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Scoring Rule Voting Games and Dominance Solvability*

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Abstract

This paper studies the dominance-solvability (by iterated deletion of weakly dominated strategies) of general scoring rule voting games. The scoring rules we study include Plurality rule, Approval voting, Negative Plurality Rule, Borda rule and Relative Utilitarianism. We provide a classification of scoring rule voting games according to whether the sufficient conditions for dominance solvability require sufficient agreement on the best alternative or on the worst alternative. We also characterise the solutions when the sufficient conditions for dominance solvability are satisfied.

Keywords: Scoring Rules, Voting Games, Dominance Solvability, Iterated Weak Dominance, Condorcet Winner

JEL Classification Numbers: C72, D71, D72

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1 Introduction

The solution concept of Nash equilibrium in voting games has the drawback of admitting predictions that seem unreasonable. For example, in voting games with more than three candidates even if every voter has the same preferences, the least preferred alternative might win in a Nash equilibrium. Indeed, this is true not only with plurality rule[1] but with *any* scoring rule[2].

The reason this problem arises with Nash equilibria is that it allows any possible beliefs on the part of voters, as long as they are consistent. For example, suppose that it is common knowledge that a candidate, A is worst for all voters. Nevertheless, there is a Nash equilibrium where every voter votes for A because he believes that all other voters will vote for A. One easy way of eliminating this equilibrium is to require voting strategies to be weakly undominated. Unfortunately this requirement is not sufficient to give a unique prediction. This problem was motivated first in a preceding article (Dhillon and Lockwood, 1999) for the case of Plurality Rule only.

However, all the problems that arise with Nash equilibria in plurality rule games also arise in other scoring rule games. We therefore study the application of the iterated elimination of weakly dominated strategies to all scoring rules. A game that yields a unique result after the iterated elimination of dominated strategies is called Dominance Solvable (DS). Farquarson (1969) called this procedure "sophisticated voting", and he called a voting game "determinate" if sophisticated voting led to a unique outcome.

Why study Dominance Solvability? If we consider the fact that most of the scoring rules we consider (except Negative Plurality Rule) choose the Condorcet winner whenever the sufficient conditions for the game to be DS

[1] Voters can vote for only one candidate and the candidate with the maximum votes wins the election.

[2] A scoring rule is a voting rule which specifies the vote vectors that voters can use, and then assigns a score to each candidate based on the total number of votes that a candidate gets. The candidate(s) with the highest score wins the election.

are satisfied, we could interpret Dominance Solvability of the scoring rule voting game as being linked to the manipulability of a voting game. Thus, whenever the sufficient conditions for the game to be DS are satisfied, the outcome includes (except for Negative Plurality Rule) the Condorcet Winner, i.e. the outcome is the same as if voters voted sincerely. In this sense we are comparing different scoring rules in terms of this criterion of manipulability.

Iterated Admissibility or iterated elimination of weakly dominated strategies has been criticised by a number of authors, as a strong theoretical justification for it has been elusive. A number of recent articles however provide both learning and common knowledge justifications for it (see e.g. Marx (1999) and Gilli (2002)).

We restrict ourselves to the more realistic case of three candidates or three alternatives. Experimental studies (see for example Ho, Camerer and Weigelt, (1998)) have pointed out that although iterated dominance is one of the most basic principles in game theory, in general "...at the risk of overgeneralising across games that are too different experimental results show that subjects rarely violate dominance but usually stop after one– three levels of iteration." For reasons that will become obvious, the number of iterations in scoring rule voting games are closely linked to the number of candidates. Thus, we feel it is more relevant to study the case of three alternatives. The flavour of the results would be qualitatively the same with more alternatives[3]. The scoring rules we study in this paper are: Negative Plurality Rule (NPR), Approval Voting (AV), Borda Rule (BR) and Relative Utilitarianism (RU). We also compare the results on PR (Dhillon and Lockwood, 1999) to the results of the three other scoring rules.

Our main results are: (1) A generalisation of the results on Plurality Rule voting (Dhillon and Lockwood, 1999), in the sense that we derive sufficient conditions for scoring rule voting games to be DS in terms of one statistic

[3]Buenrostro has separately proved generalisations for some of the scoring rules studied in this paper.

of the game: the degree of agreement on the best or the worst alternative, (2) A classification of scoring rule voting games based on the strength of the conditions required for DS shows that Approval Voting performs quite well relative to other rules. Intuitively, this is because it is the least restrictive in terms of the strategies allowed to voters and (3) When the game satisfies the sufficient conditions for Dominance Solvability we investigate if the unique outcome is also the Condorcet Winner. The scoring rule games for which the Condorcet Winner is not chosen by the iterated elimination of dominated strategies even when it exists and the sufficient conditions for Dominance Solvability are satisfied are not very desirable rules according to this criterion.

The layout of the paper is as follows. Section 2 presents the model and defines concepts and notation that will be used in the rest of the paper for the one stage voting game with three alternatives. Sections 3,4 and 5 offer a general classification of scoring rule voting games according to the sufficient conditions for DS. Section 6 compares some of the well known Scoring Rule voting games like Plurality Rule, Negative Plurality Rule, Approval Voting. Section 7 concludes.

2 The Model

In the following analysis we assume there are three alternatives, and an arbitrary number, $n > 3$, of voters. This is a simplified case of the general voting game and it is the simplest case where strategic voting can occur. It is common in the literature to compare voting systems with three alternatives; for example, Myerson and Weber (1993), Myerson (2002). Also, often major political elections have no more than three candidates. We believe that the results would not be qualitatively different with more candidates.

A *scoring rule* is characterised by a set $C \subset \mathbb{R}^3$, which represents the set of feasible ballots or vote vectors a voter is permitted to submit (Myerson, 2002). A vote vector, $c = (c_1, c_2, c_3)$ represents a ballot that gives c_1 points to candidate one, c_2 points to candidate two etc. The vote-vectors of all voters are added up to obtain a total point score for each candidate. The winning set of candidates is those that get the maximum point score. If there is a tie, we allow all candidates in the winning set to be chosen with equal probability. We assume that the set C is a non empty subset of $\mathbb{R}^3$. Every voter can choose from the same feasible set C so the scoring rules we consider are anonymous and neutral. We assume w.l.o.g. scoring rules to be normalised, $0 \leq c_i \leq 1$, for all i. Thus in all feasible vote vectors, the candidate who is top ranked gets 1 point and the candidate who is worst ranked gets a point 0. Scoring rules differ only in the number of points that can be given to the middle ranked candidate.

Examples of scoring rules are: Plurality Rule (PR),Negative Plurality Rule (NPR), Approval Voting (AV), Borda Rule (BR) and Relative Utilitarianism (RU)(Dhillon and Mertens, 1999).

Among these, PR, NPR, and AV allow $c_i \in \{0, 1\}$ only, while RU allows any $c_i \in [0, 1]$. Thus, for three alternatives, normalisation and no indifferences imply that permitted vote vectors in any scoring rule $r \in \{PR, NPR, AV, BR, RU\}$ are all possible permutations of the vector $(1, s_r, 0)$ where $s_r \in S_r \subset [0, 1]$. The set S_r characterises the scoring rule r. Thus $S_{PR} = 0$, $S_{NPR} = 1$, $S_{BR} = 1/2$, $S_{AV} = \{0, 1\}$ and $S_{RU} = [0, 1]$.

The social choice literature usually considers scoring rules which allow singleton sets S_r. Our definition of a scoring rule is more general: we allow S_r to be any subset of $[0, 1]$. We introduce here the Relative Utilitarian (RU) scoring rule: RU is a social welfare function that consists of normalising individual (von-Neumann Morgentern) utilities between 0 and 1 and then adding them (Dhillon and Mertens, 1999). If interpreted as a scoring rule RU calls for voters to submit ballots that allow the middle ranked alternative to be given any point betweeen 0 and 1. We can derive any scoring rule from RU by suitably restricting the strategy space.

We assume that there are no abstentions. With costless voting, abstention is weakly dominated and therefore deleted in the first stage for all voters. We also assume strict preferences, so that vectors of the form $(c, c, ..., c)$ are not permitted in all the voting games we consider.

Let us now define the voting game Γ_r corresponding to scoring rule r. The strategies of voters are the set of possible vote vectors allowed in any scoring rule. Let $v_i^r \in V_i^r$ represent the vote vectors allowed to individual $i$in scoring rule r, where $r =$ PR,NPR,AV,BR, RU. The profile of vote vectors, one for each individual, is denoted v. The score for a particular candidate a, corresponding to a vector of votes v_r is denoted $\omega_a(v_r)$ (or just ω_a when it is clear which vote vector we are considering). A score profile (corresponding to a vote vector,v_r) is a vector $\omega_r = (\omega_1, \omega_2, \omega_3)$, where ω_i represents the total point score of candidate i. Ω_r denotes the space of scoring vectors ω_r, for scoring rule r.

Let $W(v_r)$ denote the set of winning candidates given the vote vector v_r. The payoffs are given by the expected utility over the set of winning candidates.

Let the set of alternatives be $X = \{x, y, z\}$ and the set of voters be N such that $|N| = n$.

Define $W(v_r) = \{a \in X | \omega_a(v_r) \geq \max(\omega_b(v_r), \omega_c(v_r)), \forall b, c \in X\}$ as the *Winning Set* for a given profile v_r.

Define $L(v_r) = \{a \in X | \omega_a \leq \min(\omega_b, \omega_c), \forall b, c\}$ as the *Losing set* for a given profile v_r.

We will impose the following regularity condition (Dhillon and Lockwood 1999) which ensures that the order of deletion of weakly dominated strategies does not matter(see Marx and Swinkels'(1997) Transference of Decisionmaker Indifference (TDI) condition which is sufficent to ensure that the order of deletion does not matter: If A.1 is satisfied, then TDI is satisied):

A1. *For all* v_r, v'_r *s.t.* $W(v_r) \neq W(v'_r),\ u_i(v_r) \neq u_i(v'_r),\ i \in N$

This says that no player is indifferent between any two different winsets[4].

Let $N_a(N'_a) \subset N$ with $a \in \{x, y, z\}$ represent the set of individuals that rank a as the worst (best) alternative and let $n_a(n'_a)$ be the number of voters in this set. Let $q_a = \frac{n_a}{n}$ and $q'_a = \frac{n'_a}{n}$.

In what follows we focus only on pure strategies that survive iterated deletion of weakly dominated strategies (since we are interested in dominance solvability). In the rest of the paper we suppress the subscript r in v_r, ω_r when it is clear which scoring rule is being discussed in a a particular section, and we will use $v_i, i \in N$, to denote the vote vector for voter i under the voting rule that is being analysed. Let v_{-i} represent the voting profile of all players except player i. Let $\omega_a(v_{-i})$ denote the total points that $a \in X$ gets in the profile v_{-i}. Finally, say that in game Γ, preferences are *polarized over alternative* $a \in X$ if there is an $M \subset N$ such that all $i \in M$ rank a highest, and $i \in N/M$ rank a lowest. Preferences over alternative a are *non-polarized* otherwise. Finally, let Γ_i denote the reduced game after i rounds of the elimination of weakly dominated strategies and W^∞ denote the outcome when no further iterated elimination is possible.

Fix a scoring rule, r. Let $p_a(v)$ denote the probability that alternative a is in the winset given profile v. We define a voter i to be *pivotal on a set* $S \subset X$, if $\forall a \in S, \exists$ a strategy $v_i(a) \in V_i$ such that $p_a(v_{-i}) \neq p_a(v)$, where v is the profile $(v_{-i}, v_i(a))$ (note that the order of vote vectors in the profile does not matter as scoring rule voting games are anonymous), and $\forall a \notin S$, for all strategies $v'_i \in V_i$, $p(a)(v_{-i}) = p_a(v')$, where $v' = (v_{-i}, v'_i)$. We say that a strategy v_i is *at least as good as* strategy v'_i for voter i if strategy v_i does not decrease the probability that an alternative which is higher ranked by voter i is in the winset, relative to strategy v'_i, for any profile of (pure) strategies of other voters. We denote the set of alternatives which are condorcet winners as X^{CW} and a CW is denoted as $a^{CW} \in X^{CW}$.

Finally we often use the short form UBR for Unique Best Response.

[4]A1 implies that no voter is indifferent between any pair of alternatives.

We focus on sufficient conditions that require only *ordinal* information on preferences[5]. In the next section we show that most scoring rules have very similar sufficient conditions for DS. In particular we show that most scoring rules can be categorised according to whether the sufficient conditions can be expressed in terms of sufficient agreement on the best alternative or sufficient agreement on the worst alternative.

3 A classification of scoring rule voting games

The idea behind strategic voting is that individuals try to differentiate maximally between the alternatives that are tied given the vote vectors of all other voters. This has to be consistent with the type of vote vectors they are permitted in the scoring rule. We can thus deduce something about the undominated vectors. In PR, voters have a single vote (i.e. $c_i = 1$ for any i implies $c_j = 0, \forall j \neq 1.$). Thus if they are pivotal over any set involving the worst ranked alternative, they must give it zero, and if not pivotal on this alternative they may as well give it zero points. Therefore the search for sufficient conditions for dominance solvability of the PR voting game is essentially a search for conditions under which we can reduce the set of possible winning candidates. This idea extends to other scoring rules as well.

For scoring rule r let $\overline{s}_r$ denote $\max s \in S_r$, and $\underline{s}_r$ denote $\min s \in S_r$. Obviously if S_r is a singleton then $\overline{s}_r = \underline{s}_r = s_r$. Denote $\Sigma(v_r) = \omega_x(v_r) + \omega_y(v_r) + \omega_z(v_r)$. Note that for any scoring rule $n(1+\underline{s}_r) \leq \Sigma(v_r) \leq n(1+\overline{s}_r)$.

It is quite intuitive that with three alternatives, voters would never give less than $\overline{s}_r$ to their best alternative and never give more than $\underline{s}_r$ to their worst alternative. This is what the next proposition shows.

Proposition 1 *In the game Γ_r the only strategies that are undominated for a voter i are those that give $c_j \geq \overline{s}_r$ to his top ranked alternative $j \in X$ and*

[5] Although we shall see later that this is not quite true for Borda Rule.

$c_k \leq \underline{s}_r$ to his worst ranked alternative $k \in X$.

See Appendix Section A.1 for the proof.

Thus the first stage of iterated elimination leads to the reduced game denoted by Γ_{1r} for scoring rule r, where the strategies are of the form $(\overline{s}_r, 1, 0)$, $(1, s_r, 0)$ and $(1, 0, \underline{s}_r)$.

Our sufficient conditions for DS revolve around finding the conditions under which we can reduce the set of possible candidates that can win the election. This could happen in two ways: either we can eliminate the candidate who is worst ranked by most voters or we can say something about the candidates who cannot lose if there is sufficient agreement on the best and then use that to reduce the possible outcomes. We call these two sets of sufficient conditions *Agreement on the Worst* and *Agreement on the Best* respectively. We show that if $\underline{s}_r < 1/2$, then a scoring rule voting game is DS if there is "sufficient" agreement on the worst and if $\overline{s}_r > 1/2$ then a scoring rule voting game may be DS if there is sufficient agreement on the best. The conditions for Agreement on the best and Agreement on the worst are not symmetric– the reason is that agreement on the best only helps us to eliminate candidates in the *Losing Set* while we are interested in reducing the possible winning outcomes.

4 Agreement on the worst:

4.1 Scoring rules with $\underline{s}_r < 1/2$

W.l.o.g let z be the candidate that most voters rank worst[6]. Recall that n_z is the number of voters who rank z worst. In what follows we will drop the subscript r from v_r and use W synonymously with $W(v)$, L synonymously

[6]This is uniquely defined if z is a Condorcet Loser. Whenever our sufficient conditions are satisfied, this is indeed the case.

with $L(v)$. This makes the notation simpler.

In the next theorem we derive the sufficient conditions for Dominance Solvability of Scoring Rule voting games with $\underline{s}_r < 1/2$. We know from Proposition 1 that voters will give a minimal score to their worst candidate in the undominated game. Consider a voter i who has z as his worst candidate. If there are sufficiently many voters who rank z worst, then z can get at most $n_z\underline{s}_r + (n - n_z)$. Now if $\underline{s}_r < \frac{1}{2}$, this means that z can never be in the winning set, so that even the voters who do not have z as the worst candidate will not waste their votes on z. Thus the gameis reduced to a game between x and y and the CW must win. This is what we show in this section.

Theorem 1 *(A) If $\underline{s}_r < 1/2$, then the game Γ_r is DS if $n_z > \frac{n(2-\underline{s}_r)}{3(1-\underline{s}_r)}$. (B)Also whenever the sufficient conditions for DS are satisfied, (i) W^∞ contains at most two alternatives,(ii) if n is odd, a unique CW, a^{CW} exists and $W^\infty = \{a^{CW}\}$; (iii) if n is even, at least one CW exists and $W^\infty = X^{CW}$.*

Before we prove this theorem we need a few lemmas. Proofs of all lemmas are in the Appendix.

Lemma 2: Assume $\underline{s}_r < 1/2$. If $n_z > \frac{n(2-\underline{s}_r)}{3(1-\underline{s}_r)}$ then $W \subset \{\{x\}, \{y\}, \{xy\}\}$.

Corollary to Lemma 2: If $\underline{s}_r = \frac{1}{2}$, and $n_z = n$ then $W(v) \in \{\{x\}, \{x, y\}, \{y\}, \{x, y, z\}\}$, for all $v \in V$.

Lemma 3: Consider a voter $i \in N_z$ such that $x \succ_i y \succ_i z$. Assume $W \subset \{\{x\}, \{y\}, \{xy\}\}$. Then either (i) the strategy $(1, 0, \underline{s}_r)$ weakly dominates strategies $(1, s_r, 0)$ and $(\overline{s}_r, 1, 0)$ for $s_r > 0$, or (ii)all strategies of i are equivalent.

Lemma 4: Consider a voter $i \notin N_z$ such that $z \succ_i x \succ_i y$. Assume $W \subset \{\{x\}, \{y\}, \{xy\}\}$. Then either (i) the strategy $(1, 0, \overline{s}_r)$ (denoted v_i) weakly dominates strategies $(0, \underline{s}_r, 1)$ (denoted v_i') and $(s_r, 0, 1)$ (denoted v_i'') for $s_r > 0$, or (ii) all strategies of i are equivalent.

Proof of Theorem 1 (A): Consider $i \in N_z$. W.l.o.g assume $x \succ_i y \succ_i z$. By

Proposition 1, his undominated strategies (or his strategies in the game Γ_{1r} are of the form $(\overline{s}_r, 1, 0)$, $(1, s_r, 0)$ and $(1, 0, \underline{s}_r)$. Consider a voter $j \notin N_z$ such that $z \succ_i x \succ_i y$. By Proposition 1, his undominated strategies are $(1, 0, \overline{s}_r)$, $(0, \underline{s}_r, 1)$ and $(s_r, 0, 1)$ By Lemma 2, we know that $W \subset \{\{x\}, \{y\}, \{x, y\}\}$.

Consider voter i above. By lemma 3, if strategy $(1, 0, \underline{s}_r)$ weakly dominates any of the strategies $(1, s_r, 0)$ and $(\overline{s}_r, 1, 0)$ for $s_r \neq 0$, then we can remove all such strategies . Similarly for voter j above if strategy $(1, 0, \overline{s}_r)$ weakly dominates strategies $(0, \underline{s}_r, 1)$ and $(s_r, 0, 1)$ for some s_r then we can remove these strategies in the second round of iterated deletion to reach the reduced game Γ_{2r}.

In the game Γ_{2r}, either some voters have only one strategy remaining and this is the one that gives 1 to the best candidate out of x, y and 0 to the other, or all remaining strategies for all remaining voters are equivalent. This means that we can choose the strategy that gives 1 to the best candidate out of x, y and 0 to the other for all voters (since whichever strategy we choose the outcome is the same for all voters). Thus all voters have only one strategy left and the game is DS. □

Proof of Theorem 1(B): For the second part, observe that there z is the unique Condorcet Loser in these voting games. Hence there must be a CW, which is unique if n is odd. By the proof of theorem 1, we know that all voters vote sincerely between x and y in the reduced game Γ_{2r}, while z gets $n(\underline{s}_r)$ votes. Thus $\max(\omega_z) < \frac{n}{2}$ while if there is a unique CW (say x) then $\min(\omega_x) > \frac{n}{2}$. If n is even and both x and y are CW then $\omega_x = \omega_y = \frac{n}{2}$. Hence the winning set is the set of CW's.

□

4.2 Scoring Rules with $\underline{s}_r = \frac{1}{2}$

Let, w.l.o.g, n_z be the largest number of voters who have the same worst alternative. As in Dhillon and Lockwood (1999) we say that $i \in N$ has *dominated middle alternative* (DMA) preferences if he prefers a lottery with equal probabilities over all three alternatives to his middle-ranked alternative.

This section essentially shows that when $\underline{s}_r$ increases to $\frac{1}{2}$ the degree of agreement required on the same worst alternative increases. The reason is that our sufficient conditions are conditions that ensure that z will never be in the winning set if voters use iteratively undominated strategies.

Theorem 2 *(A)If $\underline{s}_r = \frac{1}{2}$, the game Γ_r is DS if $n_z = n$ and either (i) n is odd, (ii) n is even and $n'_x \neq n'_y$ or (iii) all voters have DMA preferences. (B) If the conditions in (A) hold then at least one CW exists and W^∞ includes X^{CW}.*

Proof of Theorem 2(A) Using the Corollary to Lemma 2, $W(v) \in \{\{x\},\{x,y\},\{y\},\{x,y,z\}\}$, for all $v \in V$.

Note that if $n_z = n$ Proposition 1 implies that $\max(\omega_z(v)) = \frac{n}{2}$.

Suppose that $\exists\ v$ such that $W(v) = \{x, y, z\}$. This implies that $\omega_x(v) = \omega_y(v) = \omega_z(v)$. Such a profile exists iff all $i \in N$ give $\frac{1}{2}$ to z. But if all $i \in N$ give $\frac{1}{2}$ to z, they must all be using the strategy which gives 1 to the best alternative and 0 to the second ranked.Since $\omega_x(v) = \omega_y(v) = \omega_z(v)$ it must be that $n'_x = n'_y = \frac{n}{2}$ and therefore n is even.

Thus if n is odd, or $n'_x \neq n'_y$, no such profile exists, and $\{x, y, z\} \notin W(v)$,for any $v \in V$. Then the result follows using Lemmas 3 and 4 and the game is DS (following the same proof as for the case $\underline{s}_r < 1/2$.). We now consider the case when n is even and $n'_x = n'_y = \frac{n}{2}$.

Let $i \in N_z$ be such that (w.l.o.g) $x \succ_i y \succ_i z$.. By Proposition 1 the remaining strategies for such a player in the game Γ_{1r} are: $v_i = (1, 0, 1/2)$, $\widetilde{v}_i = (1, s_r, 0)$ and $\widehat{v}_i = (\overline{s}_r, 1, 0)$. It is sufficient to show that v_i weakly dominates the two other strategies in the case $W(v) = \{x, y, z\}$ (since the rest follows from the proof of Theorem 1.).

Now assume that n is even, $n'_x = n'_y$ and all the voters have DMA preferences. Let v_{-i} be such that $W(v) = \{x, y, z\}$, this implies that $\omega_x(v_{-i}) + 1 = \omega_y(v_{-i}) = \omega_z(v_{-i}) + \frac{1}{2}$. Therefore, the only possible outcome for $\widetilde{v}$ is $W(\widetilde{v}) = \{y\}$ and since i has DMA preferences strategy v_i is at least as good as $\widetilde{v}_i$. Analogously, the only possible outcome for $\widehat{v}$ is $W(\widehat{v}) = \{y\}$ and since i has DMA preferences strategy v_i is at least as good as $\widehat{v}_i$.

If there exists a profile v_{-i} such that strategy $(1, 0, \frac{1}{2})$ is strictly better than strategy $(1, s_r, 0)$ for any $s_r \in S_r$ and there exists a profile such that $(1, 0, \frac{1}{2})$ is strictly better than strategy $(\overline{s}_r, 1, 0)$, then we can eliminate the dominated strategies. Otherwise all strategies are equivalent and we can choose $(1, 0, \frac{1}{2})$.

Therefore, if $n_z = n$ and either (i), (ii) or (iii) in Theorem 2 hold then the only strategy that is undominated in Γ_{1r} for all voters is the strategy that gives 1 to the best and 1/2 to the worst alternative.

□

Proof of Theorem 2(B): If the sufficient conditions for DS are satisfied, there is a Condorcet Loser (z), so clearly there is a unique CW if n is odd and if n is even then both x and y are CW's. The proof of part (A) shows that all voters are left with the strategy that gives 1 to the best and $\frac{1}{2}$ to z. Thus, the winning set coincides with the CW if n is odd, and if n is even all three alternatives get equal votes so the winning set includes the Condorcet winners.

□

5 Agreement on the Best

W.l.o.g let x be the alternative that a plurality of voters rank best. As shown by Proposition 1 the reduced game Γ_{1r} gives a maximum of $\underline{s}_r$ to the worst alternative and a minimum of $\overline{s}_r$ to the best alternative. Thus $\min(\omega_x) = n'_x \overline{s}_r$.

Recall $n(1+\underline{s}_r) \leq \Sigma(v) \leq n(1+\overline{s}_r)$. Recall too that the strategies that survive in Γ_{1r} for i such that $x \succ_i y \succ_i z$ are of the form $(1, 0, \underline{s}_r)$; $(1, s_r, 0)$; $(\overline{s}_r, 1, 0)$.

When a sufficiently large number of voters agree that x is the best candidate, then we might conjecture that x is always in the winning set, and the only question is whether it is uniquely in the winning set or not. This is indeed the case: consider Negative Plurality Rule which is the worst scoring rule in the sense that $\underline{s}_r = 1$. Even if *all* voters agree that x is the best each voter still has two undominated strategies remaining, so that the outcome could be $\{x, y\}$ or $\{x, z\}$. Indeed, this is true *even if all voters have exactly the same preferences.* The co-ordination problem is particularly bad with NPR. We derive some sufficient conditions for NPR in Section 6. On the other hand when $\underline{s}_r < 1$, there is some hope that the game is DS if sufficiently many voters agree on the same best alternative. The exact conditions are derived below. If $\overline{s}_r$ is less than $\frac{1}{2}$, then even if all voters agreed on the same best candidate, we still could not ensure that that candidate would always be in the winning set. That is why we need $\overline{s}_r \geq \frac{1}{2}$, to be able to derive sufficient conditions based on agreement on the best.

5.1 Scoring Rules when $\overline{s}_r > \frac{1}{2}$:

Theorem 3:*(A) Let* $\overline{s}_r > 1/2$ *and* $\underline{s}_r < 1$. *If* $n'_x > \max[\frac{n(1+\overline{s}_r)}{3\overline{s}_r}, \frac{n}{2-\underline{s}_r}]$ *and* $\overline{s}_r \geq \underline{s}_r$ *then the Scoring rule game is DS and* x *is the unique winner. (B) If the sufficient conditions are satisfed, the unique CW is* x.

Before we prove this theorem, we need a few lemmas. We assume from now on that $\overline{s}_r > 1/2$.

Lemma 5: If $n'_x > \frac{n(1+\overline{s}_r)}{3\overline{s}_r}$ *then* $L \in \{\{y\},\{z\},\{y,z\}\}$.

Comment 1: Thus we can deduce that $\omega_x > min(\omega_y, \omega_z)$ for all profiles v.

Corollary 1 to Lemma 5: If $n'_x > \frac{n(1+\overline{s}_r)}{3\overline{s}_r}$ *then* $W(v) \in \{\{x\},\{y\},\{z\},\{x,y\}\{x,z\}\}, \forall v$.

The proof is obvious using Comment 1 above: $\{y,z\}$ and $\{x,y,z\}$ cannot be in the winning set for any profile v.

Corollary 2 to Lemma 5: If $\overline{s}_r = \frac{1}{2}$ *and* $n'_x = n$ *then* $L \in \{\{y\},\{z\},\{x,y,z\},\{y,z\}\}$.

Lemma 6: Let $i \in N'_x$*, such that* $x \succ_i y \succ_i z$*. If* $L \subset \{\{y\},\{z\},\{y,z\}\}$*, then either (a) strategies* $v_i = (\overline{s}_r, 1, 0)$ *and* $v'_i = (1, s_r, 0)$ *are both weakly dominated by strategy* $\tilde{v}_i = (1, \underline{s}_r, 0)$ *or (b) all strategies are equivalent for* i.

If strategy $\tilde{v}_i$ weakly dominates strategy v_i (v'_i) then we can eliminate $v_i(v'_i)$. Otherwise, the two are equivalent and we can choose $\tilde{v}_i$.

Thus the reduced game Γ_{2r} is the game where all $i \in N'_x$ have strategies $(1, 0, \underline{s}_r)$ and $(1, \underline{s}_r, 0)$ remaining.

Lemma 7: In the game Γ_{2r}*, if* $\underline{s}_r < 1$*, and* $n'_x > \frac{n}{2-\underline{s}_r}$ *then the Scoring rule game is DS and* x *is the unique winner.*

Corollary to Lemma 7: In the game Γ_{2r} *if* $\underline{s}_r < \frac{1}{2}$*, then if* $n'_x > \frac{2n}{3}$*, the game is DS and* x *is the unique winner.*

Proof of Theorem 3: Denote $n'_T = \frac{n(1+\overline{s}_r)}{3\overline{s}_r}$. The Corollary to Lemma 5 shows that whenever $n'_x > n'_T$ then $\{y,z\}$,$\{x,y,z\} \notin W(v)$, for any $v \in V$. Then Lemma 6 tells us that in this case, all $i \in N'_x$ will give 1 to x. Lemma 7 shows that if $n'_x > n'_t$, then $W = \{x\}$ and the game is DS. Thus if $n'_x > \max[n'_T, n'_t]$ all conditions are satisfied so the game is DS and x is the unique winner. $\square$

Corollary 1 to Theorem 3: *Let* $\overline{s}_r > 1/2$ *and* $\underline{s}_r < \frac{1}{2}$*. If* $n'_x > \frac{n(1+\overline{s}_r)}{3\overline{s}_r}$ *then the Scoring rule game is DS and* x *is the unique winner.*

Proof of Corollary 1 to Theorem 3: Since $\overline{s}_r \in (\frac{1}{2}, 1]$ and $\underline{s}_r \in [0, \frac{1}{2})$, we have $\inf(n'_T) = \frac{2n}{3} = \sup(n'_t) = \frac{2n}{3}$, so if $n'_x > \frac{2n}{3}$ the game is DS and x is the unique winner for all $\overline{s}_r, \underline{s}_r$, in the intervals considered.

□

Corollary 2 to Theorem 3: If $\overline{s}_r > 1/2$ and $\underline{s}_r < \frac{1}{2}$, and $n'_a > \frac{n(1+\overline{s}_r)}{3\overline{s}_r}$ then a is the unique Condorcet Winner and is the only determinate outcome.

The proof is obvious. □

5.2 Scoring Rules with $\overline{s}_r = \frac{1}{2}$.

Theorem 4:*If $\overline{s}_r = \frac{1}{2}$, and $n'_x = n$, the scoring rule game is DS and the unique winner is the Condorcet winner.*

Proof of Theorem 4: The Corollary to Lemma 5 shows that $L(v) \in \{\{y\}, \{z\}, \{y, z\}, \{x, y, z\}\}$. Thus $\min(\omega_x)(v) \geq \min(\omega_y(v), \omega_z(v))$ for any profile v. This means that $W(v) \in \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{x, y, z\}\}, \forall v$. Consider an individual $i \in N'_x$ such that $x \succ_i y \succ_i z$. It is easy to see that Lemma 6 applies in this case as well when $L(v) \in \{\{y\}, \{z\}, \{y, z\}\}$. It is sufficient, therefore to show that Lemma 6 holds when $L(v) = \{x, y, z\}$. Thus, let $v_i = (\overline{s}_r, 1, 0)$ and $v'_i = (1, s_r, 0)$ and $\tilde{v}_i = (1, \underline{s}_r, 0)$. Let $(v_i, v_{-i}) = v$, $(v'_i, v_{-i}) = v'$, $(\tilde{v}_i, v_{-i}) = \tilde{v}$. If $W(v) = W(v') = \{x, y, z\}$ then $W(\tilde{v}) = \{x\}$. Thus by the proof of Lemma 6, we can eliminate strategies $(\overline{s}_r, 1, 0)$ and $(1, s_r, 0)$. The only strategies that remain are those that give 1 to x. Since $\omega_x(v) \geq n'_x = n$, for all v while $max_{v \in V}(\omega_y(v), \omega_z(v)) = n'_x \underline{s}_r = n\underline{s}_r$, the game is DS and x is the unique winner, since $\underline{s}_r \leq \overline{s}_r = \frac{1}{2}$.

□

6 Results for Scoring Rules: PR,AV,RU,BR, NPR

What can we say about sufficient conditions for DS of the scoring rules that are familiar in the literature? Well, our first result is that the sufficient conditions for DS for both AV and RU are exactly the same! Thus, there is no loss in restricting strategies to be $s_r \in \{0, 1\}$.

Corollary to Theorems 1 and 3: *The PR game is DS if $n_z > \frac{2n}{3}$. The AV and RU voting games are DS if either (i)$n_z > \frac{2n}{3}$ or (ii) $n'_x > \frac{2n}{3}$. If the sufficient conditions for DS are satisfied, there exists a CW and the winning set coincides with the set of CW's for all three scoring rules.*

For Borda Rule we have the following result:

Corollary to Theorems 2 and 4: *The BR game is DS if EITHER (A) $n_z = n$ and either (i) n is odd (ii) n is even and $n'_x \neq n'_y$ or (ii) all voters have DMA preferences, OR (B)$n'_x = n$. If these sufficient conditions are satisfied, in case A(i) there is a unique CW and it is the unique winning alternative. In case A(ii) if all voters are equally divided between x and y then all three alternatives will be in the winning set. In case (B) the unique CW coincides with the winning set.*

6.1 Sufficient Conditions for non Dominance Solvability of PR, AV, BR, RU

Although we cannot say whether the sufficient conditions derived above are also necessary[7], we could try and derive sufficient conditions for non DS that rely only on ordinal information. Dhillon and Lockwood (1999) follow pre-

[7]This is because the game could be DS even if the sufficient conditions are not satisfied – recall that we wanted conditions that used information only on ordinal preferences– e.g. with some utility functions and not with others.

cisely this approach in the case of PR. We are able to derive these conditions for NPR but not for the other voting rules.

The sufficient conditions for non Dominance Solvability for AV, BR and RU are very complicated to characterise. The reason for this is that unlike in PR and NPR the map from the space of voting profiles V to the space of Scoring Vectors Ω is not one to one, while it is one to one (upto permutations between individuals) in the case of PR and NPR. Thus the inverse function does not exist.

For example $\omega_x = \omega_y = n; \omega_z = 0$ clearly corresponds to voting profiles $n(1,0,0) + n(0,1,0)$ under PR but in AV it could correspond to either $n(1,0,0) + n(0,1,0)$ or to $n(1,1,0)$.

We believe that the sufficient conditions will need to take into account more information than simply the degree of agreement on the best and worst alternatives. We construct an example for AV using the Condorcet cycle, to show that the game is not DS when there is sufficient heterogeneity in preferences:

Example 1 (Non DS of AV)

Let $n = 6, n'_x = n'_y = n'_z = 2, n_z = n_y = n_x$ and the preferences such that:

$$\begin{aligned} 1 &: x \succ y \succ z \\ 2 &: x \succ y \succ z \\ 3 &: y \succ z \succ x \\ 4 &: y \succ z \succ x \\ 5 &: z \succ x \succ y \\ 6 &: z \succ x \succ y \end{aligned}$$

In Γ_1 the strategies that survive are $(1,1,0)$ and $(1,0,0)$ for 1 and 2; $(0,1,0)$ and $(0,1,1)$ for 3 and 4 and $(0,0,1)$ and $(1,0,1)$ for 5 and 6. Consider 1 and 2: strategy $(1,1,0)$ is a UBR to $(1,0,0)+(0,1,0)+(0,1,1))+2(0,0,1)$. Strategy $(1,0,0)$ is a UBR to e.g $(1,0,0) + 2(0,1,0) + (1,0,1) + (0,0,1)$. For players

3 and 4, strategy $(0,1,1)$ is a UBR to $2(1,0,0)+(0,1,0)+(1,0,1)+(0,0,1)$ and strategy $(0,1,0)$ is a UBR to $2(1,1,0)+(0,1,1)+(0,0,1)+(1,0,1)$. Finally for players 5 and 6, strategy $(1,0,1)$ is a UBR to $(1,0,0)+(1,1,0)+2(0,1,0)+(0,0,1)$ while strategy $(0,0,1)$ is a UBR to $(1,1,0)+(1,0,0)+2(0,1,1)+(1,0,1)$. Thus, the game is not DS. □

NPR does not come under either of the categories we studied. Indeed, the conditions of Theorem 3 are not satisfied for NPR (since $\underline{s}_r = 1$.). Thus we find the sufficient conditions for this scoring rule as a separate case:

6.2 Negative plurality rule (NPR)

In the NPR voting game players have the following pure strategies: $S_i = \{(0,1,1),(1,0,1) \text{ and } (1,1,0)\}$. The following lemma characterises the weakly dominated strategies in this game.

Theorem 5 (Sufficient conditions for Dominance Solvability): *The NPR game with three alternatives is DS if* $\frac{2n}{3} - \frac{2}{3} \leq n'_x \leq \frac{2n}{3} - \frac{1}{3}$, *and preferences in* x *are polarised.*

To prove this result, observe that Proposition 1 implies that Γ_{1r} for $r = NPR$ consists of strategies that give 1 to the best alternative for all voters. Thus the undominated game Γ_{1r} has two strategies for each player.

Lemma 8: Consider the undominated game Γ_{1NPR}. *If* $n'_x \geq \frac{2n}{3} - \frac{2}{3}$ *and preferences in* x *are polarised, the only undominated strategy for any* $i \notin N'_x$ *is* $(0,1,1)$.

Thus, only strategy $(0,1,1)$ remains for all $i \notin N'_x$, and strategies $(1,1,0)$ and $(1,0,1)$ for $i \in N'_x$ in the reduced game.

From the above, all $i \in N'_x$ give 1 to x (Lemma 1) and all $i \notin N'_x$ give 0 to x (Lemma 2), hence in the game Γ_{2r}, $\omega_x = n'_x$.

Lemma 9: Consider the undominated game Γ_{2r} *where* $\omega_x = n'_x$. *If* $n'_x \leq \frac{2n}{3} - \frac{1}{3}$

the only undominated strategy for any $i \in N'_x$ is one which gives 1 to the best and 0 to the worst alternative.

Proof of Theorem 5: Proposition 1 shows that the first stage of iterated deletion leads to the game Γ_{1r}, where only strategies that give 1 to the best alternative survive. Then Lemma 8 shows that in Γ_1 for all $i \notin N'_x$ the only strategy that is not weakly dominated is $(0, 1, 1)$. Finally Lemma 9 shows that in Γ_{2r} for the remaining voters, $i \in N'_x$, the strategy that gives 1 to the best and 0 to the worst is the only one that survives. Hence all voters have exactly one strategy remaining and the game is DS. □

The proposition above gives sufficient conditions for the NPR game to be DS. Are these conditions necessary as well? As the next proposition shows, this is not the case. We cannot completely classify the game when preferences are not polarised.

Theorem 6 (Sufficient condition for Non Dominance Solvability): *(i) If $n'_x \leq \frac{2n}{3} - \frac{5}{3}$ the NPR game is not DS, (ii) If $n'_x \geq \frac{2n}{3} + \frac{2}{3}$ and preferences in x are polarised, the NPR game is not DS.*

Of course these two propositions do not establish the necessity of the conditions for DS. Indeed, the polarisation condition in Theorems 5 and 6 only makes it easier to classify games but is not a necessary condition.

We present an example that shows that the conditions for non Dominance Solvability are not necessary conditions. Thus the game is not DS even when the conditions are not satisfied.

Example 2 (NPR): $q'_x \geq \frac{2}{3} + \frac{2}{3n}$, preferences are not polarised and the game is not DS.

Let $n = 6, n'_x = 5$ and the preferences such that:

$$\begin{aligned} 1 &: x \succ y \succ z \\ 2 &: x \succ y \succ z \\ 3 &: x \succ y \succ z \\ 4 &: x \succ y \succ z \\ 5 &: x \succ y \succ z \\ 6 &: y \succ x \succ z \end{aligned}$$

In the game Γ_1 the following strategies survive: players $1-5:(1,1,0)$ and $(1,0,1)$ and player 6:$(1,1,0)$ and $(0,1,1)$. For player 6: strategy $(0,1,1)$ is a UBR to $4(1,1,0)+(1,0,1)$ while $(1,1,0)$ is a UBR to $5(1,0,1)$. For players 1-5, strategy$(1,1,0)$ is a UBR to $4(1,0,1)+(0,1,1)$ while $(1,0,1)$ is a UBR to $4(1,1,0)+(0,1,1)$. Hence the game is not DS. □

Thus, to summarise, the NPR game is not DS when $n'_x \leq \frac{2n}{3} - \frac{5}{3}$, it is DS when preferences in x are polarised and $\frac{2n}{3} - \frac{1}{3} \geq n'_x \geq \frac{2n}{3} - \frac{2}{3}$ and it is not DS when $n'_x \geq \frac{2n}{3} + \frac{2}{3}$, and preferences in x are polarised. Asymptotically, however the game is DS iff $n'_x = \frac{2n}{3}$ and preferences are polarised in x, but obviously there is also the requirement that n'_x be an integer. Indeed, we can show that even when n is "small", very few NPR games can be classified as DS. The following proposition shows this:

Taking the integer condition into consideration, note that we need $\lfloor \frac{2n-1}{3} \rfloor \geq n'_x \geq \lceil \frac{2n-2}{3} \rceil$ and preferences polarised, for Dominance Solvability. Let $2n-1 = w$. We can write w as $3m+r$ where m is an integer and $r = 0,1,2$.

Proposition 2: *There exists an n'_x such that $\lfloor \frac{w}{3} \rfloor \geq n'_x \geq \lceil \frac{w-1}{3} \rceil$ iff either $r = 0$, or $r = 1$.*

Proposition 3:*If the conditions stated in Theorem 5 hold, then (i) if $n \geq 5$ a unique CW a^{cw} exists but a^{cw} is never in the winset (ii) if $n = 4$ at least one CW exists and the alternative (s) in the winset is (are) CW.*

7 Conclusion

In this paper we found conditions for three-alternatives voting games to be DS under NPR, AV, BR and RU voting rules. For NPR game we also found conditions for the game not to be DS. These conditions are stated in terms of the largest proportion of voters who agree on which alternative is the worst (best).

Our results show that Approval Voting performs quite well. The intuition is that voters have much more flexibility under this rule. Ideally voters need to be able to choose to maximally differentiate between any two alternatives. BR does not allow the maximal differentiation. RU does allow it, but also allows other strategies which turn out never to be needed. The 'good" properties of AV have been recently studied from another point of view by Brams and Sanver (2003).

A natural question that arises at this stage is: can we say something more precise about the relations between the conditions required for Dominance Solvability of these different scoring rules? The next result tries to answer this question.

Theorem 7: Whenever the sufficient conditions for Dominance Solvability for PR are satisfied, so are those for RU and AV. Whenever the sufficient conditions for Dominance Solvability for BR are satisfied, so are those for RU and AV. The sufficient conditions for Dominance Solvability for AV and RU are the same.

The proof is obvious.

From the results in the last section we can compare plurality, negative plurality approval and Borda voting rules using as a criterion the conditions for the associated voting games to be DS.

The worst in terms of our criterion seems to be NPR. While we cannot show analytically that whenever the sufficient conditions for Dominance Solvability are satisfied for NPR they are also satisfied for AV and RU, we have not found a counterexample either. Many examples can be constructed where the NPR game is DS and it turns out that the corresponding AV game (i.e. with the same preferences) is also DS.

An attractive feature about the PR,AV and RU games is that the iterated elimination procedure takes a very intuitive form: we use a sequence of iterations that corresponds closely to the reasoning in voters's minds when they vote strategically: i.e. the iteration proceeds by elimination of candidates that everyone knows are effectively not in the race. Thus the steps of iterated elimination correspond to reducing the set of outcomes that can occur! At every step all voters have the same strategy set. This feature of Plurality Rule (and AV,RU) makes it eminently suitable to be used in experimental settings for testing the powers of subjects with regard to iterated dominance reasoning.

Obviously the main part missing in this paper is necessary conditions for Dominance Solvability (or at least sufficient conditions for non Dominance Solvability) so that we could characterise the necessary and sufficient conditions at least for large groups of voters. This proved to be difficult to do when we move away from scoring rules where the number of strategies is equal to the number of alternatives.

References

[1] Dhillon, A. and Lockwood, B., When are Plurality Rule Voting Games Dominance-Solvable?, forthcoming, Games and Economic Behaviour.

[2] Dhillon, A. and J.F.Mertens, Relative Utilitarianism,*Econometrica*, Vol. 67, No.3, 1999.

[3] Brams, Steven J. and M.Remzvi Sanver, Voter Sovereignty and Election Outcomes, NYU Working Paper, 2003.

[4] Farquharson, R., *Theory of Voting,* New Haven: Yale University Press, 1969.

[5] Gilli, M., Iterated Admissibility as Solution Concept in Game Theory, Working Paper, Department of Economics, University of Milan–Bicocca, 2002.

[6] Ho, T. Camerer, C. and Weigelt, K., Iterated Dominance and Iterated Best Response in Experimental "p–Beauty Contests", *The American Economic Review* **88** (1998), 947–969.

[7] Marx, L., Adaptive Learning and Iterated Weak Dominance, *Games and Economic Behavior* **26** (1999), 253–278.

[8] Marx, L. and Swinkels, J., Order Independence for Iterated Weak Dominance, *Games and Economic Behavior* **18** (1997), 219–245.

[9] Moulin, H., *The Strategy of Social Choice*, North Holland, 1983.

[10] Myerson, R., Comparison of Scoring Rules in Poisson Voting Games, *Journal of Economic Theory* **103** (2002), 219–251.

[11] Myerson, R. and Weber,R., A Theory of Voting Equilibria, *American Political Science Review* **87** (1993), 102–114.

A. Appendix

A.1. Proposition 1

Proposition 1 *In the game Γ_r the only strategies that are undominated for a voter i are those that give $c_j \geq \overline{s}_r$ to his top ranked alternative $j \in X$ and $c_k \leq \underline{s}_r$ to his worst ranked alternative $k \in X$.*

Before we prove this proposition, we introduce a lemma:

Lemma 1: Consider a voter i such that $x \succ_i y \succ_i z$. Consider two strategies for such a voter: $v_i = (c_x, c_y, c_z)$ and $v_i' = (c_x', c_y', c_z')$. Then strategy v_i is at least as good as v_i' iff $c_x - c_y \geq c_x' - c_y'$ and $c_x - c_z \geq c_x' - c_z'$ and $c_y - c_z \geq c_y' - c_z'$. The proof is obvious.

Proof of Proposition 1: Assume, w.l.o.g that $x \succ_i y \succ_i z$. The strategies of voter i are partitioned (upto duplication when $s_r \in \{0, 1\}$) into the following: $\{(1, s_r, 0); (0, s_r, 1); (s_r, 1, 0); (s_r, 0, 1); (1, 0, s_r); (0, 1, s_r)\}$, for $s_r \in S_r$. We show (i) that any strategy of the form $(0, s_r, 1)$ for a fixed s_r is weakly dominated by strategy $(1, s_r, 0)$. By Lemma 1 $(1, s_r, 0)$ is at least as good as $(0, s_r, 1)$: when v_{-i} is chosen so that 1/2 of the voters (except i) use $(1, 0, s_r)$ and 1/2 use $(s_r, 0, 1)$ (if n is even then let the extra voter use $(s_r, 0, 1)$), then $\omega_z(v_{-i}) - \omega_x(v_{-i}) \leq 1 - s_r$, $\omega_y(v_{-i}) = 0$. Thus for this profile, $(1, s_r, 0)$ is strictly better than $(0, s_r, 1)$. (ii) Assume $s_r \neq 0$ (otherwise the strategies are not distinct): Any strategy of the form $(0, 1, s_r)$ is weakly dominated by $(s_r, 1, 0)$. By Lemma 1 $(s_r, 1, 0)$ is at least as good as $(0, 1, s_r)$ and we can construct the same profile v_{-i} as for (i), except that if n is even, let the extra voter use $(0, 1, s_r)$ and all others are evenly divided between the strategies $(1, 0, s_r)$ and $(s_r, 0, 1)$ so that $\omega_z(v_{-i}) - \omega_x(v_{-i}) \leq s_r$ and $\omega_y(v_{-i}) \leq 1$. On this profile, $(s_r, 1, 0)$ is strictly better than $(0, 1, s_r)$. Moreover, if $n > 3$, y can never be in the winset with the profile v_{-i}. (iii) Assume $s_r \neq 1$, (otherwise the two strategies are not distinct). Any strategy of the form $(s_r, 0, 1)$ is weakly dominated by $(1, 0, s_r)$. Again, by lemma 1, strategy $(1, 0, s_r)$ is at least as good as strategy $(s_r, 0, 1)$. It is strictly better for v_{-i} constructed above for (i).

This leaves us with strategies of the form $\{(1, s_r, 0); (s_r, 1, 0); (1, 0, s_r)\}$, for $s_r \in S_r$. We now show that (i) either strategy $(\overline{s}_r, 1, 0)$ weakly dominates strategy $(s_r, 1, 0)$ for some $s_r \neq \overline{s}_r$ or all such strategies are equivalent. By lemma 1, $(\overline{s}_r, 1, 0)$ is at least as good as $(s_r, 1, 0)$ for all $s_r \neq \overline{s}_r$. Either there exists a profile and an s_r such that $\overline{s}_r = \omega_z(v_{-i}) - \omega_x(v_{-i}) > s_r$ and $\omega_y(v_{-i}) = 0$: In this case, strategy $(\overline{s}_r, 1, 0)$ is strictly better than $(s_r, 1, 0)$. Thus all such strategies can be removed. Or all such strategies $(s_r, 1, 0)$ are equivalent. The removal of such *redundant* strategies does not change the set of outcomes that survive iterated elimination (see Marx and Swinkels, 1997): thus we are left with strategy $(\overline{s}_r, 1, 0)$ in any case. (ii) Either strategy $(1, 0, \underline{s}_r)$ weakly dominates strategy $(1, 0, s_r)$ for some $s_r \neq \underline{s}_r$ or all such strategies $(1, 0, s_r)$ are equivalent. By lemma 1 strategy $(1, 0, \underline{s}_r)$ is at least as good as all strategies $(1, 0, s_r)$ for all $s_r \neq \underline{s}_r$. If there exists a profile v_{-i} such that $1 - \underline{s}_r = \omega_z(v_{-i}) - \omega_x(v_{-i}) > 1 - s_r$ and $\omega_y(v_{-i}) = 0$, then $(1, 0, \underline{s}_r)$ weakly dominates $(1, 0, s_r)$. We can thus remove such strategies. If there does not exist any such profile for any $s_r \neq \underline{s}_r$, then all such strategies are equivalent, and we can choose to eliminate all except $(1, 0, \underline{s}_r)$. (iii) Finally (since we can choose the order of elimination) we choose to let strategies $(1, s, 0)$ remain, even if some might be dominated.

$\square$.

A.2.Scoring Rules with $\underline{s}_r < \frac{1}{2}$ *Lemma 2: Assume* $\underline{s}_r < 1/2$. *If* $n_z > \frac{n(2-\underline{s}_r)}{3(1-\underline{s}_r)}$ *then* $W \subset \{\{x\}, \{y\}, \{xy\}\}$.

Proof of Lemma 2: By proposition 1 the maximum score that z can get in any profile is $\omega_z = n_z(\underline{s}_r) + (n - n_z)1$. Suppose to the contrary that z was in the winning set for some profile v: the minimal score z requires would be in the case that it ties with x and y. Note that the minimum sum of scores over profiles is $\Sigma_r = n(1 + \underline{s}_r)$, while the maximum sum of scores possible when z ties with x and y is $\omega_x(v) + \omega_y(v) + \omega_z(v) = 3(n_z(\underline{s}_r) + (n - n_z)1)$. If $n_z > \frac{n(2-\underline{s}_r)}{3(1-\underline{s}_r)}$ then $\omega_x(v) + \omega_y(v) + \omega_z(v) < \Sigma_r$, a contradiction. Moreover $n_z \leq n$ implies that $\underline{s}_r \leq 1/2$. The case $\underline{s}_r = 1/2$, is discussed in Section 3 (Borda Rule). Hence we need $\underline{s}_r < 1/2$, for the Lemma to hold.

$\square$

Corollary to Lemma 2: If $\underline{s}_r = \frac{1}{2}$, *and* $n_z = n$ *then* $W(v) \in \{\{x\}, \{x, y\}, \{y\},$

$\{x, y, z\}\}$, *for all* $v \in V$.

Proof of Corollary to Lemma 2: When $\underline{s}_r = \frac{1}{2}$, then it is possible that z is in the winning set.But this is only possible if it ties with x and y. Suppose not: it is sufficient to look at cases where $W(v) = \{x, z\}$ or $W(v) = \{y, z\}$. W.l.o.g let $W(v) = \{x, z\}$ for some $v \in V$ then $\max(\omega_x(v) + \omega_z(v)) = 2(\max(\omega_z)) = 2(\frac{n}{2}) = n < n(1+\frac{1}{2})$. Since $n(1+\frac{1}{2}) \leq \Sigma(v) \leq n(1+\overline{s}_r)$ for all profiles $v \in V$, this is a contradiction.

□

Lemma 3: Consider a voter $i \in N_z$ *such that* $x \succ_i y \succ_i z$. *Assume* $W \subset \{\{x\}, \{y\}, \{xy\}\}$. *Then either (i) the strategy* $(1, 0, \underline{s}_r)$ *weakly dominates strategies* $(1, s_r, 0)$ *and* $(\overline{s}_r, 1, 0)$ *for* $s_r > 0$, *or (ii)all strategies of* i *are equivalent.*

Proof of Lemma 3: (i) Let $v_i = (1, 0, \underline{s}_r)$, $v'_i = (1, s_r, 0)$, and $v''_i = (\overline{s}_r, 1, 0)$. Let $v = (v_i, v_{-i})$, $v' = (v'_i, v_{-i})$, $v'' = (v''_i, v_{-i})$. If $W(v) = \{x\}$, then clearly $W(v') \subset \{\{x\}, \{y\}, \{x, y\}\}$, since the score for x must be the same while that for y may increase if $s_r > 0$. Also if $W(v) = \{x\}$, then clearly $W(v'') \subset \{\{x\}, \{y\}, \{x, y\}\}$, since the score for x goes down given v_{-i} if $\overline{s}_r < 1$ while that for y increases. If $W(v) = \{y\}$, then clearly $W(v') = \{y\}$, since the score for x must be the same while that for y may increase if $s_r > 0$. Also if $W(v) = \{y\}$, then clearly $W(v'') = \{y\}$, since the score for x does not increase given v_{-i} if $\overline{s}_r \leq 1$ while that for y increases. Finally if $W(v) = \{xy\}$, then $W(v') \subset \{\{y\}, \{x, y\}\}$, since the score for x must be the same while that for y may increase if $s_r > 0$. Also if $W(v) = \{xy\}$, then clearly $W(v'') \subset \{\{y\}, \{x, y\}\}$, since the score for x goes down given v_{-i} if $\overline{s}_r < 1$ while that for y increases.

Thus no matter what v_{-i} is, the strategy $(1, 0, \underline{s}_r)$ is weakly better for i given that $W \subset \{\{x\}, \{y\}, \{xy\}\}$. The strategy $(1, 0, \underline{s}_r)$ is strictly better than v'_i (that is, if $s_r \neq 0$) if there exists a v_{-i} such that $1 > \omega_y(v_{-i}) - \omega_x(v_{-i}) \geq 1 - s_r$, or if there exists a v_{-i} such that $1 = \omega_y(v_{-i}) - \omega_x(v_{-i}) > 1 - s_r$.

Similarly the strategy v_i is strictly better than the strategy v''_i if there exists a profile such that $1 > \omega_y(v_{-i}) - \omega_x(v_{-i}) \geq \overline{s}_r - 1$, or if there exists a v_{-i} such that $1 = \omega_y(v_{-i}) - \omega_x(v_{-i}) > \overline{s}_r - 1$. If any of these strategies exists in the game Γ_{1r} then strategy v_i weakly dominates strategy v_i' and v_i'' respectively.

If such profiles do not exist, then any pair of such strategies must be equivalent in the sense that $W(v_i, v_{-i}) = W(v_i', v_{-i}), \forall v_i, v_i' \in V_i$.

$\square$

Lemma 4: Consider a voter $i \notin N_z$ *such that* $z \succ_i x \succ_i y$. *Assume* $W \subset \{\{x\}, \{y\}, \{xy\}\}$. *Then either (i) the strategy* $(1, 0, \overline{s}_r)$ *(denoted* v_i*) weakly dominates strategies* $(0, \underline{s}_r, 1)$ *(denoted* v_i'*) and* $(s_r, 0, 1)$ *(denoted* v_i''*) for* $s_r > 0$*, or (ii) all strategies of* i *are equivalent.*

Proof of Lemma 4: (i) Let $v = (v_i, v_{-i})$, $v' = (v_i', v_{-i})$, $v'' = (v_i'', v_{-i})$.

(A)If $W(v) = \{x\}$, (i.e. $\omega_y(v_{-i}) - \omega_x(v_{-i}) < 1$) then clearly $W(v') = \{x\}$, if (i)$\omega_x(v_{-i}) - \omega_y(v_{-i}) > \underline{s}_r$, $W(v') = \{x, y\}$, if (ii) $\omega_x(v_{-i}) - \omega_y(v_{-i}) = \underline{s}_r$,and (iii)$W(v') = \{y\}$, if $\omega_x(v_{-i}) - \omega_y(v_{-i}) < \underline{s}_r$.

(B)If $W(v) = \{x, y\}$, (i.e. $\omega_y(v_{-i}) - \omega_x(v_{-i}) = 1$) then clearly$W(v') = \{y\}$, since given that $\omega_y(v_{-i}) - \omega_x(v_{-i}) = 1$, $\omega_x(v_{-i}) < \omega_y + \underline{s}_r$, for any $\underline{s}_r \in S_r$.

(C)If $W(v) = \{y\}$, (i.e. $\omega_y(v_{-i}) - \omega_x(v_{-i}) > 1$) then clearly $W(v') = \{y\}$, since given that $\omega_y(v_{-i}) - \omega_x(v_{-i}) > 1$, $\omega_x(v_{-i}) < \omega_y + \underline{s}_r$, for any $\underline{s}_r \in S_r$.

Thus no matter what v_{-i} is, the strategy v_i is weakly better for i given that $W \subset \{\{x\}, \{y\}, \{xy\}\}$. The strategy v_i is strictly better than v_i' if there exists a v_{-i} such that $\omega_y(v_{-i}) - \omega_x(v_{-i}) < 1$ and $\omega_x(v_{-i}) - \omega_y(v_{-i}) = \underline{s}_r$, or $\omega_x(v_{-i}) - \omega_y(v_{-i}) < \underline{s}_r$ (Cases A (ii) and (iii) respectively), or if $\omega_y(v_{-i}) - \omega_x(v_{-i}) = 1$ (Case (B).) If any of these profiles exist then strategy v_i weakly dominates strategy v_i'. If no such profile exists the two strategies are equivalent.

Now we consider strategy v_i'':

(A')If $W(v) = \{x\}$, (i.e. $\omega_y(v_{-i}) - \omega_x(v_{-i}) < 1$) then clearly $W(v'') = \{x\}$, if $\omega_y(v_{-i}) - \omega_x(v_{-i}) < s_r$, $W(v') = \{x, y\}$, if $\omega_y(v_{-i}) - \omega_x(v_{-i}) = s_r$, (for $s_r < 1$), and $W(v') = \{y\}$, if $\omega_y(v_{-i}) - \omega_x(v_{-i}) > \underline{s}_r$.

(B')If $W(v) = \{x, y\}$, (i.e. $\omega_y(v_{-i}) - \omega_x(v_{-i}) = 1$) then clearly $W(v') =$

$\{y\}$, since given that $\omega_y(v_{-i}) - \omega_x(v_{-i}) = 1$, $\omega_y(v_{-i}) - \omega_x(v_{-i} > s_r$, for any $s_r < 1$.

(C')Finally if $W(v) = \{y\}$, (i.e. $\omega_y(v_{-i}) - \omega_x(v_{-i}) > 1$) then clearly $W(v') = \{y\}$, since given that $\omega_y(v_{-i}) - \omega_x(v_{-i}) > 1$, $\omega_y(v_{-i}) - \omega_x > s_r$, for any $s_r \in S_r$.

The strategy v_i is strictly better than the strategy v_i'' if there exists a profile such that $\omega_y(v_{-i}) - \omega_x(v_{-i}) < 1$ and $\omega_y(v_{-i}) - \omega_x(v_{-i}) = s_r$, (for $s_r < 1$), or $\omega_y(v_{-i}) - \omega_x(v_{-i}) > \underline{s}_r$ (Cases (A')(ii) and (iii) respectively) or $\omega_y(v_{-i}) - \omega_x(v_{-i}) = 1$ (Case (B')). If any of these profiles exists strategy v_i weakly domimates strategy v_i''. If no such profile exists then the two strategies are clearly equivalent, i.e. they give the same outcome for all profiles v_{-i}. □

A.3.Scoring Rules with $\overline{s}_r > \frac{1}{2}$

Lemma 5: If $n'_x > \frac{n(1+\overline{s}_r)}{3\overline{s}_r}$ *then* $L \in \{\{y\}, \{z\}, \{y, z\}\}$.

Proof of Lemma 5: Suppose to the contrary that $\exists v \in V$ such that $x \in L(v)$. Then, since all $i \in N'_x$ give a minimum of $\overline{s}_r$ to x we have: $\min(\omega_x) = n'_x\overline{s}_r$. If $x \in L$, $\omega_y \geq \omega_x, \omega_z \geq \omega_x$. Thus, $\min(\omega_x + \omega_y + \omega_z) \geq 3n'_x\overline{s}_r > n(1+\overline{s}_r)$ since $n'_x > \frac{n(1+\overline{s}_r)}{3\overline{s}_r}$, a contradiction.

□

Corollary 2 to Lemma 5: If $\overline{s}_r = \frac{1}{2}$ *and* $n'_x = n$ *then* $L \in \{\{y\}, \{z\},\{x, y, z\},\{y, z\}\}$.

Proof of Corollary to Lemma 5: It is sufficient to show that $\{x, y\}$ and $\{x, z\}$ cannot be in the losing set. W.l.o.g suppose $L(v) = \{x, y\}$ for some $v \in V$. Then $\omega_x(v) = \omega_y(v) < \omega_z(v)$. Since $n'_x = n$, $\min(\omega_x(v)) = \frac{n}{2}$, so that $\min(\omega_x(v) + \omega_y(v) + \omega_z(v) > \frac{3n}{2}$, a contradiction, since $\Sigma(v) \leq \frac{3n}{2}$ when $\overline{s}_r = \frac{1}{2}$. □

Lemma 6: Let $i \in N'_x$, *such that* $x \succ_i y \succ_i z$. *If* $L \subset \{\{y\}, \{z\}, \{y, z\}\}$, *then either (a) strategies* $v_i = (\overline{s}_r, 1, 0)$ *and* $v'_i = (1, s_r, 0)$ *are both weakly dominated by strategy* $\tilde{v}_i = (1, \underline{s}_r, 0)$ *or (b) all strategies are equivalent for* i.

Proof of Lemma 6: By Proposition 1, strategies for such an individual i in Γ_{1r} are $(1, 0, \underline{s}_r)$; $(1, s_r, 0)$; $(\overline{s}_r, 1, 0)$, where $s_r \in [\underline{s}_r, \overline{s}_r]$.

Using Lemma 1, we can deduce that the probability that x is in the winning set cannot decrease relative to other alternatives in the winning set when i uses strategy $(1, \underline{s}_r, 0)$ instead of strategy $(\overline{s}_r, 1, 0)$ or $(1, s_r, 0)$ (since $c_x - c_y$ and $c_x - c_z$ are both greater with strategy $\tilde{v}_i$ than with strategies v_i and v_i').

This means that whenever $W(v) = \{x\}$, then $W(\tilde{v}) = \{x\}$, when $W(v) = \{x, y\}$ then $W(\tilde{v}) \in \{\{x, y\}, \{x\}\}$; when $W(v) = \{x, z\}$, then $W(\tilde{v}) \in \{\{x\}, \{x, z\}\}$.

Similarly, whenever $W(v') = \{x\}$, then $W(\tilde{v}) = \{x\}$, when $W(v') = \{x, y\}$, then $W(\tilde{v}) \in \{\{x, y\}, \{x\}\}$; when $W(v') = \{x, z\}$, then $W(\tilde{v}) \in \{\{x\}, \{x, z\}\}$.

So the only cases to check are $W \in \{\{y\}, \{z\}\}$. If $W(v) = W(v') = \{z\}$ when i uses strategy v_i, or v_i' then i can do no worse with strategy $\tilde{v}_i$. Thus the only case to check is when $W = \{y\}$ when i uses strategy v_i or v_i'.

Thus consider strategy v_i first. If $W(v) = \{y\}$, then $\omega_y(v_{-i}) + 1 > \omega_x(v_{-i}) + \overline{s}_r$, and $\omega_y(v_{-i}) + 1 > \omega_z(v_{-i})$. If he uses $\tilde{v}_i$ then the outcome could be $\{y\}$ (if $\omega_y(v_{-i}) + \underline{s}_r > \omega_x(v_{-i}) + 1$ or $\{x, y\}$ (if $\omega_y(v_{-i}) + \underline{s}_r = \omega_x(v_{-i}) + 1)$, or $\{x\}$. The outcomes $\{z\}$ or $\{y, z\}$, are ruled out since $\omega_x + \underline{s}_r > \omega_z$ by Comment 1.

If a profile v_{-i} exists such that $W(v) = \{x, y\}$ and $W(\tilde{v}) = \{x\}$; or $W(v) = \{x, z\}$, and $W(\tilde{v}) = \{x\}$ or $W(v) = \{z\}$ and $W(\tilde{v}) = \{x\}$ or $W(\tilde{v}) = \{x, z\}$ then strategy $\tilde{v}_i$ weakly dominates strategy v_i.Otherwise the two are equivalent.

The same argument applies to v_i'. The outcomes $\{z\}$ or $\{z, y\}$ are ruled out by Comment 1, so if $W(v') = \{y\}$ then $W(\tilde{v}) \in \{\{x\}, \{x, y\}\}$.

If a profile v_{-i} exists such that $W(v') = \{x, y\}$ and $W(\tilde{v}) = \{\{x\}\}$ or $W(v') = \{x, z\}$, and $W(\tilde{v}) = \{\{x\}\}$ or $W(v') = \{z\}$ and $W(\tilde{v}) = \{x\}$ or $W(\tilde{v}) = \{x, z\}$ then strategy $\tilde{v}_i$ weakly dominates strategy v_i.Otherwise the two are equivalent.

$\square$

Lemma 7: In the game Γ_{2r}, if $\underline{s}_r < 1$, and $n'_x > \frac{n}{2-\underline{s}_r}$ then the Scoring rule game is DS and x is the unique winner.

Proof: The reduced game Γ_{2r} is the game where all $i \in N'_x$ have strategies $(1, 0, \underline{s}_r)$ and $(1, \underline{s}_r, 0)$ remaining. This implies that $\min(\omega_x) = n'_x$, $\max(\omega_y, \omega_z) = n'_x(\underline{s}_r) + (n - n'_x)$. Thus $W = \{x\}$ if $n'_x > n'_x\underline{s}_r + n - n'_x$, i.e. if $n'_x > \frac{n}{2-\underline{s}_r}$. $\square$

Corollary to Lemma 7: In the game Γ_{2r} if $\underline{s}_r < \frac{1}{2}$, then if $n'_x > \frac{2n}{3}$, the game is DS and x is the unique winner.

Proof: Denote $n'_t = \frac{n}{2-\underline{s}_r}$. By Lemma 7 the game is DS and x is the unique winner if $n'_x > n'_t$. Since $\underline{s}_r \in [0, \frac{1}{2})$, $\sup n'_t = \frac{2n}{3}$. Thus the condition $n'_x > n'_t$ is always satisfied if $n'_x > \frac{2n}{3}$. $\square$

A.4 Negative Plurality Rule

Lemma 8: Consider the undominated game Γ_{1NPR}. If $n'_x \geq \frac{2n}{3} - \frac{2}{3}$ and preferences in x are polarised, the only undominated strategy for any $i \notin N'_x$ is $(0, 1, 1)$.

Proof of Lemma 8: W.l.o.g. consider $i \in N'_z$. Preferences are polarised so we have $z \succ_i y \succ_i x$. By Proposition 1 his remaining strategies in Γ_{1r} are $(1, 0, 1)$ and $(0, 1, 1)$. The strategy $(1, 0, 1)$ is a UBR whenever i is pivotal between candidates z and y only[8], i.e. the strategy $(1, 0, 1)$ is a UBR to the following profiles: (i) $\omega_y = \omega_z \geq \omega_x + 1$, or (ii) $\omega_y - \omega_z = 1$ and $\omega_z \geq \omega_x + 1$, or possibly (iii) $\omega_y - \omega_z = 1$ and $\omega_z \geq \omega_x$, (if the lottery over (x, y, z) is preferred by i to y).

The strategy $(0, 1, 1)$ is a UBR whenever i is pivotal between candidates x and y only[9], i.e. the strategy $(0, 1, 1)$ is a UBR to the following profiles: (i)' $\omega_x = \omega_y \geq \omega_z + 1$, or (ii)' $\omega_x - \omega_y = 1$ and $\omega_y \geq \omega_z + 1$, or possibly (iii)' $\omega_y - \omega_x = 1$ and $\omega_x \geq \omega_z$, (if the lottery over (x, y, z) is less preferred by i to y). Similarly the strategy $(0, 1, 1)$ is a UBR to the following profiles: (iv)' $\omega_x = \omega_z \geq \omega_y + 1$ or (v)' $\omega_x - \omega_z = 1$ and $\omega_z \geq \omega_y$ or (vi)'$\omega_x = \omega_y = \omega_z$. In

[8] I.e. he cannot achieve the outcome x for sure with his vote, only in a lottery.

[9] I.e. he cannot achieve his best outcome z with his vote but he can achieve either outcome x or y.

all other profiles v_{-i}, the two strategies give the same payoff. Since there are only two pure strategies, it is sufficient to show that $\nexists\, v_{-i}$ such that strategy $(1,0,1)$ is a UBR and $\exists\, v_{-i}$ such that $(0,1,1)$ is a UBR. Claim 1 proves the former and Claim 2 proves the latter.

Claim 1 (Lemma 8): If $n'_x \geq \frac{2n}{3} - \frac{2}{3}$, then $\nexists\, v_{-i}$ such that either of (i), (ii) or (iii) holds.

Proof: If such a profile exists, it must exist for the minimum ω_x possible i.e. when $\omega_x = n'_x$. Thus it is sufficient to consider this case. Assume $\exists\, v_{-i}$ such that either of (i), (ii) or (iii) holds. Case (i) implies that $\min \Sigma(v_{-i}) = 3n'_x + 2$, case (ii) implies that $\min \Sigma(v_{-i}) = 3n'_x + 1$, and case (iii) implies that $\min \Sigma(v_{-i}) = 3n'_x + 3$. If $n'_x \geq \frac{2n}{3} - \frac{2}{3}$ for cases (i), (ii) and (iii) we have that $\min \Sigma(v_{-i}) > 2n - 1 > 2n - 2$, a contradiction.

□

Claim 2 (Lemma 8). If $n'_x \geq \frac{2n}{3} - \frac{2}{3}$ then $\exists\, v_{-i}$ such that strategy $(0,1,1)$ is a UBR.

Proof: We construct a strategy v_{-i} such that $(0,1,1)$ is a UBR. In particular, we construct a strategy v_{-i} such that either (i)' or (ii)' holds.

Let $\omega_x = n'_x + n'_y + \lceil \frac{n'_z - 1}{2} \rceil$, $\omega_y = n'_x + n'_y + \lfloor \frac{n'_z - 1}{2} \rfloor$, $\omega_z = n'_z - 1$, then clearly $\Sigma(v_{-i}) = 2n - 2$, hence this profile is feasible[10]. So the only thing to check is that strategy $(0,1,1)$ is a UBR. Suppose n'_z is odd: then $\omega_x = \omega_y$ and we need (from (i)'): $\omega_x \geq \omega_z + 1$, i.e. $n'_z \leq \frac{2n}{3} - \frac{1}{3}$. If n'_z is even: $\omega_x - \omega_y = 1$, and we need (from (ii)') $\omega_y \geq \omega_z$, i.e $n'_z \leq \frac{2n}{3}$. Thus, if $n'_z \leq \frac{2n}{3} - \frac{1}{3}$, then this profile exists for both cases.

Thus, it remains to show that if $n'_x \geq \frac{2n}{3} - \frac{2}{3}$ then $n'_z \leq \frac{2n}{3} - \frac{1}{3}$. Observe that $n'_x + n'_y + n'_z = n$. Hence, if $n'_x \geq \frac{2n}{3} - \frac{2}{3}$, we have that $\max(n'_z) \leq \frac{1n}{3} + \frac{2}{3}$. Moreover, since $n \geq 3$, note that $\frac{n}{3} + \frac{2}{3} \leq \frac{2n}{3} - \frac{1}{3}$.

□

[10] Take $(1,1,0)$ for all $i \in N'_x$, N'_y, and if $n'_z - 1$ is even divide voters in N'_z equally in to those using $(0,1,1)$ and those using $(1,0,1)$, and if $n'_z - 1$ is odd take $\frac{n'_z}{2}$ voters in N'_z using $(1,1,0)$ and $\frac{n'_z}{2} - 1$ using $(1,0,1)$.

Lemma 9: Consider the undominated game Γ_{2r} where $\omega_x = n'_x$. If $n'_x \leq \frac{2n}{3} - \frac{1}{3}$ the only undominated strategy for any $i \in N'_x$ is one which gives 1 to the best and 0 to the worst alternative.

Proof of Lemma 9: Let $i \in N'_x$. W.l.o.g let $x \succ_i y \succ_i z$. We show that strategy $(1, 0, 1)$ is dominated by $(1, 1, 0)$. The strategy $(1, 0, 1)$ is a UBR to the following profiles: (i) $\omega_x = \omega_y \geq \omega_z + 1$, or (ii) $\omega_y - \omega_x = 1$ and $\omega_x \geq \omega_z + 1$, or possibly (iii) $\omega_y - \omega_x = 1$ and $\omega_x \geq \omega_z$, (if the lottery over (x, y, z) is preferred by i to y).

The strategy $(1, 1, 0)$ is a UBR to the following profiles: (i)' $\omega_x = \omega_z \geq \omega_y + 1$, or (ii)' $\omega_y - \omega_x = 1$ and $\omega_x \geq \omega_z$, (if the lottery over (x, y, z) is less preferred to y.) or (iii)' $\omega_z = \omega_y$ and $\omega_y \geq \omega_x + 1$, (iv)' $\omega_z - \omega_y = 1$ and $\omega_y \geq \omega_x$, (v)' $\omega_z - \omega_y = 1$ and $\omega_y \geq \omega_x + 1$,and (vi)' $\omega_z = \omega_y = \omega_x$. In all other profiles v_{-i}, the two strategies give the same payoff. It is sufficient to show (since there are only two pure strategies) that $\nexists\, v_{-i}$ such that strategy $(1, 0, 1)$ is a UBR and $\exists\, v_{-i}$ such that $(1, 1, 0)$ is a UBR. Claim 1 proves the former and Claim 2 proves the latter.

Claim 1 (Lemma 9): If $n'_x \leq \frac{2n}{3} - \frac{1}{3}$, then $\nexists\, v_{-i}$ such that either of (i), (ii) or (iii) holds.

Proof: Since $i \in N'_x$, we know that $\omega_x = n'_x - 1$. Assume $\exists\, v_{-i}$ such that either of (i), (ii) or (iii) holds. Case (i) implies that $\max \Sigma(v_{-i}) = 3n'_x - 4$, case (ii) implies that $\max \Sigma(v_{-i}) = 3n'_x - 3$, and case (iii) implies that $\max \Sigma(v_{-i}) = 3n'_x - 2$. If $q'_x \leq \frac{2}{3} - \frac{2}{3n}$ for cases (i), (ii) and (iii) we have that $\max \Sigma(v_{-i}) < 2n - 4 < 2n - 2$, a contradiction.$\square$

Claim 2 (Lemma 9). If $n'_x \leq \frac{2n}{3} - \frac{1}{3}$ then $\exists\, v_{-i}$ such that strategy $(1, 1, 0)$ is a UBR.

Proof: We construct a strategy v_{-i} such that $(1, 1, 0)$ is a UBR. In particular we construct v_{-i} such that one of (iii)'or (iv)'holds.

Let $\omega_z = n'_y + n'_z + \lceil \frac{n'_x - 1}{2} \rceil$, $\omega_y = n'_y + n'_z + \lfloor \frac{n'_x - 1}{2} \rfloor$, $\omega_x = n'_x - 1$, Then clearly $\Sigma(v_{-i}) = 2n - 2$, hence this profile is feasible[11]. So the only thing to check

[11] Take $(0, 1, 1)$ (the only surviving strategy) for all $i \notin N'_x$ and if $n'_x - 1$ is even divide voters in N'_x equally in to those using $(1, 1, 0)$ and those using $(1, 0, 1)$, and if $n'_x - 1$ is

is that strategy $(1,1,0)$ is a UBR. Suppose n'_x is odd: then $\omega_y = \omega_z$ and we need (from (iii)'): $\omega_y \geq \omega_x + 1$, i.e. $n'_x \leq \frac{2n}{3} - \frac{1}{3}$. If n'_x is even, then we need (from (iv)') $\omega_y \geq \omega_x$, i.e $n'_x \leq \frac{2n}{3}$. Thus, if $n'_x \leq \frac{2n}{3} - \frac{1}{3}$, then at least one of these profiles exists and $(1,1,0)$ is a UBR.

□

Theorem 6 (Sufficient condition for Non Dominance Solvability): *(i) If $n'_x \leq \frac{2n}{3} - \frac{5}{3}$ the NPR game is not DS, (ii) If $n'_x \geq \frac{2n}{3} + \frac{2}{3}$ and preferences in x are polarised, the NPR game is not DS.*

Proof of Theorem 6: (i) It is sufficient to show that every strategy in Γ_{1r} is a UBR to some profile v_{-i}. We show this w.l.o.g for $i \in N'_x$[12]. W.l.o.g assume $x \succ_i y \succ_i z$. Strategy $(1,0,1)$ is a UBR to the following profile: $\omega_x = n'_x - 1 + n'_y + \lfloor \frac{n'_z}{2} \rfloor$, $\omega_y = n'_x - 1 + n'_y + \lceil \frac{n'_z}{2} \rceil$, $\omega_z = n'_z$. This is clearly feasible and it remains to check that $\omega_x \geq \omega_z + 1$. If n'_z is even, then this is so if $n'_z \leq \frac{2n}{3} - \frac{4}{3}$, and if n'_z is odd, it requires $n'_z \leq \frac{2n}{3} - \frac{5}{3}$. Thus if $n'_z \leq n'_x \leq \frac{2n}{3} - \frac{5}{3}$, then strategy $(1,0,1)$ is a UBR. Similarly strategy $(1,1,0)$ is a UBR to the following profile: $\omega_x = n'_x - 1 + n'_z + \lfloor \frac{n'_y}{2} \rfloor$, $\omega_z = n'_x - 1 + n'_z + \lceil \frac{n'_y}{2} \rceil$, $\omega_y = n'_y$. This is clearly feasible and it remains to check that $\omega_x \geq \omega_y + 1$. If n'_y is even, then this is so if $n'_y \leq \frac{2n}{3} - \frac{4}{3}$, and if n'_y is odd, it requires $n'_z \leq \frac{2n}{3} - \frac{5}{3}$. Thus if $n'_y \leq n'_x \leq \frac{2n}{3} - \frac{5}{3}$, then strategy $(1,1,0)$ is a UBR.□

(ii) Since $n'_x > \frac{2n}{3} - \frac{2}{3}$, by Proposition 1 and Lemma 8, we are in game Γ_{2r} where all $i \notin N'_x$ have only strategy $(0,1,1)$ remaining (recall that preferences are polarised in x). It is sufficient to show that for all $i \in N'_x$, strategies $(1,1,0)$ and $(1,0,1)$ are both UBR to some profile v_{-i}, in Γ_2. We show this w.l.o.g for $i \in N'_x$ such that $x \succ_i y \succ_i z$. Strategy $(1,0,1)$ is a UBR to the following profile: Let $n'_x - 1 - (n'_y + n'_z)$ of $i \in N'_x$ vote $(1,1,0)$; and the remaining $n'_y + n'_z$ of them vote $(1,0,1)$. Obviously, all $i \notin N'_x$ vote $(0,1,1)$. Note that $n'_x - 1 \geq (n'_y + n'_z)$, (since $n'_x \geq \frac{2n}{3} + \frac{2}{3} \geq \frac{n}{2} + \frac{1}{2}$) and $\omega_x + \omega_y + \omega_z = 2n - 2$ so this profile is feasible. Moreover in this profile, $\omega_x = \omega_y \geq \omega_z + 1$ iff $n'_x \geq \frac{2n}{3} + \frac{2}{3}$. Strategy $(1,1,0)$ is a UBR to the following profile: Let $n'_x - 1 - (n'_y + n'_z)$ of $i \in N'_x$ vote $(1,0,1)$; and the remaining

odd take $\frac{n'_x}{2}$ voters in N'_x using $(1,0,1)$ and $\frac{n'_x}{2} - 1$ using $(1,1,0)$.

[12]Since $n'_x = \max_{a=x,y,z}(n'_a)$.

$n'_y + n'_z$ of them vote $(1,1,0)$. Obviously, all $i \notin N'_x$ vote $(0,1,1)$. Note that $n'_x - 1 \geq (n'_y + n'_z)$, and $\omega_x + \omega_y + \omega_z = 2n - 2$ so this profile is feasible. Moreover in this profile, $\omega_x = \omega_z \geq \omega_y + 1$ iff $n'_x \geq \frac{2n}{3} + \frac{2}{3}$.

□

Proposition 2: *There exists an n'_x such that $\lfloor \frac{w}{3} \rfloor \geq n'_x \geq \lceil \frac{w-1}{3} \rceil$ iff either $r = 0$, or $r = 1$.*

Proof of proposition 2: First we show that if $r = 0, 1$, there exists an n'_x satisfying the required inequality. If $r = 0$, $\lfloor \frac{w}{3} \rfloor = m = \lceil \frac{w-1}{3} \rceil$, since $\lceil \frac{w-1}{3} \rceil = m - \lfloor \frac{1}{3} \rfloor = m$.

If $r = 1$, $\lfloor \frac{w}{3} \rfloor = m + \lfloor \frac{1}{3} \rfloor = m$. $\lceil \frac{w-1}{3} \rceil = m$, since $w - 1 = 3m$.

It remains to show that when $r = 2$, there does not exist an n'_x satisfying the above:Suppose $\exists$ an n'_x that satisfies the above. Then we need $\lfloor \frac{\omega}{3} \rfloor \geq \lceil \frac{\omega}{3} \rceil$.

If $r = 2$, we have $\lfloor \frac{w}{3} \rfloor = m + \lfloor \frac{2}{3} \rfloor = m$, and $\lceil \frac{w-1}{3} \rceil = m + \lceil \frac{1}{3} \rceil = m + 1$, since $w - 1 = 3m + 1$, a contradiction.

□.

Proposition 3: *If the conditions stated in Theorem 5 hold, then (i) if $n \geq 5$ a unique CW a^{cw} exists but a^{cw} is never in the winset (ii) if $n = 4$ at least one CW exists and the alternative (s) in the winset is (are) CW.*

Proof of Proposition 3. Let $\frac{2n}{3} - \frac{2}{3} \leq n'_x \leq \frac{2n}{3} - \frac{1}{3}$, and preferences in x be polarised. (i) It is easy to check that if $n'_x \geq \frac{2n}{3} - \frac{2}{3}$ and $n \geq 5$ then $n'_x > \frac{n}{2}$ therefore, x is the unique CW. Lemma 2 and lemma 3 imply that : $\omega_x = n'_x$, $\omega_y = n - n'_x + n_z$ and $\omega_z = n - n'_x + n_y$ where $n_a(n'_a)$ is the number of voters who rank a as the worst (best) alternative. Now assume that x is in the winset. If x is in the winset $\Rightarrow \omega_x > \omega_y$ and $\omega_x > \omega_z$ since $n_x + n_y + n_z = n$ and $n_x = n - n'_x$ (preferences in x are polarised) this implies that $n'_x > \frac{2n}{3}$ which is a contradiction. Thus, x is never in the winset.

□