Credibility and Strategic Learning in Networks

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Credibility and Strategic Learning in Networks *

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Abstract

This paper studies a model of diffusion in a fixed, finite connected network. There is an interested party that knows the quality of the product or idea being propagated and chooses an implant in the network to influence other agents to buy or adopt. Agents are either "innovators", who adopt immediately, or rational. Rational consumers buy if buying rather than waiting maximizes expected utility. We consider the conditions on the network under which efficient diffusion of the good product with probability one is a perfect Bayes equilibrium. Centrality measures and the structure of the entire network are both important. We also discuss various inefficient equilibria.

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1 Introduction

1.1 Main features

This paper studies a model of diffusion and social learning (of what we call broadly “technology”) in a connected network with the following essential features:

1. An individual, known henceforth as the firm, has private information about the quality of the technology it seeks to diffuse to the greatest extent possible. This firm is outside the network but might choose to pay some agent in the network to propagate its product or idea.

2. The network is populated by agents or players, one at each node. Each agent observes the actions of his or her neighbors over time and makes a decision on whether to adopt the technology or not. These players come in two varieties, innovators, who always adopt the technology, and standard players, each of whom is fully rational and makes a decision on whether to adopt or not based on intertemporal expected utility maximisation. A consequence of rationality is that these agents take into account what their observations about their neighbors imply about parts of the network that they cannot observe.

3. The network is fixed, finite and exogenously given; agents with direct links can engage in restricted communication with each other, however agents who are not directly linked can have no communication with each other.

4. The structure of the network is common knowledge to the firm and to all agents.

As we shall discuss later in this section, when we mention related literature, the main novelty of this paper is that the players rationally decide whether the communications they receive from their neighbors are credible or not. Learning is therefore a strategic choice by agents and this affects whether diffusion occurs to the whole population or dies out within some finite distance of the origin. As far as we know, ours is the first paper to study this issue in the context of networks in any detail.

1.2 Motivation

There are several different economic problems that motivated us in studying this problem, though the model we end up with does not fit every aspect of these motivating problems. One example comes from a a New York Times magazine article about viral marketing. The article discusses a company called bzz.com. The article mentions that the company would “implant" agents with good connections to sell products like books, CDs or party food items to their friends and social “neighbors”. The company would provide talking points to the agents, who would then “recommend" the product to their neighbors.

Another example is suggested by Munshi’s [17] empirical study of social learning in the Green Revolution in India. In small communities, it is not unreasonable to suppose that the structure of communication among farmers is common knowledge. The government or its representatives want to push new high-yielding varieties of seeds for wheat and rice.[1] It chooses an individual in the community whose adoption of the new technology will have the most widespread impact.[2] Neighbors of this individual observe the percentage of the farm acreage he devotes to the new varieties and each neighbor then makes a similar decision for his or her own farm, which is then observed by neighbors of neighbors and so on. Munshi observes that the diffusion occurred quickly for wheat cultivation, but at least in the initial stages, not for rice.

Another example of a new technology is the use of high velocity fluid jets to decompress herniated discs in the spine, which was *recommended*, among others, by Dr Jerald Vizzone, who treated TV comedian Stephen Colbert for a broken wrist and obtained widespread name recognition among Colbert's large TV audience.[3]

More generally, doctors make recommendations of technologies or medicines that have been useful in treating patients, to other doctors they know who face similar problems. Here the focus is on communicating to one's neighbor some supposedly effective treatment for a common malady.

As a final motivating example, consider a web page with a button saying "Like". If you are a member of the online network Facebook and press that button, your direct "friends" or neighbors will be informed about your positive feelings for the web page, which could advertise a product, an idea in a blog or a book or piece of music.

Of course, our model is not going to capture all the characteristics associated with these examples. For instance, it is a stretch to assume (as we do in our model) that the structure of the network is common knowledge, except perhaps in small communities. We have been forced to make this assumption in order to analyse what turns out to be a complicated phenomenon.

1.3 The model, a brief verbal description

The players in this model are the *seller* or the *firm,* which knows the quality of the product it seeks to sell and potential buyers who are arranged in a *fixed exogenous network,* whose structure is common knowledge. We shall use this terminology to describe the motivating examples in the last subsection. For instance, the seller could be a medical technology company seeking adoption by doctors of a particular method of surgery, with the buyers being the doctors. The given network is the exogenous structure of social communication among the doctors in a particular specialty.

The number of buyers (also referred to as agents) is finite. A buyer can inspect or otherwise obtain information about the product, determine its quality and communicate a recommendation of the quality to his or her neighbors. A doctor "recommends" by passing on what has worked for her patients to other doctors. Each buyer can receive a recommendation

[1] This example may seem inappropriate if one views the government as a benevolent agent because that would rule out its promoting a bad technology. However, governments in developing countries are sometimes said to be "bought over" by "big" business houses. Also, there is no reason why private entities such as companies producing new types of agricultural inputs cannot replace the government in Munshi's story.

[2] Munshi does not discuss this; however, it seems natural to think about such a choice in the context of his work.

[3] See www.hydrocision.com/hydro/home/documents/Endoresector8-20-07.pdf.

from his or her *neighbors* in the network, that is the other buyers whom he knows directly. Though buyers who are neighbors of each other can communicate recommendations to each other, none of them can observe whether his or her neighbor has, in fact, inspected the product and, therefore, what he or she has discovered about its quality. Notice that we rule out the possibility of *negative* recommendations. In many contexts, this is not a bad assumption. For instance, a doctor may not want to publicize the fact that a particular method of treatment has not worked. Thus "no recommendation" is noisy bad news and a "recommendation" is possibly good news.[4]

The seller can choose to "seed" the network by paying an agent at any given node in the network to give a positive recommendation about the seller's product. As mentioned earlier, the seller knows in advance whether his product is of good or bad quality, a starker version of the usual reality that the seller has better information than the buyers. Only one node in the network can be so "seeded" and is referred to in the paper as an implant. In the Extensions section, we discuss multiple implants as well.

Buyers have ex ante beliefs about whether the product is good and can be one of two types. There is some probability that a buyer at a given node is an "innovator", who will try the new product immediately; with the complementary probability a buyer is normal, in that she makes a rational decision on whether to buy or not. We assume (to avoid trivialities) that the ex ante belief is below the threshold required to induce the second type of buyers to purchase the product. We shall see later that the threshold depends on the position of the buyer in the network and the time at which the recommendation is received, though the finiteness of the game ensures that there is a lowest threshold.

Buyers who are not innovators and who receive recommendations from their neighbors have to form posterior beliefs about the quality of the product, and then decide whether to buy the product. Each purchase gives the seller a unit profit (prices are assumed to be fixed) and future payoffs are discounted; so if buying is optimal, buying now is better than waiting.

The game proceeds as follows: In the first period, the type of the seller is first drawn. Given the type, good or bad, the seller chooses at most one implant. At the same time, each buyer observes whether he or she is an innovator or not (each buyer is an innovator or not independently of other buyers.) Each player observes only his or her own type, though the probability distributions are common knowledge. So no one else knows whether the buyer at site i is an innovator or not. Innovators then buy and make recommendations or not. The implant, if there is one, may decide to make a recommendation immediately to his or her neighbors, or to wait until the recommendation is more credible. Each buyer, implant or not, can speak only once.

In the next period, buyers who did not buy in the first period observe recommendations or the absence of recommendations from their neighbors, make Bayesian inferences from these events and decide whether to buy or not. Since the network is finite, all information will percolate through the network in finite time, so the game is finite. That is, any credible recommendation and purchase must take place before a finite upper bound on time.

We assume that the seller profits by 1 unit for every buyer who buys the product and

[4] In the Extensions section, we briefly discuss what happens if we allow agents to make negative recommendations.

the buyer gets a positive utility from a good product and a disutility from a bad product and that all payoffs are discounted with a fixed, common discount factor.

This is therefore a game of incomplete information and we focus on the Perfect Bayesian Equilibria of this game, which are known to exist. Our principal interest is in studying the conditions under which the product of good quality will diffuse with probability one throughout the network in the fastest possible time - we call this the efficient diffusion equilibrium (EDE). In particular, we investigate the types of network structures that are consistent with the existence of an EDE.

Our analysis shows that the two types of firm will follow very different targeting strategies in equilibrium. A firm that produces a good quality product will be interested in the speed of diffusion over time, and will want to place its implant at a node that maximizes the measure of centrality known as "decay centrality". On the other hand, a firm producing a bad quality product is more myopic, because it knows that its product will not sell beyond one period. So, this firm will want to place its implant at nodes that have the highest number of connections. However, the bad quality firm must also ensure that its agent's recommendation is credible. For instance, if consumers know that the bad quality firm implants a particular node with probability one, then that node's recommendation is less likely to be credible.

The optimal behavior of the two types of firm determine whether an EDE exists or not. Our analysis reveals some counter-intuitive results. For instance, a popular theme in the existing literature on diffusion of products or viruses (in epidemiology, transferred over to the analysis of "viral" marketing) suggests that it is optimal for the seller to choose an "influential" member of the population to be its representative.[5] A "naive" view is that highly connected individuals are more likely to be influential. However, our analysis shows that if any individual is "too influential" in the sense of being connected to *everyone*, then an EDE cannot exist. This is because both types of firms would target such an individual, thereby destroying the credibility of her recommendation. Of course, efficient diffusion would typically be guaranteed in contexts where the *credibility of recommendations* is not at stake. This is one point in which the strategic element in the problem has bite. It also turns out for somewhat subtler reasons that *larger* (in a sense to be described later) networks are more likely to support efficient diffusion.

We provide a partial characterization of networks which support EDEs. We also have some "comparative statics" results. In particular, we focus on the role of the network structure. We formalize the sense in which "too well connected" networks cannot sustain efficient diffusion. We also compare the circle network and tree networks in terms of the *minimum* size of the network required to sustain efficient diffusion. We also discuss the role of the probability that consumers are innovators. Of course, there can be other types of inefficient equilibria and we illustrate the nature of these equilibria for the special case where the network is a *line*. We also briefly discuss the robustness of our results if the model is changed to allow for innovators to make *negative* recommendations. Obviously, the possibility of negative recommendations will lower the expected profits from employing an "implant" for the bad quality firm, and thus make the existence of an EDE more likely. However, conditional on the bad quality firm employing an implant, there is very little qualitative change in our

[5] This has given rise to the development of algorithms to locate the most influential members in a network. See Richardson and Domingos [19] and Kempe [15].

results.

We conclude this subsection with a point about the sequential rationality requirement. We can go very far with just Bayesian updating, especially if the probability that a buyer is an innovator is positive. However, there are cases in which probability zero events have to be considered (off the proposed equilibrium path) and the way these are handled will determine what the equilibrium is. Usually, we will assume that unexpected events, if they occur, signal a bad product. Most of our more general results, however, are not affected by these beliefs, though these could play an important role in particular examples.

1.4 Related literature

There has been a voluminous literature on diffusion of innovations arising from different causes; for a game-theoretic analysis on how these different causes could lead to different observed patterns of diffusion, see Peyton Young [23]. However, Young does not explicitly consider a network structure. Draief and Massoulie [5] and Durrett [6] are texts that analyse diffusion in networks of non-strategic "players" and the influence of connectivity on how fast a virus spreads. Conley and Udry [4] is an empirical study of diffusion in an explicitly displayed network of farmers in a village. We have already mentioned Munshi's work on the importance of social learning.

The paper that considers what seems at first sight to be a closely related problem of a monopolist "seeding" a network in order to spread information about his product is Galeotti and Goyal [11]. This paper considers a monopolist who can choose a fraction x of the population (modeled as the unit interval [0,1]) at a cost of $c(x)$). Each individual picks a finite number of neighbors from the unit interval, so the neighbors of different individuals are in independent subsets (the probability of a common neighbor is 0). The number of neighbors varies according to a probability distribution; each individual can get information about the product either directly from the monopolist or indirectly from one of her neighbors, who is himself directly informed. The results obtained are essentially about properties of the degree distribution that facilitate spread of the monopolist's information. (Example 2 in Section 4 of this paper shows that two networks could have the same degree distribution in our model, but very different properties relating to efficient diffusion.)

Unlike our work, Galeotti and Goyal do not consider a seller with private information or Bayesian buyers who take into account how their current state of information reveals what is happening in unobserved parts of the network. Our results relate primarily to different notions of centrality in a given network structure, rather than to the average degree or the variance of the degree distribution as in Galeotti and Goyal. A continuum of agents from which a finite number are drawn obviously affords independence assumptions that do not hold for our analysis of a fixed finite network, where local dependence is an issue.

Our work is also related to social learning in networks, as in Bala and Goyal [1], Chatterjee and Xu [2].[6] The main difference between our work and these papers is that we have a strategic seller who has private information about quality and is trying to manipulate the diffusion, whilst none of the other papers do. Also our buyers are fully rational Bayesians;

[6] See Goyal[12] for an illuminating survey of papers on learning in networks. Other related papers are Ellison[7], Ellison and Fudenberg[8], Ellison and Fudenberg[9].

even [1] do not model buyers who infer events in unobserved parts of the network, as our buyers do. It is not surprising therefore that our equilibria are quite different.

Since the main novelty about this paper is the set-up with a motivated, privately-informed seller and rational buyers, we briefly explain here the features that arise from this modeling choice. First, there is the credibility of recommendations, which depends on the equilibrium decisions of good and bad types of seller. Second, there is the implant having to take into account not only the number of possible recipients of the recommendation, but also the neighbors of these recipients. (If none of these neighbors of neighbors makes a recommendation, this is noisy bad news and affects credibility differently for different individuals.) Third, there is intertemporal optimality in that a potential buyer could choose to wait for more information, reasoning that, if the product is good, good news will be traveling through the network with a probability depending on the equilibrium strategy of the good type of seller and the acceptance probabilities of recipients of recommendations. There is thus a tradeoff between buying and waiting depending on the position of the individual.

One effect of these three factors is that the *entire* network is important. That is, we cannot describe the results in terms of simple parameters such as the degree distribution or the diameter or the connectivity. These do not provide detailed enough structure to account fully for rational behavior.

The next section describes the model more formally. We then illustrate the workings of our model for two specific network structures - the line and the star. This is followed by a partial characterization result on the type of networks that sustain an EDE. Our comparative statics results are contained in the subsequent section. The following sections describe the nature of inefficient equilibria on the line and extensions to the basic model.

2 The formal model

In this section, we describe the basic model. The set $N = \{1, 2, \ldots, n\}$ represents the set of consumers. The structure of interactions between the set of consumers is represented by means of a graph Γ in which the nodes are elements of N and $ij \in \Gamma$ if consumers i and j can communicate with each other. There is a firm F, which is interested in selling its product. The product is either of type $G(ood)$ or $B(ad)$. Firm F knows the type of its product and other agents only know the common knowledge probability that the product is good, p. We assume the graph is connected, so there is a path through which diffusion can occur between any two individuals.

All buyers have an initial probability $p^0 \equiv p$ that the product is of the good type. There are *two* types of consumers. Buyers of the first type - we refer to them as the *innovators*- get utility g' from the G product and utility $-b'$ from the B product. These numbers are such that it is a dominant strategy for innovators to buy immediately. On the other hand, the second type of buyers (the *normal* types) get utilities g and $-b$ from the G and B type products. Throughout the paper, we retain the following assumption.

Assumption 1

$$p < \bar{p} \equiv \frac{b}{b+g}$$

Hence, the second type of consumer will not buy the good unless she revises her probability belief about the good in subsequent periods. If her updated belief in some period t is $p^t = \bar{p}$, she will be indifferent between buying and not buying the good if she does not expect to receive any further information about the product in future periods. Since there are a finite number of agents, there must be a time period in which this is so (take the maximum distance from a node to any other node and suppose that information, if transmitted, takes a number of periods equal to the distance to traverse the path). If future information is expected, in general a buyer i is indifferent between buying and not buying in period t for $p^t = \bar{p}_i^t$, where $\bar{p}_i^t$ could depend on the time period t, the position of the agent and the strategies of the good and bad types and is in general strictly greater than $\bar{p}$. When $t = 1$, we will typically write $\bar{p}_i$ instead of $\bar{p}_i^t$. We illustrate the dependence of $\bar{p}_i^t$ on these factors when the network is a line in Section 3.

In one of our results - comparison of circle and trees in terms of the minimum size of the network required to sustain efficient diffusion - we will make the simplifying assumption that the second type of consumers do not wait for further confirmatory information. This may, for instance, be due to some form of bounded rationality.

Assumption BR : For all i, for all t, $\bar{p}_i^t = \bar{p}$.

Each consumer buys the product at most once. The firm gets 1 unit for each item purchased and 0 if an item is not purchased. There are no capacity constraints on the number of items sold.

Future payoffs are discounted by δ for F and for the consumers.

The time line: Nature draws the type of F and this is revealed only to F. F chooses a site i to place one "implant" at a "small" cost c or decides not to use any implants. The implant, if any, is paid to pass on a recommendation to his neighbors in Γ. If i is not an implant, she can be an "innovator" in which case she tries the new product immediately. The probability that any site i is an innovator is $\rho < 1$ and the event that "i is an innovator" is independent of other events "$j \neq i$ is an innovator". All this takes place, in sequence, at $t = 0$. At $t = 1$, any i who is an innovator, and who has obtained a utility of g', makes a recommendation to his neighbors. An implant always makes a recommendation, but may choose the time at which she makes the recommendation for strategic reasons. The neighbors receiving the recommendation might choose to buy the product or not to buy it. At $t = \tau$, any site who is either an implant or has tried the product and found it good, after receiving a recommendation in $\tau - 1$, can make a recommendation to neighbors. A site i does not observe if neighbors have received recommendations or have chosen to buy the product- she only observes whether recommendations are made by the neighbors themselves.

There is no exogenous time limit on the game; however, since there are a finite number of neighbors and each speaks at most once, the game must end in finite time.

Strategies and equilibrium

1. The players in this game are the firm with a product, with private information about its quality G or B;an implant, whose preferences are the same as the firm's, and the consumers who are not innovators, at site i. *An innovator is assumed to recommend*

the good product to her neighbors and makes no recommendation if the product is bad.[7]

2. At time 1, G or B chooses whether to use an implant at site i. Let the probabilities with which i is chosen be α_i for G and β_i for B, respectively, conditional on there being an implant. Let $\ominus$ represent the decision not to use an implant by either type of firm.

3. At time t, $t \geq 1$, let h_{it} denote the private history of the non-innovating agent at site i, the set of whose direct neighbors is denoted $N_i(\Gamma)$. If agent i is not an implant, the history consists of a vector $(x_{j\tau})$, $\tau < t$, $j \in N_i$, where, for a particular $\tau, x_{j\tau} \in \{0, 1\}$ and a value of 1 means that j has made a positive recommendation about the new product at time τ. Given the history h_{it}, a strategy for agent i is to choose a time $t + t'$ to buy the product. Formally, it is a sequence $t'(h_{it})$ for all t. Here $t' = 0$ means the player buys the product immediately and $t' = \infty$ indicates "Never buy". (Note node i can only buy once.) A mixed strategy is defined in the usual way. We focus on behavioral strategies, consisting of actions $t' = 0$ ("Buy now") or $t' > 0$ ("Wait"), with randomization possible between these actions.

4. Let i now be the implant of the firm. Player i has a private history consisting of her type and $h_{it} = (x_{j\tau}, s_{i\tau})$, $\tau < t$, where $s_{i\tau}$ is 1 if implant i makes a recommendation (or "speaks") in period τ and 0 otherwise. A strategy for the implant at i is a map from (her type, h_{it}) to a probability of speaking in period t, *i.e.* a probability that $s_{it} = 1$. Note that the implant and the firm are assumed to have identical payoffs, conditional on entry.

   Note that the firm enters (buys an implant) or not at the beginning of the game. We assume that entry is not possible at a later time, though we discuss this assumption in the Extensions section. However, the firm/implant can *speak* at any time. Since "speaking"/recommending is, for us, possibly interpretable as being observed using the product, any individual can speak only once.

5. *Equilibrium* in this game is to be interpreted as Perfect Bayes Equilibrium. The requirements are: (i) Each agent, including the firm and implants but not including the innovators, updates beliefs according to Bayes' Theorem whenever possible and (ii) each agent maximizes her expected payoff at each (private) history given these beliefs. Out-of-equilibrium beliefs will be explicitly described when necessary.

Updating Beliefs

Let α and β be the mixed equilibrium strategies of types G and B respectively. Suppose consumer i receives a recommendation from her neighbor $i - 1$ in period 1. If she receives no other recommendation, what is the probability that the product is G? Let us denote this by $\eta^1_{i,i-1}$, where the superscript refers to the time the recommendation is received and the subscripts to the recipient and the sender of the recommendation.

Let $d_i(\Gamma) = |N_i(\Gamma)|$ be the *degree* of i in Γ. (Henceforth, whenever there is no ambiguity about Γ, we will simply write d_i, d_j, etc.)

[7] We discuss briefly the implications of "negative" recommendations in the Extensions section.

Since the derivation of the probability $\eta^1_{i,i-1}$ is somewhat tedious to check, we reproduce the actual calculation. The probability required is: Prob. [product is G| i is not an implant or innovator and none of the other neighbors other than $i-1$ has made a recommendation and $i-1$ has made a recommendation]

Let's call the conditioned event A and the conditioning event B.

Then, by Bayes' Theorem, P($A \mid B$) $= P(B \mid A)P(A)/[P(B \mid A)P(A) + P(B \mid A^C)P(A^C)]$.

$=\frac{P(A \text{ and } B)}{P(A \text{ and } B)+P(A^C \text{ and } B)}$

The numerator is then $p(1-\alpha_i)(1-\rho)^{d_i-1}[\frac{\alpha_{i-1}}{1-\alpha_i} + \rho(1-\frac{\Sigma_{j\in N_i}\alpha_j}{1-\alpha_i})]$

$=p(1-\rho)^{d_i-1}[\alpha_{i-1} + \rho(1-\Sigma_{j\in N_i \cup i}\alpha_j)]$ for $\alpha_i \neq 1$. (We can take limits if $\alpha_i = 1$)

The denominator is this quantity plus another term $P(B \mid A^C)P(A^C)]$. This is the probability of all this happening if the product is a bad one. The second term in the denominator is therefore $(1-p)(1-\beta_i)\frac{\beta_{i-1}}{1-\beta_i}$ again assuming $\beta_i < 1$. Hence,

$$\eta^1_{i,i-1} = \frac{p(1-\rho)^{d_i-1}\left[\alpha_{i-1} + (1-\sum_{j\in N_i}\alpha_j - \alpha_i)\rho\right]}{p(1-\rho)^{d_i-1}\left[\alpha_{i-1} + (1-\sum_{j\in N_i}\alpha_j - \alpha_i)\rho\right] + (1-p)\beta_{i-1}} \quad (1)$$

Some "special" cases illustrate the nature of the updating process. Suppose that the type G firm uses a pure strategy so that for some site m, $\alpha_m = 1$. Suppose i is a neighbor of m, but receives only one recommendation from some $j \neq m$. Then, i must conclude that j is a bad implant - if the product had been good, then there would have been a good implant at m who would then have passed on a recommendation to her. This argument generalizes even when the type G firm uses a strategy whose support is some set M containing more than one node. Suppose now that i is a *common* neighbor of all nodes in M. Again, if i does not receive a recommendation from some member of M, she will conclude that any other recommendation comes from a bad implant. Next, suppose again that $\alpha_m = 1$ and that m receives a recommendation from some neighbor. Of course, such recommendations are not credible to m - she would have been used as an implant by the type G firm if the product was good. These inferences are confirmed by equation (1) - in all cases, the numerator is 0.

Of course, if i receives a recommendation from *two* or more neighbors, then i concludes that the product is G with probability one- if there is a bad implant at j, then there cannot be a bad implant at $j' \neq j$.

Suppose next that i receives a recommendation from $i-1$ in some period $t > 1$, but no recommendation from any other neighbor. If i has not received any recommendations before period t and receives one from $i-1$ in period t, this can happen because the product is Bad, there is an implant at $i-1$ and the implant chooses to speak at period t. Alternatively, the product is Good, $i-1$ heard a recommendation from one of her neighbors in the previous period, but none of i's other neighbors received a recommendation from any of their neighbors in period $t-1$. Explicit computations of these probabilities are hard to describe since these depend on the structure of the network. But, notice that some recommendations are easy to dismiss. For instance, suppose Γ is a *line*, and let i be an extreme point of Γ, with degree one. Then, any recommendation from i coming in period $t > 1$ is not credible to i's neighbor since i could not have received a recommendation in period $t-1$.

Efficient Diffusion Equilibrium

We shall mainly, though not exclusively, limit ourselves to the consideration of "efficient diffusion equilibria" (EDE), adopting the viewpoint of, say, a development agency (or social planner) that wants a *good* idea or product to be spread through the entire population as quickly as possible, given the constraints of the network structure and the technology of diffusion. Note that the B type in our model can expect to sell only to the initial adopters, since there will not be any further recommendations from the initial adopters once they find that the product is bad. Therefore, the study of diffusion through the network is most relevant for the G type, and hence the focus on EDE.

Consider an environment in which Firm G is the only type of firm, so that the issue of credibility of recommendations does not arise. For instance, consumers may be initially unaware about the existence of the product, but are willing to buy the product after receiving a recommendation. Then, Firm G will want to "seed" the network by using an implant. In the absence of any issue of credibility of recommendations, the product will diffuse throughout the network with probability one if an implant is used. Since the firm discounts the future, it will want to place its implant so as to maximize the *speed* of diffusion. The optimal site(s) for a Good implant is related to a measure of centrality of network structures. Let d_{ij} denote the *geodesic distance* between i and j in the graph Γ. That is, d_{ij} is the length of the shortest path between i and j.

Definition 1 *A node i maximizes decay centrality in a graph Γ if $\sum_{j \neq i} \delta^{d(i,j)} \geq \sum_{j \neq k} \delta^{d(k,j)}$ for all $k \in N$.*

Let $D(\Gamma)$ denote the set of nodes maximizing decay centrality. While it is not easy to compute this set in general graphs, the set is easily identified in special cases. For instance, if Γ is a line, then the median(s) must be maximizing decay centrality. Or if Γ is a star, then the hub (that is a node with degree $n-1$) is obviously the node maximizing decay centrality.

Notice that since consumers also discount the future, consumers' surplus is also maximized when the G-implant is placed at a node that maximizes decay centrality.

Henceforth, we are particularly interested in a PBE (the *efficient diffusion equilibrium* (EDE)) with two properties -(i) the good product will diffuse throughout the network with probability one, and (ii) the good implant is placed at some node maximizing decay centrality.

So, if (α, β) denote the probability distributions with which the G-type and B-type implants are placed at different nodes, then the support of α must be contained in the set $D(\Gamma)$. Moreover, there must be at least one sequence of recommendations originating from all nodes with $\alpha_i > 0$ which are accepted with probability one- otherwise the good product will not diffuse through the entire network with probability one. Hence, conditional on the product being Good, an EDE maximizes consumer surplus.[8]

Of course, we also need to identify when G and B will decide to use an implant. This must depend on a comparison of the increase in expected profit resulting from an implant and the cost c incurred by employing an implant. We assume that $c > 0$, but is "small" so that the expected net gain from using an implant will be non-negative.

[8] As mentioned earlier, if the product is Bad, the upper bound on the extent of diffusion is given by the maximal degree of any node in the network.

3 Two Examples: EDE on the Line and Circle

In this section, we provide an informal discussion of two specific network structures, the *line* and *circle,* in order to illustrate the conditions required for an efficient diffusion equilibrium.

3.1 The Line

Assume, for simplicity, that n is odd.

So, let Γ be a line, with the sites ordered so that 1 and n are the end-points of the line having degree one, while all other sites have degree 2. Since n is odd, the unique median maximizes decay centrality. Hence, in any PBE (α, β), where α and β denote the equilibrium strategies of the Good and Bad types respectively, $\alpha_m = 1$. Now, if this PBE is to be an EDE, then the recommendation coming from m has to be accepted with probability one. That is, $m-1$ and $m+1$ must accept any recommendation coming from m with probability one. Since the bad type can always mimic the Good type, this implies that wherever the bad implant is placed, her recommendation must be accepted by two neighbors.

But, of course, $\beta_m \neq 1$. For, if $\beta_m = 1$, then from equation (1),

$$\eta^1_{m-1,m} - p = \eta^1_{m+1,m} - p = p[\frac{((1-\rho)-1)(1-p)}{p(1-\rho)+(1-p)}] < 0,$$

and neither of m's neighbors would buy the product after receiving a recommendation from m, which is a contradiction. So, while the support of β can *include* m, it cannot coincide with $\{m\}$. Over what set of nodes can the bad type "distribute" β? The answer of course is that the support of β must be contained in those nodes who can make "credible" recommendations to both their neighbors, since m's recommendations should be credible. This must be the set

$$S \equiv N \setminus \{1, m-2, m-1, m+1, m+2, n\}$$

It is clear that 1 and n cannot be in S since they have only one neighbor. Although $m-2$ and $m+2$ have degree 2, notice that $m-1$ (respectively $m+1$) will not believe a single recommendation from $m-2$ (respectively from $m+2$).[9] Of course, $m-1$ and $m+1$ cannot sell to m. Also, notice that if $n < 9$, then m will be the sole member of S, and there will not be any EDE.

It is also clear when the bad type will want to use an implant when there is an EDE. Let the implant be placed at i. Since i can be an innovator with probability ρ (in which case he would buy the product anyway), the effective cost of an implant at i is

$$\rho + c$$

The benefit is the additional probability that $i-1$ and $i+1$ buy the product. With probability $(1-\rho)^2$, neither is an innovator. With probability $2(1-\rho)\rho$, one of the two is an innovator. Hence, the benefit is

$$2(1-\rho)^2\delta + 2(1-\rho)\rho\delta = 2\delta(1-\rho)$$

[9] Both $m-1$ and $m+1$ know that if the product is good, then m would have made a recommendation.

So,the net gain of an implant for B is given by

$$2\delta(1-\rho)-\rho-c$$

where c is the cost of an implant. So, B will use an implant if

$$c \leq c^B(\delta,\rho) \equiv 2\delta(1-\rho)-\rho$$

Not surprisingly, the higher the value of ρ, the lower is the expected gain from employing an implant.

Consider G. Let $\underline{\pi}^G(\delta,\rho)$ denote the profit of G in the absence of an implant, and $\bar{\pi}^G(\delta,\rho)$ denote the profit of G if an implant is used.[10]

Then, G will use an implant if

$$c \leq c^G(\delta,\rho) \equiv \bar{\pi}^G(\delta,\rho) - \underline{\pi}^G(\delta,\rho)$$

The expected profits of G with or without an implant increases in ρ. However, notice that the *difference* between the two levels of profit decreases as $\delta \to 1$ since the speed of diffusion is less important as δ increases. In the limit,

$$\lim_{\delta\to 1}(\bar{\pi}^G(\delta,\rho) - \underline{\pi}^G(\delta,\rho)) = n-1-n[1-(1-\rho)^n]$$

On the other hand, an increase in δ increases the value of an implant to B. So, for "high" δ, it may well be the case that only the bad type uses an implant!

Let $c < \min\left(c^B(\delta,\rho), c^G(\delta,\rho)\right)$.

Suppose $\beta_i > 0$ for some i. One can place an upper bound on how high β_i can be in equilibrium. Since i's recommendation must be accepted with probability one by both her neighbors, their updated probability that the product is good cannot fall below the threshold values for acceptance in period 1. Note that for $m-1$ and $m+1$, this threshold value must be $\bar{p}$ itself since there must be an implant at m if the product is good. The threshold values for "acceptance" at other nodes are more tedious to compute as we explain below in Remark 2. Essentially, any node $k < m$ who receives a recommendation from $k-1$ can wait for $m-k$ periods to receive confirmation that the product is good through a series of recommendations emanating from m. She needs to trade off the cost of waiting with the expected loss in case $k-1$ is a B-implant, to compute the threshold value. With some abuse of notation, let $\bar{p}^1_{i-1}$ and $\bar{p}^1_{i+1}$ denote the threshold values associated with recommendations from i.

Equation 1 and the fact that $\alpha_m = 1$ now readily yield the upper bound on $\bar{\beta}_i$.

$$\bar{\beta}_m = \frac{p(1-\bar{p})(1-\rho)}{\bar{p}(1-p)}, \text{and for } i \neq m, \bar{\beta}_i = \frac{\rho(1-\rho)p}{1-p}\min\left[\frac{(1-\bar{p}^1_{i-1})}{\bar{p}^1_{i-1}}, \frac{(1-\bar{p}^1_{i+1})}{\bar{p}^1_{i+1}}\right]$$

It is now easy to describe what an EDE looks like when $n \geq 9$ is odd, and the cost of an implant does not exceed c.

(i) The type G firm puts its implant at the median of the line with probability one.

(ii) The support of β is contained in S, and for each i in the support of β, $\beta_i \leq \bar{\beta}_i$, $\beta_m \leq \bar{\beta}_m$ and $\beta_i = 0$ elsewhere.

[10] These expressions are difficult to compute for general n.

(iii) The implant (irrespective of type) "speaks" in period 1.

(iv) Recommendations received from each site in the support of β are accepted with probability one in period 1.

(v) In subsequent periods $t > 1$, a site $i < m$ accepts a recommendation from $i + 1$ with probability one if the equilibrium response of $i + 1$ was to accept the recommendation from $i + 2$ in period $t - 1$. Similarly, a site $i > m$ accepts a recommendation from $i - 1$ if the equilibrium response of $i - 1$ was to accept the recommendation from $i - 2$ in period $t - 1$.

As we have mentioned before, Property (ii) follows from the fact that recommendations will not be accepted unless the updated belief that the product is good reaches the threshold value of $\bar{p}^1$. Property (iv) is consistent with this since a site that receives a recommendation (and is not an innovator or an implant) is at least indifferent between buying and not buying and strictly prefers buying if the inequality is strict.

For property (iii), note that if the product is bad, then the implant can only hope to persuade two neighbors to buy the product - her recommendation will not be passed on. If the product is good, then the product will diffuse through the entire population given (v). Since an implant's recommendation is accepted with probability one by both neighbors, the implant gets the same outcome as early as possible in either case.

Property (i) follows by noting that type G cannot be indifferent between m and any other site. At m, the implant will obtain an expected payoff of $\delta.2 + \delta^2.2 + ...$ for $m - 1$ terms. At $m - k$, say, the payoff will be $\delta.2 + \delta^2.2 + ...\delta^t 2$ for $m - k - 1$ terms and $\delta^{m-k}.1 + ..$ for an additional $2k$ terms, thus taking $m + k - 1$ periods to diffuse completely rather than $m - 1$ periods, if the good implant locates at m.[11] Thus the speed of diffusion is higher by locating at m, since there are two new buyers in each period for every period the diffusion continues, whilst at $m - k$, there is only one buyer for every period after $k - 1$.

The argument for (v) includes the following: Since the B implant speaks in period 1 in equilibrium, any recommendation after period 1 must come from a relayed recommendation from an innovator or a good implant at some $i \neq m$ or m respectively. If $t \geq m - 1$, no recommendation will occur along the equilibrium path but it is assumed that off-equilibrium beliefs also induce acceptance of the recommendation.

We note that the B implant will not deviate to speaking after $m - 1$, even with this belief, because of discounting and the acceptance probabilities of 1 even if B speaks early.

In this equilibrium, a message from m in period 2 would be an event of probability 0. To check that neither type wants to deviate, we have to consider out-of-equilibrium beliefs, but all beliefs about the type sustain this equilibrium. The m implant's recommendation is accepted with probability 1 in the first period, so discounting makes it sub-optimal to wait, no matter what the belief.

Remark 1 *We note again that $\mid S \mid$ must be large enough for the B implant to put "small amounts" of probability at each site in S in his randomization. For example, if $\mid S \mid$ is very small so that there is no distribution that makes it possible for $\beta_i \leq \bar{\beta}_i$ for all sites in S and $\beta_m \leq \bar{\beta}_m$, then this equilibrium will not exist. On the other hand, notice that if a smaller "line" supports an EDE, then so must a larger line. In other words, efficient diffusion is more likely the larger the number of potential customers!*

[11] This calculation does not take ρ into account, but it is obvious this wouldn't change the ranking because sites are independently innovators or non-innovators

Remark 2 *An explicit characterization of $\bar{p}_i^t$ is tedious, but it is easy to see how it can be done. Suppose $\alpha_m = 1$ as above. Suppose site i is located at such a distance to the left of m that a message from m will arrive in period T. At period $T-1$, suppose i receives a recommendation from a neighbor on his left, which could have been relayed from an innovator $T-1$ steps to i's left. For what value of p^{T-1} would i accept this recommendation? If he waits for one more period, he will receive perfect information, either a message from his right neighbor, which will make $p^T = 1$ or no message, which will imply $p^T = 0$. His expected utility from waiting if his current belief that the product is good is p^{T-1}, is therefore $\delta p^{T-1} g$. If he buys now, he will get $p^{T-1}(g+b) - b$. The value of $\bar{p}_i^{T-1}$ is therefore $\frac{b}{b+(1-\delta)g}$. If $\delta = 0$, this is exactly $\bar{p}$; if δ is close to 1, this quantity is also close to 1. One can similarly calculate $\bar{p}_i^{T-2}$ at $T-2$ by comparing the utility of buying now versus waiting one period for imperfect information and two periods for perfect information, and so on.*

If i gets a recommendation before period T-1 from a neighbor on the path between him and m, perfect information is ruled out, since each player only speaks once and i's neighbor will not pass on any other message from the right. There is still some probability of imperfect information from the left (confirmatory if the product is good, none if the product is bad) coming from an innovator whose message takes the requisite number of periods to reach i. Once again, a value of $\bar{p}_i^t$, which makes i indifferent between buying and not buying, can be calculated. On the line, both the above possibilities cannot arise together. In more general networks, one would take the maximum value of $\bar{p}_i^t$ calculated for each possible path on which additional information could travel.

Of course, inefficient equilibria can exist when n is not large enough. These will typically involve the bad implant delaying his recommendation for strategic reasons. In such equilibria, recommendations will not be accepted with probability one.

3.2 The Circle

In this subsection, we will assume that Assumption BR holds, and derive the minimum value of n for which the circle will sustain an EDE, as a function of the parameters of the model. The circle also illustrates very well the sense in which “large" networks are more conducive in sustaining an EDE.[12]

Consider the circle on n nodes and strategies α and β where for each i, $\alpha_i = \beta_i = 1/n$. This pair of strategies will provide the lower bound on n that we want to derive. In view of our extended discussion of the structure of an EDE for the line, we will be very brief in our current discussion. Essentially, the selected pair of strategies must be such that any recommendation from any node in period 1 is accepted by both neighbors with probability one. Since we employ Assumption BR, this means that the updated belief at any node after receiving a recommendation must be at least $\bar{p}$. Also, all recommendations in subsequent periods are also accepted with probability one. This ensures that the recommendation of the type G-implant diffuses throughout the network. Since the type B implant's recommendation is also accepted by both her neighbors in period 1, the B implant has no incentive to deviate and delay its recommendation.

[12] We point out that this has nothing to do with Assumption BR, which is being assumed simply in order to derive an *explicit* characterization of the minimum size of n required to sustain an EDE.

Using equation 1,

$$\eta^1_{i,i-1} = \frac{p(1-\rho)(1+(n-3)\rho)}{p(1-\rho)(1+(n-3)\rho)+(1-p)}$$

Since $\eta^1_{i,i-1}$ must be at least $\bar{p}$ in order for these strategies to sustain an EDE, we get

$$\begin{aligned} \frac{p(1-\rho)(1+(n-3)\rho)}{p(1-\rho)(1+(n-3)\rho)+(1-p)} &\geq \bar{p} \\ [p(1-\rho)(1+(n-3)\rho)](1-\bar{p}) &\geq \bar{p}(1-p) \\ n &\geq 3+\frac{K}{\rho(1-\rho)}-1/\rho \end{aligned}$$

where $K \equiv \frac{\bar{p}(1-p)}{p(1-\bar{p})} > 1$, the latter inequality following from the fact that $\bar{p} > p$. Denote

$$\underline{n}^C(K,\rho) \equiv 3+\frac{K}{\rho(1-\rho)}-1/\rho \tag{2}$$

So, the smallest integer which is at least as large as $\underline{n}^C(K,\rho)$ is the minimum size of a circle network required to support an EDE. Obviously, any larger circle will also support an EDE - again an illustration of the "larger is better" principle.

It is clear that $\underline{n}(K,\rho)$ is increasing in K - the larger is $\bar{p}$ or smaller is p, the smaller is the probability weight that the type B-implant can put on any node in order to ensure that a recommendation is credible. Fix K, and consider how a change in ρ affects $\underline{n}(K,\rho)$. Suppose ρ increases, and $i-1$ has received a recommendation from i. Then, there are two opposite effects at work. First, the increase in ρ means that i is more likely to be an innovator and this increases the credibility of the recommendation. On the other hand, it is also more likely that $i-2$ is an innovator, and the latter's silence reduces the credibility of the recommendation. The positive effect will dominate for small values of ρ, but the negative effect will be bigger for large values of ρ. Routine calculation shows that $\underline{n}(K,\rho)$ reaches a minimum at $\rho = \sqrt{K(K-1)}-(K-1)$.

We will have more to say about the circle in section 5.1.

4 A Partial Characterization Result

In this section, we describe a sufficient condition for an EDE to exist, and then show that this condition is necessary for certain types of network structures.

Say that a link $ij \in \Gamma$ is *critical* if $\Gamma - ij$ has more components than Γ. That is, if a critical link is removed from a connected network Γ, then the network Γ no longer remains connected. Say that a node i is *critical* in Γ if *all* links of i are critical. Of course, if the network is a tree, then all links are critical, and so all nodes are critical.

Throughout this section, we restrict our attention to networks satisfying:

Assumption S: There is a unique node m maximizing decay centrality in Γ. [13]

[13] This assumption results in a somewhat more transparent result. In an earlier version, we did not use this assumption here. This is available from the authors.

Henceforth, we also drop reference to the network Γ whenever this will not cause any confusion.

For any node $i \neq m$, let $\bar{N}_i = N_i \setminus N_m$. Let $\bar{d}_i = |\bar{N}_i|$. We will refer to $\bar{d}_i$ as the *effective* degree of a node $i \neq m$. The effective degree of m will be d_m itself.

Since our interest is in EDE, Assumption S implies that $\alpha_m = 1$. Then $\bar{N}_i$ is the set of potential neighbors of i who may possibly believe that the product is good after receiving a single recommendation from i in period 1. To see this, notice that if the product is good, then a positive recommendation *must* come from m. The absence of such a recommendation signals that any other recommendation comes from a bad implant. So, any k who is a neighbor of m will find a recommendation credible only if m sends a recommendation.

Given m, partition nodes into sets $S_1, \ldots, S_K$ such that S_1 is the set of nodes maximizing $\bar{d}_i$, S_2 is the set of nodes with the next highest value of $\bar{d}_i$, and so on.

For every node i, let k_i be the site in $\bar{N}_i$ which maximizes degree, and d_{k_i} be its degree. Notice that equation 1 implies that for any given value of β_i,

$$\eta^1_{k_i,i} \leq \eta^1_{j,i} \text{ for all } j \in \bar{N}_i$$

For each $j \in \bar{N}_i$, let β_i^j be the value of β_i which sets $\eta^1_{j,i} = \bar{p}_j$.

$$\text{For each } i, \bar{\beta}_i \equiv \min_{j \in \bar{N}_i} \beta_i^j \tag{3}$$

The next theorem identifies a sufficient condition for a network structure to support an EDE. It also shows that if m is critical, then this condition is also necessary. These conditions are satisfied by the line, and so this theorem will include the line as a special case.

Theorem 1 *Suppose the cost of an implant is sufficiently low for both types of the firm to use an implant. Then, an EDE exists if*

$$\sum_{i \in S_1} \bar{\beta}_i \geq 1. \tag{4}$$

Conversely, an EDE does not exist if m is critical, $m \in S_1$ and equation 4 does not hold.[14]

Proof. Suppose equation 4 is satisfied. Let $\alpha_m = 1$ be the strategy employed by the good type, and choose β such that

$$\beta_i \leq \bar{\beta}_i \text{ for all } i \in S_1$$

and

$$\beta_i = 0 \text{ if } i \notin S_1$$

The response strategies are straightforward. All sites accept all recommendations in all periods.

It is easy to check that these strategies constitute a PBE. Clearly, the type G firm has no incentive to deviate since her implant is at some site maximizing decay centrality and recommendations are accepted with probability one. Similarly, the type B firm has no

[14]Hence, equation 4 is necessary for the existence of an EDE in all trees satisfying Assumption S and where $m \in S_1$.

incentive to deviate since she obtains a payoff of $\bar{d}_1 - c$, conditional on no innovators in $\bar{N}_1$. Clearly, no other site can yield a higher payoff. The response decisions are optimal because (i) any recommendation coming from a site not in the union of the supports of α and β must be coming from an innovator, and (ii) the updated belief of any i after receiving a recommendation from a potential implant is at least as large as the threshold value $\bar{p}$.

Consider now the necessity of this condition. First, the type G firm must be choosing an $\alpha_m = 1$ in any EDE. Since m is critical in Γ, all neighbors of m must accept a recommendation from m with probability one in an EDE. So, the type B firm by placing her implant at m can obtain a payoff of $|S_1| - c$. Hence, in equilibrium, the support of β must be contained in S_1. Moreover, if $\beta_i > 0$, every member of $\bar{N}_i(\Gamma)$ has to accept the recommendation of i. Hence, the maximum probability weight that the type B firm can put on i cannot exceed $\bar{\beta}_i$. This is not possible if equation 4 does not hold. ■

Equation 4 is easy to interpret. If there are a sufficient number of nodes maximizing effective degree, then an EDE is easy to support since the type B firm has enough "space" to distribute his probability. Conversely, if an EDE is to exist, then it must be possible for the type B firm to ensure that a recommendation from each node in the support of its mixed strategy is credible for $|S_1|$ neighbors.

Why is equation 4 not necessary without additional conditions? Suppose, that $\alpha_m = 1$, but $m \notin S_1$, but say in S_2. Also, assume that equation 4 does not hold. It is possible then to have another equilibrium in which (i) the type B uses a mixed strategy over nodes in $S_1 \cup S_2$, (ii) the probability of acceptance of recommendations coming from nodes in S_1 is adjusted below one so as to ensure that the expected payoff from an implant located in S_1 is the same as that from an implant in S_2.[15] The freedom to distribute some probability weight over nodes in S_2 may now help in ensuring existence of equilibrium. Instead of formally deriving a sufficient condition for this type of equilibrium, we illustrate such an equilibrium in the example below.

Example 1 *Let $n = K(K+1)$ where $K > 2$. Denote $I = \{i_1, \ldots, i_K\}$, and let each $i_k \in I$ be the hubs of K stars $\Gamma_1, \ldots, \Gamma_K$, each Γ_k having $K+1$ peripheral sites. Finally, let site 1 be connected to each site in I, and to no other site. Also, no site i_k in I is connected to any site in the other stars. So,*

$$\Gamma = (\bigcup_{i \in I} \Gamma_i) \cup \{1i_1, \ldots, 1i_K\}$$

We want to choose values of ρ and δ such that 1 maximizes decay centrality. Then, $\bar{d}_1 = K$. But, notice that for each $i_k \in I$, $\bar{d}_{i_k} = K+1$ since none of the peripheral sites in Γ_{i_k} are connected to 1. So, $1 \notin S_1$.

Suppose the type G firm places an implant at 1. Then, the expected benefit from the implant will be

$$B_1 = (1-\rho)\delta\left[\xi_K(1 + \delta\xi_{K+1})\right]$$

where for any $k = K, K+1$

$$\xi_k = \sum_{i=1}^{k} \binom{k}{i} i(1-\rho)^i \rho^{k-i}$$

[15] However, the value of δ cannot be too high. If δ is high, then the low discounting may induce the type B implant at some site in S_1 to strategically postpone her recommendation to a later period.

On the other hand, if the type G firm places an implant at any of the sites in $\{i_1, \ldots, i_K\}$*, then the expected payoff is*

$$B_2 = (1-\rho)\delta\left[\xi_{K+1} + (1-\rho)\left(1 + \delta\xi_{K-1}(1+\delta\xi_{K+1})\right)\right]$$

Evaluating the two expressions at $\rho = 0$*, it is easy to check that*

$$B_1 > B_2 \text{ if } \delta > \frac{2}{K^2-1}$$

That is, if $\delta > \frac{2}{K^2-1}$*, then* 1 *maximizes decay centrality for* $\rho = 0$ *and hence for "small" values of* ρ*. Assume that* $\rho > 0$ *is such that* $B_1 > B_2$*. Also,*

$$\delta \in \left(\frac{2}{K^2-1}, \frac{\xi_K}{\xi_{K+1}}\right)$$

Notice that since $\lim_{\rho\to 0} \frac{\xi_K}{\xi_{K+1}} = \frac{K}{K+1}$*,* δ *is well-defined for* $K > 2$ *and for small values of* ρ*.*

Let

$$\bar{\beta}_1 = \frac{(1-\rho)^K p(1-\bar{p})}{\bar{p}(1-p)}, \bar{\beta}_i = \frac{p\rho(1-\bar{p})}{\bar{p}(1-p)} \text{for each } i \in I,$$

Using equation 1, it is easy to verify that if $\alpha_1 = 1$*, then for each* $i \in I \cup \{1\}$*,* $\eta^1_{k,i} = \bar{p}$ *if* $\beta_i = \bar{\beta}_i$*. Suppose*

$$\sum_{i\in I} \bar{\beta}_i < 1, \text{ but } \sum_{i \in I} \bar{\beta}_i + \bar{\beta}_1 \geq 1$$

Notice that equation 4 is not satisfied. However, the pair of strategies (α, β) *along with response decisions specified below is an EDE.*

Let

(i) $\alpha_1 = 1$,

(ii) $\beta_i = \bar{\beta}_i$ *for each* $i \in I$, $\beta_1 = 1 - \sum_{i\in I} \bar{\beta}_i$.

(iii) Each $i \in I$ *accepts recommendation from* 1 *with probability one in period 1.*

(iv) Each $i \in \Gamma_{i_k}, i \neq i_k$ *accepts a recommendation from* i_k *with probability* $\frac{\xi_K}{\xi_{K+1}}$ *in period 1 and with probability 1 in period 2.*

(v) Both types of implants make their recommendations in period 1.

To check that these constitute an equilibrium, first notice that the type G firm has no incentive to deviate since (1) 1 *maximizes decay centrality, (2) the implant speaks immediately and the product diffuses throughout the network in 2 periods in view of (iii) and (iv) above. Consider now the type B implant. Her expected payoff from any site* $i \in I$ *is* ξ_K *given the acceptance probabilities of the peripheral sites in the star. This is also the expected payoff from the implant at* 1 *since* 1*'s recommendation is accepted with probability one. Also, note that since* $\delta \leq \frac{\xi_K}{\xi_{K+1}}$*, the implant at* $i \in I$ *has no incentive to strategically postpone her recommendation to period 2, even though her recommendation in this period would be accepted with probability one. Finally, we check that the response decisions described in (iii) and (iv) are optimal. To see this, note that the relevant agents in each case do not expect any further information flows and so* $\bar{p}$ *is the appropriate threshold for acceptance.*

If $m \in S_1$ but is not critical, then there could be an equilibrium of the following kind. The good product may diffuse throughout the network with probability one even if some neighbors of m who do not constitute critical links with m refuse m's recommendations - the fact that some link mi is not critical obviously implies that there is some path from m to i not involving the link mi. An implication of this is that the type B firm needs fewer customers in equilibrium. Now, suppose each node $i \in S_1$ has one neighbor in $\bar{N}_i(\Gamma)$ with very high degree, say h_i, while the others have relatively low degree. Then, one option for the type B firm is to put probability weights $\tilde{\beta}_i > \bar{\beta}_i$ on each i such that all nodes in $\bar{N}_i(\Gamma)$ except h_i accept i's recommendation. In other words, the freedom to dispense with h_i as a customer helps to raise the probability that type B can put on each node i in S_1 and so it may be possible to support an EDE even when equation 4 is not satisfied.

The sufficiency condition in Theorem 4 is phrased in terms of effective degrees, and not in terms of "standard" network characteristics such as degree distribution or diameter of the network. This is almost inevitable because the type B firm will choose to place his implant at some node maximizing effective degree, and that latter does not easily translate into the familiar network characteristics. The following example shows that two networks with the same degree distribution can have quite different properties with respect to the existence of an EDE.

Example 2 *In this example both networks are trees with the same degree distribution, such that there is an EDE in one , but not in the other. Let* $n = 7$*.*

Tree 1: Node 1 is connected to nodes 2 and 3; node 2 is connected to 4 and 5, while node 3 is connected to 6 and 7.

Tree 2: Node 1 is connected to 2,3,4; 4 is connected to 5 and 6, while 6 is connected to 7.

Both trees have degree distribution $(3, 3, 2, 1, 1, 1, 1)$*. In tree 2, node 4 maximizes decay centrality for all* $\delta > 0$*. In tree 1, node 1 maximizes decay centrality for* $\delta > 1/2$*.*

Let $\delta > 1/2$*. Although both trees have the same degree distribution, the distribution of effective degrees is not identical. In tree 1, the distribution of effective degrees is* $(2, 2, 2, 1, 1, 1, 1)$*. To see this, check that nodes 1,2,3 all have effective degree 2, while the rest have effective degree 1. In tree 2, the distribution of effective degrees is* $(3, 2, 1, 1, 1, 1, 1)$*. For some parameter values, it is possible to sustain an EDE in tree 1, where* $\alpha_1 = 1$*, while the support of* β *is* $\{1, 2, 3\}$*. It is easy to see that no parameter values can sustain an EDE in tree 2.*

5 Comparative Statics

In this section, we discuss the role of different parameters and the network structure in sustaining an EDE.

It is easy to check that an increase in p makes it more likely that an EDE exists since it allows the type B firm to place more probability weight on nodes and still satisfy the requirement that the updated beliefs reach the threshold value of $\bar{p}$. In other words, if an EDE exists for some value of p and then p increases, there must continue to be an EDE. In what follows, we focus on the role of the network structure and of ρ.

5.1 The Role of the Network Structure

We first show that no site i can be too well-connected if the network is to sustain an EDE. In particular, no network that contains a star encompassing all nodes can support an EDE.

In what follows, for any set $M \subset N$, and for any $i \notin M$, let $\bar{N}_i(M) = N_i \setminus \cap_{j \in M} N_j$, and $\bar{d}_i(M) = |\bar{N}_i|$. The interpretation of $\bar{N}_i$ is that if M is the support of α, then $\bar{N}_i$ is the set of potential customers of i. Since individuals in $\cap_{j \in M} N_j$ are connected to all nodes in M, they will not accept any recommendation from a node $i \notin M$.

Theorem 2 *Suppose* Γ *contains a star as a subgraph. Then,* Γ *cannot support an EDE.*

Proof. Let $M = \{i \in N | d_i(\Gamma) = n - 1\}$. If Γ contains a star, this set is non-empty. Then, all members of M maximize decay centrality. If an EDE exists, the support of α is some $M' \subseteq M$. Take any site $i \notin M'$. Then, $\bar{d}_i(M') = 0$ since any site $j \neq i$ is connected to all nodes in M'. So, the support of β is also M'. Then, it follows from equation 1 that

$$\eta^1_{j,i} \leq p < \bar{p}$$

So, no neighbor of the bad implant at i buys the product after receiving a recommendation from i. Since the bad type is indifferent between all sites in the support of β, no site in the support of β can get her recommendation accepted. This implies that only the good type employs an implant. However, this cannot be an equilibrium since the bad type would then deviate and place an implant at some site in M'. ■

Remark 3 *Note that Theorem 2 implies that the complete graph cannot support an EDE!*

Under Assumption S, we can also place an upper bound on the degree of m.

Theorem 3 *Let Assumption S hold, with* m *the unique node maximizing decay centrality in* Γ*. Then, if* Γ *is to support an EDE,* $d_m \leq \frac{n-1}{2}$*.*

Proof. Let Γ support an EDE, and let $d_m = k$. Then, for all $i \neq m$, $\bar{d}_i \leq n - k - 1$. Suppose $d_m > \bar{d}_i$ for all $i \neq m$. Then the bad type would prefer to put her implant at m since recommendations from m are accepted with probability one. But, if $\beta_m = 1$, then $\eta^1_{m,j} < p < \bar{p}$ and this is not consistent with m's recommendations being accepted. Hence, $d_m = k \leq \bar{d}_i \leq n - k - 1$ for some $i \neq m$. This implies $d_m \leq \frac{n-1}{2}$. ■

Theorems 2 and 3 describe some network structures that cannot support an EDE. In particular, nodes maximizing decay centrality cannot be too well-connected since their connections tend to reduce the effective degree of those nodes which are "close" to them.

In previous results, we have indicated that there is a sense in which larger networks are more likely to support efficient diffusion. The minimum size of network needed to support an EDE will, of course, depend upon the structure of the network as well as on the parameters $(\rho, p, \bar{p})$. Here, we ask the following question. Is a circle or a tree more likely to support an EDE if the number of nodes is fixed? We provide some answers to this question under the assumption of Bounded Rationality defined earlier and for all trees satisfying Assumption S. We show that the value of ρ is crucial for such comparisons for reasons described below.

As we have mentioned earlier, the minimum size circle network is obtained when both the G and B types use mixed strategies putting equal probability weights on all nodes. So, every node in the network can be in the support of β. In contrast, we show first that there are at least *four* nodes which cannot be in the support of β when the network is a tree satisfying Assumption S. This tends to increase the minimuum size of a tree which can support an EDE. The circle also has the advantage that each node has only two neighbors. So, when node i receives only *one* recommendation, the fact that its other neighbor did not make a recommendation is not a “big" negative signal. Consider a tree where every node i which is in the support of β has to have its recommendation accepted by at least one node with two or more neighbors. Then, the “negative " signal of receiving only one recommendation "cancels out" for the tree and the circle, but the minimum size of the circle sustaining an EDE is smaller than the analogous lower bound for the tree because there are at least four nodes in the tree which cannot be in the support of β. But, there can be trees which have the property that some node(s) i in the support of β need only get their recommendations accepted by *leaves* who have no other neighbors. Then, the fact that such nodes have no other neighbors allows the B-type to put relatively larger probability weights on node(s) i. This gives an advantage which becomes larger as ρ increases. So, for “large" values of ρ, such trees are more likely to support efficient diffusion than circles on the same number of nodes. However, we show below that if $\rho \leq 1/2$, then the minimum size of the circle supporting an EDE is smaller than the corresponding minimum size for a tree.

Let T denote the set of all trees satisfying Assumption S.

Recall that equation 2 specifies $\underline{n}^c(K,\rho)$, the minimum size of the circle network as a function of ρ, for a fixed value of $K \equiv \frac{\bar{p}(1-p)}{p(1-\bar{p})}$. Let $\underline{n}^L(\rho,K)$ and $\underline{n}^T(\rho,K)$ denote the corresponding minimum sizes for a line network and the class of tree networks satisfying Assumption S.

For any g, let $L(g) = \{i \in N | d_i(g) = 1\}$. That is, elements of $L(g)$ are the *leaves* of g. Finally, we define a subset of trees T^* below.

$$T^* = \{g \in T || N_i(g) \cap L(g)| < d_m(g) \forall i\}$$

For any $i,j \in N$, the path from i to j, denoted $p(i,j)$, is the set of nodes $\{i_0, i_1, \ldots, i_K\}$ with $i_0 = i, i_K = j$ and $i_k i_{k+1} \in g$.

The following lemma will be useful.

Lemma 1 *Let $g \in T$. If g supports an EDE, then $J \equiv \{i \in N | \beta_i = 0\}$ has at least four elements.*

Proof. We first show that if $g \in T$ and g supports an EDE,then $n > 5$. For suppose $n = 5$. There must be $i \neq m$ such that $\beta_i > 0$ since $\beta_m < 1$. Then the only possibility is that m is connected to i and a leaf l, while $N_i(g) = \{m, l_1, l_2\}$ where l_1, l_2 are leaves which are distinct from l. The decay centrality at m is $2(\delta + \delta^2)$, while that at i is $3\delta + \delta^2$. So, m cannot be maximizing decay centrality which is a contradiction. Hence, $n \geq 6$.

Obviously, $L(g) \subset J$. So, if there are four or more leaves, then the lemma is true.

Suppose there are exactly three leaves. Then, $d_m(g) \in \{2, 3\}$. It is easy to check that at least one of m's neighbors is in J but not in $L(g)$. Again, the lemma is true.

Suppose g has exactly two leaves. Then, both of m's neighbors are in $J \setminus L(g)$. ■

Theorem 4 *Let Assumption BR hold. Consider any* $K \equiv \frac{\bar{p}(1-p)}{p(1-\bar{p})}$. *Then, the following are true.*

(i) For all $\rho > 0$, $\underline{n}^{T^*}(\rho, K) \geq \underline{n}^C(\rho, K)$.

(ii) There exists $\bar{\rho} \geq 1/2$ *such that for all* $\rho \leq \bar{\rho}$, $\underline{n}^T(\rho, K) \geq \underline{n}^C(\rho, K)$.

(iii) For all $(\rho, p, \bar{p})$, $\underline{n}^L(\rho, K) \geq \underline{n}^C(\rho, K)$.

Proof.

(i) Suppose $g \in T^*$, and it sustains an EDE with β being the equilibrium strategy of type B firm. Using equation 1,we calculate the maximum possible weights on different nodes subject to retaining credibility of recommendations. So,

$$\begin{aligned} \frac{p(1-\rho)^k}{p(1-\rho)^k + (1-p)\beta_m} &\geq \bar{p} \\ \text{or } p(1-\rho)^k(1-\bar{p}) &\geq \bar{p}(1-p)\beta_m \\ \text{or } \frac{(1-\rho)^k}{K} &\geq \beta_m \end{aligned} \tag{5}$$

where $k+1 \geq 2$ is the maximum degree of some $j \in N_m(g)$.

Consider other nodes in the support of β. Let $i \neq m, i \notin J$. It follows that

$$\begin{aligned} \frac{p\rho(1-\rho)^q}{p\rho(1-\rho)^q + (1-p)\beta_i} &\geq \bar{p} \\ \text{or } p\rho(1-\rho)^q(1-\bar{p}) &\geq \bar{p}(1-p)\beta_i \\ \text{or } \frac{\rho(1-\rho)^q}{K} &\geq \beta_i \end{aligned} \tag{6}$$

where $q+1 \geq 2$ is the maximum degree of some $j \in \bar{N}_i(g)$. Here, in asserting that some node in $\bar{N}_i(g)$ which accepts i's recommendation has degree at least two, we use the fact that $g \in T^*$ - so some node which is *not* a leaf has to accept i's recommendation.

In view of Lemma 1, the maximum number of nodes that can be in the support of β is $n-4$, including node m. Since $\sum_{i \in N} \beta_i = 1$, we have

$$\begin{aligned} \frac{(1-\rho)^k}{K} + (n-5)\left[\frac{\rho(1-\rho)^q}{K}\right] &\geq 1 \\ \text{or since } \rho < 1, \frac{(1-\rho)}{K} + (n-5)\left[\frac{\rho(1-\rho)}{K}\right] &\geq 1 \\ \text{or } n &\geq 5 + \frac{K}{\rho(1-\rho)} - \frac{1}{\rho} \end{aligned}$$

Hence,

$$\underline{n}^{T^*}(K, \rho) \geq 5 + \frac{K}{\rho(1-\rho)} - \frac{1}{\rho} \tag{7}$$

Using equation 2

$$\underline{n}^{T^*}(K, \rho) - \underline{n}^C(K, \rho) \geq 2 \tag{8}$$

This proves (i).

(ii) Suppose $g \in T \setminus T^*$, and it supports an EDE. Suppose there is some equilibrium strategy of the type B-firm such that there is i with $\beta_i > 0$ *and* $\bar{N}_i(g) \cap L(g)$ is sufficiently large for i not to need any $j \in \bar{N}_i(g) \setminus L(g)$ (if nonempty) to accept i's recommendation.[16] Then, since all leaves who accept i's recommendation have no other neighbor other than i, equation 4 readily yields the maximum probability weight that can be put on i.

$$\frac{\rho}{K} \geq \beta_1$$

Let

$$M(g, \beta) = \{i \in N | \beta_i = \frac{\rho}{K}\}$$

Notice that if $M(g, \beta)$ is empty, then the proof in (i) suffices. If $M(g, \beta)$ is non-empty, then we divide the proof into two cases.

Case 1 : Suppose $M(g, \beta)$ contains exactly one node.

Then, in view of lemma 1, the maximum number of nodes *not* in $M(g, \beta)$ but which are in the support of β can be $(n-5)$. Since this number includes m, we have

$$\begin{aligned}
\frac{\rho}{K} + \frac{(1-\rho)^k}{K} + (n-6)\left[\frac{\rho(1-\rho)^q}{K}\right] &\geq 1 \\
\text{or since } \rho < 1,\ \frac{\rho}{K} + \frac{(1-\rho)}{K} + (n-6)\left[\frac{\rho(1-\rho)}{K}\right] &\geq 1 \\
\text{or } n &\geq 6 + \frac{K}{\rho(1-\rho)} - \frac{1}{\rho(1-\rho)}
\end{aligned}$$

Hence,

$$\underline{n}^T(K, \rho) \geq 6 + \frac{K}{\rho(1-\rho)} - \frac{1}{\rho(1-\rho)} \tag{9}$$

Using equation 2

$$\begin{aligned}
\underline{n}^T(K, \rho) - \underline{n}^C(K, \rho) &\geq 3 + \frac{1}{\rho} - \frac{1}{\rho(1-\rho)} \\
&\geq 3 - \frac{1}{1-\rho} \\
&\geq 0 \text{ if } \rho \leq 2/3
\end{aligned} \tag{10}$$

Case 2 : Suppose $M(g, \beta)$ contains ℓ nodes where $\ell > 1$. Since $d_m \geq 2$, $|L(g)| \geq 2\ell$. Hence, there can be at most $(n - 3\ell - 1)$ nodes i which are in the support of β that are neither in $M(g)$ nor m. So,

$$\begin{aligned}
\frac{\ell\rho}{K} + \frac{(1-\rho)^k}{K} + (n-3\ell-1)\left[\frac{\rho(1-\rho)^q}{K}\right] &\geq 1 \\
\text{or since } \rho < 1,\ \frac{\ell\rho}{K} + \frac{(1-\rho)}{K} + (n-3\ell-1)\left[\frac{\rho(1-\rho)}{K}\right] &\geq 1 \\
\text{or } n &\geq 3\ell + 1 + \frac{K}{\rho(1-\rho)} - \frac{1 + (\ell-1)\rho}{\rho(1-\rho)}
\end{aligned}$$

[16] A necessary condition is that $|\bar{N}_i(g) \cap L(g)| \geq d_m(g)$.

So,

$$\underline{n}^T(K,\rho) \geq 3\ell + 1 + \frac{K}{\rho(1-\rho)} - \frac{1+(\ell-1)\rho}{\rho(1-\rho)} \tag{11}$$

and

$$\begin{aligned}\underline{n}^T(K,\rho) - \underline{n}^C(K,\rho) &\geq 3\ell - 2 + \frac{1}{\rho} - \frac{1+(\ell-1)\rho}{\rho(1-\rho)} \\ &\geq 0 \text{ if } \rho \leq \tfrac{2\ell-2}{3\ell-2} \end{aligned} \tag{12}$$

Since $\frac{2\ell-2}{3\ell-2} = 1/2$ at $\ell = 2$ and is strictly increasing in ℓ, this establishes the bound of $\bar{\rho} = 1/2$.

(iii) Suppose g is a line supporting an EDE. The support of α contains either two nodes (if n is even) or just one node. In either case, the support of β cannot include the extreme points of the line since they have degree one. Since all nodes in the circle have degree two, the set of nodes which can possibly be in the support of β is strictly larger for the circle, irrespective of whether n is odd or even. Hence, the circle can support an EDE whenever a line with the same number of nodes supports an EDE. ■

Remark 4 *The bound of $\bar{\rho} = 1/2$ in part (ii) is reached for $K = (1+\rho)$ in the network where say node 1 is connected to nodes 2 and 3, with nodes 2 and 3 being connected to two leaves each. In this case, $\underline{n}^T(K,\rho) = 7$, while $\underline{n}^C(K,\rho) \leq 7$ iff $\rho \leq 1/2$. In all other cases, the appropriate upper bound for ρ is greater than 1/2.*

5.2 The Influence of Innovators

Recall that ρ is the probability of an innovator. Here, we discuss the role of ρ on an EDE. In order to simplify the discussion, we assume throughout this section that Assumption S is satisfied. Thus, the type G firm must be using a pure strategy of placing her implant at m with probability one, where m is the unique node in $D(\Gamma)$.

Consider first the case where $\rho = 0$. This is a particularly stark case to analyze, where the equilibria of the previous sections do not exist-specifically only m can speak credibly in the first period.[17] A bad implant at other sites will have to wait to speak. For instance, a neighbor of m can speak in period 2 by pretending to have received a recommendation from m in period 1, a site at a distance of 2 from m can speak in period 3 and so on.

The specific structure of the network will determine whether the good product will diffuse throughout the network.

Theorem 5 *Suppose $\rho = 0$, Assumption S is satisfied*[18] *and the unique node maximizing decay centrality is both critical and is in S_1. Then, Γ cannot support an EDE.*

Proof. Let m be the unique node maximizing decay centrality. If an EDE exists, $\alpha_m = 1$. Also, if m is critical, then all of m's neighbors must accept m's recommendations. If some neighbor j does not accept m's recommendation with probability one, then the criticality

[17] There is also a trivial "no trade" equilibrium, where neither G nor B enter, and any recommendation is believed to come from the B type firm.

[18] This assumption can be relaxed but is maintained here for ease of exposition.

of m implies that the good product will not diffuse to some segment of the network. Since $m \in S_1$, the type B firm can get $|S_1|$ acceptances by putting an implant at m. Of course, $\beta_m = 1$ is not possible since all neighbors of m would then not revise their beliefs about the product. On the other hand, when $\rho = 0$, recommendations from no other site are credible initially. Hence, the expected payoff to type B from an implant at $i \neq m$ is strictly less than the payoff at m if the recommendation from m were credible. This shows that the type B firm does not have an equilibrium strategy that sustains efficient diffusion. ■

This theorem of course immediately implies that when $\rho = 0$, the line or, more generally, a tree where the node maximizing decay centrality also has maximal effective degree cannot support an EDE. It is easy to construct examples of trees which support an EDE if m does not have maximal effective degree. Consider the following example.

Example 3 *Let $n \geq 9$ and n be odd. Let Γ be as follows. Individual 1 has just 2 links to 2 and 3. Divide the set $\{4, \ldots, n\}$ into two equal subsets, let 2 be connected to all individuals in the first subset, each of whom have no other link. Similarly, let,individual 3 be connected to all agents in the second subset, each of whom have no other link. Figure 1 illustrates the graph for the case $n = 9$.*

Assume that

$$\delta \geq \max\left(\frac{4}{n-3}, \frac{n-5}{n-3}\right)$$

First, notice that if type G places his implant at 1 and all subsequent recommendations are accepted with probability one, then his payoff is $2\delta + (n-3)\delta^2 - c$. On the other hand, if he places his implant at 2 or 3, then his payoff is $(((n-3)/2) + 1)\delta + \delta^2 + ((n-3)/2)\delta^3 - c$. The inequality $\delta \geq \frac{n-5}{n-3}$ ensures that the first sum is at least as large.

Second, assume also that p and $\bar{p}$ are such that the type B can put probability 1/2 on each of the nodes 2 and 3, and still make a credible recommendation in period 2. That is, the neighbors of 2 and 3 have to infer whether sites 2 and 3 have received a recommendation from the implant of type G placed at 1 in the previous period or whether it is the type B implant who is speaking in period 2, having strategically kept silent in period 1. A probability weight of $\beta_2 = \beta_3 = 1/2$ brings their updated belief that the recommendation is being passed on from 1 to the threshold $\overline{p}$, given the initial parameter values.

Then, the following is an EDE. The type G puts his implant at 1 with probability 1, the type B randomizes between 2 and 3 with equal probability. The type G implant speaks immediately while the type B implant speaks in period 2. All recommendations are accepted with probability one.

These constitute an equilibrium because if $\delta \geq \frac{4}{n-3}$, then the type B implant has no incentive to deviate and place his implant at 1 - he gets $\delta^2(n-3)/2 - c$ in equilibrium, whereas he would get $2\delta - c$ by placing his implant at 1. The response decisions are optimal because updated beliefs are not below the threshold.

How does a positive ρ impact on the possibility of efficient diffusion? Fix all other parameters and the network structure Γ. Now, ρ influences the nature of equilibrium in two ways. First, the higher the value of ρ, the lower is the net gain from having an implant for both firm types. So, there will be some value of $\bar{\rho}$ such that if $\rho \geq \bar{\rho}$,then one or both types of firm F will refrain from employing an implant.

Second, the value of ρ influences the updating process according to equation 1. Suppose ρ changes. How does this affect $\eta^1_{i,i-1}$ for a fixed value of β_i and distribution α? An increase in ρ makes it more likely that $i-1$ is an innovator, but also makes it less likely that none of the *other* neighbors of i are innovators. These effects move in opposite directions and an unambiguous answer is difficult to provide.

Suppose, however, that Assumption S holds, and an EDE exists where type B's equilibrium strategy places no probability weight on m, the unique node maximizing decay centrality. Then, the trade-offs are somewhat easier to discern.

In this case, the expression for $\eta^1_{i,j}$ simplifies considerably for $j \neq m$ and becomes

$$\eta^1_{i,j} = \frac{p\rho(1-\rho)^{d_i-1}}{p\rho(1-\rho)^{d_i-1} + (1-p)\beta_i}$$

Now,

$$\text{sign } \frac{\partial \eta^1_{i,j}}{\partial \rho} = \text{ sign } \frac{\partial p\rho(1-\rho)^{d_i-1}}{\partial \rho}$$

Hence,

$$\frac{\partial \eta^1_{i,j}}{\partial \rho} \geq 0 \text{ iff } (1-d_i\rho) \geq 0$$

So, if the initial value of ρ for which an equilibrium exists is "low", then an increase in ρ does not decrease $\eta^1_{i,j}$ and so the same strategies continue to be an equilibrium. On the other hand, an increase in ρ at higher values decreases $\eta^1_{i,j}$ and so the same strategy may not be an equilibrium. These suggest that there is some threshold value of ρ above which an equilibrium of this type is not possible.

6 Inefficient Equilibria

From the previous sections, it is clear that some network structures may not support efficient diffusion equilibria. In fact, two well-studied network structures - the *line* (unless it is large enough) and the *star* - do not support EDE. Since a PBE must always exist in our model, there must be at least one "inefficient" equilibrium. There can be two types of inefficient equilibria in this model. In the first type, the good product diffuses throughout the network with probability one, but the G-implant is not located at the node maximizing decay centrality. In the second type of equilibrium, perhaps the good type locates at the node maximizing decay centrality, but the good product does not diffuse through the network with probability one. The size and type of network structure determine the kind of inefficient equilibrium that will exist.

For instance, there cannot be complete diffusion with probability one in the *star* unless the future is discounted heavily. A brief sketch of the argument follows. Suppose there is a PBE where the good type product diffuses with probability one. Since we know that recommendations from m cannot be accepted with probability one in period 1, the support of α is contained in the set of peripheral nodes. Then, the equilibrium must involve m accepting the G-implant's recommendation in period 1, and then successfully passing on the recommendation to other peripheral nodes in period 2. So, the B-implant can locate at m, speak one period later, and get at least $\delta(n-1)$. But, if the B-implant locates at a peripheral node, then he can get only 1. So, the type B firm will want to place his implant at m with probability one. Then, a recommendation from m even in period 2 is not credible. Hence, there cannot be any such PBE.

On the other hand, the *line* can support both types of inefficient equilibrium. We first discuss the general issues involved in the construction of such equilibria for the line, and then illustrate them with two examples. Considerations for general networks are similar but are difficult to see explicitly.

Recall from Section 3 that

$$S = N \setminus \{1, n, m-1, m+1, m-2, m+2\}$$

is the set of sites which can possibly have two consumers, and that an EDE may not be possible if S contains too few members for the type B firm to distribute its probability weight for acceptance probabilities of 1 to be best responses.

Possible ways out of this problem involve changing the equilibrium acceptance probabilities so that some or all nodes in $\{1, n, m-1, m+1, m-2, m+2\}$ on the line can be part of the support of β. We must consider the following, however:

1. The equilibrium acceptance probabilities have to be indexed now by time. Suppose some node i makes recommendations to nodes j in period 1 and these are accepted with probabilities less than 1. Node i, if it contains an implant from B, must not, in equilibrium, gain by deviating to being silent in period 1 but speaking in period 2. Such a deviation need not be known to be off the equilibrium path, since i might be passing on recommendations he received and accepted the preceding period. If an implant is supposed to speak in the first period with probability one, a recommendation from that site in period 2 will be accepted for sure. For players with two neighbors, the expected payoff in equilibrium, without considering the cost of an implant or a payoff from an innovator at the same site, must therefore be 2δ. Thus the incentives for implants to delay speaking must be considered seriously. (This is not an issue if all recommendations are accepted with probability 1, since in that case an implant with two neighbors would clearly do better speaking in the first period.) However, for each node i, there is a maximal time i_T at which he can speak credibly to all his neighbors. After this time the probability of acceptance goes to zero for at least one neighbor. We discuss this last point in detail later in this section. If a message is received from $m, 1$ or n in period 2, we assume this is from a B type and it is rejected. Messages at other sites in period 2 have positive probability. Later periods are covered similarly-though other beliefs could sustain the equilibrium.

2. The second major issue that arises with equilibrium acceptance probabilities in $(0, 1)$ is that they affect the updating of probabilities. Consider $n = 9$, and an equilibrium where the G type places an implant at site 5 with probability 1 (as in the EDE). Take the case of individual 3, who is not an innovator or an implant and who does not receive any recommendation from either neighbor in period 1. Suppose this player 3 gets a recommendation in period 2. The conditional probability that the product is good given a recommendation from 2 depends on (i) the value of ρ, the probability

a good recommendation can originate in 1, and (ii) the probability that 2 accepts a recommendation from 1. If the recommendation comes from 4 in period 2, the relevant probabilities include the acceptance probability with which 4 accepts 5's recommendation. If the acceptance probabilities are different, the conditional probabilities for recommendations from left or right will be different. The higher the acceptance probabilities, the more is not receiving a recommendation "bad news" for Player 2. This means that not receiving a recommendation from the right could be worse news than not getting one from the left.

3. For small n and for given c, it might be optimal for both types to locate at m. In this case, acceptance probabilities have to make the bad type indifferent between entering and not entering and the good type strictly prefer to enter. We do not discuss this further because we consider the cost c to be relatively small so that both types will find it optimal to enter.

We now discuss i_T in more detail. Consider $i \in S$. Suppose $i = 2$, and there is an implant at 2. Suppose this implant at 2 makes a recommendation in period 2. Then, 3 can believe that site 1 is an innovator and has passed on a positive recommendation in period 1, which is then passed on by 2 in period 2. But, suppose 2 makes a recommendation in period 3. Then, 3 knows that 2 must be an implant who has not made a recommendation in an earlier period.

In general, suppose $i < m$, and is at a distance of k_l from 1 and k_r from m with $k_r < k_l$. Then, if i makes a recommendation in period $t = k_r$, then $i + 1$ will know that conditional on the product being good, a positive recommendation should have come from the right in no more $k_r - 1$ periods.

Hence, for each $i \in S$, there is a time period i_T up to which a bad implant can make recommendations and still have a positive probability of getting her recommendation accepted by both her neighbors. If the maximum time period for a credible recommendation is different for the left and right neighbor, we set i_T to be the minimum of the two. Note that $m_T = 1$.

For each $i \in S$, let s_i^t denote the probability with which the implant at i makes a recommendation in period t for $1 \leq t \leq i_T$. In equilibrium, the implant mixes over the timing of the recommendation only if the sum of the acceptance probabilities of his neighbors in period $t-1$ equals δ times the corresponding sum in period t. If i speaks in period t,then the player to $i's$ left assesses the probability that the product is good to be $\eta_{i-1,i}^t$ and the person to $i's$ right as $\eta_{i+1,i}^t$.The explicit expressions for these are complicated because they have to take into account possibly different acceptance probabilities along the way (from agents who received recommendations at prior time periods from a possible innovator or good implant to i) depending on whether the assumed path to i is from $i's$ left or right. Using the values of $\eta_{j,i}^t$, where j is $i's$ neighbor, is not an innovator and has received no recommendation from any of her other neighbors up to this point, we set each $\eta_{j,i}^t = \bar{p}_j^t$ (if j is to randomise), where $\bar{p}_j^t$ is the value that makes j indifferent between buying now or waiting.

We can similarly calculate i_T for other nodes. Note that a recommendation is credible for *both* neighbors, if a site has two, if $t \leq i_T$, but it could remain credible for one neighbor and not for the other for $t > i_T$.

We now present an example which illustrates some of the features of an inefficient equilibrium in which the good implant is located at the median, but the good type product does not diffuse through the network with probability one.

Let $n = 9$ and Γ be a line. Let $\delta \geq 1/2, \rho = p = 1/2$, and $\bar{p}^0 \equiv \frac{b}{b+g} = 0.6$.

The constructed equilibrium has the following features:

(i) $\alpha_5 = 1$.

(ii) $\beta_1 = \beta_9 = 0$, and all other $\beta_i > 0$. The values are specified below.

(iii) The implant of the G-type firm "speaks" in period 1. The B-type implant at nodes 3,4,5,6,7 also makes a recommendation immediately, while the implants at nodes 2 and 8 randomize between speaking in periods 1 and 2.

(iv) Nodes 4 and 6 accept the recommendation from 5 with probability 1/2 in period 1. For $i = 3, 4$, the recommendation of i is accepted with probability one by $i - 1$ in period 1. Similarly, for $i = 6, 7$, the recommendation of i is accepted by $i + 1$ in period 1.

(v) Nodes 1 and 9 accept the recommendations of 2 and 8 respectively with probability one in periods 1 and 2.

(vi) Nodes 3 and 7 reject the recommendations of 2 and 8 respectively with probability one in period 1. They accept these recommendations with probability $\gamma = \frac{1-\delta}{\delta}$ in period 2.

(vii) All other "credible" recommendations are accepted with probability one. [19]

We now provide some detailed calculations to show that these constitute an equilibrium. First, note that

$$\eta^1_{45} = \eta^1_{65} = \frac{p(1-\rho)}{p(1-\rho) + (1-p)\beta_5}$$

Also, the threshold values of 4 and 6 for accepting a recommendation from 5 must be $\bar{p} = 0.6$. Setting $\eta^1_{45} = \eta^1_{65} = \bar{p}$, we get

$$\bar{\beta}_5 = 0.333$$

Note that this value of $\bar{\beta}_5$ ensures that there cannot be an EDE in this example. This is because if an EDE is to exist, all the probability mass of β has to be placed on 2,5 and 8, but since $\beta_5 \leq 1/3$ and $\beta_2 < 1/3$, $\beta_8 < 1/3$, the sum will be less than 1 and therefore an EDE is not possible.

Now,

$$\eta^1_{23} = \frac{p(1-\rho)}{p(1-\rho) + (1-p)\beta_3}$$

Moreover, 2 cannot hope to get any further information in later periods - if 1 were an innovator and had to make a recommendation, she would have done so in period 1 itself. So,

[19] A "credible recommendation is the following. Any recommendation in period $t \leq i_T$ is credible for both neighbours of $i \in S$. However, a recommendation from 2 in period 4 is credible for 1(since 2 may be passing on a recommendation originating from 5, but *not* credible for 3.

2's threshold probability value for acceptance after receiving a recommendation from 3 is $\bar{p}$. Equating $\bar{p} = \eta^1_{23}$, we get

$$\bar{\beta}_3 = .166$$

For exactly similar reasons, we get

$$\bar{\beta}_7 = .166$$

We now calculate the threshold probability value for which 3 is indifferent between accepting and rejecting a recommendation received solely from 4 in period 1. Denote this as $\bar{p}^1_{34}$. If 3 accepts this recommendation at this value, then his expected utility is $\bar{p}^1_{34}(b+g) - b$. But, 3 could postpone his decision to purchase by one more period in the hope of receiving another recommendation from 2 next period.[20] Since 2 accepts 1 's recommendation with probability one, the probability of receiving such a recommendation equals $\bar{p}^1_{34}\rho$, and so the expected utility from waiting is $\bar{p}^1_{34}\rho\delta g$.[21] Hence,

$$\bar{p}^1_{34} = \frac{b}{b+g-\delta\rho g} = \frac{3}{5-\delta}$$

Since

$$\eta^1_{34} = \frac{p(1-\rho)}{p(1-\rho)+(1-p)\beta_4}$$

the value of β_4 which makes 3 indifferent between accepting and rejecting 4 's recommendation is

$$\bar{\beta}_4 = \frac{0.5 - 0.25\delta}{3}$$

Similarly,

$$\bar{\beta}_6 = \frac{0.5 - 0.25\delta}{3}$$

We now calculate $\bar{\beta}_2$ and $\bar{\beta}_8$. Let s^1_i, s^2_i be the probabilities with which $i = 2, 8$ make recommendations over periods 1 and 2. We first calculate $\bar{p}^1_{32}$, the threshold probability which makes 3 indifferent between accepting and rejecting 2's recommendation in period 2. If 3 waits one period, then she may receive another recommendation from 4 (who would be passing on the recommendation from 5). Since 4 accepts 5's recommendation with probability 1/2, the probability that 3 receives a recommendation from 4 in period is $1/2\bar{p}^1_{34}$. Equating expected utilities, we get $\bar{p}^1_{32} = \frac{3}{5-\delta}$.[22] Hence,

$$\bar{\beta}_2 s^1_2 = \frac{0.5 - 0.25\delta}{3}$$

In period 2, the threshold probability value of acceptance for 3 must be $\bar{p}$ since she cannot get any further recommendations. Using this, routine calculation yields

$$\bar{\beta}_2 s^2_2 = 0.021$$

[20] Conditional on receiving a recommendation from 4 in period 1, a second recommendation from 2 in period 2 must imply that 2 is passing on a recommendation from 1 since 2 cannot be either a bad implant or an innovator.

[21] We can check that if there is no recommendation from 2, the probability of G drops to $\frac{3}{7-\delta}$, which is less than $\bar{p}$.

[22] Here again, no recommendation from 4 is sufficiently bad news that 2 then does not buy. The conditional probability of G is again $\frac{3}{7-\delta}$.

Adding, and using the fact that $s_2^1 + s_2^2 = 1$, we get

$$\bar{\beta}_2 \in [0.10, 0.145] \text{ for } \delta \in [1/2, 1]$$

Similarly

$$\bar{\beta}_8 \in [0.10, 0.145] \text{ for } \delta \in [1/2, 1]$$

Hence, for all values of $\delta \geq 1/2$,

$$\sum_{i \neq 1,9} \bar{\beta}_i > 1$$

Now, for any value of $\delta \geq 1/2$, choose the probability distribution β such that

$$\beta_2 = \beta_8 = \bar{\beta}_2, \beta_i < \bar{\beta}_i \text{for } i = 3, 4, 5, 6, 7, \beta_1 = \beta_9 = 0$$

Also, choose s_i^1, s_i^2 for $i = 2, 8$ so that $\eta_{32}^1 = \bar{p}_{32}^1$, $\eta_{32}^2 = \bar{p}$, $\eta_{78}^1 = \bar{p}_{78}^1$, $\eta_{78}^2 = \bar{p}_{78}^2$. Clearly, these can be done given the choice of $\bar{\beta}_i$.

We first check that the actions specified for both the G-type and B-type implant are optimal. Note that for each $i \neq 1, 9$, the sum of the acceptance probabilities equals 1 if $i \in S$, while exactly one neighbor of $i \notin S$ accepts the recommendation with probability one. Also, for $i = 2, 8$, the expected payoffs are equal in periods 1 and 2. It is easy to check that these ensure that the B-type has no profitable deviation. Similarly, the G-type is also maximizing expected payoff by placing his implant at the median with probability one.

The relevant probability weights β_i and s_i^1, s_i^2(for $i = 2, 8$) have been chosen so that except for nodes 1 and 9, all other nodes receiving credible recommendations are indifferent between accepting and rejecting recommendations. Hence, their stipulated responses are optimal. Of course, $\bar{p}_{12}^1 == \bar{p}_{98}^1 \bar{p}$ since neither 1 nor 9 can expect to get any other recommendation. This ensures that 1 and 9 have to accept the recommendations of 2 and 8 with probability one in period 1.

Suppose now that $\delta < 1/2$. Then, the foregoing is not an equilibrium because 2 and 8 have no incentive to make a recommendation in period 2. For instance, even if both 1 and 3 accept 2's recommendation in period 2 with probability one, the (gross) expected payoff is still less than 1 - the expected payoff from recommending in period 1. However, it is easy to check that the following modification to the preceding specification is an equilibrium

(i) The B-type implant at nodes 2 and 8 set $s_i^1 = 1$. That is, they recommend with probability 1 in period 1.

(ii) Set $\beta_2 = \beta_8 = \frac{0.5 - 0.25\delta}{3}$. This ensures that 3 and 7 are indifferent between accepting and rejecting first-period proposals from 2 and 8 respectively. They both reject with probability one, while 1 and 9 accept with probability one.

(iii) Set $\beta_m = \bar{\beta}_m = 1/3$.

(iv) Set $\beta_i < \bar{\beta}_i$ for $i = 3, 4, 5, 6$.

(v) Other response decisions are the same as before.

Notice that this specification would not be an equilibrium when $\delta > 1/2$ because 2 (or 8) would then have an incentive to deviate by postponing a recommendation to period 2. This would be accepted with probability 1 by both neighbors, hence giving an expected utility of $2\delta > 1$.

It is possible to construct an equilibrium which has complete diffusion with probability one, but with the G-implant located at one extreme of the line. We indicate the nature of such equilibria without explicitly constructing an example. Let $n = 7$. [23] Let $\alpha_1 = 1$, while $\beta_4 = \beta_5 = \beta_6 = 1/3$. Let the values of (δ, ρ, p) be such that there exist (s_i^1, s_i^2) for $i = 4, 5, 6$ so that that $s_i^1 = 3\bar{\beta}_i$ and $s_i^2 \leq 3\bar{\beta}_i$. Then, a PBE can be constructed with the following features

(i) $\alpha_1 = 1$, while $\beta_4 = \beta_5 = \beta_6 = 1/3$.

(ii) The B-implant "speaks" with probability s_i^1 in period 1 and with probability s_i^2 in period 2.

(iii) Node 2 accepts the recommendation from 1 with probability 1 in period 1. In general, all nodes i accept recommendations from $(i-1)$ in period $(i-1)$. Notice that this ensures that the recommendation from the G-implant diffuses throughout the network.

(iv) For $i = 4, 5, 6$, nodes $i-1$ and $i+1$ accept i's recommendation with probability δ in period 1 and with probability 1 in period 2.

(v) Node $i = 2$ does not accept a recommendation from 3 in any period; an action sustained by the out-of-equilibrium belief that any recommendation from the right must come from a B-implant. This belief is justified since a recommendation should have come from 1 if the product was of the G-type.

So, the B-type firm gets a gross expected payoff of $2\delta > 1$ provided $\delta > 1/2$. This ensures that the B-type does not want to deviate to any node with effective degree 1. To check that the G-type firm does not want to deviate to 4, one can use the fact that 2 does not accept 3's recommendation.

In such an example, complete diffusion is possible because the G-type chooses a location which is not very attractive to the B-type. This example suggests a general principle for other network structures - complete diffusion may be possible if the G-type can credibly locate to nodes which do not have the highest possible degree since these are the nodes which are likely to attract the B-type implant.

7 Extensions

We consider some possible extensions of the basic model.

7.1 Negative recommendations

We have assumed that "recommendations" can only be positive. The intuition motivating this is that a neighbor can observe someone using the product over some period of time; if the product has been tried and found unacceptable, it will be discarded. The New York

[23] We choose small n in order to illustrate that complete diffusion with probability one is possible in networks which do not support EDE.

Times article cited in the introduction mentions implants, who have been paid to propagate a food item at pot luck parties, prominently displaying the brand packaging as evidence of use. Also, individuals may not want to admit that they have been "duped" into buying a bad product, and so may not pass on negative information.

However, one needs to consider negative recommendations as well if only to consider the robustness of the model. It is easy to check that Theorems 2, 3 and 5 continue to remain valid.

It also turns out that the possibility of negative recommendations will actually simplify calculations in one respect in that the probability calculations would not now depend on the potential recipient's degree. So, the analogue of equation 1 will now be

$$\eta^1_{i,i-1} = \frac{p\left[\alpha_{i-1} + (1 - \sum_{j\in N_i}\alpha_j - \alpha_i)\rho\right]}{p\left[\alpha_{i-1} + (1 - \sum_{j\in N_i}\alpha_j - \alpha_i)\rho\right] + (1-p)\beta_{i-1}} \tag{13}$$

However, it would complicate expected payoff calculations for B, where a high degree recipient of an implant's (positive) recommendation would be more likely to have countervailing negative information than one of low degree. Such a problem would not arise for regular graphs, but the general issue is illustrated below.

So, suppose B places implants at i and j with some positive probability. Then, his expected payoff from i is

$$E_i = \sum_{k\in\bar{N}_i} (1-\rho)^{d_k - 1} P_k$$

where P_k is the probability with which the offer is accepted by k. The updated belief for all $k \in \bar{N}_i(\Gamma)$ will now be the same - it will just depend on β_i and not on the degrees of k.

So, since $E_i = E_j$, we need

$$\sum_{k\in\bar{N}_j} (1-\rho)^{d_k - 1} P_k = \sum_{k\in\bar{N}_i} (1-\rho)^{d_k - 1} P_k$$

The specification of a general sufficient condition is now more difficult because the derivation of the support of β is now more complicated. However, the qualitative result that a larger network is more conducive for efficient diffusion remains unchanged for the line and regular networks.

The possibility of negative recommendations may also help in sustaining efficient diffusion, particularly in dense networks where nodes have high degree. This is because the expected payoff of the type B firm will now be lower- and it will be lower the larger are the degrees of different sites that can be recipients of implant recommendations. So, the type B firm may simply not employ implants.

7.2 Multiple implants

If the firm can choose multiple implants, the qualitative features of the analysis will be similar. Clearly it does not make sense for the multiple implants to have overlapping supports (for the firm's randomized strategies). This suggests that for large networks, the firm will partition the networks in such a way as to have one implant randomly located (for B) in each element of the partition. If δ is close to 1 and an EDE exists as above, there is very little incentive for G to incur the cost of an additional implant, since this can only speed up the diffusion and the benefit from this might be low compared to the cost. Therefore, for low discounting, we would expect to have several B implants but only one G implant. This suggests that the B implants would either have to rely on a relatively high ρ for credibility or speak only at sufficiently late time periods to mimic a message transmitted along the network from a supposed good implant, which might be located some distance away.

7.3 The bad type's probability of producing a good product.

We have assumed so far that Firm F knows its type, where *type* is identified with the quality of the product that is produced. Let us redefine type as follows. The type G firm produces a good product with probability one, while type B produces a good product with small positive probability ϵ and the bad product with probability $1-\epsilon$. Suppose as before that firm F knows its type in the modified sense.[24]

In this case, the following cases could arise (this is not an exhaustive description):

(i) There is a unique node m maximizing decay centrality, which also maximizes degree centrality. In this case, an EDE will not exist. The reason is that both G and B will care about speed of diffusion, though B will care less, and therefore both will prefer to locate at m rather than at any other node. As pointed out earlier, both types locating with probability 1 at m cannot be an equilibrium.

(ii) There is a unique node m maximizing decay centrality but it does not maximize effective degree centrality. Now B will be better off not locating at m for ε small enough. If he locates at a site that has effective degree at least 1 more than m, he can get some additional payoff. If he locates at m, he loses at least 1 for sure and obtains some additional payoff depending on ε and δ. For ϵ small enough, this is not a best response for B. In this case, the analysis from the firm's point of view will not change from that discussed earlier in this paper. Hence, an EDE will exist under the same conditions as before.

7.4 Implants can be chosen after innovators have moved

We have assumed that the firm had to choose whether to have an implant or not in period 1, before innovators moved. However, if the firm were to observe innovators trying the product, the type G firm might wish to wait to see the configuration of innovators first before deciding whether to use an implant and where to place the implant. Suppose the realisation of the random configuration of innovators is denoted by C. Then, given C, G will choose a site i to locate an implant or might decide not to have an implant, if he has decided to wait. Let the optimal site for an implant be $i(C)$. If there is more than one, suppose one is chosen

[24] Alternatively, suppose type is identified with quality of the product as before, but consumers who buy the bad quality product make a "mistake" with small probability - they make a positive recommendation with probability ϵ.

randomly. Type $G's$ optimal choice as a function of the random configuration of innovators then induces a probability distribution over possible sites, which we denote by $\alpha(i)$. Note that $\Sigma_i \alpha(i)$ could be strictly less than 1.

If the value of the information about innovators is sufficiently large compared with the cost (given by discounting) of waiting one period (it is clear that the firm of type G will not wait longer than one period), what will the effect be on the existence of complete diffusion of the good product? Suppose that an EDE exists in the model where implants have to be chosen before innovators move. Now G waits until period 2. For diffusion with probability 1 to take place, there must be a sequence of recommendations that is accepted with probability 1 starting from some node. If B also waits until period 2, in equilibrium, to take advantage of $G's$ period 2 entry, any recommendation in period 1 must come from an innovator. Therefore $B's$ implant should deviate to speaking in period 1 and thereby getting a payoff that is not discounted by δ.

Therefore, suppose that B moves in period 1 and G in period 2. Then, for a period 1 recommendation to be credible, the most weight B can put on any site is limited by the probability of an innovator there, whilst in the original problem, B could put more weight on each site by free riding off both innovators and $G's$ implant. Thus, even if an EDE were to exist in the original model, it need not in this modified model, if the equilibrium is of this nature.

Therefore, suppose that the equilibrium has B randomizing between entering in period 1 and period 2. Then, if recommendations get accepted with probability 1 in period 2, $B's$ expected payoff in period 1 must be δ times the period 2 expected payoff. This means recommendations in period 1 have to be rejected with positive probability. Given that each node speaks only once, now an equilibrium with complete diffusion with probability 1 will not exist.

8 Conclusions

In this paper, we have explored the implications for diffusion of a product or a technology to a network of rational consumers and "innovators", where the seller of the product or technology has private information about its quality. Consumers are aware that firms may "seed" the network, and also know that both "good" and "bad" quality firms may do so. As a result, agents cannot take recommendations from their social neighbors at face value - the credibility of recommendations has to be evaluated with beliefs updated using Bayes' Rule. Within this framework, we show that a priori notions about what network structure is conducive to efficient diffusion may be misleading. In particular, "small" networks and highly-connected agents may actually impede the diffusion of the good product. The requirement of credibility of learning has bite; these results would not hold in a model without rationality. Also the entire structure of the network is important in determining whether efficient diffusion is possible or not, though the nodes with the highest degree and those that maximize "decay centrality", a notion of centrality that takes into account whether an agent is connected to other agents who are themselves central, play special roles in the equilibrium pattern of implants and diffusion.

References

[1] Bala, V. and S. Goyal, (1998) "Learning from Neighbours", *Review of Economic Studies,* **65**, 3, 595-621

[2] Chatterjee, K. and S.Hong Xu, (2004)**,** 'Technology Diffusion by Learning from Neighbours'**,** *Adv. in Appl. Probab.*, **36**, 2, 355–376

[3] Coleman, J.S., E.Katz and H.Menzel (1966), *Medical Innovation: A Diffusion Study*, Indianopolis, Indiana, Bobbs-Merrill.

[4] Conley, T.G., and C. Udry, (2010) "Learning About a New Technology: Pineapple in Ghana", *American Economic Review*, **100**, 1, 35-69.

[5] Draief, Moez and Laurent Massoulie (2010), *Epidemics and Rumours in Complex Networks,* London Mathematical Society Lecture Notes no. 369, Cambridge University Press, Cambridge, UK.

[6] Durrett, Rick T. (2007), *Random Graph Dynamics,* Cambridge University Press, New York and Cambridge.

[7] Ellison, Glenn, (1993), "Learning, Local Interaction and Coordination", *Econometrica*, **71**, 1047-1071.

[8] Ellison, Glenn and Drew Fudenberg(1993), "Rules of Thumb for Social Learning", *Journal of Political Economy*, **101**, 612-644.

[9] Ellison, Glenn and Drew Fudenberg(1995), "Word-of-mouth Communication and Social Learning", *Quarterly Journal of Economics*, **109**, 93-125.

[10] Foster, A.D. and M.Rosenzweig (1995), "Learning by Doing and Learning from Others: Human Capital and Technological Change in Agriculture", *Journal of Political Economy*, **103**, 1176-1209.

[11] Galeotti, A. and S. Goyal, (2009), "Influencing the Influencers: a Theory of Strategic Diffusion", *RAND Journal of Economics,* , **40**, 3, 509-532.

[12] Goyal, S., (2005), "Learning in Networks", in G.Demange and M.Wooders (ed) *Group Formation in Economics*, Cambridge, Cambridge University Press.

[13] Goyal, S. and A. Vigier (2010), "Robust Networks", mimeo, University of Cambridge, Faculty of Economics.

[14] Jackson, M. and L. Yaariv (2010), "Diffusion, Strategic Interaction and Social Structure", to be published in *Handbook of Social Economics*, edited by J.Benhabib, A.Bisin and M. Jackson.

[15] Kempe, D., Kleinberg, J., and Tardos, E. (2003), "Maximizing the Spread of Influence in a Social Network", *Proceeding of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining.*

[16] Leskovec, J., Adamic, L.A. and Huberman B.A. (2007), "The Dynamics of Viral Marketing", *ACM Transactions on the Web* **1**, articl 5.

[17] Munshi, K., (2004), "Social Learning in a Heterogeneous Population: Technology Diffusion in the Indian Green Revolution," *Journal of Development Economics*, **73** ,1, 185-213.

[18] Rogers, E. (2003), *Diffusion of Innovations*, New York: Free Press.

[19] Richardson,M., and P.Domingos (2002) "Mining Knowledge-Sharing and Sites for Viral Marketing", *Proceeding of the Eighth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining.*

[20] Silverman, G (2001), *The Secrets of Word-of-Mouth Marketing: How to Trigger Exponential Sales through Runaway Word of Mouth*, New York, Amacom books.

[21] Conley,T and C. Udry. (2009) "Learning about a New Technology", to appear in *American Economioc Review.*

[22] Walker, Rob, (2004) "The Hidden (in Plain Sight) Persuaders", *The New York Times Sunday Magazine,* December 5.

[23] Young, H. Peyton (2009), "Innovation Diffusion in Heterogeneous Populations; Contagion, Social Influence and Social Learning", *American Economic Review,* **99**, 1899-1924.