline{p}\right)\right)K} + \frac{(1+d)^4 - 1}{(1+d)^3} \right)^{-1}$$

where (following the notation in Propositions 2 and 3)

$$K \equiv \left(b\,(1+d)\,A_1(r) \int_P \left(1 + A_1(r)^{-\frac{1}{r}} p^{1-\frac{1}{r}} \right)^r dF(p) \right)^{-1/r} \underline{p}^{1-\frac{1}{r}}$$

then

$$\max_p \hat{w}_2(p) = \hat{w}_2(\overline{p}) < 0 \,,$$

and borrowing is optimal for any price draw, throughout the payment cycle.

□

Proof: Shown in the Appendix.

If this condition holds, then the wealth level following the repayment of the maximum debt, $-(1+d)\overline{d}$, from income, *y*, is sufficiently low that as a consequence of optimal consumption in week 1, borrowing must occur. Borrowing is optimal in all subsequent weeks of the payment cycle, and the borrowing constraint binds in week 4. Table 2 presents numerical values for the lower bound on g given above. That bound is decreasing in risk aversion and uncertainty. Hence for a given level of uncertainty, a more risk averse individual will borrow throughout the payment cycle for a lower level of the borrowing limit than a less risk averse individual. Similarly, greater uncertainty results in borrowing through the payment cycle at a lower level of the borrowing limit, given risk aversion.

The assumption that d applies through the payment cycle simplifies our analysis substantially because it ensures that the interest rate applied between periods does not depend on the wealth or price level. If this were the case, the derivation of the analytical solutions would be unreasonably complex, although not impossible. An alternative to this assumption is to suppose that $\overline{d}$ is *less* than or equal to some fraction g of income y, where g is sufficiently small that *positive* wealth is always carried over the payment cycle until the final week. Hence r is the applicable interest rate. The analytical expression in this case is obtained following identical arguments to the one we describe earlier. This assumption requires that the minimum week 4 kink, $\min_p \hat{w}_4(p)$, is positive, and gives an upper bound on the level of the borrowing limit

$$g < \left(\frac{1}{K} + \frac{(1+d)(1+r)^3 - 1}{(1+r)^3} \right)^{-1},$$

where K is defined as before. Numerical calculations for the value of the parameters used previously show that this upper bound on g is between 0.2 and 0.3. For values of g such that

$$\left(\frac{1}{K} + \frac{(1+d)(1+r)^3 - 1}{(1+r)^3} \right)^{-1} < g < \left(\frac{1}{(1+C_2(\overline{p}))(1+C_3(\underline{p}))K} + \frac{(1+d)^4 - 1}{(1+d)^3} \right)^{-1},$$

our analytical solution must be modified so that the applicable interest rate is a function of price and wealth. However, our results still provide a good qualitative guide for consumption behaviour.

4. Comparative Statics

So far our formal results describe the partial analytical solution for the week specific consumption function, establish their range of definition and the relative position of these weekly functions over the payment cycle. One question of interest in the literature on life-cycle consumption is the difference between the effects of the borrowing limit and uncertainty on optimal behaviour, when individuals have precautionary motives. The analytical solutions in Proposition 2 allow us to calculate (we believe for the first time in the literature) the exact effect on optimal behaviour of *changes* in these parameters. Thus far the literature has, in contrast, focussed on the effects of *introducing* incomplete capital and insurance markets on optimal consumption and precautionary saving.[8] These results are now presented in Propositions 5 and 6.[9]

Proposition 5

Assume the conditions in Proposition 1 and Proposition 2 hold, therefore the week specific expenditure functions are such that, i.e.

$$g_i(w,p)=a_i(p)w+b_i(p),\ \textit{for}\ \underline{w}_i \le w \le \hat{w}_i(p),\ \textit{and}\ i = 2,3,4.$$

with $b_2(p)=a_2(p)\bar{d}/(1+d)^2$, $b_3(p)=a_3(p)\bar{d}/(1+d)$, $b_4(p)=\bar{d}$.

The effect on the expenditure function of a change in the borrowing limit for each week is

Week 2: $$\frac{\partial}{\partial \bar{d}}\left(a_2(p)w+b_2(p)\right)=a_2(p)/(1+d)^2;\quad 0<a_2(p)/(1+d)^2<1. \qquad 5.i$$

Week 3: $$\frac{\partial}{\partial \bar{d}}\left(a_3(p)w+b_3(p)\right)=a_3(p)/(1+d);\quad 0<a_3(p)/(1+d)<1 \qquad 5.ii$$

Week 4: $$\frac{\partial}{\partial \bar{d}}\left(w+\bar{d}\right)=1. \qquad 5.iii$$

In addition, note the size of this effect is independent of wealth;

$$\frac{\partial^2}{\partial \bar{d}\partial w}\left(a_i(p)w+b_i(p)\right)=0. \qquad 5.iv$$

The bounds on wealth for which the analytical functions are defined are decreasing in the borrowing limit;

$$\frac{\partial \hat{w}_i(p)}{\partial \bar{d}}<0,\qquad -1\le\frac{\partial \underline{w}_i}{\partial \bar{d}}<0,\qquad \frac{\partial \hat{w}_i(p)}{\partial \bar{d}}\le\frac{\partial \underline{w}_i}{\partial \bar{d}}. \qquad 5.v$$

The upper bound is increasing in price and the lower bound is independent of price;

[8] However, our results are limited to the consumption function (in Proposition 2), which applies only over the range of wealth defined in Proposition 3 and therefore may not generalise to the consumption function defined over the range of wealth $[\underline{w}_i,\infty)$.

[9] Despite many attempts we have not been able to obtain conclusive results concerning the effect a change in the risk aversion parameter on the expenditure functions.

$$\frac{\partial \hat{w}_i(p)}{\partial p} > 0 . \qquad \frac{\partial \underline{w}_i}{\partial p} = 0 . \qquad 5.vi$$

□

Proof: Shown in the Appendix.

A relaxation of the borrowing limit increases only the intercept of the consumption function. Given $b_i(p) = a_i(p)\overline{d}_i$, this increase depends on the size of the slope of the consumption function but is independent of the wealth level. Hence the increase is smallest in week 2, largest in week 4 but is constant in each week for all wealth levels $\underline{w}_i \leq w \leq \hat{w}_i(p)$. The range of definition for the analytical solution in Proposition 2 is shifted leftward (since both bounds of the range of definition are decreasing in the credit limit), but it shrinks in size (because the rate of decrease of the upper bound is greater than that of the lower bound). A higher price draw causes the constraint to bind for a larger range of wealth, cet. par. Increasing the borrowing limit shifts the range of definition over negative wealth levels to the left, and simultaneously causes an increase in optimal consumption. These combined effects imply a parallel leftward shift of the consumption function in wealth-consumption space.[10]

The weekly consumption functions are non differentiable at the kink points $\hat{w}_i(p)$. Hence the results in *5.i – iii* apply only to wealth levels strictly below the kink and show that a change in the borrowing limit has no effect on the MPC. However, for values of wealth at or close to the kink point, this non-differentiability has an important effect because an increase in the borrowing limit will reduce the kink point, $\hat{w}_i(p)$. Obviously if the wealth level *w*, is above the (new) smaller threshold value the analytical derivatives no longer apply. Nevertheless the MPC is lower at *w*, and consumption is increased. Similarly, a reduction in

[10] These results are the analytical equivalent in the context of our model of the numerical results in Carroll (2001) who shows that an increase in the borrowing limit causes the consumption function to shift to the left by the percentage increase in the borrowing limit.

the borrowing limit will increase the kink point, increasing the MPC for wealth levels just above the initial kink point, but below the (new) higher kink.

[Insert Figure 2 here]

These effects are best illustrated numerically for a large change in the borrowing limit: Figure 2 shows the numerical solution for each week with the borrowing limit at 66% and 90% of income. The consumption functions in each week are parallel at levels of wealth below the kink (e.g. $\tilde{w}_a$) indicating that the increase in consumption is independent of wealth and the MPC is unchanged when the borrowing limit increases. Observe that at wealth levels just below the initial kink point, (e.g. $\tilde{w}_b$), the consumption functions are no longer parallel and the MPC is reduced following the increase in the borrowing limit. While we cannot calculate analytically the increase in consumption in this second case, the numerical results suggest that it is less than the increase when the MPC is unchanged.

We turn now to the effect of a mean preserving increase in spread (MPS) in the distribution of uncertainty. The results in Proposition 6 allow us to contrast the effect on the MPC of changing the level of uncertainty with those in the previous proposition for the borrowing limit. [11]

Proposition 6

Assume the distribution of prices $\tilde{F}(p)$ *represent a MPS of the distribution of prices* $F(p)$.

(i) $r > 2$ *is a necessary and sufficient condition for*

$$a_3(p)_{|F(p)} > a_3(p)_{|\tilde{F}(p)} \ ,$$

and

$$b_3(p)_{|F(p)} > b_3(p)_{|\tilde{F}(p)}$$

[11] Unfortunately the techniques that we use in the proof of this proposition do not allow us to say whether a MPS has a positive or negative effect on $\hat{w}_i(p)$. We can observe numerically that both effects are possible and that the effects can differ in sign across weeks depending on the parameters of the model.

i.e. the MPS reduces the slope and intercept of the consumption function in week 3.

(ii) $r > \max(2, \hat{r})$ *is a sufficient condition for*

$$a_2(p)_{|F(p)} > a_2(p)_{|\tilde{F}(p)}$$

and

$$b_2(p)_{|F(p)} > b_2(p)_{|\tilde{F}(p)}$$

i.e. the MPC reduces the slope and intercept of the consumption function in week 2, where $\hat{r}$ *denotes the minimum value of* r *for which* $\left(r(r-2)\left(b(1+d)^{1-r}\int_P p^{r-1}dF(p)\right)^{\frac{1}{r}}\right)^{\frac{r}{r-1}} > \bar{p}$ *for a given distribution* $F(p)$ *with support* P *over the closed interval* $[\underline{p}, \bar{p}]$.

□

Proof: Shown in the Appendix.

Proposition 6 shows that if $r > \max(2, \hat{r})$ an increase in the level of uncertainty (in a mean preserving sense), will generally lead to a reduction in the MPC in weeks 2 and 3 for wealth levels below the threshold, $\hat{w}_i(p)$. Because the minimum wealth level is unchanged, this implies a fall in the level of consumption (i.e. the value of the intercepts decrease) in weeks 2 and 3 for all price draws. The consumption function in week 4 is unaffected by the level of uncertainty.

[Insert Figure 3 here]

Numerical analysis for some reasonable parameter values of the effect of a MPS reveals that $\hat{r}$, the minimum value of r that fulfils the sufficient condition for a MPS to reduce $a_2(p)$ given in *6.ii*, is greater than the value of r where $a_2(p)$ actually falls with a MPS. Figure 3 shows that $\hat{r}$ lies between 2.4 and 2.5, while a MPS reduces $a_2(p)$ when r is close to 2.16. To gain an intuition for these results, it is necessary to decompose the effect of the MPS on $a_2(p)$ into two parts. The first component relates to expectations taken over the price distribution in week 4 and is negative if $r > 2$. The second term relates to expectations taken over the price distribution in week 3 and is negative when $r > \hat{r}$. The relative size of these terms is such that for a value of r where $2 < r < \hat{r}$, the first element dominates the second and hence the total effect of the MPS on $a_2(p)$ is negative, despite the sufficient condition being violated. Clearly for all $r > 2$, a MPS will reduce $a_3(p)$, while some values of r within the range, $2 < r < \hat{r}$, it will increase $a_2(p)$. For $r > \hat{r}$, a MPS has an unambiguously negative effect on both slopes.

It is useful to contrast the results from Propositions 5 and 6 with those of Carroll and Kimball (2001). They focus on the concavity of the value and consumption functions induced by both uncertainty and liquidity constraints, relative to those functions when there are no capital market imperfections or uncertainty. Hence, the cause of precautionary saving and high MPC out of predictable changes in income at low values of wealth cannot be identified. They then conclude that with CRRA preferences and a pre-existing liquidity constraint the effect on precautionary saving and consumption of introducing an additional borrowing constraint is the same as introducing risk to income. With our model, where both a borrowing constraint and uncertainty exist, the effect on the MPC of introducing a tighter constraint is different from that of changing the level of uncertainty.[12] Relaxing the borrowing limit causes

[12] Carroll and Kimball (2001) show that with quadratic preferences, the effect of an increase in uncertainty (in a mean preserving sense) when a liquidity constraint pre-exists depends on whether the change in the support of the distribution affects the probability that the liquidity constraint will bind. If not, then there is no effect on the optimal consumption function because marginal utility is linear each side of the kink point induced by the liquidity constraint. The analytical solution in Proposition 2 is derived by assuming that the liquidity constraint binds in the final week of the month for every price with probability equal to 1 and this is independent of the support of the price distribution. We show in Proposition 6 that despite that assumption, a MPS of the price distribution reduces the MPC in each

a leftward shift of the consumption function, increases the level of consumption and for wealth levels below the kink, has no effect on the MPC. For values of wealth where the increase in the borrowing limit results in optimal consumption changing from regime 4 to 3, the MPC is reduced. Conversely, tightening the liquidity constraint either increases or has no effect on the MPC. In contrast, if $r > \max(2, \hat{r})$, the MPC is decreasing in the level of uncertainty (as measured by a MPS) for all wealth levels below $\hat{w}_i(p)$.

5. Conclusion

In this paper, we presented a model of short-run consumption smoothing that incorporates features that are common to both long and short run problems (uncertainty and imperfect capital markets). We also allow for the periodic receipt of income and assume that the source of uncertainty is variation in prices, which are both characteristics of the short-run consumption problem. Our modelling approach follows the inventory models of commodity prices (see Deaton and Laroque, 1995) with periodic harvests (see Chambers and Bailey, 1996).

We show that, for a given price, it is never an optimal solution to the model, to set marginal utility at any given wealth equal in any two weeks. This property allows us to order the week specific consumption functions. This result does not depend on either functional form assumptions or on restrictions such as those used in Chambers and Bailey (1996).

We calculate the analytical solution for marginal utility and consumption for low levels of wealth when the borrowing constraint always binds. This allows us to show the relative position of the consumption functions across all weeks and given that any ordering that exists is always preserved, the same relative positions apply to the solutions over the whole range of wealth. At a given wealth and price, consumption is highest in the week of

week. This difference arises because we assume CRRA preferences and hence the marginal utility is everywhere non linear.

income receipt, lowest in the next week and increasing until the next receipt of income. This ordering over all wealth levels, is confirmed by the numerical solution to the model, calculated assuming a CRRA felicity function and acceptable values of the other parameters.

We also characterise the comparative statics for the model and show that tightening the borrowing constraint will either increase or have no effect on the MPC and raise the level of consumption, whilst increasing the level of uncertainty (in a mean preserving sense) reduces both the MPC and the level of consumption. Our calculations differ from the existing literature because we examine the effect of changing the level of these parameters rather than the effect of introducing them. Hence, in contrast to previous results (Carroll, 2001, Carroll and Kimball 2001), we show that the effects of these parameters on optimal behaviour can be distinguished by examining the MPC.

Appendix.

Proof of Proposition 1.

For any p in the price support, suppose that for the wealth level $\underset{\sim}{w}(p)$ and $\underset{\sim}{w}(p) > -\bar{d}_3$, the solutions to (4) for the third and fourth week, i.e. $q_3(w,p)$ and $q_4(w,p)$, are such that:

$$w \neq \underset{\sim}{w}(p),\; w > -\bar{d}_3;\; q_3(w,p) \neq q_4(w,p), \tag{A1}$$

$$w = \underset{\sim}{w}(p); \qquad q_3(w,p) = q_4(w,p). \tag{A2}$$

$q_3\left(\underset{\sim}{w}(p), p\right)$ is determined by (4) above, and it easy to see that (A2) above implies

$$q_4\left(\underset{\sim}{w}(p), p\right) = \max \left\{ \begin{array}{l} b(1+r)\int_P q_4\left((1+r)\left(\underset{\sim}{w}(p) - p\left(pq_4\left(\underset{\sim}{w}(p),p\right)\right)^{-\frac{1}{r}}\right), p\right) dF(p), \\ \min\left\{ \frac{1}{p}\left(\frac{\underset{\sim}{w}(p)}{p}\right)^{-r},\; b(1+d)\int_P q_4\left((1+d)\left(\underset{\sim}{w}(p) - p\left(pq_4\left(\underset{\sim}{w}(p),p\right)\right)^{-\frac{1}{r}}\right), p\right) dF(p) \right\} \\ \frac{1}{p}\left(\frac{\underset{\sim}{w}(p)+\bar{d}_3}{p}\right)^{-r} \end{array} \right\}. \tag{A3}$$

Consider the following consumption problem with $w > -\bar{d}_3$ and $p \in P$:

$$W(w,p) = \max_{\substack{pc \leq w+d \\ 0 \leq d \leq \bar{d}_3}} u(c) + b \left\{ \begin{array}{ll} E_P\left(W\left((1+r)(w-pc), p\right)\right) & if \quad w - pc > 0 \\ E_P\left(W\left((1+d)(w-pc), p\right)\right) & otherwise \end{array} \right\}, \tag{A4}$$

which corresponds to the case where the individual is endowed with some wealth level but does not receive any income thereafter and must plan her consumption in the future. Note that the problem is well defined for all wealth levels, even negative, as long as $w > -\bar{d}_3$. Clearly the first order condition that define the unique optimal solution (given the assumptions concerning the admissible solution we made earlier in the text) for this problem is given by a function $m(w,p)$ such that

$$m(w,p)=$$

$$\max\left\{\begin{array}{l} b(1+r)\int_P m\left((1+r)\left(w-p\left(pm(w,p)\right)^{-\frac{1}{r}}\right),p\right)\,dF(p), \\ \min\left\{\frac{1}{p}\left(\frac{w}{p}\right)^{-r},\ b(1+d)\int_P m\left((1+d)\left(w-p\left(pm(w,p)\right)^{-\frac{1}{r}}\right),p\right)\,dF(p)\right\} \\ \frac{1}{p}\left(\frac{w+\bar{d}_3}{p}\right)^{-r} \end{array}\right\}, \tag{A5}$$

Hence at $\left(\underset{\sim}{w}(p),p\right)$ the solution to (A3) is an optimal solution for (A4).

However the budget set for the problem described in (A4) is strictly included in the budget set of the original periodic income receipt problem considered in (3) and therefore the optimal solution defined by (A4) yields a lower level of expected discounted utility than the optimal solution to (3). Hence an optimal solution to (4) in week 3 cannot have property (A2) , i.e. for any given p, and $w>-\bar{d}_3$, the optimal solutions $q_3(w,p)$ and $q_4(w,p)$ cannot meet. An identical argument establishes the same property for $q_2(w,p)$ and $q_3(w,p)$, and $q_4(w,p)$ and $q_1(w,p)$.

Consider now $q_1(w,p)$ and $q_2(w,p)$. For any p in the price support, suppose that for the wealth level $\underset{\sim}{w}(p)$ and $\underset{\sim}{w}(p)>-\bar{d}_1$, the solutions to (4) for the first and second week are such that:

$$w\neq\underset{\sim}{w}(p),\ w>-\bar{d}_1;\ q_1(w,p)\neq q_2(w,p), \tag{A6}$$

$$w=\underset{\sim}{w}(p);\qquad q_1(w,p)=q_2(w,p). \tag{A7}$$

Then (A7) and (4) together imply that at $\left(\underset{\sim}{w}(p),p\right)$

$$q_2\left(\underset{\sim}{w}(p),p\right)=$$

$$\max\left\{\begin{array}{l} b(1+r)\int_P q_2\left((1+r)\left(\underset{\sim}{w}(p)+y-p\left(pq_2\left(\underset{\sim}{w}(p),p\right)\right)^{-\frac{1}{r}}\right),p\right)dF(p), \\ \min\left\{\frac{1}{p}\left(\frac{\underset{\sim}{w}(p)+y}{p}\right)^{-r},\ b(1+d)\int_P q_2\left((1+d)\left(\underset{\sim}{w}(p)+y-p\left(pq_2\left(\underset{\sim}{w}(p),p\right)\right)^{-\frac{1}{r}}\right)p\right)dF(p)\right\} \\ \frac{1}{p}\left(\frac{\underset{\sim}{w}(p)+y+\bar{d}}{p}\right)^{-r} \end{array}\right\} \tag{A8}$$

which is the solution at $\left(\underset{\sim}{w}(p), p\right)$ to a consumption problem comparable to (A4) where a positive income *y* is received every period. The budget associated with the problem solved by (3) (where income receipt is periodic) is strictly within the budget set implicit in (A8) and as a consequence of monotonicity of $u(c)$ the solution to (A8) can not be a solution to (3). By implication, (A7) cannot be true for solutions to (4). For any given p, where $w > -\bar{d}_1$, $q_1(w,p)$ and $q_2(w,p)$ cannot meet.

Proof of Proposition 2.

We know that for all $p, p \in P$ and for $\underline{w}_4 \leq w \leq \hat{w}_4(p)$

$$q_4(w,p) = \frac{1}{p} l\left(\frac{w+\bar{d}}{p}\right) \qquad \text{(A9)}$$

For all $p \in P$, and for $\underline{w}_3 \leq w \leq \hat{w}_3(p)$, assume that the solution $q_3(w,p)$ is such that

$$q_3(w,p) = \frac{1}{p} l\left(\frac{a_3(p)w + b_3(p)}{p}\right). \qquad \text{(A10)}$$

Clearly this implies that expenditure is a linear function of wealth $g_3(w,p) = p\, l^{-1}\left(pq_3(w,p)\right) = a_3(p)w + b_3(p)$. From (4) observe that

$$q_3(w,p) = b(1+d)\int_P q_4\left(\left((1+d)\left(w - p\, l^{-1}\left(pq_3(w,p)\right)\right)\right), p\right) dF(p) \qquad \text{(A11)}$$

Denote $\min_p \hat{w}_i(p) \equiv \underline{\hat{w}}_i$. In addition, note

$$\left((1+d)\left(w - p\, l^{-1}\left(pq_3(w,p)\right)\right)\right) = \left((1+d)\left((1-a_3(p))w - b_3(p)\right)\right) \leq \underline{\hat{w}}_4$$

implying that the liquidity constraint is always binding in week 4, in which case (A9) holds. Therefore applying marginal utility, $l(x) = x^{-r}$, the solution for q_4 must be

$$q_4\left((1+d)\left((1-a_3(p))w - b_3(p)\right), p\right) = \left((1+d)\left((1-a_3(p))w - b_3(p)\right) + \bar{d}\right)^{-r} p^{r-1}.$$

Then integrating over the distribution of prices *F* gives

$$\int_P q_4\left((1+d)\left((1-a_3(p))w-b_3(p)\right),p\right)dF(p)$$
$$=\left((1+d)\left((1-a_3(p))w-b_3(p)\right)+\bar{d}\right)^{-r}\int_P p^{r-1}dF(p).$$

Substituting this into the right hand side of (A11) gives

$$\begin{aligned} q_3(w,p) &= \frac{1}{p}\left(\frac{a_3(p)w+b_3(p)}{p}\right)^{-r} \\ &= b\,(1+d)\left((1+d)\left((1-a_3(p))w-b_3(p)\right)+\bar{d}\right)^{-r}\int_P p^{r-1}dF(p). \end{aligned} \tag{A12}$$

For ease of notation, let

$$A_3(r)=b\,(1+d)^{1-r}\int_P p^{r-1}dF(p) \text{ and } B(p)=p^{r-1},$$

then (A12) becomes

$$B(p)\left(a_3(p)w+b_3(p)\right)^{-r}=A_3(r)\left((1-a_3(p))w-b_3(p)+\bar{d}/(1+d)\right)^{-r}. \tag{A13}$$

The inverse of marginal utility is given by $l^{-1}(x)=x^{-1/r}$, and using this to rewrite (A13) gives

$$a_3(p)w+b_3(p)=C_3(p)\left((1-a_3(p))w-b_3(p)+\bar{d}/(1+d)\right),$$

where $C_3(p)=\left(\frac{A_3(r)}{B(p)}\right)^{-1/r}$. Then over the range $\underline{w}_3\le w\le \hat{w}_3(p)$ for which (A10) is defined, it must be the case that

$$a_3(p)w=C_3(p)(1-a_3(p))w \text{, and } b_3(p)=C_3(p)\left(\bar{d}/(1+d)-b_3(p)\right).$$

Together these expressions imply

$$a_3(p)=\frac{C_3(p)}{1+C_3(p)},\qquad b_3(p)=\frac{C_3(p)}{1+C_3(p)}\frac{\bar{d}}{(1+d)}=a_3(p)\frac{\bar{d}}{1+d}.$$

Finally substituting these solutions into (A9) yields Proposition 2(i).

Proposition 2(ii) follows the same arguments where for $\underline{w}_3\le w\le \hat{w}_3(p)$ and for all $p\in P$

$$q_3(w,p)=\frac{1}{p}\,l\left(\frac{a_3(p)w+b_3(p)}{p}\right)$$

$$= b(1+d)\int_P q_4\left(\left((1+d)\left((1-a_3(p))w - b_3(p)\right)\right), p\right) dF(p) .$$

Then for all $p \in P$, for $\underline{w}_2 \le w \le \hat{w}_2(p)$, the solution $q_2(w,p)$ is such that

$$q_2(w,p) = \frac{1}{p} l\left(\frac{a_2(p)w + b_2(p)}{p}\right)$$

$$= b(1+d)\int_P q_3\left(\left((1+d)\left(w - p l^{-1}(pq_2(w,p))\right)\right), p\right) dF(p) . \qquad \text{(A14)}$$

Using a linear expenditure function $g_2(w,p) = p l^{-1}(pq_2(w,p)) = a_2(p)w + b_2(p)$

and assuming that the wealth carried into week 3 satisfies

$$\left((1+d)\left(w - p l^{-1}(pq_2(w,p))\right)\right) = (1+d)\left(w - (a_2(p)w + b_2(p))\right) \le \underline{\hat{w}}_3$$

Substituting the solution for $g_2(w,p)$, and noting $b_3(p) = a_3(p)\bar{d}/(1+d)$ this becomes

$$q_3\left((1+d)\left(w - p l^{-1}(pq_2(w,p))\right), p\right)$$
$$= \left(a_3(p)\left\{(1+d)\left((1-a_2(p))w - b_2(p)\right)\right\} + a_3(p)\bar{d}/(1+d)\right)^{-r} p^{r-1}.$$

Integrating over the distribution of prices, F, and substituting this into the right hand side of (A14) gives

$$q_2(w,p) = \frac{1}{p}\left(\frac{a_2(p)w + b_2(p)}{p}\right)^{-r}$$
$$= b(1+d)\left\{(1+d)(1-a_2(p))w - (1+d)b_2(p) + \frac{\bar{d}}{(1+d)}\right\}^{-r} \int_P p^{r-1} a_3(p)^{-r}\, dF(p)$$

Let $A_2(r) = b(1+d)^{1-r}\int_P p^{r-1} a_3(p)^{-r}\, dF(p)$, and $B(p) = p^{r-1}$. This simplifies to

$$B(p)\left(a_2(p)w + b_2(p)\right)^{-r} = A_2(r)\left((1-a_2(p))w - b_2(p) + \bar{d}/(1+d)^2\right)^{-r},$$

and defining $C_2(p) = \left(\frac{A_2(r)}{B(p)}\right)^{-1/r}$, this gives

$$a_2(p)w + b_2(p) = C_2(p)\left((1-a_2(p))w - b_2(p) + \bar{d}/(1+d)^2\right).$$

For the range $\underline{w}_2 \le w \le \hat{w}_2(p)$ for which (A13) holds,

$$a_2(p) = \frac{C_2(p)}{1+C_2(p)}, \qquad b_2(p) = \frac{C_2(p)}{1+C_2(p)}\frac{\bar{d}}{(1+d)^2},$$

defining the analytical solution for $q_2(w,p)$ and $g_2(w,p)$.

The $C_i(p)$ terms are positive (giving $0 \le a_i(p) \le 1$) and differ only by a factor of $a_3(p)^{-r}$ in the function that is integrated over the distribution of prices F. Given $a_3(p)^{-r} \ge 1$, then $C_2(p) \le C_3(p)$ which is a sufficient condition for $a_2(p) \le a_3(p)$. In addition $b_i(p) = a_i(p)\overline{d}_i$, and therefore $b_2(p) \le b_3(p) \le \overline{d}$.

Given $\underline{w}_4 \equiv -\overline{d} > -(1+d)\overline{d}$ and $\overline{d}_4 \equiv \overline{d}$, the maximum possible consumption in week 4 for wealth $-(1+d)\overline{d}$ is $-(1+d)\overline{d} + \overline{d}$ and this is clearly negative. Hence the marginal utility cost of $-(1+d)\overline{d}$ is infinite in week 4 i.e. $q_4(-(1+d)\overline{d}, p) = +\infty$, and hence $q_1(-(1+d)\overline{d}, p) < q_4(-(1+d)\overline{d}, p)$.

□

Proof of Corollary 1.

Proposition 1 implies that if an ordering exists between the solutions at any level of wealth, then that ordering is preserved at all levels of wealth. Proposition 2 provides an ordering of the marginal utility and consumption functions over the weeks at low levels of wealth. Hence the ordering shown in Proposition 2 is preserved over all wealth levels.

Proof of Proposition 3

Let $\hat{w}_4(p)$ be the maximum value of wealth such that the optimal choice of consumption involves maximum borrowing and wealth is at the minimum at the start of week 1. Then

$$q_4(\hat{w}_4(p), p) = b(1+d)\int_P q_1(-(1+d)\overline{d}, p)\,dF(p) \qquad \text{(A15)}$$

First to calculate $q_1(-(1+d)\overline{d}, p)$, the optimal solution at minimum wealth. Denote $\underline{w}_1 = -(1+d)\overline{d}$ and $\underline{q}_1(p) = q_1(\underline{w}_1, p)$. Assume

$$\underline{q}_1(p) = b\,(1+d)\int_P q_2\Big((1+d)\big(\underline{w}_1 + y - p\, l^{-1}\big(p\underline{q}_1(p)\big)\big), p\Big) dF(p) \qquad \text{(A16)}$$

For wealth level $\underline{w}_1$, the corresponding expenditure is $g_1(\underline{w}_1, p) = p\, l^{-1}\big(p\underline{q}_1(p)\big)$. Then using $l^{-1}(x) = x^{-1/r}$, the wealth carried into week two can be written as $(1+d)\Big(\underline{w}_1 + y - p\big(p\underline{q}_1(p)\big)^{-1/r}\Big) \le \underline{\hat{w}}_2$. Using the results from Proposition 1 and $b_2(p) = a_2(p)\bar{d}/(1+d)^2$, gives

$$q_2\left((1+d)\left(\underline{w}_1 + y - p\big(p\underline{q}_1(p)\big)^{-\frac{1}{r}}\right), p\right)$$
$$= p^{\,r-1}\left(a_2(p)(1+d)\left(\underline{w}_1 + y - p\big(p\underline{q}_1(p)\big)^{-\frac{1}{r}}\right) + a_2(p)\bar{d}/(1+d)^2\right)^{-r}$$

Integrating over the distribution of prices, substituting this into the right hand side of (A16) and simplifying gives

$$\underline{q}_1(p) = b\,(1+d)^{1-r}\left(\underline{w}_1 + y - p\big(p\underline{q}_1(p)\big)^{-\frac{1}{r}} + \bar{d}/(1+d)^3\right)^{-r}\int_P a_2(p)^{-r} p^{\,r-1} dF(p).$$

Let

$$A_1(r) = b(1+d)^{1-r}\int_P p^{\,r-1} a_2(p)^{-r}\, dF(p),$$

and rewriting $\underline{q}_1(p)$ gives

$$\underline{q}_1(p) = A_1(r)\left(\underline{w}_1 + y - p^{1-\frac{1}{r}}\big(\underline{q}_1(p)\big)^{-\frac{1}{r}} + \bar{d}/(1+d)^3\right)^{-r}.$$

Applying the inverse of marginal utility to both sides and taking $\underline{q}_1(p)^{-\frac{1}{r}}$ terms together

$$q_1\big(-(1+d)\bar{d}, p\big) = A_1(r)\Big(-(1+d)\bar{d} + y + \bar{d}/(1+d)^3\Big)^{-r}\left(1 + A_1(r)^{-\frac{1}{r}} p^{1-\frac{1}{r}}\right)^{r} \qquad \text{(A17)}$$

Then (A15), the solution in week four can be written as

$$\frac{1}{p}\, l\left(\frac{\hat{w}_4(p) + \bar{d}}{p}\right) = b\,(1+d)\int_P q_1(\underline{w}_1, p)\, dF(p)$$

$$= b\,(1+d)A_1(r)\left(-(1+d)\bar{d}+y+\bar{d}/(1+d)^3\right)^{-r}\int_P\left(1+A_1(r)^{-\frac{1}{r}}p^{1-\frac{1}{r}}\right)^r dF(p) \equiv D_1(r)$$

Clearly from this $\hat{w}_4(p) = p.\left(pD_1(r)\right)^{-\frac{1}{r}} - \bar{d}$.

(ii)

Now consider that $\hat{w}_3(p)$ is the upper bound for wealth such that for each price $p \in P$ in week 3, the wealth carried over to week 4 is always less than $\min_p \hat{w}_4(p)$, for all possible price outcomes $p \in P$ i.e. $(1+d)\left(w-a_3(p)w-b_3(p)\right) \le \underline{\hat{w}}_4$. Then $\hat{w}_3(p)$ can be calculated as

$$\hat{w}_3(p) = \frac{\underline{\hat{w}}_4 + a_3(p)\bar{d}}{(1+d)\left(1-a_3(p)\right)} .$$

(iii)

Consider that $\hat{w}_2(p)$ is the upper bound for wealth such that for each price $p \in P$ in week 2, the wealth carried over to week three is always less than $\min_p \hat{w}_3(p)$, for all possible price outcomes $p \in P$ i.e. $(1+d)\left(w-a_2(p)w-b_2(p)\right) \le \underline{\hat{w}}_3$. Then $\hat{w}_2(p)$ can be calculated as

$$\hat{w}_2(p) = \frac{\underline{\hat{w}}_3(1+d)+a_2(p)\bar{d}}{(1+d)^2\left(1-a_2(p)\right)} .$$

□

Thus once the kink in week four is calculated, the kinks in the other weeks can be easily derived.

Proof of Proposition 4

Recall the definition of $\hat{w}_2(p)$ above:

$$\hat{w}_2(p) = \frac{\underline{\hat{w}}_3(1+d) + a_2(p)\overline{d}}{(1+d)^2(1-a_2(p))}.$$

The kink is clearly increasing in price and hence the maximum is negative if

$$\frac{\underline{\hat{w}}_3}{(1-a_2(\overline{p}))} < -C_2(\overline{p})\frac{\overline{d}}{(1+d)}.$$

Substituting the definition of $\underline{\hat{w}}_3$, using the equality $(1-a_i(p)) = 1/(1+C_i(p))$, and simplifying gives

$$\underline{\hat{w}}_4 < -\overline{d}\frac{C_2(\overline{p}) + C_3(\underline{p})(1+C_2(\overline{p}))}{(1+C_2(\overline{p}))(1+C_3(\underline{p}))}.$$

Defining

$$K \equiv \left(b\,(1+d)\,A_1(r)\int_P \left(1 + A_1(r)^{-\frac{1}{r}} p^{1-\frac{1}{r}}\right)^r dF(p) \right)^{-1/r} \underline{p}^{1-\frac{1}{r}},$$

and substituting in the definition of $\hat{w}_4(\underline{p})$, this becomes

$$K\left(-(1+d) + y/\overline{d} + 1/(1+d)^3\right) < 1 - \frac{C_2(\overline{p}) + C_3(\underline{p})(1+C_2(\overline{p}))}{(1+C_2(\overline{p}))(1+C_3(\underline{p}))},$$

which simplifies to

$$\frac{y}{\overline{d}} < \left(\frac{1}{(1+C_2(\overline{p}))(1+C_3(\underline{p}))K}\right) + \frac{(1+d)^4 - 1}{(1+d)^3}.$$

Given $g = \overline{d}/y$, we get Proposition 4.

Proof of Proposition 5

The proofs of *5 i – iv* follow directly from the functional form of the analytical solutions.

(v) Recalling the definition of $\hat{w}_4(p)$ from Proposition 4, this can be written as

$$\hat{w}_4(p) = p.z^{-\frac{1}{r}}\left(-(1+d)\overline{d} + y + \overline{d}/(1+d)^3\right) - \overline{d}$$

where z is independent of $\overline{d}$ and given by

$$z \equiv pb(1+d)A_1(r)\int_P \left(1+A_1(r)^{-\frac{1}{r}}p^{1-\frac{1}{r}}\right)^r dF(p)$$

Then

$$\frac{\partial \hat{w}_4(p)}{\partial \overline{d}} = pz^{-\frac{1}{r}}\left(\frac{1-(1+d)^4}{(1+d)^3}\right)-1.$$

Clearly $\frac{\partial \hat{w}_4(p)}{\partial \overline{d}} \leq 0$ because $(1+d)^4 > 1$. Following on from this

$$\frac{\partial \hat{w}_3(p)}{\partial \overline{d}} = \frac{1}{(1-a_3(p))(1+d)}\left(a_3(p) - \frac{\partial \hat{w}_4(p)}{\partial \overline{d}}\right).$$

Clearly $\frac{\partial \hat{w}_3(p)}{\partial \overline{d}} \leq 0$ because $a_3(p) < \left| pz^{-\frac{1}{r}}\left(\frac{1-(1+d)^4}{(1+d)^3}-1\right)\right|$,

and using a similar argument, we can show $\frac{\partial \hat{w}_2(p)}{\partial \overline{d}} \leq 0$.

Consider the value of minimum income in the final week of the cycle: $\underline{w}_i = -\overline{d}$. Clearly this is independent of price and $\frac{\partial \underline{w}_i}{\partial \overline{d}} = -1$. Hence it is trivial to show that $pz^{-\frac{1}{r}}\left(\frac{1-(1+d)^4}{(1+d)^3}\right) \leq 0$ because $(1+d)^4 > 1$. A similar argument establishes the same property for $\frac{\partial \hat{w}_3(p)}{\partial \overline{d}}$ and $\frac{\partial \hat{w}_2(p)}{\partial \overline{d}}$.

(vi) Recalling again the definition of $\hat{w}_4(p)$ from Proposition 3,

$$\frac{\partial \hat{w}_4(p)}{\partial p} = (pD_1(r))^{-\frac{1}{r}}\left(1-\frac{1}{r}\right),$$

and hence $\frac{\partial \hat{w}_4(p)}{\partial p} > 0$ because $r > 1$.

Using the definition of $\hat{w}_3(p)$ from Proposition 3, we get

$$\frac{\partial \hat{w}_3(p)}{\partial p} = \frac{(1+d)(1-a_3(p))\left(\frac{\partial a_3(p)}{\partial p}\right)\bar{d} + \left(\underline{\hat{w}}_4 + a_3(p)\bar{d}\right)\left((1+d)\frac{\partial a_3(p)}{\partial p}\right)}{(1+d)^2(1-a_3(p))^2}$$

which simplifies to

$$\frac{\partial \hat{w}_3(p)}{\partial p} = \frac{\partial a_3(p)}{\partial p}\frac{\underline{\hat{w}}_4 + \bar{d}}{(1+d)(1-a_3(p))^2}. \qquad \text{(A18)}$$

Manipulation of $C_3(p)$ and $a_3(p)$ from Proposition 2 shows

$$\frac{\partial a_3(p)}{\partial p} = \frac{\partial C_3(p)}{\partial p}\frac{1}{(1+C_3(p))^2} \text{ and } \frac{\partial C_3(p)}{\partial p} = \left(\frac{r-1}{r}\right)(pv)^{\frac{-1}{r}} \qquad \text{(A19)}$$

where v is independent of price and defined as

$$v \equiv b(1+d)^{1-r}\int_P p^{r-1}dF(p)$$

giving

$$\frac{\partial C_3(p)}{\partial p} > 0, \text{ because } r > 1.$$

From (A18) and (A19); $\frac{\partial \hat{w}_3(p)}{\partial p} \propto \frac{\partial a_3(p)}{\partial p} \propto \frac{\partial C_3(p)}{\partial p}$, giving $\frac{\partial \hat{w}_3(p)}{\partial p} > 0$.

A similar argument shows

$$\frac{\partial \hat{w}_2(p)}{\partial p} = \frac{\partial a_2(p)}{\partial p}\frac{(1+d)\underline{\hat{w}}_3 + \bar{d}}{(1+d)^2(1-a_3(p))^2},$$

$$\frac{\partial C_2(p)}{\partial p} > 0, \text{ because } r > 1,$$

$$\frac{\partial \hat{w}_2(p)}{\partial p} \propto \frac{\partial a_2(p)}{\partial p} \propto \frac{\partial C_2(p)}{\partial p}, \text{ giving } \frac{\partial \hat{w}_2(p)}{\partial p} > 0.$$

Proof of Proposition 6

The proof of Proposition 6 is based on the convexity of the functions over which the expectation of price is taken. This determines whether these expectations increase or decrease in value with a mean preserving increase in spread (MPS). This is straightforward when dealing with the expenditure function in week 3. However the proof for week 2 requires that we hypothetically separate the effect of a change in distribution in each week and examine whether each of these components is a convex function of price. Hence throughout we abuse slightly the previous notation to make explicit the dependence of each function on the price distribution. The proof draws heavily on Hirschleifer and Riley (1992), Section 3.4.

Define the functions

$$g(p) = p^{r-1},$$

$$h(p,E) = g(p)\, a_3(p,E)^{-r},$$

$$R(E) = \left(b\,(1+d)^{1-r} \int_P g(p)\, dE(p) \right)^{-\frac{1}{r}}$$

$$k(x,E) = x \left(\frac{x^{\frac{1}{r}} R(E)}{1 + x^{\frac{1}{r}} R(E)} \right)^{-r},$$

where E denotes any distribution of prices.

(i) Consider first the slope of the expenditure function in week 3, $a_3(p,F)$. Using the solutions in Proposition 2, the notation from above and making explicit the dependence of $C_3(p)$ on any price distribution F,

$$C_3(p,F) = p^{\frac{r-1}{r}} R(F) = p^{\frac{r-1}{r}} \left(b\,(1+d)^{1-r} \right)^{-\frac{1}{r}} \left(\int_P g(p)\, dF(p) \right)^{-\frac{1}{r}},$$

and thus whether $C_3(p,F)$ and therefore $a_3(p,F)$ increases or decreases with a mean preserving spread of any distribution F is determined by whether $g(p)$ is a convex function of p. Calculating derivatives of $g(p)$ with respect to p gives

$$g'(p) = (r-1)p^{r-2},$$

$$g''(p) = (r-1)(r-2)p^{r-3}.$$

Clearly $r > 2$ is a sufficient condition for $g(p)$ to be a convex function giving

$$\int_P g(p)\,dF(p) < \int_P g(p)\,d\tilde{F}(p).$$

This implies that for any positive constant *t*,

$$t\left(\int_P g(p)\,dF(p)\right)^{-\frac{1}{r}} > t\left(\int_P g(p)\,d\tilde{F}(p)\right)^{-\frac{1}{r}}$$

and hence $C_3(p,F) > C_3(p,\tilde{F})$. This is a sufficient condition for $a_3(p,F) > a_3(p,\tilde{F})$. The same result applies to the intercept term given $b_3(p) = a_3(p)\bar{d}_3$.

(ii) The slope of the week 2 expenditure function, $a_2(p,F)$, can be written as a composite of the functions above. Firstly, denote

$$C_3(p,E) = \left(p^{r-1}\right)^{\frac{1}{r}} R(E),$$

in which case $a_3(p,E)$ can be written as

$$a_3(p,E) = \left(\frac{C_3(p,E)}{1+C_3(p,E)}\right) = \left(\frac{\left(p^{r-1}\right)^{\frac{1}{r}} R(E)}{1+\left(p^{r-1}\right)^{\frac{1}{r}} R(E)}\right).$$

Then substituting this into the definition for $h(p,E)$ gives

$$h(p,E) = p^{r-1}\left(\frac{p^{\frac{r-1}{r}} R(E)}{1+p^{\frac{r-1}{r}} R(E)}\right)^{-r} = g(p)\left(\frac{g(p)^{\frac{1}{r}} R(E)}{1+g(p)^{\frac{1}{r}} R(E)}\right)^{-r} = k(p,E)\circ g(p).$$

Define

$$H(E,F) = \left(\int_P h(p,E)\,dF(p)\right)^{-\frac{1}{r}}$$

and again making explicit the dependence of $C_2(p)$ on the distributions E and F, denote

$$C_2(p,E,F) = H(E,F)\left(b\left(1+d\right)^{1-r}\right)^{-\frac{1}{r}} p^{\frac{r-1}{r}}.$$

The sign of the effect of the MPS on $C_2(p,E,F)$ and hence on $a_2(p)$ depends on whether each component of *H*, which is integrated over the price distribution, is convex or concave.

This purpose of making explicit the dependence on the price distribution, is to separate the different components of $C_2(p,E,F)$ which are affected by the MPS. Essentially we imagine that there are different price distributions in each week. Then the effect of a MPS on *H(E,F)* depends on the effect of the MPS in each week and therefore can be decomposed into

$$H(E,F)-H(\tilde{E},\tilde{F})=H(E,F)-H(E,\tilde{F})+\left(H\left(E,\tilde{F}\right)-H(\tilde{E},\tilde{F})\right). \qquad \text{(A20)}$$

We firstly examine $H(E,F)-H(E,\tilde{F})$ i.e. treat the distribution E as constant, and show the conditions under which the function $h(p,E)$, which is integrated over the distribution F, is convex. Recalling that $h(p,E)=k(p,E)\circ g(p)$, the derivatives are given by

$$k'(p,E)=\left(\frac{x^{\frac{1}{r}}R(E)}{1+x^{\frac{1}{r}}R(E)}\right)^{1-r},$$

$$k''(p,E)=\frac{1-r}{r}\frac{1}{p\left(1+p^{\frac{1}{r}}R(E)\right)}\left(\frac{x^{\frac{1}{r}}R(E)}{1+x^{\frac{1}{r}}R(E)}\right)^{1-r},$$

$$h''(p,E)=\left(k''(p,E)\circ g(p)\right)\left(g'(p)\right)^2+\left(k'(p,E)\circ g(p)\right)g''(p).$$

If we assume $r>2$, then $g''(p)>0$. Then convexity of $h(p,E)$ requires

$$\frac{\left(k''(p,E)\circ g(p)\right)}{\left(k'(p,E)\circ g(p)\right)}>-\frac{g''(p)}{\left(g'(p)\right)^2}.$$

Substituting the relevant terms gives

$$\frac{1-r}{r}\frac{1}{p^{r-1}\left(1+p^{\frac{r-1}{r}}R(E)\right)}>-\frac{(r-2)}{(r-1)}\frac{1}{p^{r-1}},$$

which simplifies to

$$\left(r(r-2)R(E)\right)^{\frac{r}{r-1}}>\bar{p}.$$

Denote the minimum value of r for which this condition holds at the upper bound of the price support $\bar{p}$ as $\hat{r}$. Treating the distribution E as constant, if $r>\hat{r}$, then $h(p,E)$ is a convex function and thus an MPS will increase $h(p,E)$ but reduce $H(E,F)$. Thus $H(E,F)-H(E,\tilde{F})<0$.

To examine the second term in (A20), $H(E,\tilde{F})-H(\tilde{E},\tilde{F})$, recall the definition of $h(p,E)$,

$$h(p,E)=g(p)\left(\frac{g(p)^{\frac{1}{r}}R(E)}{1+g(p)^{\frac{1}{r}}R(E)}\right)^{-r}.$$

The effect of a change in E arises through the function

$$R(E)=\left(b(1+d)^{1-r}\int_P g(p)dE(p)\right)^{-\frac{1}{r}}.$$

If $r>2$ then $g(p)$ is convex and for any positive constant t

$$t\left(\int_P g(p)dF(p)\right)^{-\frac{1}{r}}>t\left(\int_P g(p)dG(p)\right)^{-\frac{1}{r}}$$

giving $R(E)>R(\tilde{E})$, implying $h(p,E)>h(p,\tilde{E})$ and hence $H(E,\tilde{F})<H(\tilde{E},\tilde{F})$. Thus if $r>\max(2,\hat{r})$, $H(E,F)-H(\tilde{E},\tilde{F})<0$ and a MPS reduces the slope of the expenditure function in week 2. The same effect holds for the intercept given $b_2(p)=a_2(p)\overline{d}_2$.

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TABLE 1

Numerical Values of Slopes

$d = 0.4\%$, $r = 0$, $b = 0.95$, $p = 1 = E_P[p]$

	N(1,0.01^2)	N(1,0.1^2)	N(1,0.2^2)
$r = 2.5$			
$a_2(p)$	0.341	0.341	0.340
$a_3(p)$	0.506	0.505	0.504
$r = 5$			
$a_2(p)$	0.338	0.336	0.331
$a_3(p)$	0.503	0.501	0.494

TABLE 2

Numerical Values of g where $\overline{d} = g\,y$, such that borrowing throughout the payment cycle is optimal.

r	N(1,0.01^2)	N(1,0.1^2)	N(1,0.2^2)
2	0.804	0.738	0.656
2.5	0.790	0.711	0.615
3	0.781	0.693	0.587
3.5	0.774	0.680	0.566
4	0.769	0.670	0.550

Figure 1 (a): Numerical Solution in Consumption – Wealth space

$r = 3$, $d = 0.4\%$, $r = 0$, $b = 0.95$, $p = 1 = E_P[p]$

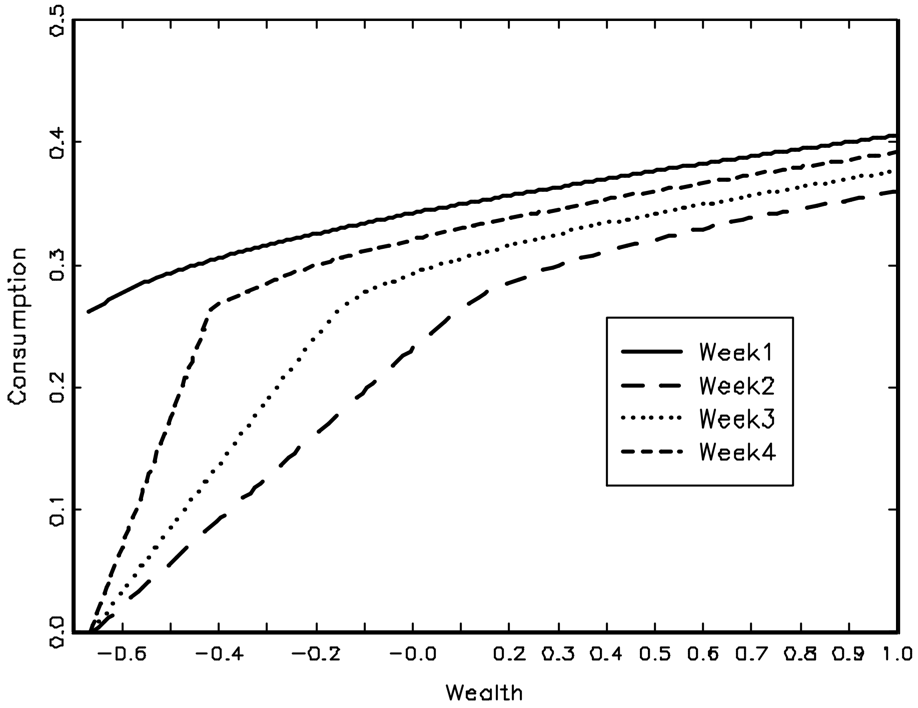


Figure 1 (b): Numerical Solution in Marginal Utility – Wealth space

$r = 3$, $d = 0.4\%$, $r = 0$, $b = 0.95$, $p = 1 = E_P[p]$

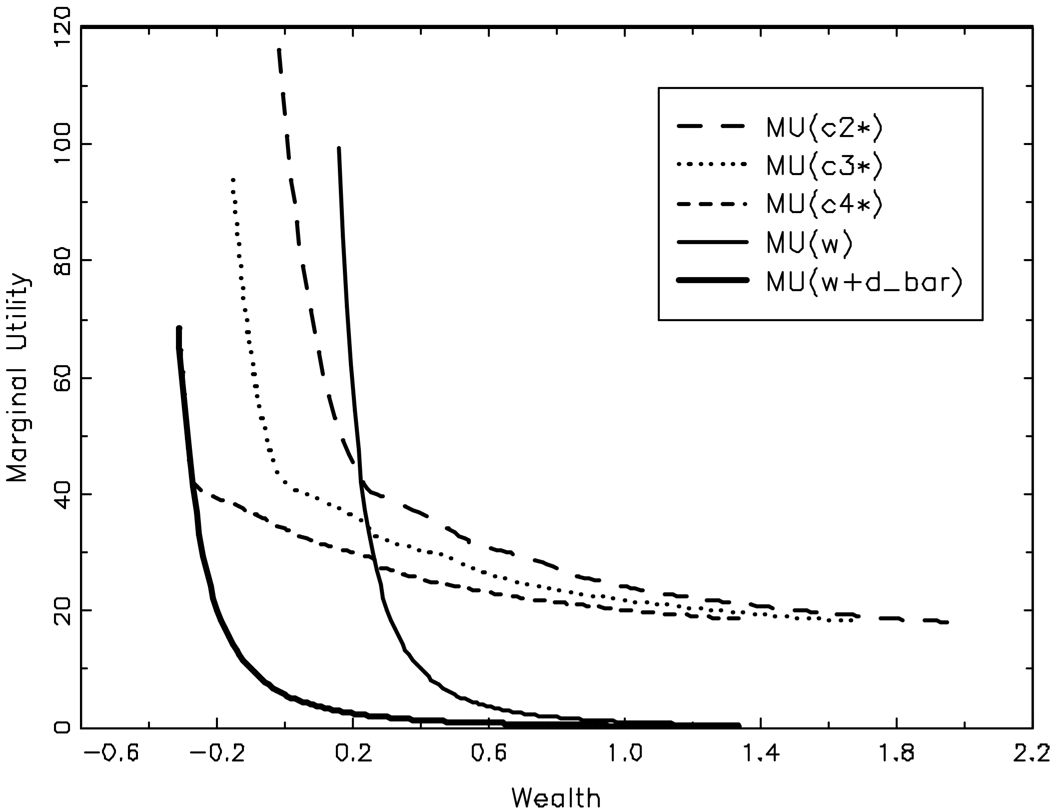
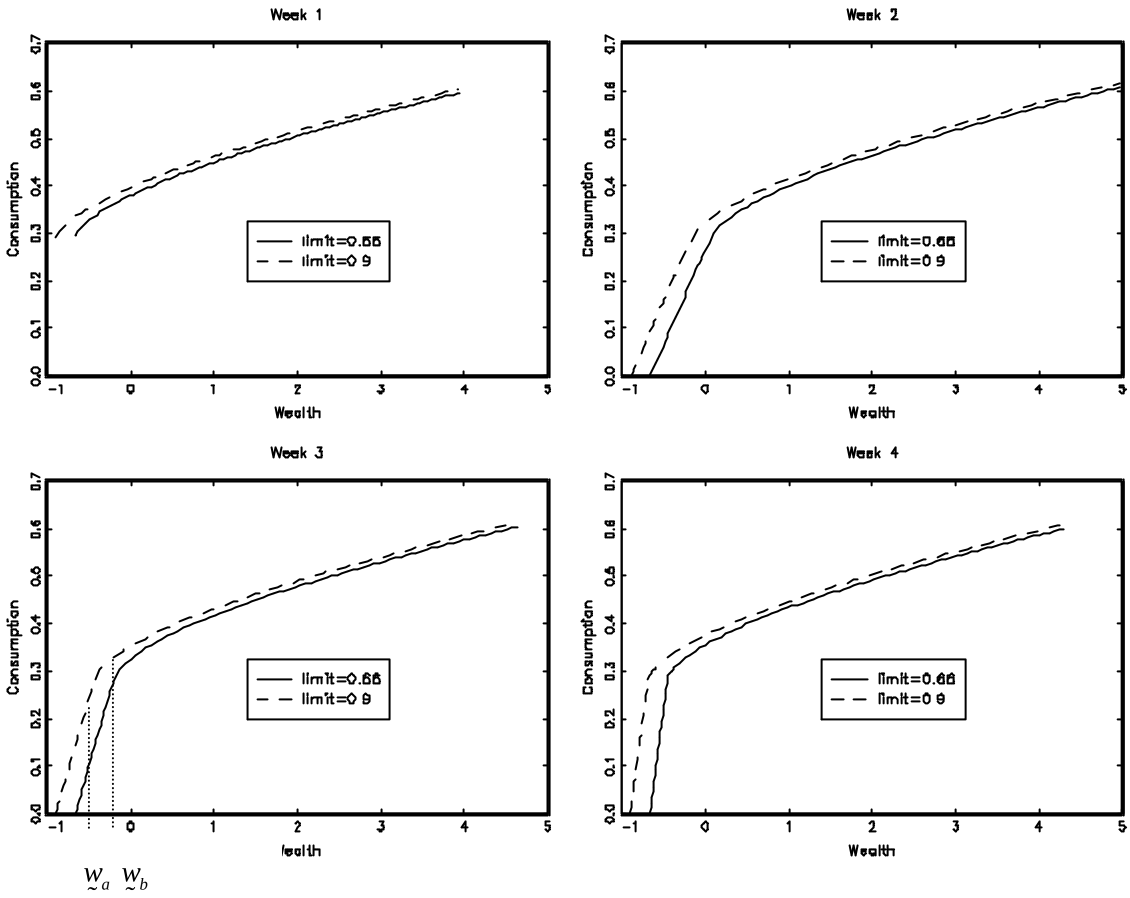
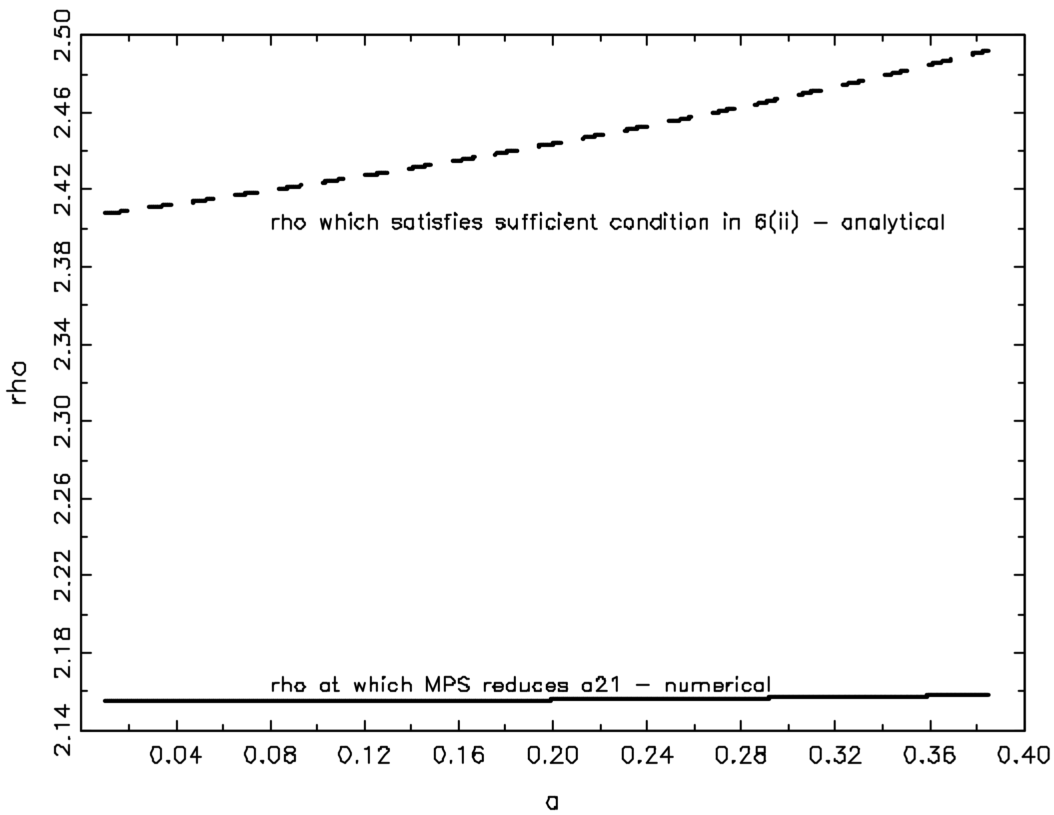


Figure 2: Effect of an increase in the borrowing limit from 66% of income to 90% of income.



The parameters used here are the same as the ones used in Figure 1.

Figure 3: Comparison of Sufficient and Necessary conditions from Proposition 6(ii).



We assume that prices are uniformly distributed with support $\left[\underline{p}, \overline{p}\right] = \left[1-a, 1+a\right]$. All other parameters are the same as in Figure 1.