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Abstract

We derive closed-form solutions for the Rubinstein alternating offers game for cases where the two players have (possibly asymmetric) utility functions that belong to the HARA class and discount the future at a constant rate. We show that risk aversion may increase a bargainers payoff. This result - which contradicts Roth's 1985 theorem tying greater risk neutrality to a smaller payoff - does not rely on imperfect information or departures from expected utility maximization.

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1 Introduction

In the celebrated Rubinstein (1982) bargaining game, the parties to a bilateral negotiation make alternating offers on how to split an economic surplus (normalized here to size 1). The players discount the future and are therefore impatient to conclude the negotiation. In the case of linear utility and constant discount rates, the game has a unique subgame-perfect equilibrium (SPE) in which agreement is immediate and the parties receive

$$\widetilde{q} = \frac{\delta + \delta\widetilde{\delta}h}{\delta + \widetilde{\delta} + \delta\widetilde{\delta}h} \qquad \text{and} \qquad q = \frac{\widetilde{\delta}}{\delta + \widetilde{\delta} + \delta\widetilde{\delta}h}, \tag{1}$$

respectively, where $\widetilde{\delta} \in (0,1)$ and $\delta \in (0,1)$ are the discount rates of player 1 and player 2 and h is the length of the interval between offers (player 1 makes the first offer and is denoted throughout by $\tilde{}$). The game confers a 'first-mover' advantage on player 1, but this artefact disappears as the time interval between offers shrinks to zero. In the limit, payoffs depend only on the relative impatience of the players:

$$\widetilde{q} = \frac{\delta}{\delta + \widetilde{\delta}} \qquad \text{and} \qquad q = \frac{\widetilde{\delta}}{\delta + \widetilde{\delta}}. \tag{2}$$

When discount rates are equal, the alternating offers game generates the familiar Nash Bargaining Solution (NBS) of a 50-50 split. For arbitrary discount rates $\in [0,1]$, Binmore (1987a) showed that the Rubinstein equilibrium corresponds to a generalized NBS in which the bargaining powers of the players are inversely related to their discount rates.

Rubinstein's analysis has proven its worth not only in the game theory literature but also in applied theory and empirical work (e.g., Shaked and Sutton 1984, Bulow and Rogoff 1989, Muthoo 1996 and Binmore 2007a). In an applied context, however, the assumption of linear utility is restrictive, and particularly so if the possibility of delay is viewed as central to how players behave. Broad categories of microeconomic behaviour under uncertainty – including applications that might well incorporate aspects of bilateral monopoly – cannot be understood without appeal to some form of risk aversion. At the same time, a great deal of empirical evidence in macroeconomics and modern consumption theory in particular suggests positive risk aversions.

Theorists have long since moved beyond linearity in studying the alternating offers game (Binmore, Osborne and Rubinstein 1992 survey early contributions). Our interest, however, is more specific: what is the impact of concavity on bargaining payoffs? Roth (1985 and 1989) studies this question in the alternating offers game and finds that greater risk aversion decreases a bargainers' share. [1] To our knowledge, departures from Roth's finding have relied either on the inclusion of lotteries in the set of possible outcomes (Roth and Rothblum 1982), on imperfect information (Osborne 1984), or on departures from the expected utility maximization paradigm (Volij and Winter 2002).

[1] As he notes, risk aversion in a non-stochastic environment refers purely to strategic risk - the risk that agreement is delayed - rather than probabilistic risk.

From an applied perspective, the impact of Roth's result has been limited by the absence of closed-form solutions for the risk-averse case. We show in this paper, however, that the linear case is nested within a broader class of cases incorporating alternative and possibly asymmetric degrees of risk aversion. We derive closed form solutions for the alternating offers game for cases in which the two players have utility functions that belong to the hyperbolic absolute risk aversion class (HARA) and have constant discount rates.

The analysis that is closest to ours is that of Binmore (2007b), who derives a closed-form solution to the alternating offers game when the players have iso-elastic utility functions $u(z) = z^{\sigma}$ for $0 < \sigma < 1$. Our approach differs from Binmore's, however, in some important respects. First, we solve the alternating offers game for the entire class of HARA utility functions. Second, in direct contrast to Roth (1985) and the subsequent literature (e.g. Binmore (2007b)), we find that the impact of risk aversion on payoffs can be positive. We illustrate this with a case in which one player displays decreasing absolute risk aversion (DARA). Contrary to previous studies, this contradiction of Roth does not rely on imperfect information or on departures from expected utility maximization. Third, Binmore's analysis is restricted to situations in which both players display risk aversions below one. The bulk of empirical evidence, however, places the degree of relative risk aversion above unity. Our analysis of DARA utility functions covers a much broader spectrum of risk aversions.

2 Preliminaries: linear utility

Consider the alternating offers game over a division of a pie of size one (Rubinstein 1982).

For the game starting at time t, the minimal initial offer by player 1 in any SPE, $q_t^{[1]}$, must leave player 2 indifferent between accepting that amount and rejecting it in order to make its own best counter-offer in the following period. Hence

$$u(q_t^{[1]}) = \beta(h)u(q_{t+h}^{[2]}), \tag{3}$$

where h is the (exogenous) time interval between offers, $\beta(h) = (1+\delta h)^{-1}$ is the discount factor player 2 applies to future utility, and $q_{t+h}^{[2]}$ is the largest share player 2 can hope to retain when it makes its counter-offer. By the same argument, $q_{t+h}^{[2]}$ must leave player 1 indifferent between accepting and rejecting. Hence

$$\widetilde{u}(1 - q_{t+h}^{[2]}) = \widetilde{\beta}(h)\widetilde{u}(1 - q_{t+2h}^{[1]}) \tag{4}$$

where $\widetilde{}$ refers to player 1.

When utility functions are linear, equations (3) and (4) yield the difference equation

$$q_t^{[1]} = \beta[1 - \widetilde{\beta}(1 - q_{t+2h}^{[1]})], \tag{5}$$

with stationary solution

$$q^S(h) = \frac{\beta(1-\widetilde{\beta})}{1-\beta\widetilde{\beta}} \tag{6}$$

or, equivalently, (1). When both players discount the future the transversality condition

$$\lim_{\tau\to\infty} (\beta\widetilde{\beta})^{\tau/2h} q_{t+\tau}^{[1]} = 0 \tag{7}$$

holds for any $h > 0$ and establishes $q_t^{[1]} = q^S$ as the unique solution to (5).

A straightforward argument then establishes that (6) also characterizes player 1's maximal equilibrium offer (Shaked and Sutton 1984, Rubinstein 1987, Binmore 1987b). Equilibrium is therefore unique. The players employ stationary strategies, with player 1 offering q^S whenever it has the offer, always accepting anything at least as good as $1 - q^S$, and always rejecting anything worse than $1 - q^S$ (player 2 does the reverse). Implementation is immediate: player 2 accepts player 1's first offer. Equation (6) approaches equation (2) as the time period between offers goes to zero.

3 The Rubinstein game with utility functions of the HARA class

We show in this section that linear utility is nested within a much broader class that generates closed-form solutions to the Rubinstein game. To see this, note that in the general case equations (3) and (4) imply the recursion

$$q_t^{[1]} = u^{-1}[\beta u\{1 - \widetilde{u}^{-1}(\widetilde{\beta}\widetilde{u}(1 - q_{t+2h}^{[1]}))\}]. \tag{8}$$

The properties of (8) are governed by those of the composite function $g(x) = u^{-1}(mu(x-n))$, for constants m and n. This function is linear in x for any member of the widely-used HARA class of utility functions first described by Merton 1971:

$$u(x) = \frac{\gamma}{1-\gamma}\left(\frac{ax}{\gamma}+d\right)^{1-\gamma} \qquad \text{where} \qquad \frac{ax}{\gamma}+d \geq 0. \tag{9}$$

The Arrow-Pratt measure of risk aversion $\rho(x)$ for this class of utility functions equals $a\gamma/(ax+d\gamma)$. It is easy to see that linear utility prevails whenever $\gamma = 0$. Given HARA utility, the composite function $g(\cdot)$ takes the form $g(x) = m^{1/(1-\gamma)}(x-n) + (m^{1/(1-\gamma)} - 1)\gamma d/a$.

Defining the modified discount factors $\widetilde{b} = \widetilde{\beta}^{1/(1-\widetilde{\gamma})}$ and $b = \beta^{1/(1-\gamma)}$, the shift parameters $\widetilde{k} = \widetilde{\gamma}\widetilde{d}/\widetilde{a}$ and $k = \gamma d/a$ and applying $g(\cdot)$ where needed in equation (8), we obtain a straightforward generalization of (5): [2]

$$q_t^{[1]} = b(1-\widetilde{b})[1+\widetilde{k}] + (b-1)k + (b\widetilde{b})q_{t+2h}^{[1]}. \tag{10}$$

Proposition 1 *When utility functions are in the HARA class, the Rubinstein alternating offers game with discounting has a unique SPE. As the interval between offers goes to zero, the payoff received by player 2 (with discount factor β) is given by*

$$q = \frac{\widetilde{\delta}'(1+\widetilde{k}) - \delta' k}{\delta' + \widetilde{\delta}'} \tag{11}$$

where $\delta' = \delta/(1-\gamma)$ and $\widetilde{\delta}' = \widetilde{\delta}/(1-\widetilde{\gamma})$.

[2] For a player displaying log utility (e.g., $\widetilde{\gamma} = 1$, for player 1), we replace the corresponding modified discount factor with $\widetilde{b} = \exp(\widetilde{\beta}) \in (1, e)$.

Proof. To obtain (11), take the limit of the expression

$$q^S(h) = \frac{b(1-\widetilde{b})\left(1+\widetilde{k}\right)+(b-1)k}{1-b\widetilde{b}}$$

as $h \to 0$ and apply L'Hospital's rule. ■

Note that there is a discontinuity in the 'modified discount factors' $\widetilde{\delta}'$ and δ' at $\gamma = 1$. As they become negative for $\gamma > 1$, this cases formally fall outside the framework of Rubinstein (1982). [3] Throughout, we therefore focus on the cases where $\gamma \in [0, 1]$.

The properties of (11) are intuitive. First, the solution reduces to (2) when preferences are linear (i.e. $\widetilde{\gamma} = \gamma = 0$). Second, symmetry of preferences produces a 50/50 split. Third, when both players display constant relative risk aversion (i.e. when $\widetilde{k} = k = 0$) we have $q = \widetilde{\delta}'/(\delta' + \widetilde{\delta}')$ and the risk tolerance of the players affects the solution only if it differs across players; otherwise the solution for the linear case continues to hold even when the players are both risk averse. Since γ corresponds to the relative risk aversion in the iso-elastic case, the game leads to well behaved solutions only as long as relative risk aversions stay below unity.

[3]Rubinstein required that the side payment needed to compensate a player for delay be an increasing function of the payment being delayed. Thus w would have to be increasing in x in $u(x) = \beta u(x+w)$. But for HARA $w(x) = [(1-b)/b](x+k)$, which is decreasing in x for $b > 1$. With CRRA only cases in which risk aversion is below unity respect Rubinstein's regularity condition.

4 Effects of risk aversion on payoffs

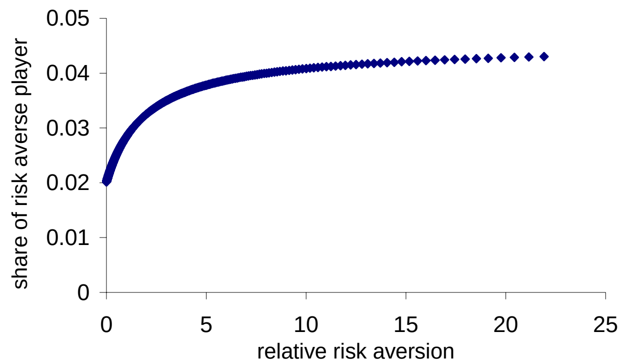
Equation (11) allows us to study the impact of risk aversion on a player's payoff. Moreover, it allows us to conclude that Roth's 1985 theorem tying greater risk neutrality to a smaller bargaining share does not hold generally. To see this, consider the case in which player 2 has a utility function that displays decreasing absolute risk aversion (DARA) while player 1 is risk neutral. This configuration pitching a risk averse player against a risk neutral is of considerable practical interest, with potential applications to insurance, credit, land tenure, and employment relationships, to mention just a few. Let $a = 1$, $d = -1$ so that we have

$$\widetilde{u}(x) = \widetilde{a}x \qquad \text{and} \qquad u(x) = \frac{\gamma}{1-\gamma}\left(\frac{x}{\gamma} - 1\right)^{1-\gamma}.$$

It is easy to verify that the Arrow-Pratt measure of risk aversion $\rho(x)$ for player 2 is strictly increasing in the preference parameter $\gamma \in [0, 1]$. Note that, in this case, the restriction in (9), that guarantees that the DARA utility function is well defined becomes $x \geq \gamma$. To ensure that utility is defined for all feasible bargains $q \in [0, 1]$, assume that players receive endowments at the rate $\widetilde{\omega}$ and ω per period. The endowments assure that the problem is well specified for all $\gamma \leq h\omega$. With the above specification the difference equations (3) and (4) simplify to

$$q_t^{[1]} = (h\omega - \gamma)(b - 1) + bq_{t+h}^{[2]}, \tag{12}$$

Figure 1: Linear vs. DARA



and

$$q^{[2]}_{t+h} = (1 + h\widetilde{\omega})(1 - \widetilde{b}) + \widetilde{b}q^{[1]}_{t+2h}. \tag{13}$$

Substituting (13) into (12) then establishes

$$q^S(h) = \frac{b(1 - \widetilde{b})(1 + h\widetilde{\omega}) + (b - 1)(h\omega - \gamma)}{1 - b\widetilde{b}} \tag{14}$$

as the unique solution to the bargaining game.

As an example, consider the case where $\widetilde{\omega} = 0$ and both players discount the future at the same rate, i.e. $\widetilde{\delta} = \delta$. We set $\omega = 0.95$, $h = 1$, and $\delta = 0.01$, with the parameter γ varying between 0 and $h\omega$. The Figure shows how the payoff of player 2 is affected by increases in γ. Starting from a payoff of 0.02, when γ and the (endogenous) risk aversion are zero, the payoff of player 2 gradually increases as γ and risk aversion grow.

The example above makes it clear that Roth's theorem linking greater risk aversion to lower payoffs is not general, and does not hold in the case of

DARA utility functions. In the Appendix we explain why his theorem only holds when the parameter d is equal to zero.

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Appendix

In his proof Roth relied on a normalization of the utility function to force it through $u(0) = 0$. He then defined an increase in risk aversion as an increasing and concave transformation of the utility function. For utility functions belonging to the HARA class, we can generate such a transformation by replacing $\gamma \in [0,1]$ by $\widehat{\gamma} > \gamma$, where $\widehat{\gamma}$ is also between zero and one. When the utility function is given by equation (9), the transformation $\widehat{u}(x) = g[u(x)]$ satisfies

$$g(z) = \frac{\widehat{\gamma}^{\widehat{\gamma}}}{1-\widehat{\gamma}} \left[f(z) + (\widehat{\gamma} - \gamma)d\right]^{1-\widehat{\gamma}} \geq 0,$$

where $f(z) = [((1-\gamma)/\gamma^{\gamma})z]^{1/(1-\gamma)} \geq 0$. It is straightforward to verify that g is an increasing function whenever $d \geq 0$ or else, whenever $\widehat{\gamma}$ is sufficiently small:

$$g'(z) = \frac{\widehat{\gamma}^{\widehat{\gamma}}}{1-\gamma} \left[f(z) + (\widehat{\gamma} - \gamma)d\right]^{-\widehat{\gamma}} \left[f(z)\right]^{\gamma} > 0.$$

Furthermore, since $\widehat{\gamma} > \gamma$, a sufficient condition for g to be concave is $d \leq 0$:

$$g''(z) = \frac{\widehat{\gamma}^{\widehat{\gamma}}}{\gamma^{\gamma}(1-\gamma)} f(z)^{2\gamma-1} \left[f(z) + (\widehat{\gamma} - \gamma)d\right]^{-\widehat{\gamma}-1} (f(z) - \gamma d)(\gamma - \widehat{\gamma}).$$

If $d = 0$, the transformation preserves $g(0) = 0$, as Roth assumed in his proof (p. 209). This normalization is not feasible, however, when d differs from zero, because in this case the transformation also involves a shift along the horizontal axis that is proportional to $(\widehat{\gamma} - \gamma)d$. Within the HARA class, therefore, Roth's proof is less general than it appears. Our counterexample shows that his normalization assumption is not without loss of generality.