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Pablo F. Beker and Emilio Espino

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Too Good to Be True: Asset Pricing Implications of Pessimism*

Pablo F. BEKER†
University of Warwick

Emilio ESPINO‡
Universidad Torcuato Di Tella

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Abstract

We evaluate whether the introduction of pessimistic homogeneous beliefs in the frictionless Lucas-Mehra-Prescott model and the Kehoe-Levine-Alvarez-Jermann model with endogenous borrowing constraints, helps explain the equity premium, the risk-free rate and the equity volatility puzzles as well as the short-term momentum and long-term reversal of excess returns. We calibrate the model to U.S. data as in Alvarez and Jermann [4] and we find that the data does not contradict the qualitative predictions of the models. When the preferences parameters are disciplined to match both the average annual risk-free rate and equity premium, the Lucas-Mehra-Prescott model gives a more quantitatively accurate explanation for short-term momentum than the Kehoe-Levine-Alvarez-Jermann model but the latter gives a more quantitatively accurate explanation for the equity volatility puzzle. Long-term reversal remains quantitatively unexplained in both models.

Keywords: Financial Markets Anomalies, Pessimism, Homogeneous Beliefs, Limited Enforceability, Endogenously Incomplete Markets.

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†Department of Economics. University of Warwick, Coventry, CV4 7AL, UK. Email: Pablo.Beker@warwick.ac.uk

‡Department of Economics, Saenz Valiente 1010 (C1428BIJ), Buenos Aires, Argentina. E-mail: eespino@utdt.edu

1 Introduction

The time series of asset returns in the US exhibits several features that are not easy to reconcile with the predictions of standard asset pricing models. Chief among them are the *early puzzles*, the risk-free rate, the equity premium and the equity volatility ones introduced by Weil [25] and Mehra and Prescott [20], and the *more recent anomalies* that suggest that asset returns can be predicted using public information, like *short-term momentum* or positive autocorrelation of asset returns in the short to medium run and *long-term reversal* or negative autocorrelation in the long run (see Moskowitz et al [19], Poterba and Summers [21] and Lo and MacKinlay [17]).

The difficulties of the standard consumption-based asset pricing model in explaining these features have challenged the profession to find new equilibrium models that generate time-series of stocks and bonds returns with the aforementioned properties. One strand of the literature studies the consequences of departing from the assumption that agents have homogeneous and correct beliefs. Most of the attention of this literature has been focused in explaining the early puzzles while substantially less attention has been paid to the more recent anomalies. Indeed, Abel [1] and Cogley and Sargent [11] show that introducing pessimism in a representative agent model, can help explain the early puzzles but they are silent about the more recent anomalies.

This paper evaluates the ability of a representative agent model with pessimistic beliefs, as in Abel [1] or Cogley and Sargent [11], to explain not only the early puzzles but also the more recent anomalies. We consider a pure exchange economy where the state of nature follows a finite first-order time homogeneous Markov process. Agents have dogmatic beliefs that are pessimistic about the persistency of the expansion state and correct otherwise.[1]

We evaluate the impact of this class of pessimism in an economy without frictions, the Lucas [18] tree model adapted to allow for stochastic growth as in Mehra and Prescott [20], as well as in an economy where credit frictions arise due to limited enforceability, the Kehoe-Levine [15] and Alvarez-Jermann [2] model. We ask how much of the early puzzles and more recent anomalies pessimism can explain.

We calibrate the stochastic process of individual income and aggregate growth rates of a two-agent economy to aggregate and US household data as in Alvarez and Jermann [3]. We use Mehra and Prescott [20] annual US data for the period 1900-2000 to compute the risk-free rate and the equity premium which equals 0.8% and 6.18%, respectively, and the 1st and 2nd order annual autocorrelations for the US stock market which equals to 13.94% and -15.28% respectively.

First, we show that the Lucas-Mehra-Prescott model extended to allow for pessimistic agents does a good job in explaining the early risk-free rate and equity premium puzzles as well as short-term momentum. We calibrate the parameters following two alternatives. In the first we fix the discount factor and the coefficient of relative of risk aversion at standard values, for instance as in Cogley and Sargent [12]. In that setting predictions are qualitatively correct even if agents have correct beliefs.

[1] Although our assumption of dogmatic priors is extreme, our model offers the same conclusions as one in which agents learn and have date-0 homogeneous non-dogmatic priors without the true distribution in their support.

In fact, pessimism not only can explain the equity premium and risk-free puzzles but also yields striking large values of short term momentum. Indeed, when the degree of pessimism is set so that the predicted levels of short-term momentum are similar to the US data, the model makes a significant progress regarding the early puzzles as the equity premium is 4.8% while the risk-free interest rate is slightly larger than 1%. The main failure is the predicted levels of long-term reversal, approximately an order of magnitude below the data. In the second exercise we calibrate the discount rate and the coefficient of risk aversion to match the average annual risk-free rate of 0.8% and equity premium of 6.18% and we find that mild degrees of pessimism are enough to match the level of short-term momentum observed in the data and to generate long-term reversal.[2] However its predicted levels of equity volatility and long-term reversal are one order of magnitude away form the US data. We conclude that, contrary to the conventional wisdom, the standard frictionless Lucas-Mehra-Prescott model with pessimistic beliefs can help resolve the risk-free and equity premium (early) puzzles and one of the more recent anomalies, i.e., short-term momentum. Its predictions regarding long-term reversal are inaccurate only in a quantitative sense.

Second, we evaluate the ability of the Kehoe-Levine-Alvarez-Jermann model to explain the early puzzles and the recent anomalies at the same time. We set the discount rate and coefficient of risk aversion to match the average annual risk-free rate of 0.8% and equity premium of 6.18%. On the one hand, the model predicts levels of short-term momentum that can be *at most* half of those in the US data and levels of long-term reversal that are more than one order of magnitude away from the US data.[3] On the other hand, the predicted levels of equity volatility are of the same order of magnitude as in the US data. We conclude that the Kehoe-Levine-Alvarez-Jermann model with pessimistic beliefs can yield predictions that are qualitative accurate as well. Compared to the Lucas-Mehra-Prescott model it gives a more accurate quantitative explanation for equity volatility at the cost of a less accurate quantitative explanation for short-term momentum.

Related Literature. The idea that distorted beliefs can help explain the behaviour of asset prices has been used before in quantitative studies of the standard frictionless Lucas-Mehra-Prescott model. Most of the studies have concentrated on three types of distortions: pessimism (i.e., agents' beliefs are first-order stochastically dominated by the true distribution), doubt (i.e., agents' beliefs are a mean-preserving spread of the true distribution) and learning (i.e., agents update their prior as time and uncertainty unfold). To isolate the effect of pessimism, some authors assume that agents have priors that place probability one on a single model, i.e., they are dogmatic. Since these agents' beliefs do not change as they observe new data, the belief distortion is constant over time. Abel (2002) combines pessimism and doubt in an economy with a dogmatic representative agent. Cogley and Sargent [11] also consider a representative agent economy but they combine pessimism and learning. They focus on the quantitative effects on the equity premium and the risk-free rate but neither of them tackle the effect of pessimism on the autocorrelations of excess returns. Adams, Marcet and Nicolini [4] study

[2]This exercise is more controversial since the discount factor can be larger than 1. See the discussion in Section 7.

[3]For this economy we also follow the strategy of calibrating the parameters at standard values but the results are basically the same.

the asset prices implications of learning when agents have identical doubtful prior on the stock price growth rate and learn as time and uncertainty unfold. They show that learning is the key driving force to make price-dividend ratios to overreact in the short-run (their notion of short-term momentum) and mean-revert in the long run (their notion of long-term reversal). The quantitative predictions of their model regarding these two features fit well their empirical counterpart while they fall short to account for the high equity premium and the low risk free rate. Cecchetti et al [10] combine pessimism and learning to explain several anomalies in a representative agent economy, including long-term reversal, but they are silent about short-term momentum. In their model the stochastic discount factor has a non stationary behaviour because they assume not only that the representative agent is pessimist but also that she believes the endowment growth follows a peculiar non-stationary process. In our model, instead, the agents correctly believe the true process is stationary and non-stationarity of the stochastic discount factor plays no key role. Our results, instead, stem from the the effect of pessimism and/or changes in the wealth distribution on the stochastic discount factor.

Cogley and Sargent [12] combine pessimism, learning and belief heterogeneity but they focus only on their effect on the market price of risk on a finite sample. Although the pessimistic agent ends up learning, they show that, for a plausible calibration of their model, it takes a long time for the effect of large pessimism on CE asset prices to be erased unless the agents with correct beliefs own a large fraction of the initial wealth.

Much less is known on the implications of distorted beliefs in economies with credit market frictions. Cao [9], Tsyrennikov [24] and Cogley et al [13] combine distorted beliefs, learning and credit frictions to study the dynamics of asset prices. Cao [9] studies the case of heterogenous beliefs to focus on survival and excess volatility of asset prices. Tsyrennikov [24] provides a numerical analysis of the effect of belief heterogeneity on the volatility of asset returns. Cogley et al focus on the wealth dynamics of a bond economy when solvency constraints are exogenously given and proportional to the agents' income. Beker and Espino [6] show that when credit frictions arise due to limited enforceability, dogmatic pessimism and belief heterogeneity together give a quantitative relevant explanation not only for the early puzzles but also for short-term momentum and long-term reversal.

The presentation is organised as follows. Section 2 describes the economy. Section 3 introduces the recursive formulation of constrained Pareto optimal allocations. Section 4 introduces the concept of competitive equilibrium and discusses decentralisation. Section 5 provides a statistical and economic characterisation of the early puzzles and the more recent anomalies. Section 6 describes the relationship between momentum and reversal. Section 7 presents the quantitative implications of pessimism. Section 8 concludes. Proofs are gathered in the Appendix.

2 The Model

We consider a one-good infinite horizon pure exchange stochastic economy. In this section we establish the basic notation and describe the main assumptions.

2.1 The Environment

Time is discrete and indexed by $t = 0, 1, 2, ...$ The set of states of nature is $S \equiv \{1, 2, 3, 4\}$. The state of nature at date zero is known and denoted by $s_0 \in S$. The set of partial histories up to date t is S^t, where $t \geq 1$, and a typical element is $s^t = (s_1, ..., s_t)$. S^∞ is the set of infinite sequences of the states of nature, $s = (s_1, s_2, \cdots)$, called a path, is a typical element and $\mathcal{F}_t$ is the $\sigma-$algebra generated by all the finite histories of length t.

We say that $\pi : S \times S \to (0, 1)$ is a transition probability matrix if $\pi(\cdot\,|\xi\,) > 0$ and $\sum_{\xi'} \pi(\xi'\,|\xi\,) = 1$ for all $\xi \in S$. If $\{s_t\}$ follows a first-order time-homogeneous Markov process with transition probability matrix π, then P^π denotes the probability measure on $(S^\infty, \mathcal{F})$ uniquely induced by π.

The following assumption will be used for the characterisation of the dynamics in Sections 5-7 where we need to be explicit about the true data generating process (henceforth, dgp).

A.0 The true dgp is given by P^{π^*} for some transition probability matrix π^*.

Definition. *A state of nature ξ is strongly persistent if $\pi^*(\xi|\,\xi) \geq \psi^*(\xi)$, where ψ^* is the invariant distribution associated with π^*.*

There is a single perishable consumption good every period. The economy is populated by two (types of) infinitely-lived agents where $i \in \{1, 2\}$ denotes an agent's name. A consumption plan is a sequence $\{c_t\}_{t=0}^\infty$ such that $c_0 \in \mathbb{R}_+$ and $c_t : S^\infty \mapsto \mathbb{R}_+$ is $\mathcal{F}_t-$measurable for all $t \geq 1$ and $\sup_{(t,s)} c_t < \infty$. Given s_0, the agent's consumption set, $\mathbb{C}(s_0)$, is the set of all consumption plans.

2.2 Beliefs, Preferences and Endowment

P_i denotes agent i's prior.[4] Throughout this paper, we assume that for each agent i and for each set of paths A

$$P_i(A) = P^\pi(A).$$

Agents' preferences are represented by

$$U^{P^\pi}(c_i) = E^{P^\pi}\left(\sum_{t=0}^{\infty} \beta^t u(c_{i,t})\right) \text{ for all } c_i \in \mathbb{C}(s_0),$$

where $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $\beta > 0$.

The aggregate endowment at date t is denoted by $y_t : S^\infty \mapsto \mathbb{R}$ and given by

$$y_t = g(s_t)\; y_{t-1} \tag{1}$$

[4] P_i is a probability measure on $(S^\infty, \mathcal{F})$

where $g(\xi')$ is the endowment growth rate in state of nature ξ', a two-point Markov Process satisfying $0 < g(1) = g(3) < 1 < g(2) = g(4)$, i.e. states 1 and 3 are the low growth (aggregate) states while 2 and 4 are the high growth (aggregate) states. The sequence of aggregate endowment is denoted by $y \equiv \{y_t\}_{t=0}^{\infty}$.

We restrict the set of preferences to consider β and π (and consequently P^π) such that[5]

$$U^{P^\pi}(y) = E^{P^\pi}\left(\sum_{t=0}^{\infty} \beta^t u(y_t)\right) < \infty. \tag{2}$$

Agents 1 and 2's endowment at date t are

$$y_{1,t} = \epsilon(s_t)\ y_t \qquad \text{and} \qquad y_{2,t} = (1 - \epsilon(s_t))\ y_t,$$

where $\epsilon(\xi')$ is agent 1's endowment share in state of nature ξ', a two-point Markov process satisfying $0 < \epsilon(1) = \epsilon(2) < \epsilon(3) = \epsilon(4) < 1$, i.e., states 1 and 2 are the (idiosyncratic) states where agent 1 is rich and states 3 and 4 are the (idiosyncratic) states where agent 2 is rich.

2.3 Feasibility, Enforceability and Constrained Optimality

Given a consumption plan $c_i \in \mathbb{C}(s_0)$, define for all t and s^t:

$$\begin{aligned} U(c_i)(s^t) &= u_i(c_{i,t}) + \beta \sum_{\xi'} \pi(\xi' \,|s_t\,) U(c_i)(s^t, \xi'), \\ U(y_i)(s^t) &= u_i(y_{i,t}) + \beta \sum_{\xi'} \pi(\xi' \,|s_t\,) U(y_i)(s^t, \xi') \end{aligned}$$

Note that agent i's autarky utility level, $U(y_i)(s^t)$, can be expressed as a function only of i, s_t and y_t. Therefore, in what follows, we simply write $U_i(s_t, y_t)$.

Let $Y(s_0)$ be the set of feasible allocations. Given s_0, a feasible allocation $\{c_1, c_2\}$ is *enforceable* if $U(c_i)(s^t) \geq U_i(s_t, y_t)$ for all t, s^t and i. Let $Y^E(s_0) \subset Y(s_0)$ be the set of enforceable allocations. A feasible allocation $\{c_1, c_2\}$ is *Pareto optimal (PO)* if there is no alternative feasible allocation $\{\tilde{c}_1, \tilde{c}_2\}$ such that $U^{P^\pi}(\tilde{c}_i) > U^{P^\pi}(c_i)$ for all $i \in \{1, 2\}$. An enforceable allocation $\{c_1, c_2\}$ is *constrained Pareto optimal (CPO) given* s_0 if there is no alternative enforceable allocation $\{\tilde{c}_1, \tilde{c}_2\}$ such that $U^{P^\pi}(\tilde{c}_i) > U^{P^\pi}(c_i)$ for all $i \in \{1, 2\}$.

Given s_0, define the *utility possibility correspondence* by

$$\mathcal{U}(s_0) = \{\tilde{u} \in \mathbb{R}^2 : \exists\ \{c_1, c_2\} \in Y(s_0)\,,\ \tilde{u}_i \leq U^{P^\pi}(c_i) \quad \forall i\},$$

and the *enforceable utility possibility correspondence* by

$$\mathcal{U}^E(s_0, y_0) = \{\tilde{u} \in \mathbb{R}^2 : \exists\ \{c_1, c_2\} \in Y^E(s_0),\ U_i(s_0, y_0) \leq \tilde{u}_i \leq U^{P^\pi}(c_i) \quad \forall i\}.$$

Given s_0, the set of CPO allocations can be characterised as the solution to the following planner's problem with welfare weight $\alpha \in [0, 1]$:

$$v^*(s_0, \alpha) \equiv \sup_{\{c_1, c_2\} \in Y^E(s_0)} \quad \alpha\ E^{P^\pi}\left(\sum_t \rho_t\ u_1(c_{1,t})\right) + (1 - \alpha)\ E^{P^\pi}\left(\sum_t \rho_t\ u_2(c_{2,t})\right). \tag{3}$$

[5]This restriction plays the role of condition 5.b in Kocherlakota [16]

3 Recursive Planner's Problem

In this section we show how to characterise recursively the set of constrained Pareto optimal allocations. We first solve a related planner's problem in which the variables are detrended and later we use that solution to construct the CPO allocations of the economy with growth.[6]

We define consumption and income shares as $\hat{c}_i = c_i/y$ and $\hat{y}_i = y_i/y$ for all i, where y denotes the current period aggregate endowment.[7] We define probabilities and stochastic discount factors as

$$\widehat{\pi}(\xi' \,|\xi\,) = \frac{\pi(\xi' \,|\xi\,)g(\xi')^{1-\sigma}}{\sum_{\tilde{\xi}} \pi(\xi' \,|\xi\,)g(\tilde{\xi})^{1-\sigma}} \quad \text{and} \quad \hat{\beta}(\xi) = \beta \sum_{\xi'} \pi(\xi' \,|\xi\,)g(\xi')^{1-\sigma}, \text{for all } \xi.$$

Beker and Espino [6] shows that the CPO consumption shares can be recursively characterised as the solution to the following problem with welfare weight $\alpha \in [0, 1]$

$$v^*(\xi, \alpha) = \max_{(\hat{c}, \hat{w}'(\xi'))} \alpha \left(u(\hat{c}_1) + \hat{\beta}(\xi) \sum_{\xi'} \hat{\pi}(\xi' \,|\xi\,)\hat{w}_1'(\xi') \right) + (1-\alpha) \left(u(\hat{c}_2) + \hat{\beta}(\xi) \sum_{\xi'} \hat{\pi}(\xi' \,|\xi\,)\hat{w}_2'(\xi') \right) \quad (4)$$

subject to

$$\hat{c}_i \geq 0 \textit{ for all } i, \qquad \hat{c}_1 + \hat{c}_2 = 1 \quad (5)$$

$$u(\widehat{c}_i) + \beta(\xi) \sum\nolimits_{\xi'} \widehat{\pi}(\xi' \,|\xi\,) \; \hat{w}_i'(\xi') \geq U_i(\xi, 1) \textit{ for all } i, \quad (6)$$

$$\widehat{w}'(\xi') \in \mathcal{U}^E(\xi'), \quad (7)$$

Let $(\widehat{c}, \; \widehat{w}'(\xi')) : S \times [0, 1] \mapsto \mathbb{R}^2$ be the policy functions solving the problem above and let $\alpha'(\xi') : S \times \; [0, 1] \mapsto \; [0, 1]$ be the next period welfare weight associated with $\widehat{w}'(\xi')$.

Given (s_0, α_0), the policy functions $(\widehat{c}, \alpha')$ generate an allocation $\{c_t\}_{t=0}^{\infty} \in \mathbb{C}(s_0) \times \mathbb{C}(s_0)$ if

$$\begin{aligned} c_{i,t} &= \widehat{c}_i(s_t, \alpha_t) y_t && \textit{for all } i, \; t \geq 0, \\ \alpha_{t+1} &= \alpha'(s_t, \alpha_t)(s_{t+1}) && \textit{for all } i, \; t \geq 0. \end{aligned}$$

Theorem 1 and Proposition 2 in Beker and Espino [6] shows that PO and CPO allocations are generated by the policy functions $\widehat{c}_i(\xi, \alpha)$, $\alpha'_{po}(\xi, \alpha)(\xi')$ and $\alpha'_{cpo}(\xi, \alpha)(\xi')$ where

$$\begin{aligned} \widehat{c}_1(\xi, \alpha) &= \frac{\alpha^{\frac{1}{\sigma}}}{\alpha^{\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}}, \qquad \widehat{c}_2(\xi, \alpha) = \frac{(1-\alpha)^{\frac{1}{\sigma}}}{\alpha^{\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}}, \\ \alpha'_{po}(\xi, \alpha)(\xi') &= \alpha, \\ \alpha'_{cpo}(\xi, \alpha)(\xi') &= \begin{cases} \underline{\alpha}(\xi') & \text{if } \alpha'_{cpo}(\xi, \alpha)(\xi') < \underline{\alpha}(\xi') \\ 1 - \underline{\alpha}(\xi') & \text{if } \alpha'_{cpo}(\xi, \alpha)(\xi') > 1 - \underline{\alpha}(\xi') \\ \alpha & \text{otherwise} \end{cases} . \end{aligned}$$

where $\underline{\alpha}(\xi')$ and $1 - \underline{\alpha}(\xi')$ are the welfare weights associated to the lifetime utility that agents 1 attains at his autarky level (the minimum enforceable weight) and at agent 2's autarky level, respectively, in state ξ'.[8]

[6] A strategy first developed by Alvarez and Jermann [3].

[7] In this section we abuse notation and let y to be a scalar and c to be a non-negative vector and c_i its i^{th} component.

[8] Note that symmetry among agents implies that the minimum enforceable weights of agents 1 and 2 in states 1(2) and 3(4), respectively, are identical.

The idea behind these laws of motion can be intuitively explained as follows. If allocations are PO, the law of motion of the welfare weights requires the next period welfare weight to be equal to the current one. This is because the planner's relative valuation of the agent's expected utility does not change as time and uncertainty unfold. If allocations are CPO, the case analysed by Alvarez and Jermann [2], the law of motion requires the next period welfare weight to be equal to the current one unless that conflicts with the need to provide incentives to avoid the agent to revert to autarky, i.e. there is some state of nature for which the current welfare weight does not lie in the interval $[\underline{\alpha}(\xi'), 1-\underline{\alpha}(\xi')]$. In that case, the planner's relative valuation of the agents' expected utility changes because it takes into account the value of avoiding the agent revert to autarky. See Beker and Espino [6] for a thorough explanation.

Invariant Distribution. In section 5 we characterise different moments of asset returns. As we argue later, asset returns depend only on consumption shares and growth rates. Thus, to understand the limit distribution of asset returns, one needs to characterise the limit distribution of welfare weights.

For PO allocations let $\Omega \equiv S \times [0,1]$ be the state space. For CPO allocations, let $\Omega \equiv \{(\xi,\alpha) \in S \times [0,1] : \underline{\alpha}(\xi) \leq \alpha \leq 1-\underline{\alpha}(\xi)\}$ be the state space. In each case $\mathcal{G}$ denotes the $\sigma-$algebra on Ω. The law of motion for the welfare weight, α'_e for $e \in \{po, cpo\}$, coupled with π^* define a time-homogeneous transition function, $F_e : \Omega \times \mathcal{G} \to [0,1]$, given by

$$F_e\left[(\xi,\alpha), \mathcal{S}\times\mathcal{A}\right] = \sum_{\xi'\in\mathcal{S},\ \alpha'_e(\xi,\alpha)(\xi')\in\mathcal{A}} \pi^*\left(\xi'\,|\xi\right) \text{ for all } (\mathcal{S}\times\mathcal{A}) \in \mathcal{G}$$

We use standard arguments to show that there is a unique invariant measure on $(\Omega,\mathcal{G})$ and that the distribution of states converges weakly to that measure.

Proposition 1. *Suppose A.0 holds. Then there exists a unique invariant measure $\psi_e : \mathcal{G} \to [0,1]$. Moreover, ψ_e is globally stable, ψ_{po} is degenerate and ψ_{cpo} is non-degenerate.*

4 Competitive Equilibrium with Solvency Constraints

In this section we define a competitive equilibrium with solvency constraints (CESC). In Section 4.1 we show that CPO allocations can be decentralised as CESC.

Every period t, after observing s^t, agents trade both the consumption good and a complete set of Arrow securities in competitive markets. Security ξ' issued at date t pays one unit of consumption if next period's state of nature is ξ' and 0 otherwise. We denote by $q_t^{\xi'}$ and $a_{i,t}^{\xi'}$ the price of Arrow security ξ' and agent i's asset holdings, respectively, at date t. Let $a_{i,-1}^{\xi'} = 0$ for all ξ', $a_{i,t} = \left\{a_{i,t}^{\xi}\right\}_{\xi\in S}$ and $a_i \equiv \{a_{i,t-1}\}_{t=0}^{\infty}$ for all i. Prices are in units of the date–t consumption good, $q_t = \left\{q_t^{\xi}\right\}_{\xi\in S}$ and a price system is given by $q \equiv \{q_t\}_{t=0}^{\infty}$. Agent i faces a state contingent solvency constraint, $B_{i,t}^{\xi'}$, that limits security ξ' holdings at date t. Let $B_{i,t} \equiv \left\{B_{i,t}^{\xi'}\right\}_{\xi'}$ and $B_i \equiv \{B_{i,t}\}_{t=0}^{\infty}$ for all i.

Given q and B_i, *agent i's problem* is

$$\max_{(c_i,a_i)} \quad E^{P^\pi}\left(\sum_{t=0}^{\infty}\beta^t\ u_i(c_{i,t})\right)$$

$$\text{s.t.} \begin{cases} c_{i,t} + \sum_{\xi'} q_t^{\xi'}\ a_{i,t}^{\xi'} = y_{i,t} + a_{i,t-1}^{s_t} & \text{for all s and t.} \\ c_{i,t} \geq 0,\ a_{i,-1} = 0,\ a_{i,t}^{\xi'} \geq B_{i,t}^{\xi'} & \text{for all } \xi'\text{, s and t.} \end{cases}$$

Markets clear if

$$\begin{aligned} c_{1,t} + c_{2,t} &= y_t \quad \text{for all t.} \\ a_{1,t}^{\xi'} + a_{2,t}^{\xi'} &= 0 \quad \text{for all } \xi' \text{ and t.} \end{aligned}$$

Definition. *A competitive equilibrium with solvency constraints (CESC) is an allocation* $\{c_1, c_2\}$, portfolios $\{a_1, a_2\}$*, a price system* q *and solvency constraints* $\{B_1, B_2\}$ *such that:*
(CESC 1) Given q *and* B_i, (c_i, a_i) *solves agent i's problem for all i.*
(CESC 2) Markets clear.

Of course, a CESC need not be CPO (see Bloise et al [7]). In what follows, however, when we refer to CESC we always mean a CESC that is CPO. A Competitive Equilibrium (CE, hereafter) is a CESC in which the corresponding allocation is PO.

4.1 Decentralisation

In this section we show how to decentralise PO and CPO allocations as CE and CESC, respectively, using the method developed by Beker and Espino [6].

Let $A_{i,e}(\xi,\alpha)$ be the unique solution to the functional equation

$$A_{i,e}(\xi,\alpha) = \widehat{c}_i(\xi,\alpha) - \widehat{y}_i(\xi) + \sum_{\xi'} \hat{Q}_e(\xi,\alpha)(\xi')\ A_{i,e}(\xi',\alpha_e'), \tag{8}$$

where

$$\hat{Q}_e(\xi,\alpha)(\xi') = \max_h \left\{ \hat{\beta}(\xi)\ \widehat{\pi}(\xi'\,|\xi)\ \left(\frac{\widehat{c}_h(\xi',\alpha_e'(\xi,\alpha)(\xi'))}{\widehat{c}_h(\xi,\alpha)}\right)^{-\sigma} \right\},$$

Expression (8) computes recursively the present discounted value of agent i's excess demand at the allocation priced by the implicit state price $\hat{Q}_e(\xi,\alpha,\mu)(\xi')$. Define $R_e^F(\xi,\alpha) = \left(\sum_{\xi'} \hat{Q}_e(\xi,\alpha)(\xi')g(\xi')^{-1}\right)^{-1}$ as the (implicit) equilibrium interest rate.

Suppose that *equilibrium interest rates are positive;* i.e. $R_e^F(\xi,\alpha) > 1$ for all (ξ,α). For this case, Theorem 4 in Beker and Espino [5], for PO allocations, and Theorem 6 in Beker and Espino [6], for CPO allocations, shows that there exists $\alpha_e(s_0) \in [0,1]$ such that for every s, t and ξ',

$$a_{i,t}^{\xi'} = A_{i,e}\left(\xi',\alpha_e'(s_t,\alpha_t)\,(\xi')\right) y(\xi)g(\xi') \tag{9}$$

$$q_t^{\xi'} = \hat{Q}_e(s_t,\alpha_t)(\xi')g(\xi')^{-1} \tag{10}$$

$$B_{i,t}^{\xi'} = A_{i,e}\left(\xi',\alpha_e'(s_t,\alpha_t)\,(\xi')\right) y(\xi)g(\xi'), \tag{11}$$

where α_t is generated by $\alpha_e'(\xi,\alpha)$ and $\alpha_0 = \alpha_e(s_0)$.

5 On Puzzles and Anomalies

In Section 5.1 we introduce a formal definition of short-term momentum and long-term reversal in terms of the empirical autocorrelations of the equity excess returns. In Section 5.2, we argue that in any CE or CESC, the empirical autocorrelations can be approximated using the population autocorrelations. In Section 5.3 we explain why pessimism helps explain the early puzzles. In Section 5.4 we provide a characterisation of the population autocorrelations in terms of the reaction of the conditional equity-premium to the realisation of the excess returns. Finally, in Section 5.5 we reinterpret the equivalent martingale measure as a market belief. We characterise the changes of the conditional equity premium to the realisation of the excess return in terms of how market pessimism changes as the market updates its belief.

5.1 Definitions

As in Mehra and Prescott [20], we study a productive unit producing the perishable consumption good. This firm issues one equity share that is competitively traded. The firm's output is the firm's dividend payment in period t. Since only one productive unit is considered, the return on this share of equity is also the return on the market.

Let d_t, p_t and r_t^f be the dividend of the asset, its ex-dividend price and the (gross) risk-free interest rate, respectively, at date t. For $t \geq 1$, let the one-period excess rate of return (the return hereafter) be defined as

$$r_t = \frac{p_t + d_t}{p_{t-1}} - r_{t-1}^f$$

where $d_t = y_t$ for all t and s.

Let P_e^D be the unique solution to the functional equation

$$P_e^D(\xi,\alpha) \quad = \quad \sum_{\xi'} \hat{Q}_e(\xi,\alpha)(\xi') \ \left(1 + P_e^D(\xi', \alpha_e'(\xi,\alpha)(\xi'))\right) \qquad \text{for all } (\xi,\alpha)$$

Therefore, the asset return and the risk-free rate can be written as

$$\begin{aligned} r_t &= R_e^E(s_{t-1},\alpha_{t-1})(s_t) \\ r_{t-1}^f &= R^F(s_{t-1},\alpha_{t-1}) \end{aligned}$$

where

$$R_e^E(\xi,\alpha)(\xi') \equiv \frac{1 + P_e^D(\xi', \alpha_e'(\xi,\alpha)(\xi'))}{P_e^D(\xi,\alpha)} g(\xi') - R_e^F(\xi,\alpha)(\xi').$$

We imagine an econometrician who observes data on returns for T consecutive periods. Let

$$\overline{r}_T \equiv \frac{1}{T}\sum_{t=1}^{T} r_t \text{ and } \sigma_T^2 \equiv \frac{1}{T}\sum_{t=1}^{T}\left(r_t - \overline{r}_T\right)^2$$

be the empirical average and variance of the returns. Let

$$cov_{k,T} \equiv \frac{1}{T}\sum_{t=1}^{T}\left(r_t - \overline{r}_T\right)\left(r_{t+k} - \overline{r}_T\right) \text{ and } \rho_{k,T} \equiv \frac{cov_{k,T}}{\sigma_T \ \sigma_T}$$

be the empirical autocovariance and autocorrelation coefficient of order $k \geq 1$.

Now we give a formal definition of the so-called *financial markets anomalies.*

Definition. *The asset displays short-term momentum on a path if* $\lim\limits_{T\to\infty} \rho_{1,T} > 0$ *on that path. The asset displays long-term reversal on a path if* $\lim\limits_{T\to\infty} \rho_{2,T} < 0$ *on that path.*

5.2 Asymptotic Approximation

The empirical mean and autocorrelations are continuous functions of the return and (CE or $CESC$) equilibrium returns are continuous functions of a Markov process with transition function F_e, where $e \in \{po, cpo\}$. If one argues that the Markov process is ergodic with invariant distribution ψ_e, then standard arguments show that the empirical moments converge to their population counterparts. That is, the following asymptotic approximation holds, $P^{\pi^*} - a.s.$, for $\tau \in \{1, 2\}$

$$\lim_{T\to\infty} \bar{r}_T = E^{P_e}(R_{1,e}), \quad \lim_{T\to\infty} cov_{\tau,T} = cov^{P_e}(R_{1,e}, R_{\tau+1,e}) \quad \text{and} \quad \lim_{T\to\infty} \sigma_T = \sigma^{P_e}(R_{1,e}), \tag{12}$$

where $P_e \equiv P^{F_e}(\psi_e, \cdot)$ and $R_{\tau+1,e} : \Omega^\infty \to \Re$ is a $\mathcal{G}_{\tau+1}-$measurable function defined by $R_{\tau,e} \equiv R_e(\xi_\tau, \alpha_\tau)(\xi_{\tau+1})$.

Theorem 2. *Suppose A.0 holds. If allocations are either PO or CPO, then the asymptotic approximation (12) holds.*

5.3 Pessimism and The Early Puzzles

The standard beta pricing formula illustrates what determines the equity premium, $E^{P_e}(R_{1,e})$, in these economies:

$$\frac{E^{P_e}(R_{1,e})}{\tilde{R}^F_{0,e}} = -\Bigg[\underbrace{cov^{P_e}\left(\max_i\left\{\frac{\beta u'(c_{i,1})}{u'(c_{i,0})}\right\}, R_{1,e}\right)}_{risk-premium} + \underbrace{cov^{P_e}\left(\frac{\pi_1}{\pi_1^*}, \max_i\left\{\frac{\beta u'(c_{i,1})}{u'(c_{i,0})}\right\} R_{1,e}\right)}_{pessimism-premium}\Bigg] \tag{13}$$

where $\pi_1^* \equiv \pi^*(\xi_1 | \xi_0)$, and $\pi_1 \equiv \pi(\xi_1 | \xi_0)$.

Expression (13) can be used to explain how pessimism helps to reconcile the predictions of the model with the data. Suppose that an econometrician observes a time series of excess returns, $\{r_t\}_{t=0}^T$, and individual consumptions, $\{c_{i,t}\}_{t=0}^T$. She computes the sample average of the multiplicative equity-premium which by Theorem 2 converges to its unconditional mean, i.e., the term $\frac{E^{P_e}(R_{1,e})}{\tilde{R}^F_{0,e}}$ in the left-hand side of (13). Now if the econometrician wants to explain the average multiplicative equity-premium, she uses the observed consumption sequence and some assumption on $\beta u'(\cdot)$ to compute the sample covariance between the stochastic discount factor, $\max_i \left\{\frac{\beta u'(c_{i,t+1})}{u'(c_{i,t})}\right\}$, and the excess return $r_{t+1,e}$. By Theorem 2, this converges to the first term in the right-hand side of (13). This term has gained a lot of attention in the literature. Indeed, the fact that it falls short by an order of magnitude to explain the left-hand side has led Mehra and Prescott to coin the term "equity premium puzzle". However, expression (13) makes clear that if agents have homogeneous but incorrect beliefs, a second covariance arises. We call this term the pessimism-premium since it reflects the implicit extra compensation that the pessimistic agents must receive to bear a risk that they consider even riskier that what it actually is. This is the key insight introduced by Abel and Cogley and Sargent, among others, to obtain higher levels of equity premium in economies with a representative pessimistic agent. In the following section we show that the key to make sense of the more recent anomalies is to understand the effect of pessimism on the conditional equity premium.

5.4 Statistical Characterisation of Momentum and Reversal

For $\tau \geq 2$, the law of iterated expectations implies that

$$cov^{P_e}(R_{1,e}, R_{\tau,e}) = E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)\right], \tag{14}$$

where $\overline{R}_{1,e} \equiv R_{1,e} - E^{P_e}(R_{1,e})$ is the *abnormal return* and $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ denotes the $\tau-$period ahead conditional (to the $\sigma-$algebra generated by the abnormal return) equity premium. We refer to $R_{2,e}$ and $R_{3,e}$ as the short-run and long-run returns, respectively. Likewise, $E^{P_e}\left(R_{2,e}|\,\overline{R}_{1,e}\right)$ and $E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e}\right)$ are the *conditional short-run equity premium and* the *conditional long-run equity premium*, respectively.

Condition (14) makes clear that the sign of the autocovariance of order τ depends on how the conditional equity premium reacts to abnormal returns. The important question is what kind of reaction of the conditional equity premium leads to short-term momentum and long-term reversal. The following definitions will be used in Proposition 3 to provide an answer to that question.

Definition. *Let $R^+ > 0$ and $R^- < 0$. For any $\tau \geq 2$,*

- $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ *trends at* (R^+, R^-) *if* $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e} = R^+\right) > E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e} = R^-\right)$.
- $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ *reverts to the mean at* (R^+, R^-) *if the reverse inequality holds above.*
- $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ *trends (reverts to the mean) if,* $P_e - a.s$, $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ *trends (reverts to the mean) at* (R^+, R^-).

This Proposition provides a characterisation for both short-term momentum and long-term reversal that follows immediately from (14), the definition above and the Markov property of the conditional equity premium.

Proposition 3. *If the τ-period ahead conditional equity premium trends, then the $\tau-$order autocorrelation is positive. If the τ-period ahead conditional equity premium reverts to the mean, then the $\tau-$order autocorrelation is negative. That is, (i) if the conditional short-run equity premium trends, then the asset displays short-term momentum and (ii) if the conditional long-run equity premium reverts to the mean, then the asset displays long-term reversal.*

5.5 The Economics of the Conditional Short-Run Equity Premium

Note that the law of iterated expectations implies that

$$E^{P_e}\left(R_{2,e}|\,\overline{R}_{1,e}\right) = E^{P_e}\left[E^{P_e}\left(R_{2,e}|\,\mathcal{G}_1\right)|\,\overline{R}_{1,e}\right].$$

For $e \in \{po, cpo\}$, let $M_e : \mathcal{G}^\infty \to [0,1]$, be the equivalent martingale measure on $(\Omega^\infty, \mathcal{G}^\infty)$ and let $m_e : \mathcal{G} \to [0,1]$ be given by

$$m_e\left(\xi' \left|\xi, \alpha\right.\right) \equiv \frac{Q_e(\xi,\alpha)(\xi')}{\sum_{\tilde{\xi}\in S} Q_e(\xi,\alpha)(\tilde{\xi})} = R_e^F\left(\xi,\alpha\right) Q_e(\xi,\alpha)(\xi') > 0.$$

Thus, m_e can be reinterpreted as the *market belief* about the states of nature next period. Let M_e be the market belief over (infinite) paths.

The rest of this section is devoted to provide conditions on the market belief so that the conditional equity premium either trends or reverts to the mean. Note that

$$E^{P_e}\left(R_{2,e}\middle|\mathcal{G}_1\right) = E^{M_e}\left(\frac{\pi_2^*}{m_{2,e}} R_{2,e}\middle|\mathcal{G}_1\right) \tag{15}$$

where $m_{2,e} \equiv m_e\left(\xi_2\left|\xi_1, \alpha_1\right.\right)$ and $\pi_2^* \equiv \pi^*\left(\xi_2\left|\xi_1\right.\right)$. When the ratio $\frac{\pi_2^*}{m_{2,e}}$ is identically equal to one, then the short-run conditional equity premium is equal to zero. When the ratio $\frac{\pi_2^*}{m_{2,e}}$ is different from one with positive probability, typically, the short-run conditional equity premium is different from zero. For example, if the ratio $\frac{\pi_2^*}{m_{2,e}}$ is greater than one when the return is positive and smaller than one otherwise, a situation where we say the market belief is pessimistic about the return, then the short-run conditional equity premium is positive.

In the PO and CPO allocations we consider later, the sign of the return at any given period depends only on the state of nature that period. The following definition gives a useful taxonomy of states when that property holds.

Definition. *A state of nature ξ' is good news if, $\psi_e - a.s$, $R_e\left(\xi,\alpha\right)\left(\xi'\right) > 0$. A state of nature ξ' is bad news if, $\psi_e - a.s$, $R_e\left(\xi,\alpha\right)\left(\xi'\right) < 0$.*

Condition (15) makes clear that the sign of $E^{P_e}\left(R_{2,e}\middle|\mathcal{G}_1\right)$ depends on how $\frac{\pi_2^*}{m_{2,e}}$ depends on the state of nature at date 2. However, to understand trending and mean reversion what we actually need to understand is how $E^{P_e}\left(R_{2,e}\middle|\mathcal{G}_1\right)$ changes as the conditioning information changes. With that purpose in mind, we introduce the following definition.

Definition. *The market is more pessimistic at state (ξ^+, α^+) than at state (ξ^-, α^-) if for every bad news ξ',*

$$\frac{\pi^*\left(\xi'\middle|\xi^+\right)}{m_e\left(\xi'\left|\xi^+, \alpha^+\right.\right)} < \frac{\pi^*\left(\xi'\middle|\xi^-\right)}{m_e\left(\xi'\left|\xi^-, \alpha^-\right.\right)}$$

where the ratio $\frac{\pi^*\left(\xi'\middle|\xi\right)}{m_e\left(\xi'\left|\xi,\alpha\right.\right)}$ denotes the market pessimism at (ξ, α).

In the i.i.d. case, i.e. $\pi^*\left(\xi'\middle|\xi\right)$ is independent of ξ, the condition for the market belief to be more pessimistic at (ξ^+, α^+) than at (ξ^-, α^-) reduces to $m_e\left(\xi'\left|\xi^+, \alpha^+\right.\right) > m_e\left(\xi'\left|\xi^-, \alpha^-\right.\right)$.

The following definition let us move from the returns space to the state space. This is necessary because Proposition 3 applies to random variables defined on the former while $E^{P_e}\left(R_{2,e}\middle|\mathcal{G}_1\right)$ and the market belief are defined on the latter.

Definition. *State $(\tilde{\xi}, \tilde{\alpha})$ is consistent with realisation R of $\overline{R}_{1,e}$ if there exists a state (ξ, α) such that $\overline{R}_e\left(\xi,\alpha\right)(\tilde{\xi}) = R$ and $\alpha_e'(\xi,\alpha)(\tilde{\xi}) = \tilde{\alpha}$.*

To grasp the difficulty in finding a sufficient condition for trending, consider the case in which, given (ξ, α), the range of $R_e(\xi, \alpha)(\cdot)$ has only two elements, say $R_e(\xi, \alpha)(L)$ and $R_e(\xi, \alpha)(H)$ and H is good news. $E^{P_e}\left(R_{2,e}|\,\overline{R}_{1,e}\right)$ trends at (R^+, R^-) if

$$\frac{\pi^*(L|\,H)\,R_e(H,\alpha^+)(L) + \pi^*(H|\,H)\,R_e(H,\alpha^+)(H)}{\pi^*(L|\,L)\,R_e(L,\alpha^-)(L) + \pi^*(H|\,L)\,R_e(L,\alpha^-)(H)} > 1$$

for states (H, α^+) and (L, α^-) consistent with realisations R^+ and R^-, respectively, of $\overline{R}_{1,e}$. Note that even if $R_e(H,\alpha^+)(\xi') > R_e(L,\alpha^-)(\xi')$ for all $\xi' \in \{L, H\}$, mean reversion might arise if $\pi^*(L|\,H)$ is sufficiently larger than $\pi^*(L|\,L)$. So, we need to find an expression that relates trending with the changes in $\pi^*(L|\,\xi)$ and $R_e(\xi,\alpha)(L)$ induced by changes in the state (ξ, α).

Since $E^{M_e}(R_{2,e}|\,\mathcal{G}_1) = 0$, then we can write

$$R_e(\xi_2,\alpha_2)(L) = -\frac{1 - m_e(L\,|\xi_2,\alpha_2)}{m_e(L\,|\xi_2,\alpha_2)} R_e(\xi_2,\alpha_2)(H) \tag{16}$$

Therefore, using (16)

$$E^{P_e}(R_{2,e}|\,\mathcal{G}_1) = \left(1 - \frac{\pi^*(L|\,\xi_\tau)}{m_e(L\,|\xi_\tau,\alpha_\tau)}\right) R_e(\xi_2,\alpha_2)(H)$$

Thus, $E^{P_e}\left(R_{2,e}|\,\overline{R}_{1,e}\right)$ trends at (R^+, R^-) if and only if

$$\left(1 - \tfrac{\pi^*(L|H)}{m_e(L|H,\alpha^+)}\right) R_e\left(H,\alpha^+\right)(H) > \left(1 - \tfrac{\pi^*(L|L)}{m_e(L|L,\alpha^-)}\right) R_e\left(L,\alpha^-\right)(H) \tag{17}$$

for all states (H, α^+) and (L, α^-) consistent with R^+ and R^-, respectively.

Definition. *The recession bias effect at* (α^+, α^-) *is*

$$\Upsilon_e\left(\alpha^+,\alpha^-\right) \equiv \frac{\pi^*(L|\,L)}{m_e(L\,|L,\alpha^-)} - \frac{\pi^*(L|\,H)}{m_e(L\,|H,\alpha^+)}.$$

The recession bias effect will be useful to characterise the reaction of the conditional equity-premium in allocations where the market belief overestimates the likelihood of a recession at every state (ξ, α). In those allocations, the recession bias effect is positive if and only if the market is more pessimistic at the expansion state than at the recession state and it is negative otherwise.

Definition. *The return inertia effect at* (α^+, α^-) *is*

$$I_e\left(\alpha^+,\alpha^-\right) \equiv R_e\left(H,\alpha^+\right)(H) - R_e\left(L,\alpha^-\right)(H).$$

The return inertia effect is positive if and only if the highest return in the following period is larger after expansions than after recessions. It follows from (17) that the conditions below are sufficient for $E^{P_e}\left(R_{2,e}|\,\overline{R}_{1,e}\right)$ to trend at (R^+, R^-):

$$\text{(i) } \Upsilon_e\left(\alpha^+,\alpha^-\right) > 0 \qquad \text{(ii) } I_e\left(\alpha^+,\alpha^-\right) > 0, \tag{18}$$

where (H, α^+) and (L, α^-) are consistent with realisations R^+ and R^-, respectively, of $\overline{R}_{1,e}$. We refer to conditions (i) and (ii) as *positive recession bias* and *positive return inertia* effects, respectively.

6 The Close Relationship Between Momentum and Reversal

In this section we present our main results about the close relationship between short-term momentum and long-term reversal of CE and CESC allocations.

6.1 CE Allocations

Theorem 4. *Suppose A0 holds. If the asset displays short-term momentum in CE, then it also displays long-term reversal if and only if expansions are not strongly persistent.*

The intuition behind this result is simple. The welfare weight is constant and, therefore, the return of the asset depends only on the current state of nature and the growth rate. Given the current state, the return increases with the growth rate and takes only two values the following period which implies that the conditional equity premium takes only two values as well. Consequently, the converse of Proposition 3 holds and short-term momentum implies the conditional short-run equity premium trends. By the Markov property, $E^{P_e}\left(R_{3,e}|\,\overline{R}_{2,e}\right)$ also trends and so it is larger than the (unconditional) equity premium if and only if an expansion occurs in period 2. The law of iterated expectations implies that $E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e}\right) = E^{P_e}\left(E^{P_e}\left(R_{3,e}|\,\overline{R}_{2,e}\right)|\,\overline{R}_{1,e}\right)$ and so one can conclude that the conditional long run equity premium after a positive abnormal return is smaller than the equity premium if and only if the probability of transiting from expansion to expansion in one period is smaller than the unconditional probability of expansions.

6.2 CESC Allocations

Theorem 5. *Suppose A0 holds. If the asset displays short-term momentum in a CESC and* $\underline{\alpha}_1(1) = \underline{\alpha}_1(2)$*, then it displays long-term reversal if and only if the economy is not persistent.*

As in Alvarez and Jermann [3], the limit distribution of welfare weights puts positive probability only on the minimum enforceable weight of the agent who is rich in each state. Our hypothesis on the minimum enforceable weights together with the symmetry of the economy, implies that, given the current state, the welfare weight depends only on the growth rate and so it takes only two values. Therefore, the asset return can be written as function of the current state of nature and the growth rate. Given the current state, the return takes only two values and increases with the growth rate as in a PO allocation. Thus, the result holds for the same reasons it holds in Theorem 4.

7 Quantitative Implications of Pessimism

In this section we calibrate the economy to US data and evaluate qualitatively and quantitatively the ability of CE and CESC allocations to generate both the early puzzles, i.e. a low risk free rate, a high equity premium and high levels of equity volatility, and the more recent anomalies, i.e. short-term momentum and long-term reversal.

7.1 Calibration

Because of symmetry, there are 10 parameters to be selected: six for π^*, $(\epsilon(1), \epsilon(3))$ and $(g(1), g(2))$. We calibrate these 10 free parameters using the same 10 moments describing the US aggregate and household income data that Alvarez and Jermann use (see Appendix B for the calibrated parameters.)

In Table 1 we report our computations of the annual averages of the risk-free interest rate and equity-premium and also the 1^{st} and 2^{nd} order empirical annual autocorrelations for the US stock market using Mehra and Prescott [20] dataset .

Table 1

Period	*Risk-Free Rate*	*Equity-Premium*	*Equity Volatility*	$1^{st}-$ *order autocorrelation*	$2^{nd}-$ *order autocorrelation*
1900-2000	*0.8%*	*6.18%*	*16.34%*	*13.94%*	*-15.28%*

We assume the agents believe the transition function belongs to the following family of transition matrices parameterised by $\varepsilon \in (-\pi^*(1|\,2), \pi^*(2|\,2)) \approx (-0.26, 0.682)$:

$$\pi^* + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \varepsilon & -\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -\varepsilon \end{bmatrix}$$

In this parametrisation, the agents have (possibly) incorrect beliefs regarding the persistency of an expansion, i.e. $\pi(2|\,2) = \pi(4|\,4) = \pi^*(2|\,2) - \varepsilon = 0.682 - \varepsilon$ and correct beliefs otherwise. In particular, they have correct beliefs regarding the idiosyncratic state.

7.2 Standard Model I: CE Allocations

In this section we present our numerical solutions for CE.

Figure 1 plots the 1^{st} and 2^{nd}-order autocorrelations of returns, on the left hand side, together with the risk-free rate and the equity premium, on the right hand side, for $\beta = (1.03)^{-1}$ and $\sigma = 2$ as in Cogley and Sargent [12]. For very optimistic beliefs, i.e., $\pi(2|2) > 0.812$, condition (2) is violated (that is, the expected utility evaluated at the aggregate endowment does not converge). Thus, we plot our numerical results for $\varepsilon \in (-0.11, 0.682)$ or, what is the same, $\pi(2|2) \in (0, 0.812)$.

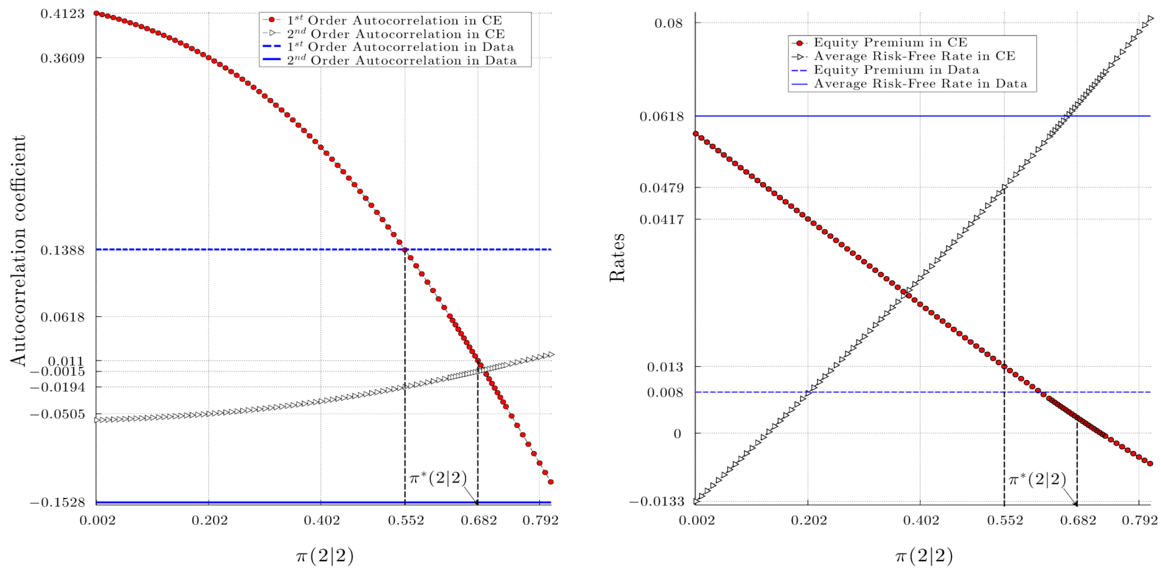

Figure 1: Autocorrelation coefficients in CE – $\beta = (1.03)^{-1}$ and $\sigma = 2$.

The right-hand side of Figure 1 illustrates Abel's [1] insight that pessimism can help explain both the equity premium and the risk-free puzzles. Indeed, observe that the risk-free rate decreases and the equity premium increases as agents become more pessimistic; i.e. as $\pi(2|2)$ becomes smaller. The salient feature of the left-hand side of Figure 1 is that, contrary to the conventional wisdom, the predictions of CE are qualitatively correct. That is, the signs of the autocorrelations are in line with those in Table 1, even if agents have correct beliefs. In fact, their absolute values are small compared to the data when agents have correct beliefs, but they increase vastly as agents become pessimistic. This is the main lesson from this exercise: pessimism can not only explain the equity premium and risk-free puzzles but also yield striking large values of short term momentum. Indeed, for $\pi(2|2) = 0.202$ the 1^{st} order autocorrelation is 0.3609 while the risk-free rate is 0.8% and the equity premium is 4.17%. Although pessimism makes the 2^{nd} order autocorrelation negative, the latter is always below -0.06 which falls too short of explaining the -0.1528 found in the US data. The degree of pessimism needed to generate levels of short-term momentum as in Table 1 is 0.552. At that level, the equity premium predicted is 4.8% while the risk-free interest rate is a bit larger than 1%, a significant progress regarding these two targets compared to the case of homogeneous correct beliefs. The main failure is the predicted level of reversal, less than 2%, approximately an order of magnitude away from the data.

The reasons why pessimism helps to explain the equity premium and the risk free rate are known; see Abel [1] and Cogley and Sargent [11]. What needs to be explained is what drives the sign of these correlations. Recall that Theorem 4 shows that short-term momentum implies long-term reversal. Thus, we need to explain the sign of the 1^{st} order autocorrelations, i.e. why short-term momentum occurs, and, by Proposition 3, it suffices to argue that trending occurs for pessimistic, correct and moderately optimistic beliefs and mean reversion for very optimistic beliefs. Since returns in CE depend only on the current state of nature and the growth rate, its range takes only two values given the current state of nature. Therefore, we can use the decomposition we introduce in (18) to explain when the conditional short-run equity premium trends.

We refer to events $\{1,3\}$ or $\{2,4\}$ as a recession (L) or an expansion (H), respectively. Note that, under our assumptions, any expansion state is good news while any recession one is bad news.

We abuse notation and denote the risk-free rate by $R^F_{po}(\tilde{\xi},\alpha_\infty)$ for $\tilde{\xi}\in\{L,H\}$. Then,

$$R^F_{po}(\tilde{\xi},\alpha_\infty)=\left[\beta\left(\pi(L|\tilde{\xi})g(L)^{-\sigma}+\pi(H|\tilde{\xi})g(H)^{-\sigma}\right)\right]^{-1}$$

where $\pi(L|L)\equiv\pi(1|1)+\pi(3|1)$ and $\pi(L|H)\equiv\pi(1|2)+\pi(3|2)$. As long as $\pi(H|H)<\pi(H|L)=\pi^*(H|L)$, the risk free rate is larger in recessions than in expansions for all $\sigma>0$. That is

$$R^F_{po}(L,\alpha_\infty)>R^F_{po}(H,\alpha_\infty)$$

Since $\pi^*(H|H)<\pi^*(H|L)$, the risk-free rate is larger in recessions that in expansions even for moderate optimism, i.e., $\pi^*(H|H)<\pi(H|H)<\pi^*(H|L)$.

The market belief about a recession at $(\tilde{\xi},\alpha_\infty)$ is $m_{po}(L|\tilde{\xi},\alpha_\infty)=R^F_{po}(\tilde{\xi},\alpha_\infty)\,\beta\,\pi(L|\tilde{\xi})\,g(L)^{-\sigma}$ and, therefore, market pessimism is given by

$$\frac{\pi^*(L|\tilde{\xi})}{m_{po}(L|\tilde{\xi},\alpha_\infty)}=\frac{\pi^*(L|\tilde{\xi})}{\pi(L|\tilde{\xi})}\frac{1}{\beta R^F_{po}(\tilde{\xi},\alpha_\infty)\,g(L)^{-\sigma}},$$

a function that is decreasing in the risk-free rate and increasing in $\frac{\pi^*(L|\tilde{\xi})}{\pi(L|\tilde{\xi})}$.

In Figure 2 we plot the recession bias (on the left) and the return inertia (on the right) effects for different levels of $\pi(2|2)$.

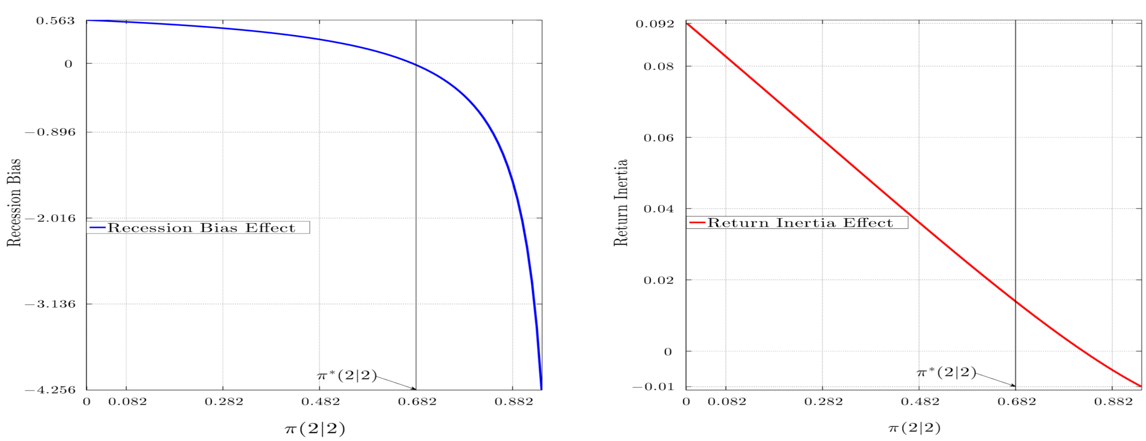

Figure 2: Recession Bias and Return Inertia Effects in CE

Figure 2 shows that the return inertia effect is positive for all $\pi(2|2)\le\pi^*(2|2)$ and the recession bias effect is positive for pessimistic beliefs and becomes negative, but small in absolute value, when beliefs are correct. Thus, the conditional short-run equity premium trends for pessimistic, correct and moderately optimistic beliefs. The conditional short-run equity premium reverts to the mean only for high levels of $\pi(2|2)$ because both effects become negative.

First, we explain the recession bias effect. For $\pi(2|2) \geq \pi^*(2|2)$, the recession bias effect is negative because $R^F_{po}(L, \alpha_\infty) > R^F_{po}(H, \alpha_\infty)$ and $\frac{\pi^*(L|H)}{\pi(L|H)} \geq 1 = \frac{\pi^*(L|L)}{\pi(L|L)}$ implies that

$$\frac{\pi^*\left(L\,|L\right)}{m_{po}\left(L\,|L,\alpha_\infty\right)} < \frac{\pi^*(L|H)}{m_{po}\left(L\,|H,\alpha_\infty\right)} \text{ for any } \pi(2|2) \geq \pi^*(2|2).$$

As $\pi(2|2)$ becomes smaller, the term $\frac{\pi^*(L|H)}{\pi(L|H)}$ becomes smaller than one and overcomes the risk-free rate effect, making the recession bias effect positive.

To explain the return inertia effect, we plot it together with the equity return and the risk-free rate in Figure 3.

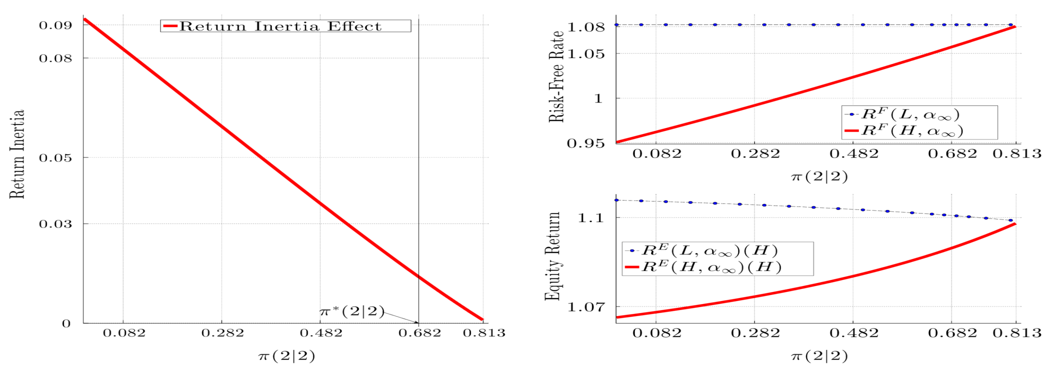

Figure 3: Determinants of the Return Inertia Effect in CE

Figure 3 shows that the return inertia effect is positive for low values of $\pi(2|2)$ because the risk-free interest rate is much lower in expansions than in recessions while equity returns are very similar across aggregate states. As $\pi(2|2)$ increases, the return inertia effect becomes negative because, on the one hand, the risk-free in the expansion state increases while, on the other hand, the equity returns and the risk-free rate in the recession state decrease and stays constant, respectively.

Figure 4 plots the 1^{st} and 2^{nd}-order autocorrelations of returns, on the left hand side, together with the risk-free rate and the equity premium, on the right hand side, when β and σ are set to match the average annual risk-free interest rate and equity premium of 0.8% and 6.18%, respectively. Once again, condition (2) is violated for very optimistic beliefs, i.e., $\pi(2|2) > 0.812$. Thus, we plot our numerical results for $\varepsilon \in (-0.11, 0.682)$ or, what is the same, $\pi(2|2) \in (0, 0.812)$.

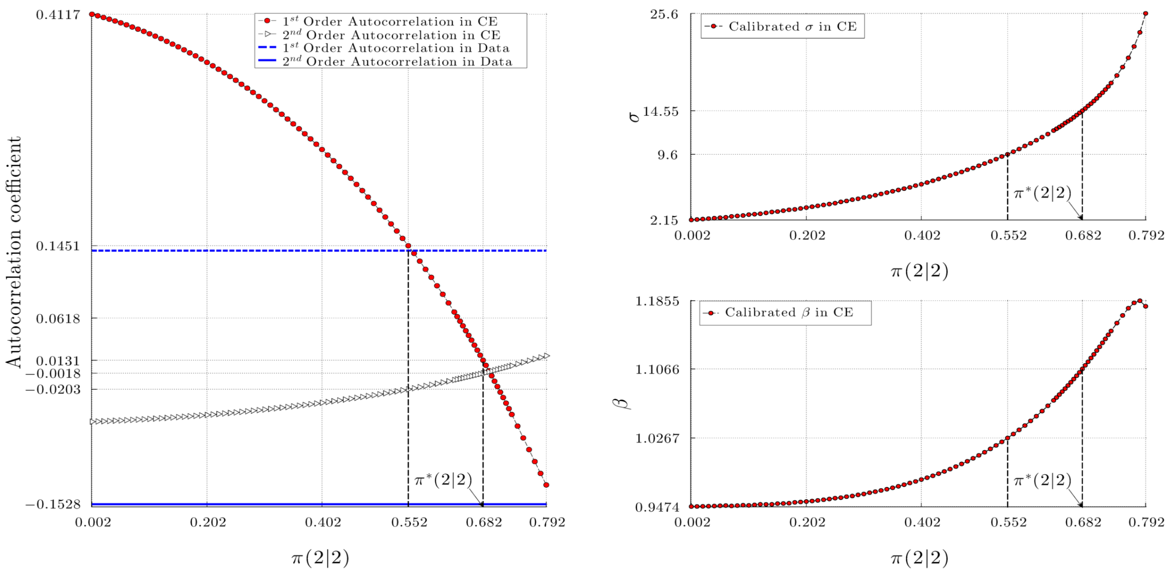

Figure 4: Autocorrelation coefficients in CE - Full Calibration

Figure 4 shows that pessimism can still yield a quantitatively relevant explanation for short term momentum even in this case. Indeed, for $\pi(2|2) = 0.552$ the 1^{st} order autocorrelation is 0.1451 which is still very close to the 0.1394 found in the US data. This sort of exercise is more controversial however.[9] As originally alerted by Weil [25] for the case of correct beliefs, the discount factor must be larger than 1 also for incorrect beliefs; in fact it could be made smaller than 1 as agents become more pessimistic but only at the cost of vastly increasing the predicted level of short-term momentum. Our exercise shows the controversy cannot be solved by the introduction of belief distortions. Finally, although the predicted 2^{nd} order autocorrelation of -0.0203 is qualitatively correct, it falls too short of explaining the -0.1528 found in the US data.

To summarise our findings in this section, we underscore that the introduction of pessimism in CE helps to explain the risk-free rate puzzle, the equity premium puzzle and the levels of short-term momentum observed in US asset returns. Even though it generates long-term reversal in asset returns, the levels predicted by the model are not quantitatively accurate.

7.3 Standard Model II: CESC Allocations-Homogeneous Beliefs

In this section we report the results of our numerical simulations for CESC.

Figure 5 plots the 1^{st} and 2^{nd}-order autocorrelations of returns, on the left-hand side, together with the risk-free rate and the equity premium, on the right-hand side, for the high-β economy of Alvarez and Jermann [3], i.e., for $\sigma = 3.5$ and β chosen to match the historic average of the risk-free interest rate.[10] For very optimistic beliefs, i.e., $\pi(2|2) > 0.812$, condition (2) is violated (that is, expected utility does not converge) while for very pessimistic beliefs, i.e., $\pi(2|2) < 0.532$, there is no β that delivers a risk-free rate of 0.8%. Thus, we plot our numerical results for $\varepsilon \in (-0.15, 0.13))$ or,

[9] See Kocherlakota [16] for a discussion.

[10] In this case the calibrated β are always smaller than one and so the conflict mentioned before is absent.

what is the same, $\pi(2|2) \in (0.532, 0.812)$.

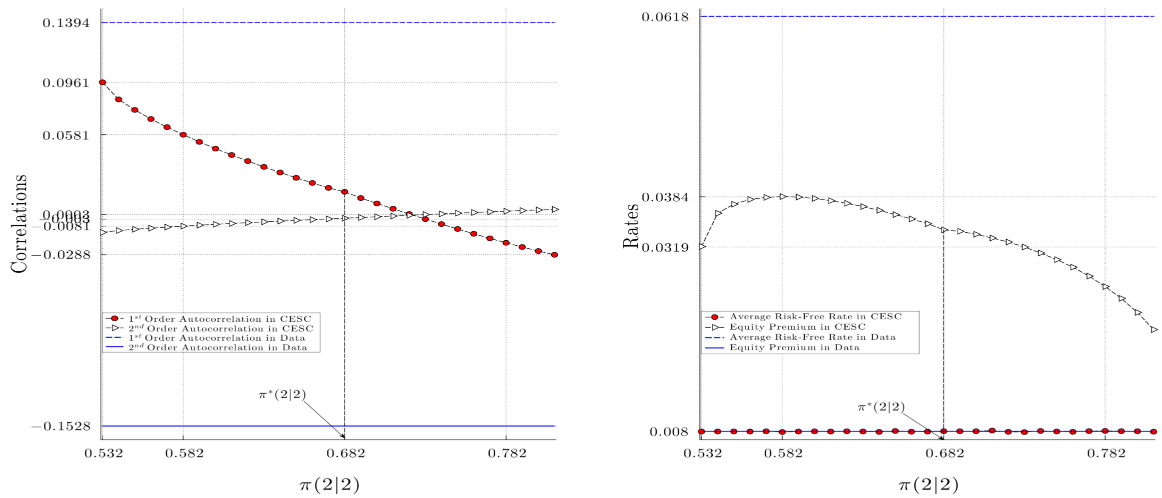

Figure 5: Autocorrelation coefficients in CESC - Calibrated β

The important lesson to draw from the left-hand side of Figure 5 is that the predictions of CESC are qualitatively correct since the asset displays short-term momentum (and long-term reversal) for pessimistic, correct and slightly optimistic beliefs. Indeed, the 1^{st}-order autocorrelation is positive for all $\pi(2|2) < 0.712$. On the other hand, the right-hand side of Figure 5 shows that the equity premium is below the target value in the data and, in particular, it decreases with the agent belief around the truth. Indeed, the largest equity premium, attained for $\pi(2|2) = 0.582$, equals 3.84% which falls short of the 6.18% found in the data. The 1^{st} and 2^{nd}-order autocorrelations are 0.0581 and -0.0081, respectively. Although both predictions are far from the targets, it is the 2^{nd} order autocorrelation the one that becomes more anomalous from the quantitative point of vew.

In order to explain the sign of these autocorrelations, we consider the CESC of a proxy economy that differs from the original economy in that its minimum enforceable weights in states 2 and 4 are set equal to those of the original economy in states 1 and 3. We use the symbol ˜ above a variable when the latter belongs to the proxy economy. That is,

$$\begin{aligned}
\underline{\tilde{\alpha}}_{i,cpo}(\xi) &= \underline{\alpha}_{i,cpo}(\xi) \qquad \text{if } \xi \in \{1,3\}, \\
\underline{\tilde{\alpha}}_{1,cpo}(2) &= \underline{\alpha}_{1,cpo}(1), \\
\underline{\tilde{\alpha}}_{2,cpo}(4) &= \underline{\alpha}_{2,cpo}(3).
\end{aligned}$$

The numerical solutions for both economies are similar because the invariant distribution places very little mass on $\left[\underline{\alpha}_{1,cpo}(2), \underline{\alpha}_{1,cpo}(1)\right)$ and $\left(\underline{\alpha}_{2,cpo}(4), \underline{\alpha}_{2,cpo}(3)\right]$. Since the minimum enforceable weights of the proxy economy are measurable with respect to the growth rate, both the market belief as well as the returns inherit that property. Furthermore, any expansion state is good news and any recession state is bad news. Since the growth rate takes only two values, so does the return given the current state, and so we are able to use the decomposition introduced in (18).

Note that the support of the invariant distribution of the (proxy economy) welfare weights consists of $(\underline{\tilde{\alpha}}_{1,cpo}(1), 1-\underline{\tilde{\alpha}}_{1,cpo}(1))$ and $(1-\underline{\tilde{\alpha}}_{2,cpo}(2), \underline{\tilde{\alpha}}_{2,cpo}(2))$. Therefore, without loss in generality, returns can be written as a function only of the state of nature. With some abuse of notation we denote the the risk-free rate of the proxy economy in state of nature $\tilde{\xi} \in \{L, H\}$ as

$$\tilde{R}^F_{cpo}(\tilde{\xi}) \equiv \beta^{-1} \left(\tilde{\pi}(L|\tilde{\xi}) g(L)^{-\sigma} + \tilde{\pi}(H|\tilde{\xi}) g(H)^{-\sigma} \right)^{-1}$$

where $\tilde{\pi}(L|L) \equiv \pi(1|1) + \pi(3|1)\tilde{\theta}_{AJ}$, $\tilde{\pi}(L|H) \equiv \pi(1|2) + \pi(3|2)\tilde{\theta}_{AJ}$ and the term $\tilde{\theta}_{AJ} > 1$, defined in Appendix B.2, reflects the increase in the marginal valuation of consumption when, as in Alvarez and Jermann [2], the enforceability constraint binds. Since the marginal valuation of consumption is highest in a recession and the latter is more likely after an expansion has occurred, i.e. $\tilde{\pi}(L|H) > \tilde{\pi}(L|L)$, then the risk-free rate of the proxy economy satisfies:

$$\tilde{R}^F_{cpo}(L) > \tilde{R}^F_{cpo}(H)$$

The market belief of the proxy economy is given by

$$\tilde{m}_{cpo}(\xi'|\xi) = \begin{cases} \beta\ \tilde{R}^F_{cpo}(\xi)\, \pi^*(\xi'|\xi)\, \frac{\pi(\xi'|\xi)}{\pi^*(\xi'|\xi)} g(\xi')^{-\sigma} & \text{if } \epsilon(\xi') = \epsilon(\xi) \\ \beta\ \tilde{R}^F_{cpo}(\xi)\, \pi^*(\xi'|\xi)\, \frac{\pi(\xi'|\xi)}{\pi^*(\xi'|\xi)} g(\xi')^{-\sigma}\ \ \tilde{\theta}_{AJ} & \text{if } \epsilon(\xi') \neq \epsilon(\xi) \end{cases}$$

This market belief has four distinct parts: (1) $\pi^*(\xi'|\xi)$, the true transition probability (2) $\beta\ \tilde{R}^F_{cpo}$ that adjusts for the equilibrium value of time, (3) $\frac{\pi_2(\xi'|\xi)}{\pi^*(\xi'|\xi)}$ that adjusts the market belief to reflect the incorrect belief of the agents, (4) $\tilde{\theta}_{AJ}$ the adjusts upwards the probability of those states where the enforceability constraint binds. Since the enforceability constraints that bind are those that correspond to states next period where the income shares change with respect to the current one, it is always the agent who is rich in that state who is constrained.

Theorem 5 implies that it suffices to explain when short-term momentum occurs in the proxy economy. By Proposition 3, it suffices to argue that trending and mean reversion occurs for low and high values, respectively, of $\pi(2|2)$.

Note that market pessimism in state of nature $\tilde{\xi} \in \{L, H\}$ is

$$\frac{\pi^*(L|\tilde{\xi})}{\tilde{m}_{cpo}(L|\tilde{\xi})} = \frac{\pi^*(L|\tilde{\xi})}{\tilde{\pi}(L|\tilde{\xi})} \frac{1}{\beta \tilde{R}^F_{cpo}(\tilde{\xi}) g(L)^{-\sigma}}$$

where $\tilde{m}_{cpo}(L|L) \equiv \tilde{m}_{cpo}(1|1) + \tilde{m}_{cpo}(3|1)$ and $\tilde{m}_{cpo}(L|L) \equiv \tilde{m}_{cpo}(2|2) + \tilde{m}_{cpo}(4|2)$.

The left- and right-hand sides of Figure 6 plots the recession bias and the return inertia effects. It shows that the recession bias and the return inertia effects are decreasing in $\pi(2|2)$. The recession bias takes positive values for pessimistic beliefs and negative ones for optimistic ones, while the return inertia effect is always positive. Consequently, for $\pi(2|2)$ small there is trending because the two effects are positive. For large $\pi(2|2)$, there is mean reversion because the return inertia effect vanishes.

To understand why the recession bias effect is negative when agents have correct beliefs, note that since $\pi^*(3|1)$ and $\pi^*(4|2)$ are small, then $\frac{\pi^*(L|\tilde{\xi})}{\tilde{\pi}(L|\tilde{\xi})}$ is close to one and so market pessimism is driven by the risk-free rate, which implies that

$$\frac{\pi^*(L|L)}{\tilde{m}_{cpo}(L|L)} < \frac{\pi^*(L|H)}{\tilde{m}_{cpo}(L|H)}$$

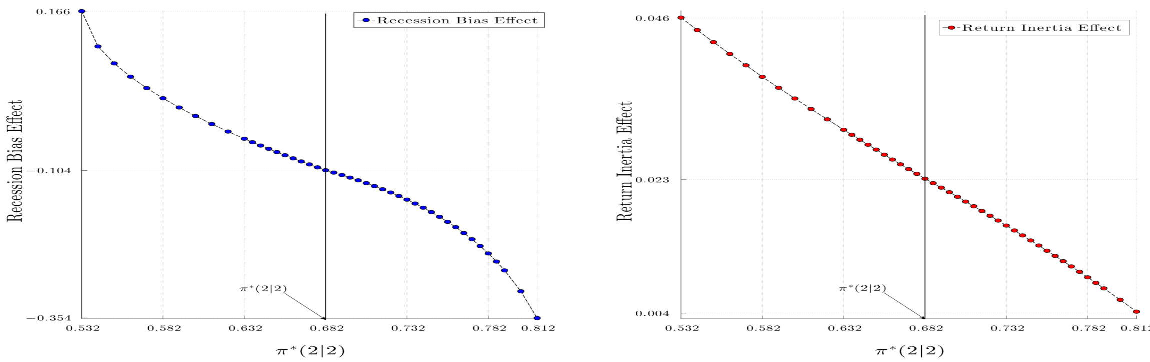

Figure 6: Recession Bias and Return Inertia Effects in CESC.

For $\pi(2|2)$ small enough, the term $\frac{\pi^*(L|\tilde{\xi})}{\tilde{\pi}(L|\tilde{\xi})}$ dominates. Thus $\frac{\pi^*(L|L)}{\tilde{\pi}(L|L)} > \frac{\pi^*(L|H)}{\tilde{\pi}(L|H)}$ and the inequality above is reversed.

To understand why the return inertia effect is positive, we plot the return inertia effect together with the equity return and the risk free rate in Figure 7.

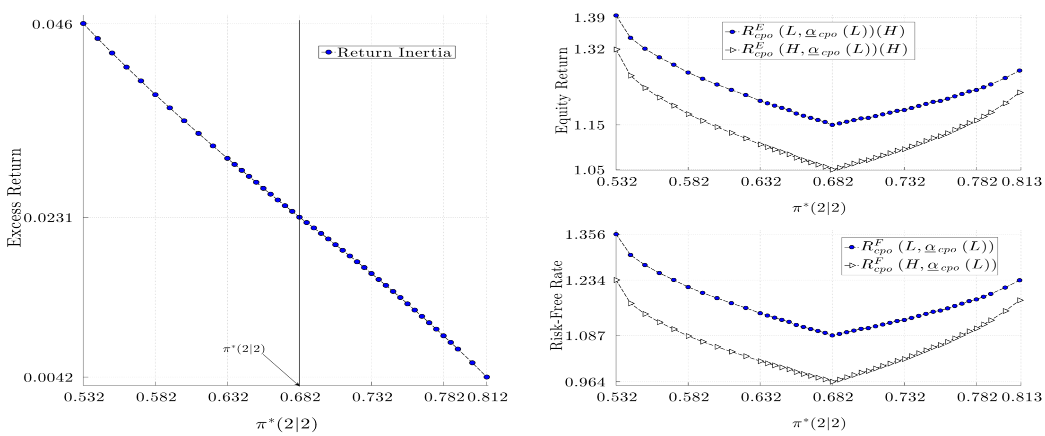

Figure 7: Return Inertia Effect in CESC.

Both $\tilde{R}_{cpo}\left(\cdot, \underline{\alpha}_{cpo}(L)\right)(H)$ and $\tilde{R}^F_{cpo}(\cdot)$ are larger after a recession than after an expansion. For pessimistic beliefs, however, the differential of the risk-free rate between a recession and an expansion is much larger than the differential of the equity return and dominates. That explains why the return inertia effect is positive for pessimistic beliefs.

Finally, Figure 8 is the counterpart of Figure 4 for CESC allocations. That is, β and σ are set to jointly match the average annual risk-free interest rate and equity premium of 0.8% and 6.18%. The 1^{st} and 2^{nd}-order autocorrelations of returns are depicted on the left hand side, while the risk-free rate and the equity premium are depicted on the right hand side. For very pessimistic beliefs, i.e., $\pi(2|2) < 0.512$ there is no combination of β and σ that delivers a risk-free rate of 0.8% and an equity premium of 6.18%. Thus, we plot our numerical results for $\varepsilon \in (-0.17, -0.26)$ or, what is the same, for $\pi(2|2) \in (0.512, 0.942)$.

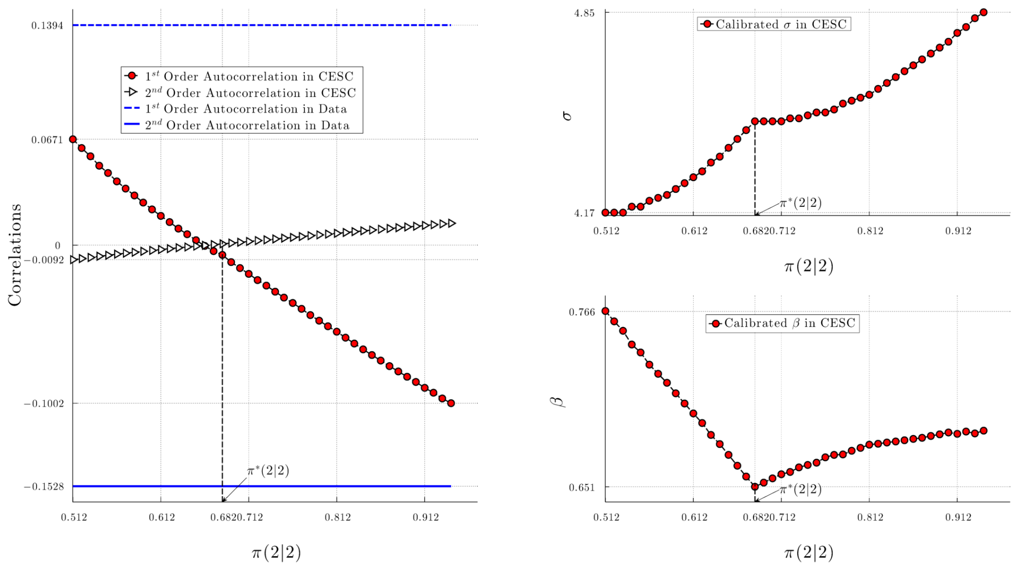

Figure 8: Autocorrelation coefficients in CESC - Calibrated β and σ

Figure 8 summarises our findings in this section and makes evident the difficulties of CESC allocations to explain both the early puzzles and the more recent anomalies at the same time. Indeed, the $1st-$ and $2nd-$ order autocorrelations predicted by CESC with correct beliefs have incorrect signs. Although those correlations can be correctly signed for sufficiently pessimistic beliefs, the model fails badly on the quantitative dimension since the predicted autocorrelations are one order of magnitude smaller that in the data at the very best.

Figure 8 forcefully show that even though pessimism makes CESC asset returns display short-term momentum and long-term reversal, its quantitative effect is smaller than in CE. Nonetheless, there are two important lessons to learn from this exercise. First, the results in Beker and Espino [6] show that when one puts together belief heterogeneity and pessimism, CESC yields a better quantitative explanation for short-term momentum and long-reversal than CE with pessimistic homogeneous beliefs. Therefore, it must be belief heterogeneity what is key to explain the quantitative relevance of CESC in Beker and Espino [6]. Second, even though under homogeneous and (possibly) incorrect beliefs , CE outperforms CESC in some dimensions, the latter generates significantly higher volatility in equity returns than the former.

Figure 9 plots the volatility (measured by the standard deviation) of the equity return in CE and CESC when β and σ are set to jointly match the average annual risk-free interest rate and equity premium of 0.8% and 6.18%.

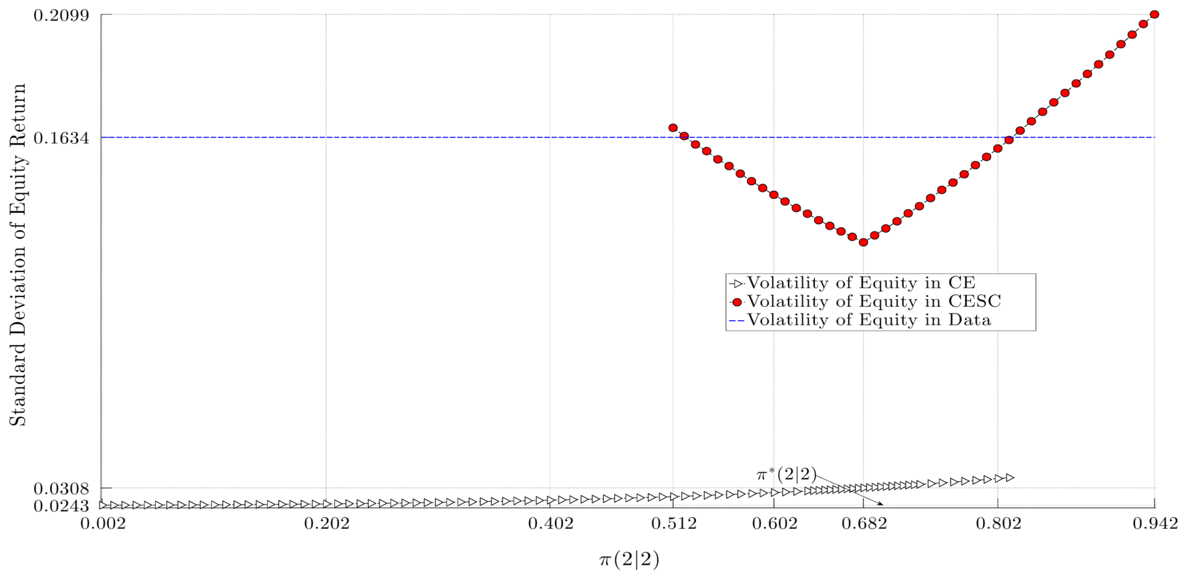

Figure 9: Volatility of Equity in CE and CESC - Calibrated β and σ

Figure 9 shows that pessimistic beliefs help explain the so-called volatility puzzle (see Campbell [8]): for $\pi(2|2) = 0.522$ the volatility of equity return in CESC is close to the 16.34% observed in the US data, several times higher than the volatility of equity return in CE.

8 Concluding Remarks

This paper shows that despite the conventional wisdom, standard asset pricing models with subject expected utility maximisers who are pessimistic can explain a large part of the the risk-free rate, equity premium and volatility puzzles and the more recent financial market anomalies like short-term momentum and long-term reversal. We find that the data does not contradict the qualitative predictions of the models; rather, it can be inconsistent with some quantitative implications of a particular economic model.

First, we show that the frictionless Lucas-Mehra-Prescott with pessimistic agents does a good job in explaining the risk-free and equity premium puzzles as well as short-term momentum while it only fails to explain quantitatively long-term reversal and the equity volatility puzzle.

Second, we show that the Kehoe-Levine-Alvarez-Jermann model with pessimistic agents does a good job in explaining the risk-free, equity premium and volatility puzzles but it only yields a qualitatively accurate explanation for the more recent anomalies. Compared to the Lucas-Mehra-Prescott model it gives a more accurate quantitative explanation for equity volatility at the cost of a less accurate quantitative explanation for short-term momentum.

Appendix A

Proof of Proposition 1. Consider first the case of PO allocations. Since the welfare weight is constant, the result holds if and only if there exists a unique invariant distribution over S that is globally stable. The latter result holds trivially from our assumptions on π^*.

For the case of CPO allocations of economies where $\cap_\xi[\underline{\alpha}(\xi), 1-\underline{\alpha}(\xi)] \neq \emptyset$, the result follows because, P^{π^*} a.s., on each path the welfare weight converges to some welfare welfare weight in the boundary of the $\cap_\xi[\underline{\alpha}(\xi), 1-\underline{\alpha}(\xi)]$.

For the case of CPO allocations of economies where $\cap_\xi[\underline{\alpha}(\xi), 1-\underline{\alpha}(\xi)] = \emptyset$, there exists some states of nature $\xi^*, \xi^{**} \in S$ such that $\underline{\alpha}(\xi^*) > 1-\underline{\alpha}(\xi^{**})$. The existence of a unique invariant distribution that is globally stable follows by Theorem 11.12 in Stokey and Lucas [22]. It suffices to show that F_{cpo} satisfies the following condition:

<u>Condition M</u>: There exists $\delta > 0$ and an integer $N \geq 1$ such that for any $A \in \mathscr{G}$, either $P^N(\omega, A) \geq \delta$, all $\omega \in \Omega$, or $P^N(s, A^c) \geq \delta$, all $\omega \in \Omega$.

Set $N \equiv 2$ and $\delta \equiv \min_\xi \pi^*(\xi^{**}|\,\xi^*)\pi^*(\xi^*|\,\xi) > 0$. Let $A \in \mathscr{G}$ and $(\xi,\alpha) \in \Omega$ be arbitrary. If $(\xi^*, \underline{\alpha}(\xi^*)) \in A$, then $P^N((\xi,\alpha),A) \geq \pi^*(\xi^{**}|\,\xi^*)\pi^*(\xi^*|\,\xi) \geq \delta$. If $(\xi^*, \underline{\alpha}(\xi^*)) \in A^c$, then $P^N((\xi,\alpha),A^c) \geq \pi^*(\xi^{**}|\,\xi^*)\pi^*(\xi^*|\,\xi) \geq \delta$.

To show the invariant distribution is not degenerate note that both $\underline{\alpha}(\xi^*)$ and $1-\underline{\alpha}(\xi^{**})$ must be part of its support since

$$\begin{aligned}
\alpha'_{cpo}(\xi^{**},\alpha)(\xi^*) &= \underline{\alpha}(\xi^*) && \text{for all } \alpha \in [\underline{\alpha}(\xi^{**}), 1-\underline{\alpha}(\xi^{**})] \\
\alpha'_{cpo}(\xi^{*},\alpha)(\xi^{**}) &= 1-\underline{\alpha}(\xi^{**}) && \text{for all } \alpha \in [\underline{\alpha}(\xi^{*}), 1-\underline{\alpha}(\xi^{*})]
\end{aligned}$$

□

The following result is a SSLN that will be used to prove Theorem 2.

Theorem A.1 (Stout [23] and Jensen and Rahbek [14])**.** *Assume $\{z_t\}_{t=0}^\infty$ is a time homogeneous Markov process with transition function F on $(Z, \mathcal{Z})$. If there exists a unique invariant distribution $\psi : \mathcal{Z} \to [0,1]$, then for any $z_0 \in Z$, any integer k and any continuous function $f : Z^k \to \Re$,*

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} f\left(z_t, ..., z_{t+k}\right) = E^{P^F(\psi,\cdot)}\left(f\left(\tilde{z}_0, ..., \tilde{z}_k\right)\right), \qquad P^F\left(z_0, \cdot\right) - a.s.$$

Proof of Theorem 2. The result follows directly from Theorem A.1 and Proposition 1. □

Proof of Proposition 3. We need to show that

$$E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)\right] > 0 \qquad \text{if } E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right) \text{ trends,} \tag{19}$$

$$E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)\right] < 0 \qquad \text{if } E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right) \text{ reverts to the mean.} \tag{20}$$

For $t \geq 0$, Ω^t is the $t-$cartesian product of Ω with typical element $\omega^t = (\xi_0, \alpha_0, ..., \xi_t, \alpha_t)$ and $\Omega^\infty = \Omega \times \Omega \times ...$ is the infinite product of the state space with typical element $\omega = (\omega_0, \omega_1, ...)$. $\mathcal{G}_{-1} \equiv \{\varnothing, \Omega^\infty\}$ is the trivial $\sigma-$algebra, $\mathcal{G}_t$ is the $\sigma-$algebra that consists of all the cylinder sets of length t. The $\sigma-$algebras $\mathcal{G}_t$ define a filtration $\mathcal{G}_{-1} \subset \mathcal{G}_0 \subset ... \subset \mathcal{G}_t \subset ... \subset \mathcal{G}^\infty$, where $\mathcal{G}^\infty \equiv \mathcal{G} \times \mathcal{G} \times ...$ is the $\sigma-$algebra on Ω^∞.

Let Ω_e be the support of P_e and $\Omega^+ \equiv \left\{\tilde{\omega} \in \Omega_e : \overline{R}_{1,e}(\tilde{\omega}) \geq 0\right\}$ and $\Omega^- \equiv \left\{\tilde{\omega} \in \Omega_e : \overline{R}_{1,e}(\tilde{\omega}) < 0\right\}$. Note that

$$
\begin{aligned}
E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right] &= P_e\left(\Omega^+\right)\ E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\middle|\Omega^+\right] + \\
&\quad P_e\left(\Omega^-\right)\ E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\middle|\Omega^-\right]
\end{aligned}
$$

and so $E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right]$ is bounded below by

$$
\begin{aligned}
&P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^+\right)\inf_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) + \\
&P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^-\right)\sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega})
\end{aligned}
$$

and above by

$$
\begin{aligned}
&P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^+\right)\sup_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) + \\
&P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^-\right)\inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega})
\end{aligned}
$$

If $E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\omega)$ trends, then

$$
\inf_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) > \sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega})
$$

and so (19) holds because

$$
\begin{aligned}
E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right](\omega) &> \left(P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^+\right)\right)\sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) + \\
&\quad \left(P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^-\right)\right)\sup_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \\
&= E^{P_e}\left(\overline{R}_{1,e}\right)\sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \\
&= 0.
\end{aligned}
$$

If $E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\omega)$ reverts to the mean, then

$$
\sup_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) < \inf_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\omega)
$$

and so (20) holds because

$$
\begin{aligned}
E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right](\omega) &< \left(P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^+\right)\right)\inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) + \\
&\quad \left(P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^-\right)\right)\inf_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \\
&= E^{P_e}\left(\overline{R}_{1,e}\right)\inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \\
&= 0.
\end{aligned}
$$

$\square$

Now we prove Theorems 4 and 5. We begin with some results on Markov Processes.

Lemma A.2. *Let $\{z_t\}_{t=0}^{\infty}$ be a two-state time homogeneous Markov process with transition function F on $(Z, \mathcal{Z})$ and invariant distribution $\psi : \mathcal{Z} \to [0,1]$, P^F be the probability measure on $(Z^\infty, \mathcal{Z}^\infty)$ uniquely induced by F and ψ and let $R : Z \times Z \to \Re$. Suppose there exists $z_+ \in Z$ such that*
(a) $E^{P^F}(R(z_1, z_2)) = 0$.
(b) $R(z, z_+) > 0$ for all z.
(c) $E^{P^F}(R(z_0, z_1)\ R(z_1, z_2)) > 0$.
Then $E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) < 0 < E^{P^F}(R(z_2, z_3)|\, z_1 = z_-)$ iff $F(z_+|\, z_+) < \psi(z_+)$.

Proof. Hypothesis (a) and the Markov property implies that $E^{P^F}(R(z_k, z_{k+1})) = 0$ for any k. Thus,

$$\psi(z-)\ E^{P^F}(R(z_{k'}, z_{k'+1})|\, z_k = z_-) = -\psi(z_+)\ E^{P^F}(R(z_{k'}, z_{k'+1})|\, z_k = z_+) \tag{21}$$

where $z_- \neq z_+$. Note also that

$$\begin{aligned} & E^{P^F}(R(z_0, z_1)\ R(z_1, z_2)) = E^{P^F}\Big(R(z_0, z_1)\ E^{P^F}(R(z_1, z_2)|\, z_1)\Big) \\ = \quad & \left[P^F(z_+, z_+) R(z_+, z_+) + P^F(z_-, z_+) R(z_-, z_+)\right]\ E^{P^F}(R(z_1, z_2)|\, z_1 = z_+) \\ & + \left[P^F(z_+, z_-) R(z_+, z_-) + P^F(z_-, z_-) R_e(z_-, z_-)\right]\ E^{P^F}(R(z_1, z_2)|\, z_1 = z_-). \end{aligned} \tag{22}$$

By hypothesis (a) and (b), $R(z, z_-) < 0$ for all z. Therefore,

$$\begin{aligned} P^F(z_+, z_-) R(z_+, z_-) + P^F(z_-, z_-) R(z_-, z_-) &< 0, \\ P^F(z_+, z_+) R(z_+, z_+) + P^F(z_-, z_+) R(z_-, z_+) &> 0. \end{aligned}$$

It follows from (21) evaluated at $k = 1$ and $k' = 1$, hypothesis (c) and (22) that

$$E^{P^F}(R(z_1, z_2)|\, z_1 = z_-) < 0 < E^{P^F}(R(z_1, z_2)|\, z_1 = z_+)$$

and the Markov Property implies

$$E^{P^F}(R(z_2, z_3)|\, z_2 = z_-) < 0 < E^{P^F}(R(z_2, z_3)|\, z_2 = z_+). \tag{23}$$

Condition (21), evaluated at $k = 1$ and $k' = 2$, implies that

$$E^{P^F}(R(z_2, z_3)|\, z_1 = z_-) < 0 < E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) \Leftrightarrow E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) > 0.$$

In addition,

$$\begin{aligned} E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) &= E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) - E^{P^F}(R(z_2, z_3)) \\ &= (F(z_2 = z_+|\, z_1 = z_+) - \psi(z_+))\, E^{P^F}(R(z_2, z_3)|\, z_2 = z_+) + \\ &\qquad (F(z_2 = z_-|\, z_1 = z_+) - \psi(z_-))\, E^{P^F}(R(z_2, z_3)|\, z_2 = z_-) \\ &= (F(z_2 = z_+|\, z_1 = z_+) - \psi(z_+)) \times \\ &\qquad \Big(E^{P^F}(R(z_2, z_3)|\, z_2 = z_+) - E^{P^F}(R(z_2, z_3)|\, z_2 = z_-)\Big). \end{aligned}$$

where the first line follows by the definition of unconditional expectation and (a). (23) implies that

$$E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) < 0 \Leftrightarrow F(z_2 = z_+|\, z_1 = z_+) - \psi(z_+) < 0.$$

□

Proof of Theorem 4. Consider any CE of an arbitrary economy. Since the allocation is PO, it follows by Theorem 2 that (12) holds and the marginal distribution of ψ_{po} over welfare weights is a point mass on α_∞. By standard arguments, there exists $\overline{R}_{po} : \{l, h\} \times \{l, h\} \to \Re$ such that for any $\tau \in \{1, 2\}$ and $\omega \in \Omega$ the (excess) returns $\overline{R}_{\tau,po}(\omega)$ is given by

$$\overline{R}_{\tau,po}(\omega) = \begin{cases} \overline{R}_{po}(l,l) & \text{if } \xi_{\tau-1}(\omega) \in \{1,3\} \text{ and } \xi_\tau(\omega) \in \{1,3\} \\ \overline{R}_{po}(l,h) & \text{if } \xi_{\tau-1}(\omega) \in \{1,3\} \text{ and } \xi_\tau(\omega) \in \{2,4\} \\ \overline{R}_{po}(h,l) & \text{if } \xi_{\tau-1}(\omega) \in \{2,4\} \text{ and } \xi_\tau(\omega) \in \{1,3\} \\ \overline{R}_{po}(h,h) & \text{if } \xi_{\tau-1}(\omega) \in \{2,4\} \text{ and } \xi_\tau(\omega) \in \{2,4\} \end{cases}$$

and

$$\overline{R}_{po}(\xi, l) < 0 < \overline{R}_{po}(\xi, h) \text{ for all } \xi \in \{l, h\}. \tag{24}$$

Let $Z = \{l, h\}$, $\mathcal{Z}$ be its finest partition, $\tilde{\pi}^*$ be the transition function on $(Z, \mathcal{Z})$ defined as the restriction of π^* to $(Z, \mathcal{Z})$ and let $\tilde{\psi}_{po}$ be the restriction of the invariant measure ψ_{po} to $(Z, \mathcal{Z})$. Let Z^∞ be the set of infinite sequences with elements in Z and $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset ... \subset \mathcal{Z}_t \subset ...\mathcal{Z}^\infty$ be the standard filtration. $P^{\tilde{\pi}^*}$ is the probability measure over $(Z^\infty, \mathcal{Z}^\infty)$ uniquely induced by $\tilde{\pi}^*$ and $\tilde{\psi}_{po}$. Let $z_t : Z^\infty \to Z$ be $\mathcal{Z}_t-$measurable. The collection $\{z_t\}_{t=0}^\infty$ on the probability space $\left(Z^\infty, \mathcal{Z}^\infty, P^{\tilde{\pi}^*}\right)$ is a two state time-homogeneous Markov process with transition function $\tilde{\pi}^*$ on $(Z, \mathcal{Z})$ and invariant distribution $\tilde{\psi}_{po} : \mathcal{Z} \times \mathcal{Z} \to [0, 1]$ satisfying

$$E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_1, z_2)\right) = 0. \tag{25}$$

First note that (25) and (24) are conditions (a) and (b), respectively, in Lemma A.2. Second, since the asset displays short-term momentum,

$$0 < E^{P_{po}}\left(\overline{R}_{1,po}\ \overline{R}_{2,po}\right) = E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_0, z_1)\ \overline{R}_{po}(z_1, z_2)\right)$$

and so condition (c) in Lemma A.2 also holds. By Lemma A.2, we conclude that

$$E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_2, z_3)\middle|\, z_1 = h\right) < 0 < E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_2, z_3)\middle|\, z_1 = l\right) \Leftrightarrow \tilde{\pi}^*(h|\,h) < \tilde{\psi}_{po}(h). \tag{26}$$

Let ω^+ and ω^- be such that $\overline{R}_{1,po}(\omega^+) > 0$ and $\overline{R}_{1,po}(\omega^-) < 0$. Then,

$$\begin{aligned} E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)(\omega^+) &= E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_2, z_3)\middle|\, z_1 = h\right), \\ E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)(\omega^-) &= E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_2, z_3)\middle|\, z_1 = l\right). \end{aligned}$$

It follows from (26), $\tilde{\pi}^*(h|\,h) = \pi^*(2|\,2) + \pi^*(4|\,2)$ and $\tilde{\psi}_{po}(h) = \psi_{po}(2) + \psi_{po}(4)$ that

$$E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)(\omega^+) < 0 < E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)(\omega^-) \Leftrightarrow \pi^*(2|\,2) + \pi^*(4|\,2) < \psi_{po}(2) + \psi_{po}(4)$$

that is, $E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)$ reverts to the mean if and only if $\pi^*(2|\,2) + \pi^*(4|\,2) < \psi_{po}(2) + \psi_{po}(4)$. By Proposition 3, the asset displays long-term reversal if $\pi^*(2|\,2) + \pi^*(4|\,2) < \psi_{po}(2) + \psi_{po}(4)$. To show the converse, suppose that $\pi^*(2|\,2) + \pi^*(4|\,2) \geq \psi_{po}(2) + \psi_{po}(4)$. Then by the argument above, $E^{P_e}\left(R_{3,po}\middle|\,\overline{R}_{1,e}\right)$ trends and it follows by Proposition 3 that the 2nd-order autocorrelation is positive and so long-run reversal fails. $\square$

Proof of Theorem 5. Consider any CESC. The price of an asset at state (ξ, α) must satisfy the Bellman equation:

$$p(\xi, \alpha) = \sum_{\xi'} Q(\xi, \alpha)(\xi')(p(\xi', \alpha'(\xi, \alpha)(\xi')) + d(\xi')) \qquad \psi_{cpo} - a.s.$$

It is easy to see that the invariant distribution places positive mass only on points (ξ, α) such that $\alpha \in \underline{\Delta} \cap \Delta(\xi, \mu^{\pi^*})$ where $\underline{\Delta} = \{(\alpha_1, \alpha_2) \in \Delta : \exists \xi \in S \text{ such that } \alpha_1 = \underline{\alpha}_1(\xi) \text{ or } \alpha_2 = \underline{\alpha}_2(\xi)\}$. The hypothesis $\underline{\alpha}_1(1) = \underline{\alpha}_1(2)$ and symmetry implies that $\underline{\alpha}_2(3) = \underline{\alpha}_2(4)$. If p_ξ, $q_{\xi\xi'}$ and d_ξ denotes $p(\xi, \underline{\alpha}(\xi))$, $Q(\xi, \underline{\alpha}(\xi))(\xi')$ and $d(\xi)$, respectively, then the Bellman equation becomes

$$p_\xi = \sum_{\xi'} q_{\xi\xi'}(p_{\xi'} + d_{\xi'}) \quad \text{for all } \xi$$

which can be written as $(I - Q) P = QD$ where Q is the 4×4 matrix with entries $q_{\xi\xi'}$, P is the 4×1 vector with entries p_ξ and D is the 4×1 vector with entries p_ξ. Note that

$$c_1(1, \underline{\alpha}(1)) = c_2(3, \underline{\alpha}(3)) \quad \text{ and } \quad c_1(2, \underline{\alpha}(2)) = c_2(4, \underline{\alpha}(4))$$

and so

$$\begin{aligned} q_{\xi 1} &= \beta(\xi, \mu)\ \pi(1\,|\xi)\ \tfrac{\partial u(c_1(1, \underline{\alpha}(1))/\partial c_1}{\partial u(c_1(\xi, \underline{\alpha}(\xi)))/\partial c_1} = \beta(\xi, \mu)\ \pi(3\,|\xi)\ \tfrac{\partial u(c_2(3, \underline{\alpha}(3))/\partial c_1}{\partial u(c_2(\xi, \underline{\alpha}(\xi)))/\partial c_1} = q_{\xi 3}, \\ q_{\xi 2} &= \beta(\xi, \mu)\ \pi(2\,|\xi)\ \tfrac{\partial u(c_1(2, \underline{\alpha}(2))/\partial c_1}{\partial u(c_1(\xi, \underline{\alpha}(\xi)))/\partial c_1} = \beta(\xi, \mu)\ \pi(4\,|\xi)\ \tfrac{\partial u(c_2(4, \underline{\alpha}(4))/\partial c_1}{\partial u(c_2(\xi, \underline{\alpha}(\xi)))/\partial c_1} = q_{\xi 4}. \end{aligned}$$

It follows that Q has rank 2. Therefore, $p_1 = p_3$ and $p_2 = p_4$.

Let $\tilde{\pi}^*$ and $(Z^\infty, \mathcal{Z}^\infty)$ be the transition matrix and the measurable space, respectively, introduced in the proof of Theorem 4. $P^{\tilde{\pi}^*}$ is the probability measure over $(Z^\infty, \mathcal{Z}^\infty)$ uniquely induced by $\tilde{\pi}^*$ and $\tilde{\psi}_{cpo}$. Let $z_t : Z^\infty \to Z$ be $\mathcal{Z}_t-$measurable. The collection $\{z_t\}_{t=0}^\infty$ on the probability space $(Z^\infty, \mathcal{Z}^\infty, P^{\tilde{\pi}^*})$ is a two state time-homogeneous Markov process with transition function $\tilde{\pi}^*$ on $(Z, \mathcal{Z})$ and invariant distribution $\tilde{\psi}_{cpo} : \mathcal{Z} \times \mathcal{Z} \to [0, 1]$.

Let $p(l) \equiv p_1$, $p(h) \equiv p_2$, $R_{cpo}(z, z') \equiv \frac{p_{z'} + d_{z'}}{p_z}$ for all $z \in \{l, h\}$ and $\overline{R}_{cpo} : \{l, h\} \times \{l, h\} \to \Re$. Then, the (excess) returns $\overline{R}_{\tau,cpo}(\omega)$ is given by

$$\overline{R}_{\tau,cpo}(\omega) = \begin{cases} \overline{R}_{cpo}(l, l) & \text{if } \xi_{\tau-1}(\omega) \in \{1, 3\} \text{ and } \xi_\tau(\omega) \in \{1, 3\} \\ \overline{R}_{cpo}(l, h) & \text{if } \xi_{\tau-1}(\omega) \in \{1, 3\} \text{ and } \xi_\tau(\omega) \in \{2, 4\} \\ \overline{R}_{cpo}(h, l) & \text{if } \xi_{\tau-1}(\omega) \in \{2, 4\} \text{ and } \xi_\tau(\omega) \in \{1, 3\} \\ \overline{R}_{cpo}(h, h) & \text{if } \xi_{\tau-1}(\omega) \in \{2, 4\} \text{ and } \xi_\tau(\omega) \in \{2, 4\} \end{cases} \tag{27}$$

Moreover,

$$\overline{R}_{cpo}(z, l) < 0 < \overline{R}_{cpo}(z, h) \text{ for all } z \in \{l, h\} \tag{28}$$

and

$$E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{cpo}(z_1, z_2)\right) = 0. \tag{29}$$

It follows from (27) that for any $k \in \{2, 3\}$

$$E^{P_{cpo}}\left(\overline{R}_{1,cpo}\ \overline{R}_{k,cpo}\right) = E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{cpo}(z_0, z_1)\ \overline{R}_{cpo}(z_1, z_k)\right).$$

Note that (29) and (28) are conditions (a) and (b) in Lemma A.2. Since the asset displays short-term momentum,

$$E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{cpo}(z_0, z_1)\ \overline{R}_{cpo}(z_1, z_2)\right) = E^{P_{cpo}}\left(\overline{R}_{1,po}\ \overline{R}_{2,po}\right) > 0,$$

and so (c) in Lemma A.2 also holds. The rest of the proof is identical to that in Theorem 4. $\square$

Appendix B

The calibrated transition matrix, π^*, governing the growth process is:

$$\pi^* = \begin{bmatrix} 0.1414 & 0.8200 & 0.0309 & 0.0077 \\ 0.2637 & 0.6820 & 0.0486 & 0.0057 \\ 0.0309 & 0.0077 & 0.1414 & 0.8200 \\ 0.0486 & 0.0057 & 0.2637 & 0.6820 \end{bmatrix}$$

and the calibrated endowment shares are:

ξ	$\epsilon(\xi)$	$g(\xi)$
1	0.6438	0.9602
2	0.6438	1.0402
3	0.3562	0.9602
4	0.3562	1.0402

B.1 Standard Model I: CE Allocations

The calibrated β is given by

$$\frac{1}{1.008} \sum_{\xi} \psi_{po}\left(\xi\right) \left(\sum_{\xi'} \pi(\xi' \,|\xi) g(\xi')^{-\sigma} \right)^{-1}.$$

B.2 Standard Model II: CESC allocations

$\tilde{\theta}_{AJ}$ is given by:

$$\tilde{\theta}_{AJ} = \left(\frac{1 - \underline{\tilde{\alpha}}_{2,cpo}\left(3, \mu^{\pi}\right)}{\underline{\tilde{\alpha}}_{1,cpo}\left(1, \mu^{\pi}\right)} \right)^{-\sigma} > 1 \text{ for } \epsilon\left(\xi'\right) \neq \epsilon\left(\xi\right)$$

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