Rationality and Dynamic Consistency under Risk and Uncertainty

Peter J. Hammond and Horst Zank

No 1033

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

Rationality and Dynamic Consistency under Risk and Uncertainty

Peter J. Hammond[a,1], Horst Zank[b,1]

[a] *Department of Economics, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, U.K.*
[b] *Department of Economics, School of Social Sciences, University of Manchester, Oxford Road, Manchester, M13 9PL, U.K.*

Abstract

For choice with deterministic consequences, the standard rationality hypothesis is ordinality — i.e., maximization of a weak preference ordering. For choice under risk (resp. uncertainty), preferences are assumed to be represented by the objectively (resp. subjectively) expected value of a von Neumann–Morgenstern utility function. For choice under risk, this implies a key independence axiom; under uncertainty, it implies some version of Savage's sure thing principle. This chapter investigates the extent to which ordinality, independence, and the sure thing principle can be derived from more fundamental axioms concerning behaviour in decision trees. Following Cubitt (1996), these principles include dynamic consistency, separability, and reduction of sequential choice, which can be derived in turn from one consequentialist hypothesis applied to continuation subtrees as well as entire decision trees. Examples of behavior violating these principles are also reviewed, as are possible explanations of why such violations are often observed in experiments.

Keywords: expected utility, consequentialism, independence axiom, consistent planning, rationality, dynamic consistency, subjective probability, sure thing principle

JEL codes: D01, D03, D81, D91, C72

Email addresses: p.j.hammond@warwick.ac.uk (Peter J. Hammond), horst.zank@manchester.ac.uk (Horst Zank)

URL: http://go.warwick.ac.uk/pjhammond (Peter J. Hammond)

Preprint for chapter 2 of Mark Machina and W. Kip Viscusi (eds.)
Handbook of Risk and Uncertainty in Economics
(Amsterdam: North-Holland, in press)

1. Introduction and Outline

1.1. Purpose of Chapter

The main subject of this chapter is single-person decision theory, especially the normative principles of decision-making under risk and uncertainty. We will pay particular attention to principles of "rationality" such as:

1. the existence of a preference ordering;
2. when discussing the choice of risky consequences, the representation of preferences by the expected value of a von Neumann–Morgenstern utility function;
3. when discussing the choice of uncertain consequences, the use of subjective or personal probabilities attached to unknown states of the world.

We will review rather thoroughly some arguments for (and against) the first two of these rationality principles of for both behavior and consistent planning. As for the third principle, space permits us only to sketch some arguments very briefly.

The arguments we will consider involve "consequentialism" and some closely related axioms put forward by, in particular, Hammond (1977, 1983, 1988a, b, 1989, 1998a, b), McClennen (1986), Seidenfeld (1988), Machina (1989), Cubitt (1996) and Steele (2010). The main issue we will focus on is whether standard principles of rational choice in a static framework, like those enunciated above, can be derived as implications of some possibly more basic hypotheses intended to describe properties of behavior and plans in a dynamic framework that includes single-person games in extensive form, more commonly referred to as decision trees.[1]

An important implication of our analysis will be that it enables a comparison between:

[1] We do not discuss Karni and Schmeidler's (1991) contrasting approach, which considers sequential decisions within an "atemporal" framework involving compound lotteries rather than decision trees. They motivate this with the example of ascending bid auctions, where decisions presumably succeed each other rather swiftly. See also Volij (1994).

1. the parsimonious *normative* or *prescriptive* approach to decision theory embodied in just one potent consequentialist principle, whose appeal is that it can be used to derive important familiar axioms such as ordinality, independence, and the sure-thing principle;

2. the *positive* or *descriptive* approach, where a factorized set of many principles is used in order that violations of the expected utility principle can be more fully analysed and their psychological properties better understood.

Indeed, when assessing a descriptive decision theory, it is important to see which of its aspects fail empirically. We also note that, in experiments involving subjects such as rats, pigeons, primates, or very young children, it is hard to observe much beyond their actual behavior; for humans with sufficient ability to communicate verbally, however, we can ask their intentions, motivations, plans, etc. At least in principle, a theory of rational behavior can therefore be expanded in order to accommodate such additional information, especially any possible mismatch between planned and realized consequences. In particular, the theory should recognize that behavior could be irrational because it is the result of irrational planning, or because behavior departs from rational plans.

1.2. The Expected Utility (or EU) Hypothesis

The famous St. Petersburg paradox relies on the *risk-neutrality* hypothesis requiring one lottery to be preferred to another iff its expected (monetary) value is greater. The paradox originated in a 1713 letter by Nicolas Bernoulli to Pierre Raymond de Montmort. It considers a lottery with an infinite sequence of prizes equal to 2^n monetary units, for $n = 1, 2, \ldots$. The nth prize is assumed to occur with probability 2^{-n}, for $n = 1, 2, \ldots$. The lottery lacks an expected value because the sum $\sum_{n=1}^{\infty} 2^{-n} 2^n$ evidently diverges to $+\infty$.

The more general *expected utility* (or EU) hypothesis, however, is that one lottery is preferred to another iff the expected *utility* of its prizes is higher. This hypothesis appeared in Cramer's (1728) suggestion to Daniel Bernoulli (1738) for resolving St. Petersburg paradox by using either $v(w) = \min\{ w, 2^{24} \}$ or $v(w) = \sqrt{w}$ as a utility function of wealth w, for $w \geq 0$. For either of these two utility functions, a routine exercise shows that the expected utility $\sum_{n=1}^{\infty} 2^{-n} v(2^n)$ converges to a finite value.

For over 200 years thereafter the EU hypothesis was rarely used and poorly understood. The situation began to change only after the publication of von Neumann and Morgenstern's (1944, 1953) treatise. Even this, however, makes little use of the distinction between expected monetary payoffs and expected utility that was later clarified in the theory of risk aversion due to Arrow (1965) and Pratt (1965). In an appendix to their treatise, von Neumann and Morgenstern did make the first attempt to formulate a preference based axiom system that would justify expected utility, with expected payoff as a special case.[2]

Fishburn and Wakker (1995) carefully discuss the incompleteness of von Neumann and Morgenstern's attempt to axiomatize the EU hypothesis. This is because, as Dalkey (1949), Marschak (1950), Nash (1950), and Malinvaud (1952) soon pointed out, von Neumann and Morgenstern had left implicit a key "independence" axiom. Perhaps more to the point, the quick succession of papers by Marschak (1950), Arrow (1951a), Samuelson (1952), and Herstein and Milnor (1953) rapidly reduced von Neumann and Morgenstern's unnecessarily complicated axiom system. Finally, the process of refining these axioms reached a much more satisfactory culmination in Jensen (1967), who based expected utility on just the three principles of *ordering*, *independence*, and *continuity* — see also Fishburn (1970).[3]

Meanwhile, the EU framework rapidly became the dominant model for choice under risk in economics, finance, insurance, game theory, and beyond.

1.3. Paradoxes

The EU hypothesis implies some rather restrictive additive separability conditions. Indeed, non-additive utility functions are rather obviously generic within the space of all possible utility functions. Thus, as Samuelson (1983, pp. 503–518) in particular emphasized in his discussion, the expected utility "dogma" was seen as determining a special case, unlikely to be empirically justified.

Even while the axioms of EU theory were still being developed, its descriptive accuracy came under severe challenge. This was largely due to the results of one ingenious experiment that Allais (1953) designed. The experiment involved subjects who typically reported preferences in clear violation

[2]See Leonard (1995) for some of the relevant history, including a possible explanation (footnote 21 on p. 753) of why this appendix was changed for the second edition.

[3]The chapter in this *Handbook* by Karni sets out these principles as axioms, but uses "weak order" to describe what we call the ordering principle, and "Archimedean" to describe one version of our continuity principle.

of the key independence property which had only recently been properly formulated. This property rightly came to be seen as typically inconsistent with observed behavior, especially in a laboratory setting. Even more damaging for the EU hypothesis, Allais's results for the case of risky consequences were then supplemented by Ellsberg (1961) for the case when risk was combined with uncertainty.[4]

These and later examples due to Allais (1979), Kahneman and Tversky (1979) and others were all used to question not only the descriptive accuracy but also the prescriptive relevance of the EU model. Indeed, these "paradoxes" ignited a fruitful debate over what rationality means, especially in the face of risk and uncertainty. This chapter is intended in part to contribute to that debate.

1.4. Non-Expected Utility

Dropping the contentious independence axiom allowed the development of *"non-expected" utility* theories of behaviour under both risk and uncertainty, such as those surveyed by Starmer (2000), as well as later chapters in this *Handbook*. In experimental studies to date, the predictions of this theory do indeed accord much better with the available data, if only because non-expected utility typically allows many more free parameters than expected utility does.

The motivation for "non-expected" utility theory, however, is precisely to provide a more accurate descriptive model. Many authors may have claimed that, because non-expected utility is descriptively more accurate, that makes it somehow prescriptively more appealing. This argument, however, strikes as philosophically suspect, not least because it crosses the fact/value divide that some philosophers refer to as Hume's Law.

Furthermore, independence is only one of Jensen's three axioms for EU. The two other ordering and continuity axioms have remained largely unquestioned, even though they actually remain as key postulates for non-expected as well as expected utility theory. In this chapter we propose to focus on normative or prescriptive decision theory, treating both ordering and independence as axioms that should be justified. If we neglect continuity, it is

[4] Ellsberg regarded his example as contradicting Savage's sure thing postulate. Yet it might be more accurate to regard it as contradicting Anscombe and Aummann's (1963) extension of that postulate, which was not even published until two years after Ellsberg (1961).

only because it is really a technical topological axiom of a kind which makes it hard to observe when the axiom is violated.

1.5. Chapter Outline

Following this introduction, the first part of the chapter provides some essential background regarding the theory of rational choice in static settings. The two Sections 2 and 3 are intended to remind the reader of the standard "static" approach to rational planning that is encountered in most microeconomics textbooks. In this approach, following von Neumann's (1928) definition of a game in normal form, the decision maker is typically assumed to choose a planned course of action once and for all. Section 2 focuses on the case when actions have deterministic consequences; Section 3 allows actions to have risky consequences described by lotteries over the consequence domain. For each of these two cases we also briefly review some of the experimental tests of the usual ordinality and independence conditions that standard economic theory imposes as postulates in these settings.

The heart of the chapter consists of Sections 4 and 5, which together introduce the dynamic considerations that arise whenever the decision maker is confronted by a non-trivial decision tree. These two sections survey the main attempts to provide axiom systems specific to decision trees that can be used to justify the usual rationality postulates in static settings, which were considered in Sections 2 and 3. As with those sections, we divide the discussion between decision trees with consequences that are:

1. purely deterministic, which are used in Section 4 to offer a possible justification for the ordinality principle (of making choices that maximize a complete and transitive preference ordering);

2. risky, which are used in Section 5 to offer a possible justification for a version of the contentious independence axiom that distinguishes expected from non-expected utility theory;[5]

3. uncertain, which are used in Section 6 to offer a possible justification for Savage's sure thing principle that plays a large role in the theory of subjective probability.

[5]Because we refrain from discussing any continuity issues, we cannot even attempt to offer a compelling justification for the existence of a utility function.

Apart from one particular "consequentialist invariance" postulate developing earlier work by the first author, we will focus on Cubitt's (1996) "factorization" of this postulate into the following set of five axioms, which together are logically equivalent to consequentialist invariance:

1. dynamic consistency, discussed in Sections 4.3.2, 5.1.4 and 6.4.4;
2. separability, discussed in Sections 4.3.3, 5.1.5 and 6.4.5;
3. reduction of compound lotteries, discussed in Section 5.1.3, and the closely related reduction of compound events, discussed in Section 6.4.3;
4. invariance to the timing of risk or uncertainty, discussed in Sections 5.1.6 and 6.4.6;
5. reduction of sequential choice, discussed in Section 4.3.4.

Actually, following Cubitt's (1996) own suggestion, we even factorize the axioms further by introducing two distinct versions of dynamic consistency, as well as two distinct versions of separability, which respectively apply at decision and chance nodes of the relevant decision tree. We confirm that a relevant subset of our version of these axioms, when applied in a very restricted domain of decision trees that contain no more than two decision nodes, imply ordinal choice; furthermore, in the case of trees with risky or uncertain consequences, the same axioms imply the contentious vNM independence axiom or the sure thing principle, even when decision trees with at most one decision node and at most two chance or event nodes are considered.

To establish logical equivalence with consequentialist invariance, in Section 4.5 we formulate this condition in a way that applies to general decision trees. This sets the stage for one elementary result establishing that consequentialist invariance implies our versions of all Cubitt's axioms that are relevant in the different cases. Closing the logical circle, however, requires demonstrating that Cubitt's axioms imply consequentialist invariance, which is rather more challenging. One obstacle is that Cubitt's axioms apply to plans, whereas consequentialist invariance applies to the realized consequences of behavior. Nevertheless, given any preference ordering on a consequence domain, in Section 4.6 on "ordinal dynamic programming" we are able to construct a behavior rule that:

1. at every decision node of every finite tree with consequences in that domain, prescribes a non-empty set of moves to an immediately succeeding node;

2. determines a consequence choice function satisfying consequentialist invariance, with the property that the chosen consequences maximize the given preference ordering.

Then in Section 5, where we allow decision trees with chance nodes and risky consequences, we are able to establish that, given any preference ordering on a lottery consequence domain that satisfies vNM independence, the same two properties hold. A similar result is shown in Section 6, where we allow decision trees with event nodes and uncertain consequences

Logic alone dictates that, among the several normatively appealing dynamic choice principles enunciated in Cubitt (1996), those that imply the ordinality and independence properties must share the same descriptive shortcomings as the EU hypothesis. It took until the late 1990s, however, before the first experiments were carried out that test these dynamic choice principles systematically. A major justification of Cubitt's choice of axioms is that, when subjects in experiments are observed to violate either ordinality, or independence, or perhaps even both, seeing which of his larger axiom set is violated can help shed light on possible psychological or other explanations of the "anomalous" behavior. With this in mind, the latter parts of Sections 4 and 5 briefly review some of the experiments which have set out to test the axioms that underlie this dynamic approach to rationality, rather than the static theory discussed in Sections 2 and 3.

2. Static Rationality with Deterministic Consequences

2.1. Preferences in Consumer Theory

In standard economics, especially consumer demand theory in microeconomics, the "ordinalist revolution" of the 1930s (see Cooter and Rapoport, 1984) saw rationality being defined as choosing, within a feasible set determined by conditions such as a budget constraint and nonnegativity conditions, a consumption bundle $\mathbf{x} = (x_g)_{g \in G}$ in a finite-dimensional Euclidean commodity space $\mathbb{R}^G$ which maximizes a (complete and transitive) preference ordering $\succsim$ on $\mathbb{R}^G$. Typically it is assumed that $\succsim$ also satisfies the monotonicity property requiring more of any good to be preferred to less, at least weakly. Moreover, it is often also assumed that the upper contour set $\{\mathbf{x} \in \mathbb{R}^G \mid \mathbf{x} \succsim \bar{\mathbf{x}}\}$ is convex, for each fixed $\bar{\mathbf{x}} \in \mathbb{R}^G$.

A little more restrictively, behavior should maximize a utility function $\mathbf{x} \to U(\mathbf{x})$ mapping $\mathbb{R}^G$ into $\mathbb{R}$ that is strictly increasing, or at least non-decreasing, in the consumption quantity x_g of each good $g \in G$. In that case behavior will be unchanged whenever the utility function to be maximized is replaced by any other that is "ordinally equivalent" in the sense of representing the same preference ordering — i.e, by any new utility function $\mathbf{x} \mapsto \tilde{U}(\mathbf{x}) = \phi(U(\mathbf{x}))$ that results from applying a strictly increasing transformation $U \mapsto \phi(U)$ to the old utility function U.

2.2. Consequence Choice Functions

Later developments in decision theory extended this preference-based approach from consumer theory, where the objects of preference are consumption bundles, to completely general choice settings, where the objects of preference are abstract consequences belonging to an arbitrary domain. As Arrow (1951b, p. 404) writes:[6]

> The point of view will be that of a theory of choice, as it is usually conceived of. The general picture of such a theory is the following: There is a set of conceivable actions which an individual could take, each of which leads to certain consequences. . . . Among the actions actually available, then, that action is chosen whose consequences are preferred to those of any other available action.

This is a rather clear statement of the doctrine for which Anscombe (1958) introduced the neologism "consequentialism" in her forceful critique. The idea, of course, is much older.

From now on, we consider an arbitrary domain Y of consequence choices that are relevant to the decision maker. For each non-empty finite *feasible set* $F \subseteq Y$, let $C(F) \subseteq F$ denote the corresponding *choice set* of consequences deemed suitable, even "rational", choices from the set F.

---

[6]We remark that Arrow went on to invoke the existence of a preference ordering over consequences. One advantage of using consequentialist invariance as a basic rationality principle is that it allows ordinality to be derived as a logical implication rather than assumed as a questionable postulate.

Definition 1. A *choice function* on Y is a mapping $F \mapsto C(F)$ which is defined on the domain $\mathcal{F}(Y)$ of all non-empty finite subsets of Y, and satisfies $C(F) \subseteq F$ for all finite $F \in \mathcal{F}(Y)$. We focus on the important special case when the choice function $F \mapsto C(F)$ is *decisive* in the sense that $C(F) \neq \emptyset$ whenever F is non-empty and finite.

2.3. Base Preference Relations

Given the choice function $F \mapsto C(F)$, we can define the associated *base relation* $\succsim^C$ as the unique binary *weak preference* relation on Y satisfying

$$a \succsim^C b \Longleftrightarrow a \in C(\{a, b\}) \tag{1}$$

for all $a, b \in Y$. Thus, $a \succsim^C b$ just in case the decision maker, when required to choose between a and b with no other consequences possible, is willing to choose a.

Note that, assuming the choice function $F \mapsto C(F)$ is indeed decisive, especially when F is a pair set $\{a, b\}$, it follows that the base relation $\succsim^C$ is *complete* in the sense that, for all $a, b \in Y$, either $a \succsim^C b$, or $b \succsim^C a$, or both. Indeed, given any pair $a, b \in Y$, decisiveness allows one to distinguish between three separate cases:

1. $C(\{a, b\}) = \{a\}$, in which case we say that a is *strictly preferred* to b and write $a \succ^C b$;

2. $C(\{a, b\}) = \{b\}$, in which case we say that a is *strictly dispreferred* to b and write $a \prec^C b$;

3. $C(\{a, b\}) = \{a, b\}$, in which case we say that a and b are *indifferent* and write $a \sim^C b$.

2.4. Arrow's Conditions for Ordinality

Definition 2. The choice function $F \mapsto C(F)$ is said to be *ordinal* when

$$C(F) = \{a \in F \mid b \in F \Longrightarrow a \succsim^C b\}. \tag{2}$$

That is, the set $C(F)$ consists of all consequences $a \in F$ that are "optimal" in the sense of being weakly preferred to any alternative $b \in F$ according to the base relation $\succsim^C$ which is derived from choice among pairs. In order to simplify (2), we make the innocuous postulate that the relation $\succsim^C$ is *reflexive* in the sense that $a \succsim^C a$ for all $a \in Y$.[7]

The base relation $\succsim^C$ is *transitive* just in case, for all $a, b, c \in Y$, it is true that $a \succsim^C b$ and $b \succsim^C c$ jointly imply that $a \succsim^C c$.

Arrow (1959) characterized ordinal choice functions as those that satisfy the condition

$$[\ G \subset F \subseteq Y \text{ and } C(F) \cap G \neq \emptyset\ ] \Longrightarrow C(F) \cap G = C(G) \tag{3}$$

that he called (C5). It is useful to break this single condition into two parts:

Contraction Consistency

$$[\ G \subset F \subseteq Y \text{ and } C(F) \cap G \neq \emptyset\ ] \Longrightarrow C(F) \cap G \subseteq C(G) \tag{4}$$

Expansion Consistency

$$[\ G \subset F \subseteq Y \text{ and } C(F) \cap G \neq \emptyset\ ] \Longrightarrow C(G) \subseteq C(F) \cap G \tag{5}$$

as discussed by Bordes (1976) and Sen (1977), who call these two conditions α and β^+ respectively.[8]

By considering the case when the subset $G \subseteq F$ is the pair $\{a, b\}$, we obtain the following "pairwise" variations of the contraction and expansion consistency conditions (4) and (5), respectively. First, the choice function $F \mapsto C(F)$ satisfies the principle of *pairwise contraction consistency* if, for any finite non-empty feasible set $F \subset Y$ one has

$$[\ a \in C(F) \text{ and } b \in F\ ] \Longrightarrow a \succsim^C b. \tag{6}$$

Second, the choice function $F \mapsto C(F)$ satisfies the principle of *pairwise expansion consistency* if, for any finite non-empty feasible set $F \subset Y$, one has

$$\left[\ b \in C(F) \text{ and } a \in F \text{ with } a \succsim^C b\ \right] \Longrightarrow a \in C(F). \tag{7}$$

[7] If the base relation $\succsim^C$ were not reflexive, we could change equation (2) to $C(F) = \{a \in F \mid b \in F \setminus \{a\} \Longrightarrow a \succsim^C b\}$. But as noted by Mas-Colell, Whinston and Green (1995, page 6 footnote 2), amongst others, it really loses no generality to assume that $\succsim^C$ is reflexive, in which case this change to equation (2) is irrelevant.

[8] The contraction consistency condition was first propounded by Chernoff (1954); sometimes it is referred to as "Chernoff's choice axiom".

Theorem 1. *Provided the choice function $F \mapsto C(F)$ is decisive, the following three conditions are equivalent:*

(a) *C satisfies both contraction consistency and expansion consistency;*

(b) *C satisfies both pairwise contraction consistency and pairwise expansion consistency;*

(c) *C is ordinal, and the base relation $\succsim^C$ is transitive.*

Proof. (c) $\Longrightarrow$ (a): Routine arguments show that, if C is ordinal (i.e., satisfies (2)) and if also the base relation $\succsim^C$ is transitive, then C must satisfy both conditions (4) and (5).

(a) $\Longrightarrow$ (b): This is an obvious implication of definitions (4) and (5) when one puts $G = \{a, b\}$.

(b) $\Longrightarrow$ (c): Suppose first that $a \in F$ and that $a \succsim^C b$ for all $b \in F$. Because C is decisive, there exists $a^* \in C(F)$. Then $a \succsim^C a^*$ and so the definition (7) of pairwise expansion consistency implies that $a \in C(F)$. On the other hand, if $a \in C(F)$ and $b \in F$, then the definition (6) of pairwise contraction consistency immediately implies that $a \succsim^C b$. It follows that (2) is satisfied, so C is ordinal.

Next, suppose that $a, b, c \in Y$ satisfy $a \succsim^C b$ and $b \succsim^C c$. Define F as the triple $\{a, b, c\}$. By the hypothesis that C is decisive, there are three logically possible cases, not necessarily disjoint:

1. $a \in C(F)$: Here pairwise contraction consistency implies directly that, because $c \in F$, so $a \succsim^C c$.

2. $b \in C(F)$: Here, because $a \succsim^C b$, pairwise expansion consistency implies that $a \in C(F)$. But then case 1 applies and so $a \succsim^C c$.

3. $c \in C(F)$: Here, because $b \succsim^C c$, pairwise expansion consistency implies that $b \in C(F)$. But then case 2 applies and so $a \succsim^C c$.

Hence $a \succsim^C c$ in every case, and so the relation $\succsim^C$ is transitive. □

2.5. Experimental Tests of Ordinality

2.5.1. Background

Many early experimental studies of economic behavior focused on the ordinality property and questioned the existence of coherent or stable preferences as such. This subsection summarizes a few puzzling empirical phenomena that emerged. We recall some of this older literature first and give references to further survey papers. Where time is involved, it is typically through consideration of timed consequences, which will be considered in Section 4.7.1. Where time plays a more crucial role by allowing the decision maker to face successive choices in a non-trivial decision tree, we defer our discussion till Section 5.5.

2.5.2. Two-Dimensional Choice Problems

The problem of choosing a single real number, when more is better than less, is not very interesting. So we consider choice among options whose consequences differ in more than one characteristic or attribute. For example, many experiments involve risky consequences in the form of a binary lottery where one prize is 0. That still leaves two dimensions: (i) a positive monetary prize; (ii) the probability of winning that prize. In marketing, the products such as cars, drinks, or other goods, typically differ in both quality and affordability, with the latter often measured as the inverse of price.

Accordingly, we start by considering choice problems where the consequences are represented by points in the two-dimensional space $\mathbb{R}^2$. We will be interested in the *dominance relation* $>_D$ defined on $\mathbb{R}^2$ by

$$(a_1, a_2) >_D (b_1, b_2) \iff a_1 > b_1 \text{ and } a_2 > b_2 \tag{8}$$

We typically assume that choice involves *undominated* options — i.e., for every non-empty finite set $F \subseteq Y$, one has $y^* \in C(F)$ only if $y \in F$ and there is no *dominant* alternative $y \in F$ such that $y >_D y^*$. We do *not* assume, however, that $C(F)$ includes every undominated option in F.

2.5.3. The Attraction Effect

In Section 2.4 it was shown that the ordinality principle of rational choice holds if and only if choice behavior satisfies both contraction and expansion consistency. These two properties were challenged by Huber et al. (1982) in an early example of choice inconsistency. Their example involves an original feasible set $G = \{c, t\}$, where c is called the competitor and t is called the target. It is presumed that neither of these two dominates the other.

Consider now expanding the set G by appending a third "decoy" alternative d that is strictly dominated by alternative t. The feasible set expands to become $F = \{c, t, d\}$.

In the experiments that Huber et al. (1982) reported in this setting, a majority of subjects' choices were observed to satisfy $C(G) = \{c\}$ and $C(F) = \{t\}$. Clearly $G \subset F$ and $C(F) \cap G = \{t\} \neq \emptyset$, yet

1. $\{c\} = C(G) \not\subseteq C(F) \cap G = \{t\}$, so expansion consistency is violated;
2. $\{t\} = C(F) \cap G \not\subseteq C(G) = \{c\}$, so contraction consistency is also violated.

Huber et al. (1982) propose an "attraction effect" as a potential explanation for such behavior. Because t dominates d, it appears superior not only to d, but more generally; in particular, t also seems superior to c, which neither dominates nor is dominated by either t or d. In contrast, when d unavailable in the original choice between c or t, this reason to perceive t as superior disappears, allowing c to be chosen.

2.5.4. The Compromise Effect

A related "compromise effect" was discussed by Simonson (1989). As before, the example starts with the feasible set $G = \{c, t\}$, where neither c nor t dominates the other. But now the third option f that is appended to G is assumed to be dominated by both c and t in an attribute for which c dominates t. Assuming this is attribute 1 in our two-dimensional model, this means that $c_1 > t_1 > f_1$. To make the example interesting, we assume that $c_2 < t_2 < f_2$. This ordering of the attributes of the different options can make t appear as a good compromise between c and f when t is feasible. This can explain why $C(\{c, t, f\}) = \{t\}$ may be observed even though originally one had $C(G) = \{c\}$.

2.5.5. Attempted Explanations

Such compromise or attraction effects preclude ordinal choice. One strand of subsequent literature seeks to explain such incoherent choice behavior and other related phenomena by means of more sophisticated psychological models (see, e.g., Tsetsos et al. 2010). Another strand of the literature appeals to heuristics such as Tversky's (1972) lexicographic procedure (see also Gigerenzer et al. 1999) or other boundedly rational choice procedures (see Manzini and Mariotti, 2007, and the references therein).

3. Static Rationality with Risky Consequences

3.1. The Mixture Space of Risky Roulette Lotteries

The EU hypothesis offers a normative principle for decision problems where the consequences y in an arbitrary domain Y are risky. Formally, this means that the objects of choice are as in the following:

Definition 3. A *consequence lottery* λ attaches to each consequence $y \in Y$ a specified "objective" probability $\lambda(y) \geq 0$, where:

1. $\lambda(y) > 0$ iff $y \in \operatorname{supp} \lambda$ for some finite *support* set $\operatorname{supp} \lambda \subseteq Y$;

2. $\sum_{y \in Y} \lambda(y) = \sum_{y \in \operatorname{supp} \lambda} \lambda(y) = 1$.

Let $\mathcal{L}$ or $\Delta(Y)$ denote the set of all such "roulette" lotteries.[9] Given any two lotteries $\lambda, \mu \in \Delta(Y)$, for each number $\alpha \in [0,1] \subset \mathbb{R}$ there exists a *mixture* $\alpha\lambda + (1-\alpha)\mu$ of the two lotteries defined by

$$[\alpha\lambda + (1-\alpha)\mu](y) := \alpha\lambda(y) + (1-\alpha)\mu(y) \quad \text{for all } y \in Y \tag{9}$$

which evidently also belongs to $\Delta(Y)$. This property of *closure under mixing* makes the set $\Delta(Y)$ of consequence lotteries a *mixture space* (see Herstein and Milnor, 1953).

For each $y \in Y$, let δ_y denote the unique *degenerate lottery* that satisfies $\delta_y(y) = 1$. Note that any $\lambda \in \Delta(Y)$ can be expressed in the form $\lambda = \sum_{y \in Y} \lambda(y)\, \delta_y$. It follows that $\Delta(Y)$ is trivially isomorphic to the convex hull of the collection of degenerate lotteries δ_y $(y \in Y)$, regarded as unit vectors in the real linear space spanned by these points. Indeed, each degenerate lottery δ_y is an extreme point of the convex set $\Delta(Y)$.

We remark finally that, because of part (1) in definition 3, the *expectation* $\mathbb{E}_\lambda f$ of any function $Y \ni y \mapsto f(y) \in \mathbb{R}$ w.r.t. any lottery $\lambda \in \Delta(Y)$ is well defined as the sum

$$\mathbb{E}_\lambda f := \sum_{y \in Y} \lambda(y) f(y) = \sum_{y \in \operatorname{supp} \lambda} \lambda(y) f(y) \tag{10}$$

of finitely many non-zero terms.

Moreover, let $\mathcal{L}^*(Y)$ denote the entire linear space spanned by the set $\{\delta_y \mid y \in Y\}$ of all degenerate lotteries on Y. Then for any fixed $f : Y \to \mathbb{R}$, the mapping $\mathcal{L}^*(Y) \ni \lambda \mapsto \mathbb{E}_\lambda f$ is still well defined by (10), and is obviously

[9] Following Anscombe and Aumann (1963), we use the term "roulette" lottery when each outcome occurs with a probability which is presumed to be objectively specified. This is in contrast to a "horse" lottery whose outcomes depend on an unknown state of the nature. This corresponds to one possible interpretation of Knight's (1921) famous distinction between risk, with objective probabilities, and uncertainty where, if probabilities exist at all, they are subjective — or perhaps better, following Savage (1954), "personal".

linear in λ. Its restriction to the domain $\Delta(Y)$ of lotteries then satisfies the *mixture preservation* property that

$$\mathbb{E}_{\alpha\lambda+(1-\alpha)\mu} f = \alpha \mathbb{E}_\lambda f + (1-\alpha)\mathbb{E}_\mu f \tag{11}$$

whenever $\lambda, \mu \in \Delta(Y)$ and $\alpha \in [0,1] \subset \mathbb{R}$.

3.2. The Expected Utility Hypothesis

In recent decades it has become commonplace for economists to extend the expected utility (or EU) hypothesis for wealth, as discussed in Section 1.2, to an arbitrary consequence domain Y, and to the set $\Delta(Y)$ of consequence roulette lotteries on Y.

Definition 4. A *utility* function $V : \mathcal{L} \to \mathbb{R}$ is said to *represent* the preference relation $\succsim$ on $\mathcal{L} = \Delta(Y)$ just in case, for any pair of lotteries $\lambda, \mu \in \mathcal{L}$, one has

$$\lambda \succsim \mu \Longleftrightarrow V(\lambda) \geq V(\mu). \tag{12}$$

The *EU hypothesis* postulates the existence of a (compete and transitive) preference ordering $\succsim$ on $\Delta(Y)$ that is represented by the *expected value*

$$V(\lambda) := \mathbb{E}_\lambda v := \sum\nolimits_{y \in Y} \lambda(y)\, v(y) \tag{13}$$

of a *von Neumann–Morgenstern utility function* (or NMUF) $v : Y \to \mathbb{R}$.

Combining (12) and (13), it is evident that the EU hypothesis entails

$$\lambda \succsim \mu \Longleftrightarrow \mathbb{E}_\lambda v \geq \mathbb{E}_\mu v \Longleftrightarrow V(\lambda) - V(\mu) = \sum\nolimits_{y \in Y} [\lambda(y) - \mu(y)]\, v(y) \geq 0. \tag{14}$$

3.3. Implications of the Expected Utility Hypothesis

3.3.1. Utility Transformations

A *strictly increasing affine transformation* of utility is a mapping $v \mapsto \alpha + \rho v$, where α and ρ are real constants, with $\rho > 0$. If any such transformation is applied to the original NMUF $y \mapsto v(y)$, thereby producing another new NMUF $y \mapsto \tilde{v}(y) \equiv \alpha + \rho v(y)$ that is "cardinally equivalent", then expected utility maximizing behavior will be the same regardless of whether one uses the old or the new NMUF.

3.3.2. Von Neumann–Morgenstern Independence

The following independence condition is the one that von Neumann and Morgenstern neglected to state. Nevertheless, it is often given their name because it is a key ingredient for their axiomatic justification of the EU hypothesis.

Definition 5. The preference ordering $\succsim$ on $\Delta(Y)$ satisfies the (vNM) *independence* principle provided that, for all lottery triples $\lambda, \mu, \nu \in \Delta(Y)$ and all scalars $\alpha \in (0, 1] \subset \mathbb{R}$, we have

$$\lambda \succsim \mu \Longleftrightarrow \alpha\lambda + (1-\alpha)\nu \succsim \alpha\mu + (1-\alpha)\nu \tag{15}$$

In particular, independence requires the preference between the two lotteries $\alpha\lambda + (1-\alpha)\nu$ and $\alpha\mu + (1-\alpha)\nu$ to be independent of the risky component ν that is common to the two lotteries.

Theorem 2. *If the EU hypothesis holds, then the preference ordering $\succsim$ on $\Delta(Y)$ satisfies (15) for all $\lambda, \mu, \nu \in \Delta(Y)$ and all $\alpha \in (0, 1]$.*

Proof. Modifying (14) appropriately, we see that

$$\begin{aligned} & V(\alpha\lambda + (1-\alpha)\nu) - V(\alpha\mu + (1-\alpha)\nu) \\ = \; & \alpha \textstyle\sum_{y \in Y}[\lambda(y) - \mu(y)]\, v(y) = \alpha[V(\lambda) - V(\mu)] \end{aligned}$$

So, because $\alpha > 0$, the equivalence (15) follows from (12) and (13). $\square$

3.3.3. Archimedean Continuity

Definition 6. The preference ordering $\succsim$ on $\Delta(Y)$ satisfies the *Archimedean continuity* principle[10] provided that, for all lottery triples $\lambda, \mu, \nu \in \Delta(Y)$ where $\lambda \succ \nu \succ \mu$, there exist scalars $\alpha', \alpha'' \in (0, 1)$ such that

$$\alpha'\lambda + (1-\alpha')\mu \succ \nu \succ \alpha''\lambda + (1-\alpha'')\mu \tag{16}$$

Theorem 3. *If the EU hypothesis holds, then the preference ordering $\succsim$ on $\Delta(Y)$ satisfies (16) for all $\lambda, \mu, \nu \in \Delta(Y)$.*

[10] The principle is also often referred to as *Jensen continuity* (cf. Jensen, 1967).

Proof. Consider any lottery triple $\lambda, \mu, \nu \in \Delta(Y)$ where $\lambda \succ \nu \succ \mu$, and so $V(\lambda) > V(\nu) > V(\mu)$. Because the mapping $\lambda \mapsto V(\lambda)$ defined by (13) is continuous in the probabilities $\lambda(y)$ of the different consequences $y \in Y$, there must exist scalars $\alpha', \alpha'' \in (0,1)$, with α' close to 1 and α' close to 0, such that

$$\alpha' V(\lambda) + (1-\alpha')V(\mu) > V(\nu) > \alpha'' V(\lambda) + (1-\alpha'')V(\mu) \tag{17}$$

Because V satisfies mixture preservation, this implies that

$$V(\alpha'\lambda + (1-\alpha')\mu) > V(\nu) > V(\alpha''\lambda + (1-\alpha'')\mu)$$

and so (16) must hold. □

3.4. Jensen's Three Axioms

Jensen's (1967) three axioms can be stated as follows:

Ordering: The binary relation $\succsim$ on $\Delta(Y)$ is an *ordering* — i.e., it satisfies the following three properties:

1. for all $\lambda \in \Delta(Y)$, one has $\lambda \succsim \lambda$ (so $\succsim$ is *reflexive*);
2. for all $\lambda, \mu \in \Delta(Y)$, one has $\lambda \succsim \mu$, or $\mu \succsim \lambda$, or both (so $\succsim$ is *complete*);
3. for all $\lambda, \mu, \nu \in \Delta(Y)$, if $\lambda \succsim \mu$ and $\mu \succsim \nu$, then $\lambda \succsim \nu$ (so $\succsim$ is *transitive*).

Independence: For all $\lambda, \mu, \nu \in \Delta(Y)$ and $\alpha \in (0,1)$, one has

$$\alpha\lambda + (1-\alpha)\nu \succsim \alpha\mu + (1-\alpha)\nu \Longleftrightarrow \lambda \succsim \mu.$$

Continuity: For all $\lambda, \mu, \nu \in \Delta(Y)$ with $\lambda \succ \mu$ and $\mu \succ \nu$, the two sets

$$\{\alpha \in [0,1] \mid \alpha\lambda + (1-\alpha)\nu \succsim \mu\} \text{ and } \{\alpha \in [0,1] \mid \alpha\lambda + (1-\alpha)\nu \precsim \mu\}$$

of mixtures of λ and ν which are weakly preferred (resp. dispreferred) to μ are both closed.

The following important characterization theorem states that Jensen's three axioms are not only necessary, but also sufficient, for EU to hold for preferences on the mixture space $\Delta(Y)$.

Theorem 4. *Assume that $\succsim$ is an arbitrary binary preference relation on the set $\mathcal{L} := \Delta(Y)$ of simple roulette lotteries over consequences in the domain Y. The following two statements are equivalent:*

(i) The preference relation $\succsim$ on $\Delta(Y)$ is represented by the expected value of each von Neumann Morgenstern utility function $Y \ni y \mapsto v(y)$ in a cardinal equivalence class. Moreover, this equivalence class is unique except in the trivial case where $\succsim$ induces at most two indifference classes among the set δ_y $(y \in Y)$ of degenerate lotteries.

(ii) The preference relation $\succsim$ on $\Delta(Y)$ satisfies Jensen's three axioms of ordering, independence and continuity.

Proof. See elsewhere in this *Handbook*, or else consult Hammond (1998a). □

3.5. Experimental Tests

3.5.1. Preference Reversal under Risk

Consider the often discussed case when the consequence domain Y consists of real numbers y representing monetary amounts, and more money is preferred to less. In this case, assuming there is a continuous preference ordering $\succsim$ over $\Delta(Y)$ the *certainty equivalent* of any lottery $\lambda \in \Delta(Y)$ is defined as the unique $y(\lambda) \in \mathbb{R}$ such that λ is indifferent to the degenerate lottery $\delta_{y(\lambda)}$. Note that uniqueness is assured because with more money always being preferred to less, one always has $\delta_{y'} \succ \delta_{y''} \Longleftrightarrow y' > y''$.

Starting with Lichtenstein and Slovic (1971), several experimenters have asked subjects to report their certainty equivalents for different lotteries, and have then noticed a "preference reversal phenomenon" (see also Grether and Plott, 1979; Tversky et al., 1990; List, 2002; Butler and Loomes, 2007; for a review see Seidl, 2002). This phenomenon occurs when there are two lotteries $\lambda, \mu \in \Delta(Y)$ such that the subject claims that $\lambda \succ \mu$, and yet the same subject's reported certainty equivalents satisfy $y(\lambda) < y(\mu)$. For example, the lottery λ might offer a high probability of winning a moderate monetary prize, which is often preferred to a lottery μ that offers a moderate probability of winning a high monetary prize. Yet often the elicited certainty equivalent of λ is lower than that of μ. The opposite reversal is also observed, but less often.

If all these claims made by a typical subject were valid, one would of course have the preference cycle

$$\lambda \succ \mu \sim \delta_{y(\mu)} \succ \delta_{y(\lambda)} \sim \lambda$$

which contradicts ordinality. Such reversals may indicate that preferences are context dependent in a way that makes the elicited certainty equivalents induce a ranking which differs from preferences (Tversky et al. 1988). This is still an area of active research (Loomes et al., 2010; Plott and Zeiler, 2005; Isoni et al., 2011).

3.5.2. The Allais Paradox

Preference reversal calls into question the existence of a single preference ordering that explains statements regarding both preference and certainty equivalents. We now move on to a test of the independence condition, based on the following challenge for decision theorists originally issued by Allais (1953, p. 527):[11]

1. Do you prefer situation A to situation B?

   SITUATION A: The certainy of receiving 100 million.

   $$\textbf{SITUATION B} \begin{cases} \text{10 chances out of 100 of winning 500 million.} \\ \text{89 chances out of 100 of winning 100 million.} \\ \text{1 chance out of 100 of winning nothing.} \end{cases}$$

2. Do you prefer situation C to situation D?

   $$\textbf{SITUATION C} \begin{cases} \text{11 chances out of 100 of winning 100 million.} \\ \text{89 chances out of 100 of winning nothing.} \end{cases}$$

   $$\textbf{SITUATION D} \begin{cases} \text{10 chances out of 100 of winning 500 million.} \\ \text{90 chances out of 100 of winning nothing.} \end{cases}$$

He reports the results of an informal survey as follows:

> Now, and precisely for the majority of very prudent people, . . . whom common opinion considers very rational, the observed responses are $A > B$, $C < D$.

Thus did Allais's subjects (and many others since) express their unwillingness to move from A to B by giving up a 0.11 chance of winning 100 million in exchange for the same chance of a winning a lottery ticket offering a conditional probability $\frac{10}{11}$ of winning 500 million, but a conditional probability

[11] The translation from the French original is our own. The monetary unit was the old French franc, whose exchange rate during 1953 was about 350 to the US dollar. So these were large hypothetical gambles.

$\frac{1}{11}$ of not winning anything. In preferring D to C, however, they are willing to have the probability of winning any prize fall from 0.11 to 0.10 provided that the size of that prize rises from 100 million to 500 million.

The preference domain here is $\Delta(Y)$, which consists of lotteries over the consequence domain

$$Y = \{a, b, c\} = \{5, 1, 0\} \cdot 10^8$$

of monetary prizes. We recall the notation δ_y for the degenerate lottery which yields y with probability 1. Then Allais's two lottery comparisons A vs. B and C vs. D can be expressed in the form

$$\begin{array}{rclcrcl} \lambda_A & = & \delta_b & \text{vs.} & \lambda_B & = & (1-\alpha)\delta_b + \alpha\mu \\ \lambda_C & = & (1-\alpha)\delta_c + \alpha\delta_b & \text{vs.} & \lambda_D & = & (1-\alpha)\delta_c + \alpha\mu \end{array} \tag{18}$$

respectively, where $\alpha := 0.11$ and $\mu := (1-\alpha')\delta_a + \alpha'\delta_c$ with $\alpha' := \frac{1}{11}$. Now the independence axiom gives the chain of logical equivalences

$$\lambda_D \succ \lambda_C \iff \mu \succ \delta_b \iff \lambda_B \succ (1-\alpha)\delta_b + \alpha\delta_b = \delta_b = \lambda_A \tag{19}$$

which violates the preferences $\lambda_A \succ \lambda_B$ and $\lambda_D \succ \lambda_C$ that Allais reports.[12]

3.5.3. The Common Consequence Effect

The Allais paradox is a particular instance of the *common consequence effect* concerning three lotteries $\lambda, \mu, \nu \in \Delta(Y)$, and the observation that, given $\alpha \in (0, 1)$ sufficiently small, a decision maker's preferences often seem to satisfy:

1. when the "common consequence" ν is sufficiently close to λ, where the left-hand lottery collapses to just λ, then

$$\alpha\lambda + (1-\alpha)\nu \succ \alpha\mu + (1-\alpha)\nu \tag{20}$$

   which would imply $\lambda \succ \mu$ if the independence axiom were satisfied;

2. when ν is sufficiently worse than λ, then

$$\alpha\lambda + (1-\alpha)\nu \prec \alpha\mu + (1-\alpha)\nu \tag{21}$$

   which would imply $\lambda \prec \mu$ if the independence axiom were satisfied.

[12] Allais (1953) regards this as violating "Savage's postulate", though he also writes of "Samuelson's substitutability principle", which seems more accurate.

These preferences are of course entirely consistent with (18) provided that we take $\lambda = \delta_b$ and either $\nu = \delta_b$ in case (20) or $\nu = \delta_c$ in case (21).

The common consequence effect occurs when the first pair of lotteries being compared involve mixtures sharing a common good consequence that gets replaced by a common bad consequence in forming the second pair of lotteries being compared. Machina (1989) interprets this as violating a separate principle that he calls "replacement separability".

3.5.4. The Common Ratio Effect

Like the Allais paradox of Section 3.5.2, the common ratio effect involves three distinct consequences $a, b, c \in Y$ such that preferences over the corresponding degenerate lotteries satisfy $\delta_a \succ \delta_b \succ \delta_c$. Given any two constants $p, q \in (0, 1)$, consider the following two choices between pairs of alternative lotteries:

$$\begin{array}{rclcrcl} \lambda & := & \delta_b & \text{vs.} & \mu & := & p\delta_a + (1-p)\delta_c \\ \lambda' & := & q\delta_b + (1-q)\delta_c & \text{vs.} & \mu' & := & qp\delta_a + (1-qp)\delta_c \end{array} \tag{22}$$

Note that the Allais paradox is a special case where the consequences a, b, c are three monetary prizes respectively equal to 500 million, 100 million, and 0 old French francs, whereas the numerical mixture weights are $p = \frac{10}{11} = 0.90909\ldots$ and $q = 0.11$ (implying that $pq = 0.1$).

The example owes its name to the existence of a *common ratio*

$$q = \lambda'(b)/\lambda(b) = \mu'(a)/\mu(a) \tag{23}$$

between the probabilities of the most favorable two outcomes. Note too that

$$\lambda' = q\lambda + (1-q)\delta_c \quad \text{and} \quad \mu' = q\mu + (1-q)\delta_c. \tag{24}$$

In particular, the common ratio q is the weight attached to both lotteries λ and μ in forming the respective mixtures λ' and μ' of λ and μ with δ_c.

Of course the vNM independence axiom implies that

$$\lambda \succ \mu \iff q\lambda + (1-q)\delta_c \succ q\mu + (1-q)\delta_c \iff \lambda' \succ \mu'$$

So the preferences $\lambda \succ \mu$ and $\mu' \succ \lambda'$ that many of Allais's subjects reported do contradict the vNM independence axiom.

The common ratio effect occurs when different lotteries are mixed with a common bad consequence. Machina (1989) interprets this as violating a principle that he calls "mixture separability".

4. Dynamic Rationality with Deterministic Consequences

4.1. Dynamic Inconsistency

Strotz (1956) started his famous article on inconsistent dynamic choice with the following quotation from Homer's *Odyssey*:

> but you must bind me hard and fast, so that I cannot stir from the spot where you will stand me ... and if I beg you to release me, you must tighten and add to my bonds.

Thus does Strotz (1956) recognize that, going back to the mists of time before Homer's *Odyssey* was ever set down in writing, humanity has recognized the important distinction between: (i) the intention to make a sequence of successive rational decisions; (ii) actually carrying out those plans. Using contemporary language, Strotz (1956) began to explore what would happen if an intertemporal utility maximizing agent could experience changing tastes.

Indeed, when decision makers can re-evaluate their plans, changing preferences typically lead to *dynamic inconsistency* in the sense that the eventually chosen course of action deviates from the one that was originally planned. Consistency, on the other hand, requires choices at later stages to conform with those that were planned at earlier stages. Later, the logical link between consistent planning and Selten's (1967) "subgame perfect equilibrium" refinement of Nash equilibrium in game theory also became readily apparent.

4.1.1. Naïve Behavior

When considering games with one player, economists have followed Strotz (1956) and Pollak (1968) in describing choice as "naive" or "myopic" if the agent, faced with a succession of several decisions to make, simply maximizes a current objective function at each stage, without heeding how that objective may change in the future. In particular, the naïve decision maker's actual choices at later stages of any decision tree differ from earlier planned choices.

4.1.2. Sophisticated Behavior

By contrast, the sophisticated decision maker works backward through the overall decision problem, as in the subgame perfect equilibrium of an extensive form game (Selten, 1965) with a different player each period, whose payoffs match the decision maker's variable preferences. This subgame perfect equilibrium outcome coincides with the result of applying backward induction outcome to a game of perfect information of the kind investigated by Farquharson (1969), Moulin (1979), and many successors. Like dynamic programming, backward induction starts in the last period, where an optimal choice is made myopically. In all subsequent induction steps, which apply to earlier stages of the decision problem, a choice is identified so that an optimal plan of action results for both the current and all following periods. The backward recursion process concludes with an optimal plan of action for the whole problem, starting with the first decision.

Strotz (1956, p. 173) described this problem as follows:

> "Since precommitment is not always a feasible solution to the problem of intertemporal conflict, the man with insight into his future unreliability may adopt a different strategy and reject any plan which he will not follow through. His problem is then to find the best plan among those that he will actually be following."

And Pollak (1968, p. 203) as follows:

> "A sophisticated individual, recognizing his inability to precommit his future behaviour beyond the next decision point, would adopt a strategy of consistent planning and choose the best plan among those he will actually follow."

Issues such as the existence and characterization of optimal sophisticated plans, or of non-empty valued choice functions, were discussed for growth models by, amongst others, Phelps and Pollak (1968) Pollak (1968), Peleg and Yaari (1973), Inagaki (1970) and, for an approach to "Rawlsian" just savings rules, by Dasgupta (1974).[13]

Another issue is whether any ordinal intertemporal utility function could represent the result of optimal sophisticated planning. In fact, in the microeconomic context of demand theory, Blackorby et al. (1973) showed how a consumer with changing preferences would generally have demand functions that violate the usual Slutsky conditions for rationality. Then Hammond (1976) showed how the "potential addict" example, already suggested by Strotz (1956) and analysed in Section 4.2, would lead to choices that violate the ordering principle. Indeed, naïve choice is ordinal iff sophisticated choice

[13] One of the main motivations for the use of infinite-horizon planning in Hammond (1973) was to avoid the inconsistencies that would arise in any finite-horizon planning approach.

is ordinal, and both these hold iff naïve and sophisticated choice coincide, which in turn holds iff there is no "essential inconsistency" of the kind that characterizes the "potential addict" example.

4.1.3. Commitment Devices

Similar ideas emerged in what most macroeconomists now like to call the "time (in)consistency" problem, especially in the work that follows Kydland and Prescott (1977) in distinguishing between successive policy choices that follow a fixed rule from those that exhibit discretion in adapting to circumstances. Indeed, Kydland and Prescott typically presume that, like Odysseus alerted to the dangers presented by the Sirens, it is worth investing in some sort of commitment device which can prevent any departures from the original plan that the agent may be tempted to make — see also McClennen (1990) among philosophers and Klein (1990) among legal scholars.

4.2. Example: A Strict Preference Cycle and the Potential Addict

4.2.1. Three Decision Trees

The first example concerns choice under certainty when there is a strict preference cycle. Specifically, suppose that a, b, c are three different consequences in the domain Y. Consider the choice function $F \mapsto C(F)$ defined on $\mathcal{F}(Y)$, the family of non-empty subsets of Y. Suppose that $F \mapsto C(F)$ induces a base relation $\succsim^C$ on Y for which there is a *strict preference cycle* on the triple $Z := \{a, b, c\}$ — i.e., one has $a \succ^C b$, $b \succ^C c$, and $c \succ^C a$. This will be true, of course, if and only if the choice function $F \mapsto C(F)$ applied to pair sets $F \subset S$ satisfies

$$C(\{a,b\}) = \{a\}, \quad C(\{b,c\}) = \{b\}, \quad \text{and} \quad C(\{a,c\}) = \{c\}. \tag{25}$$

Suppose too that $C(\{a,b,c\}) = \{b\}$, because b is deemed to be the "best" of the three consequences in S.

Consider now the triple of decision trees in Fig. 1, each of which starts with an initial node that is also a decision node, indicated by a square. Furthermore, in each tree the terminal nodes are indicated by dots and labelled by their respective consequences a, b, c.

Of these three trees, the leftmost tree T has a second decision node n_1. The decision at node n_0 is therefore:

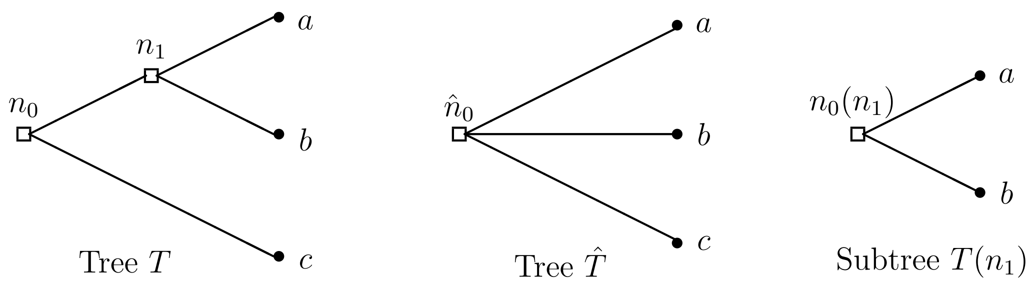

Figure 1: Three Decision Trees Associated with a Strict Preference Cycle

either to move down, leading directly to the terminal node whose consequence is c;

or to move up, leading to the decision node n_1 and the later choice between the two terminal nodes whose consequences are a and b respectively.

4.2.2. A Potential Addict

The "potential addict" example of Hammond (1976) involves the particular decision tree T in Figure 1, with the three respective consequences in the set $Z = \{a, b, c\}$ interpreted as follows:

a is **addiction**, which is the worst *ex ante* of the three possible consequences in S;

b is **bliss**, which is the best *ex ante* of the three possible consequences in S, allowing the decision-maker to enjoy some of the pleasures of addiction without experiencing any long-lasting harm;

c results from the **commitment** to avoid any possibility of addiction, thus denying oneself all the benefits of b, as well as sparing oneself all the costs of a.

After reaching the second decision node n_1 of tree T, however, the decision maker has become addicted. This implies that preferences change *ex post* so that a becomes strictly preferred to b.

The presumed potential addict's preferences $a \succ^C b$, $b \succ^C c$, and $c \succ^C a$ are those of an agent who prefers addiction to bliss after becoming addicted, but prefers bliss to commitment and commitment to addiction. A naïve agent plans b but gets diverted to a, whereas a sophisticated agent plans c and realizes that plan.

The potential addict example, with its changing preferences and even a strict preference cycle, will be ruled out by the following axioms that are inspired by Cubitt (1996).

4.3. Axioms for Ordinality

4.3.1. A General Tree with Two Decision Nodes

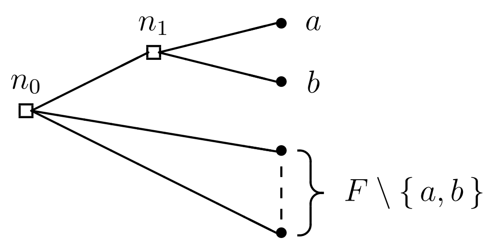

Figure 2: Decision Tree $T_{F,a,b}$ Illustrating Ordinality

The potential addict example is a particular case of a special kind of decision tree, of the form shown in Figure 2. The single lower branch in the leftmost tree of Figure 1, whose consequence is c, has been replaced in the decision tree $T_{F,a,b}$ of Figure 2 by an arbitrary finite set of branches, each with its own separate consequence in the set $F \setminus \{a, b\}$. Here $F \subseteq Y$ is an arbitrary finite set of consequences that includes both a and b as distinct members, and has at least one other distinct member besides.

At each of the two decision nodes n_0 and n_1 of the decision tree $T_{F,a,b}$, we will be especially interested in the *feasible sets of consequences* after reaching that decision node, which are evidently $F(T_{F,a,b}, n_0) = F$ and $F(T_{F,a,b}, n_1) = \{a, b\}$ respectively. We will also represent plans at those two nodes by the two *planned consequence sets*[14]

$$\Psi(T_{F,a,b}, n_0) \subseteq F(T_{F,a,b}, n_0) = F \text{ and } \Psi(T_{F,a,b}, n_1) \subseteq F(T_{F,a,b}, n_1) = \{a, b\}$$

Another piece of useful notation is

$$N_{+1}(T_{F,a,b}, n_0) = \{n_1\} \cup (F \setminus \{a, b\}) \quad \text{and} \quad N_{+1}(T_{F,a,b}, n_1) = \{a, b\}$$

[14]Cubitt (1996) defines a plan as a chosen set of terminal nodes. Because we identify each terminal node with the consequence that is obtained there, it loses no generality to define $\Psi(T_{F,a,b}, n)$ as a subset of the feasible set of consequences $F(T_{F,a,b}, n)$.

for the sets of nodes that *immediately succeed* the two decision nodes n_0 and n_1 respectively in the tree $T_{F,a,b}$.

Finally, given the planned consequence set $\Psi(T_{F,a,b}, n_0)$ at n_0, let

$$\psi(T_{F,a,b}, n_0) := \{n' \in N_{+1}(T_{F,a,b}, n_0) \mid \Psi(T_{F,a,b}, n_0) \cap F(T_{F,a,b}, n') \neq \emptyset\}$$

be the *planned move set* at node n_0; it represents those nodes in $N_{+1}(T_{F,a,b}, n_0)$ which the decision maker must be willing to go to in order not to rule out any consequence in $\Psi(T_{F,a,b}, n_0)$.

The following definition applies to general decision trees T.

Definition 7. Given any node n of any finite decision tree T, let:

1. $N_{+1}(T, n)$ denote the set of nodes that *immediately succeed* n in T;
2. $F(T, n)$ denote the *feasible set* of consequences given that node n has been reached in decision tree T;
3. $\Psi(T, n) \subseteq F(T, n)$ denote the *planned consequence set* at node n.

When n is a terminal node leading to the consequence $y \in Y$, then

$$N_{+1}(T, n) = \emptyset \quad \text{and} \quad F(T, n) = \{y\} \tag{26}$$

4.3.2. Dynamic Consistency at a Decision Node

Whenever the decision maker at node n_0 of tree $T_{F,a,b}$ plans to achieve a consequence a or b in $F(T_{F,a,b}, n_1)$, this entails arriving at node n_1. In fact one must have $n_1 \in \psi(T_{F,a,b}, n_0)$. Then "dynamic consistency at n_0" requires the plan at n_1 to involve choosing all the consequences that were both planned at n_0 and are still feasible at n_1. These are precisely the consequences in $\Psi(T_{F,a,b}, n_0) \cap F(T_{F,a,b}, n_1)$. Thus dynamic consistency at n_0 is satisfied if and only if $\Psi(T_{F,a,b}, n_0) = \Psi(T, n_0) \cap F(T_{F,a,b}, n_1)$ whenever $\Psi(T_{F,a,b}, n_0) \cap F(T_{F,a,b}, n_1) \neq \emptyset$.

More generally:

Definition 8. Let n be any decision node of any decision tree T, with $N_{+1}(T, n)$ as the set of immediately succeeding nodes. Then there is *dynamic consistency at the decision node* n provided that, whenever the planned consequence sets $\Psi(T, n)$ and $\Psi(T, n')$ at nodes n and $n' \in N_{+1}(T, n)$ satisfy $\Psi(T, n) \cap F(T, n') \neq \emptyset$, then $\Psi(T, n') = \Psi(T, n) \cap F(T, n')$.

4.3.3. Separability after a Decision Node

The next condition requires the "continuation subtree" $T(n_1)$ of $T_{F,a,b}$ that starts at node n_1 to be treated as if it were a full decision tree. Formally:

Definition 9. Let $T(n_1)$ denote the *continuation subtree* of tree T, which is the subtree whose initial node is n_1 and whose other nodes are a, b, the two successors of n_1 in tree T. The planned set $\Psi(T, n_1)$ of consequences at node n_1 satisfies *separability* provided that it equals the planned set $\Psi(T(n_1), n_1)$ of consequences at the initial node n_1 of the continuation subtree $T(n_1)$.

4.3.4. Reduction of Sequential Choice

The next condition requires that, given a decision tree that allows a sequence of choices at its two decision nodes, transforming that tree to its "reduced form" with just one decision node has no effect on the planned set of consequences. Formally:

Definition 10. Let $\hat{T}_F$ be the *reduced form* of the tree $T_{F,a,b}$, defined so that its initial node $\hat{n}_0$ is the only decision node, and also so that the feasible sets of consequences are the same, with $F(T, n_0) = F(\hat{T}_F, \hat{n}_0) = F$. Then reduction of sequential choice requires that the planned sets of consequences satisfy $\Psi(T, n_0) = \Psi(\hat{T}_F, \hat{n}_0)$.

4.4. Ordinality

Theorem 5. *Suppose that, given any finite feasible set $F \subseteq Y$ consisting of at least 3 consequences, as well as any pair of consequences $a, b \in F$, the planned consequence sets $\Psi(T, n_0)$ and $\Psi(T, n_1)$ in the decision tree $T_{F,a,b}$ of Figure 2 satisfy dynamic consistency, separability, and reduction of sequential choice.*

Let $\mathcal{F}(Y) \ni F \mapsto C(F) := \Psi(\hat{T}_F, \hat{n}_0)$ be the consequence choice function *defined on the domain $\mathcal{F}(Y)$ of all non-empty finite subsets of Y whose value is the planned consequence set in the reduced form decision tree $\hat{T}_F$ with one decision node $\hat{n}_0$ such that $F(\hat{T}_F, \hat{n}_0) = F$.*

Let $\succsim$ denote the binary relation defined on Y by the requirement that $a \succsim b \iff a \in C(\{a, b\})$ for each pair $a, b \in Y$. Then:

1. *the relation $\succsim$ is complete and transitive;*
2. *$C(F) = C^{\succsim}(F) := \{a \in F \mid b \in F \Longrightarrow a \succsim b\}$ for each $F \in \mathcal{F}(Y)$.*

Proof. Because $\Psi(\tilde{T}, \tilde{n}_0) \neq \emptyset$ for any tree $\tilde{T}$ with $F(\tilde{T}, \tilde{n}_0) = \{a, b\}$, the definition of $\succsim$ implies that the relation is complete.

Given any triple (F, a, b) with $F \in \mathcal{F}(Y)$, $a \in C(F)$ and $b \in F$, construct the decision tree $T_{F,a,b}$ shown in Figure 2. Then $F(T, n_0) = F = F(\hat{T}_F, \hat{n}_0)$, where $\hat{T}_F$ is the reduced form of F with $\hat{n}_0$ as its only decision node. By reduction of sequential choice, it follows that $C(F) = \Psi(\hat{T}, \hat{n}_0) = \Psi(T, n_0)$. In particular, because $a \in C(F)$ by hypothesis, and also $a \in F(T, n_1)$ by construction of the tree T, it follows that

$$a \in C(F) \Longrightarrow a \in \Psi(T, n_0) \cap F(T, n_1) \tag{27}$$

From dynamic consistency and then separability, it follows that

$$\Psi(T, n_0) \cap F(T, n_1) = \Psi(T, n_1) = \Psi(T(n_1), n_1) \tag{28}$$

Because $T(n_1)$ is a reduced form decision tree, the definition of $F \mapsto C(F)$ implies that

$$\Psi(T(n_1), n_1) = C(F(T(n_1), n_1)) = C(\{a, b\}) \tag{29}$$

Combining (27) (28), and (29), we see that $a \in C(F) \Longrightarrow a \succsim b$. Since this is true whenever $a, b \in F \subseteq Y$ where F is finite, it follows that the consequence choice function $F \mapsto C(F)$ satisfies binary contraction consistency.

Next, suppose that not only $a \in C(F)$, but also $b \succsim a$. The above construction shows that in the tree $T_{F,a,b}$ shown in Figure 2, one has

$$b \in C(\{a, b\}) = C(F(T(n_1), n_1) = \Psi(T, n_1) = \Psi(T, n_0) \cap F(T, n_1)$$

and so

$$b \in \Psi(T, n_0) = C(F(T, n_0)) = C(F)$$

It follows that the consequence choice function $F \mapsto C(F)$ satisfies binary expansion consistency.

We have proved that $F \mapsto C(F)$ satisfies both binary contraction consistency and binary expansion consistency. It follows from Theorem 1 that the choice function $F \mapsto C(F)$ is ordinal and its base relation $\succsim$ is transitive. $\square$

4.5. Consequentialist Invariance

4.5.1. Behavior and Its Consequences

Definition 11. In the decision tree $T_{F,a,b}$ of Figure 2, for each of the two decision nodes $n \in \{n_0, n_1\}$, let:

1. $N_{+1}(T,n)$ denote the set of *immediately succeeding nodes* of node n in decision tree T; evidently $N_{+1}(T,n_0) = \{n_1\} \cup (F \setminus \{a,b\})$ and $N_{+1}(T,n_1) = \{a,b\}$;

2. $\beta(T,n) \subseteq N_{+1}(T,n)$ denote the non-empty *behavior set* of moves that it would be acceptable for the decision maker to make at node n;

3. $\Phi(T,n) \subseteq F(T,n)$ denote the *consequence choice set* induced by behavior $\beta(T,n')$ at all nodes n' of the continuation subtree $T(n)$ whose initial node is n.

Instead of the planned consequences $\Psi(T,n)$ of reaching node n in decision tree T, the "consequentialist" approach focuses on the consequence choice set $\Phi(T,n)$ generated by the behavior $\beta(T,n)$. This set can be found by backward induction, otherwise known as "folding back" (Raiffa, 1968, Sarin and Wakker, 1994) or even "rolling back" (LaValle and Wapman 1986). The idea is, for each node $n' \in N_{+1}(T,n)$, to fold back the corresponding continuation subtree $T(n')$ into the single node n', and attach to it the consequence choice set $\Phi(T,n') \subseteq F(T,n')$ that applies after reaching node n'. Along with the feasible sets $F(T,n)$, the consequence choice sets $\Phi(T,n)$ are constructed by backward recursion, based on the respective equations

$$F(T,n) = \bigcup\nolimits_{n' \in N_{+1}(T,n)} F(T,n') \text{ and } \Phi(T,n) = \bigcup\nolimits_{n' \in \beta(T(n),n)} \Phi(T,n') \quad (30)$$

The second equation states that, for any $y \in F(T,n)$ to be a possible consequence of behavior starting from node n in tree T, there must be a node $n' \in \beta(T(n),n)$ such that $y \in \Phi(T,n')$.

Especially noteworthy here is that definition (30), in order to determine each consequence choice set $\Phi(T,n)$ generated by behavior in the tree T, depends on behavior only at the *initial node of the continuation subtree* $T(n)$. This relates to the old English adage "don't cross your bridges before you come to them", as cited by Savage (1954) in particular. That is, behavior at node n is undetermined until node n itself is reached and the "bridge" one needs to cross is to one of the immediately succeeding nodes

$n' \in N_{+1}(T, n) = N_{+1}(T(n), n)$. As will be seen, definition (30) obviates the need to assume that behavior satisfies the dynamic consistency requirement $\beta(T, n) = \beta(T(n), n)$. It is one important feature that distinguishes actual from planned behavior.

4.5.2. Consequentialist Invariance and Its Implications

The following definition is deliberately very general because in future sections we will want to apply them in several different contexts that allow decision trees with chance or event nodes as well as, in some cases, more than two decision nodes.

Definition 12. A *behavior rule* is a mapping $(T, n) \mapsto \beta(T, n)$ defined at every decision node n of all decision trees T in a specified domain $\mathcal{T}$, and satisfying $\emptyset \neq \beta(T, n) \subseteq N_{+1}(T, n)$. A behavior rule satisfies *consequentialist invariance* provided that, whenever $n, \tilde{n}$ are decision nodes of the respective trees $T, \tilde{T} \in \mathcal{T}$ at which the respective feasible sets satisfy $F(T, n) = F(\tilde{T}, \tilde{n})$, the corresponding consequence choice sets satisfy $\Phi(T, n) = \Phi(\tilde{T}, \tilde{n})$.

The following result demonstrates that, when applied to the decision tree $T_{F,a,b}$ shown in Figure 2, consequentialist invariance implies that the consequence choice sets $\Phi(T_{F,a,b}, n)$ satisfy the three assumptions on the planned consequence sets $\Psi(T_{F,a,b}, n)$ that were imposed in Section 4.3.

Theorem 6. *Given the decision tree $T_{F,a,b}$ shown in Figure 2, suppose that behavior $\beta(T_{F,a,b}, n)$ and its consequence choice sets $\Phi(T_{F,a,b}, n)$ at the two decision nodes $n \in \{n_0, n_1\}$ satisfy consequentialist invariance. Then the consequence choice sets $\Phi(T_{F,a,b}, n)$ also satisfy dynamic consistency at the decision node n_0, separability after the decision node n_0, and reduction of sequential choice.*

Proof. Suppose that $\Phi(T, n_0) \cap F(T, n_1) \neq \emptyset$. Because of (30), this is only possible if $n_1 \in \beta(T, n_0)$, in which case (30) implies that $\Phi(T, n_1) \subseteq \Phi(T, n_0)$. In particular, $\Phi(T, n_1) = \Phi(T, n_0) \cap F(T, n_1)$, as required for dynamic consistency.

Equation (30) also implies that $\Phi(T(n_1), n_1) = \Phi(T, n_1) = \beta(T, n_1)$, as required for separability.

Finally, the reduced form $\hat{T}_F$ of the tree $T_{F,a,b}$ obviously has the property that $F(T, n_0) = F(\hat{T}_F, \hat{n}_0) = F$, so consequentialist invariance implies that $\Phi(T, n_0) = \Phi(\hat{T}_F, \hat{n}_0)$, as required for reduction of sequential choice. □

4.6. Ordinal Dynamic Programming

So far we have shown that if our versions of Cubitt's axioms (dynamic consistency at a decision node, separability after a decision node, and reduction of sequential choice) hold only in each tree $T_{F,a,b}$, this is enough to ensure that the planned choice sets must maximize a preference ordering. We have also shown that if consequentialist invariance holds in each tree $T_{F,a,b}$, then our versions of Cubitt's axioms are implied. To close the logical circle we now establish the following result, which is inspired by Sobel's (1975) work on "ordinal dynamic programming". The result is stated and proved for a general finite decision tree.

Theorem 7. *Let $\succsim$ be any preference ordering on Y. Let $C^{\succsim}$ denote the ordinal choice function on the domain $\mathcal{F}(Y)$ of non-empty finite subsets of Y that is induced by $\succsim$. Then there exists a behavior rule $(T,n) \mapsto \beta(T,n)$ defined on the domain $\mathcal{T}(Y)$ of all finite decision trees with consequences in Y with the property that the induced consequence choice sets, which are calculated by folding back rule (30), satisfy consequentialist invariance with*

$$\Psi(T,n) = C^{\succsim}(F(T,n)) \tag{31}$$

at every node n of every tree T in $\mathcal{T}(Y)$.

Proof. Given any decision node n of any tree T in $\mathcal{T}(Y)$, construct the set

$$\beta(T,n) := \{n' \in N_{+1}(T,n) \mid C^{\succsim}(F(T,n)) \cap F(T,n') \neq \emptyset\} \tag{32}$$

of moves at n allowing some consequence in the choice set $C^{\succsim}(F(T,n))$ to be reached. Because the choice function $F \mapsto C^{\succsim}(F)$ is ordinal on the domain $\mathcal{F}(Y)$, theorem 1 implies that it must satisfy both expansion and contraction consistency. Hence

$$\begin{aligned} n' \in \beta(T,n) &\iff C^{\succsim}(F(T,n)) \cap F(T,n') \neq \emptyset \\ &\iff C^{\succsim}(F(T,n')) = C^{\succsim}(F(T,n)) \cap F(T,n') \end{aligned}$$

But this implies the chain of equalities

$$\begin{aligned} \textstyle\bigcup_{n' \in \beta(T,n)} C^{\succsim}(F(T,n')) &= \textstyle\bigcup_{n' \in \beta(T,n)} [C^{\succsim}(F(T,n)) \cap F(T,n')] \\ &= \textstyle\bigcup_{n' \in N_{+1}(T,n)} [C^{\succsim}(F(T,n)) \cap F(T,n')] \\ &= C^{\succsim}(F(T,n)) \cap \left[\textstyle\bigcup_{n' \in N_{+1}(T,n)} F(T,n')\right] \\ &= C^{\succsim}(F(T,n)) \cap F(T,n) \;=\; C^{\succsim}(F(T,n)) \end{aligned}$$

from which the equality

$$C^{\succsim}(F(T,n)) = \bigcup\nolimits_{n' \in \beta(T,n)} C^{\succsim}(F(T,n')) \tag{33}$$

follows trivially.

We now prove by backward induction that (31) holds at every node n of T. At any terminal node n with a consequence $y \in Y$ one has

$$\Psi(T,n) = F(T,n) = C^{\succsim}(F(T,n)) = \{y\}$$

so (31) holds trivially.

As the induction hypothesis, suppose that $\Psi(T,n') = C^{\succsim}(F(T,n'))$ for every $n' \in N_{+1}(T,n)$. Now, the folding back rule (30) states that $\Psi(T,n) = \cup_{n' \in \beta(T,n)} \Psi(T,n')$. Together with (33) and the induction hypothesis, this implies that

$$\Psi(T,n) = \bigcup\nolimits_{n' \in \beta(T,n)} \Psi(T,n') = \bigcup\nolimits_{n' \in \beta(T,n)} C^{\succsim}(F(T,n')) = C^{\succsim}(F(T,n))$$

This proves the relevant backward induction step, so (31) holds for all nodes n in tree T. □

4.7. Time Inconsistency and Hyperbolic Discounting

4.7.1. Timed Consequences and Discounting

Following Samuelson (1937), Koopmans (1960), and many others the early literature on discounting future utilities typically considers an entire *consumption stream* $\mathbf{c}$ in the form of a function $T \ni t \mapsto c(t) \in \mathbb{R}_+$, where (just in this section) $T \subseteq \mathbb{R}_+$ is the relevant *time domain*. Sometimes time was discrete, in which case a typical intertemporal utility function would take the form $\sum_{t \in T} D_t u(c_t)$, where $c \mapsto u(c)$ is a *time-independent* utility function, and D_t denotes a *discount factor* which is assumed to decrease with time. Sometimes time was continuous, in which case T would become an interval and the sum would be replaced with an integral of the form $\int_T D(t)u(c(t))dt$. A special case of some importance arose with *exponential discounting*, implying that $D(t) = d^t$ for some constant $d \in (0,1)$ in the discrete time case, and that $D(t) = e^{-\delta t}$ for some negative constant δ in the continuous time case. Strotz's (1956) work in particular considered the possible dynamic inconsistency of choice in the continuous time framework, especially when non-exponential discounting was combined with a particular kind of stationarity that required the plan chosen at any time $s > 0$ to maximize $\int_s^\infty D(t-s)u(c(t))dt$ so that future utility was effectively discounted back to the date at which the plan was made, rather than to a fixed time like 0.

The experimental literature has focused on simpler decision problems where, instead of entire consumption streams, the objects of choice are *timed consequences* of the form $(t, y) \in \mathbb{R}_+ \times Y$, where t denotes the time at which the consequence $y \in Y$ is experienced. Following Fishburn and Rubinstein (1982), it is usually assumed that preferences over timed consequences are respresented by just one term $D(t)u(y)$ of the intertemporal sum, with time t treated as part of the chosen consequence.

In case $Y = \mathbb{R}_+$ and one assumes that there is complete indifference over the timing of the 0 consequence, then one can impose the convenient normalization $u(0) = 0$. In this setting Fishburn and Rubinstein (1982) have provided further results concerning what transformations of the two functions $t \mapsto D(t)$ and $y \mapsto u(y)$ leave invariant the preferences over timed consequences that are represented by the function $(t, y) \mapsto D(t)u(y)$.

4.7.2. Experimental Tests of Exponential Discounting

A parsimonious special case occurs when there is the exponential discounting, meaning that $D(t) = d^t$ as with preferences for intertemporal consumption streams. Following Strotz's (1956) insight, exponential discounting is a necessary and sufficient condition for consistency between the planned choices today, at time 0, and future choices at time s when the objective over pairs (t, y) with $t \geq s$ shifts to $D(t-s)u(y)$.

One of the first experimental tests of exponential discounting was reported in Thaler (1981). Subjects's stated preferences for timed consequences in the near future were compared with their preferences when these timed consequences were shifted out into the far future. For example, the preferences compared were between: (i) one apple today vs. two apples tomorrow; (ii) one apple in a year from now vs. two apples in a year and a day from now. Many people state a preference in situation (i) for one apple today, but in situation (ii) they state a preference for two apples in a year and a day from now, which is inconsistent with exponential discounting. and. Such preferences are usually attributed to decreasing impatience. They also suggest that subjects fail to foresee that in one year from now situation (ii) will have become situation (i). This allows them liable to violate their originally stated preferences by preferring one apple as soon as possible to two apples one day later. It also makes them have inconsistent preferences like Strotz's naïve planners.

4.7.3. Hyperbolic Discounting and Beyond

The literature on time preferences received particular attention in Akerlof's (1991) lecture on the naïve planner's propensity to procrastinate.[15] Many theoretical models of naïve planning that followed have extended the exponential discounting model. Laibson (1997, 1998) studies economic applications of the "hyperbolic" discounting model that had earlier been suggested by psychologists (Ainslie, 1992). For example, Phelps and Pollak (1968) had proposed "quasi-hyperbolic discounting" in discrete time with $D_0 = 1$ and $D_t = bd^t$, for an additional parameter $b \in (0, 1)$. Such quasi-hyperbolic discounting implies that there is more impatience in determining preferences between successive pairs $(0, y), (1, y')$ than corresponding pairs $(t, y), (t+1, y')$ at later times, but there is constant impatience otherwise. A more general general form of *hyperbolic discounting* that implies decreasing impatience at all times comes from taking $D(t) = (1 + dt)^{-b/d}$ with parameters $d \geq 0$ and $b > 0$. As d approaches 0, this form of discounting approaches exponential discounting.

More recent works on applications of time preferences include Barro (1999), O'Donoghue and Rabin (1999a, 1999b, 2001), Harris and Laibson (2001), as well as Krusell and Smith (2003). Bernheim and Rangel (2007) further explore the policy relevance of "non-constant" discounting models; Hayashi (2003) along with Bleichrodt et al. (2009) provide extensions to non-hyperbolic discounting. These works have been motivated and complemented by the many experimental studies of the dynamic consistency of time preferences, that have been surveyed by Loewenstein and Prelec (1992) and by Frederick et al. (2002). Overall these studies provide plenty of evidence to support the hypothesis that discounting is often not exponential.

[15] We do not refrain from remarking that the link between procrastination and naïve plans was briefly discussed in Hammond (1973).

5. Dynamic Rationality with Risky Consequences

5.1. Axioms for Independence

5.1.1. A Family of Simple Trees with Risky Consequences

We are going to introduce a parametric family of decision trees, with the idea of using some simple relationships between them in order to explore the implications of a set of axioms inspired by Cubitt (1996). Especially interesting will be a condition that is related to the vNM independence condition (15) in Section 3.3.2.

The parameters are precisely those involved in the statement of the vNM independence condition. Specifically, they consist of a variable triple of lotteries $\lambda, \mu, \nu \in \Delta(Y)$, together with a variable probability $\alpha \in (0, 1)$ of moving "up" rather than "down". For each value of the parameter vector $(\lambda, \mu, \nu; \alpha)$ there will be four closely related trees $\bar{T}_{\lambda,\mu,\nu;\alpha}$, $T'_{\lambda,\mu,\nu;\alpha}$, $\bar{T}_{\lambda,\mu,\nu;\alpha}(n_1)$, and $\hat{T}_{\lambda,\mu,\nu;\alpha}$. Often we will simplify notation by omitting the parameter vector $(\lambda, \mu, \nu; \alpha)$ when this creates no ambiguity.

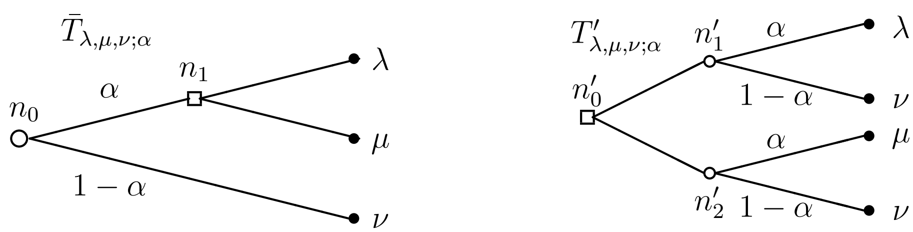

Figure 3: The Decision Tree $\bar{T}_{\lambda,\mu,\nu;\alpha}$ and the Variation $T'_{\lambda,\mu,\nu;\alpha}$

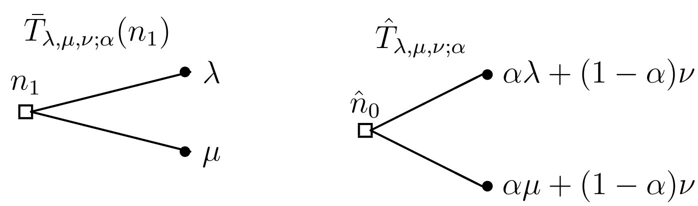

Figure 4: The Continuation $\bar{T}_{\lambda,\mu,\nu;\alpha}(n_1)$ and the Reduction $\hat{T}_{\lambda,\mu,\nu;\alpha}$

Following the notation that was introduced in Section 4.3.1 for decision trees with deterministic consequences:

Definition 13. In each of the four decision trees T of Figures 3 and 4, and for each of the nodes n of T, let:

1. $N_{+1}(T, n)$ denote the set of nodes that immediately succeed n in the tree T;

2. $F(T, n) \subset \Delta(Y)$ denote the *feasible set* of consequences given that node n has been reached in decision tree T;

3. $\Psi(T, n) \subseteq F(T, n)$ denote the *planned set* of consequences at node n.[16]

5.1.2. Folding Back Feasible Sets

The feasible sets $F(T, n)$ at some nodes of the trees in Figures 3 and 4 are obvious. Specifically,

$$
\begin{array}{rrcl}
& F(\bar{T}_{\lambda,\mu,\nu;\alpha}, n_1) & = & F(\bar{T}_{\lambda,\mu,\nu;\alpha}(n_1), n_1) = \{\lambda, \mu\} \\
\text{and} & F(\hat{T}_{\lambda,\mu,\nu;\alpha}, \hat{n}_0) & = & \{\alpha\lambda + (1-\alpha)\nu, \alpha\mu + (1-\alpha)\nu\} \\
& & = & \alpha\{\lambda, \mu\} + (1-\alpha)\{\nu\}
\end{array}
\tag{34}
$$

where the last equality follows from an obvious definition of the mixture of the two sets of lotteries $\{\lambda, \mu\}$ and $\{\nu\}$, which are subsets of $\Delta(Y)$. Furthermore, invoking the first equation in (30) at the decision node n'_0 of $T'_{\lambda,\mu,\nu;\alpha}$ yields

$$F(T'_{\lambda,\mu,\nu;\alpha}, n'_0) = F(T'_{\lambda,\mu,\nu;\alpha}, n'_1) \cup F(T'_{\lambda,\mu,\nu;\alpha}, n'_2) \tag{35}$$

At the chance nodes n_0 of $\bar{T}_{\lambda,\mu,\nu;\alpha}$ and n'_1, n'_2 of $T'_{\lambda,\mu,\nu;\alpha}$, however, and indeed for any decision node n of any finite decision tree T, we invoke:

Definition 14. Let n be any chance node of a decision tree T. For each immediately succeeding node $n' \in N_{+1}(T, n)$ of n in T, let $\pi(n'|n)$ denote the specified *conditional probability* of reaching each immediately succeeding node $n' \in N_{+1}(T, n)$ of n in T. The feasible set $F(T, n)$ at n satisfies the *folding back rule* provided it is given by the corresponding mixture

$$F(T, n) = \sum\nolimits_{n' \in N_{+1}(T,n)} \pi(n'|n)\, F(T, n') \tag{36}$$

of the immediately succeeding feasible sets $F(T, n')$.

[16] Cubitt (1996) restricts the definition of planned sets to *decision* nodes. We do not, for two reasons. First, it seems perfectly reasonable that wholly rational decision makers should be able to report their plans even when they are not about to make a decision. Second, we could in any case adopt Cubitt's device in trees like $\bar{T}_{\lambda,\mu,\nu;\alpha}$ of introducing immediately before the original initial node n_0 an extra "dummy" decision node n_{dummy} whose only immediate successor is n_0. Then the plans that we are attaching to chance (and event) nodes in our framework could be attached instead to an extra dummy decision node that precedes each chance (or event) node.

Applying (36) at the three different chance nodes in the two decision trees of Figure 3 yields

$$
\begin{array}{rrcl}
 & F(\bar{T}_{\lambda,\mu,\nu;\alpha}, n_0) & = & \alpha F(\bar{T}_{\lambda,\mu,\nu;\alpha}, n_1) + (1-\alpha)\{\nu\} \\
 & & = & \alpha\{\lambda,\mu\} + (1-\alpha)\{\nu\} \\
\text{as well as} & F(T'_{\lambda,\mu,\nu;\alpha}, n'_1) & = & \alpha\{\lambda\} + (1-\alpha)\{\nu\} \\
\text{and} & F(T'_{\lambda,\mu,\nu;\alpha}, n'_2) & = & \alpha\{\mu\} + (1-\alpha)\{\nu\}
\end{array}
\tag{37}
$$

Along with (34) and (35), this implies that

$$
F(\bar{T}_{\lambda,\mu,\nu;\alpha}, n_0) = F(T'_{\lambda,\mu,\nu;\alpha}, n'_0) = F(\hat{T}_{\lambda,\mu,\nu;\alpha}, \hat{n}_0) = \alpha\{\lambda,\mu\}+(1-\alpha)\{\nu\} \tag{38}
$$

We note in passing that rule (36) for folding back feasible sets is really no more than an implication of the laws of probability, together with the obvious requirement that the random moves at different chance nodes of any decision tree should be stochastically independent. After all, any possible source of dependence between different random moves should be modelled somewhere within the tree itself.

5.1.3. Reduction of Compound Lotteries

Our first substantive assumption on planning involves the planned consequence sets in two of the decision trees in Figures 3 and 4. Note that $\hat{T}$ is a reduction of T' in the sense that the only difference is that there is an extra step at node n'_1 in resolving the lotteries; this makes no difference to the two feasible sets $F(T', n'_0)$ and $F(\hat{T}, \hat{n}_0)$, which by (38) are both equal to $\alpha\{\lambda,\mu\} + (1-\alpha)\{\nu\}$.

Definition 15. The planned consequence sets $\Psi(T', n'_0)$ and $\Psi(\hat{T}, \hat{n}_0)$ at the initial nodes n'_0 and $\hat{n}_0$ of the two trees $T' = T'_{\lambda,\mu,\nu;\alpha}$ and $\hat{T} = \hat{T}_{\lambda,\mu,\nu;\alpha}$ shown in Figures 3 and 4 satisfy *reduction of compound lotteries* provided they are equal.

5.1.4. Dynamic Consistency at a Chance Node

Consider again the decision tree $\bar{T} = \bar{T}_{\lambda,\mu,\nu;\alpha}$ shown in Figure 3. Whatever the planned consequence set at node n_0 may be, it entails arriving at node n_1 with probability $\alpha > 0$, and also anticipating that a specific non-empty subset $\Psi(\bar{T}, n_1) \subseteq \{\lambda, \mu\}$ will be selected at that node. Now, there is an obvious bijection $j : F(\bar{T}, n_1) \to F(\bar{T}, n_0)$ whereby

$$
\lambda \mapsto j(\lambda) = \alpha\lambda + (1-\alpha)\nu \quad \text{and} \quad \mu \mapsto j(\mu) = \alpha\mu + (1-\alpha)\nu \tag{39}
$$

The following dynamic consistency condition requires this bijection to induce a correspondence between the two planned consequence sets $\Psi(\bar{T}, n_1)$ and $\Psi(\bar{T}, n_0)$. Formally:

Definition 16. The planned consequence sets $\Psi(\bar{T}, n_0)$ and $\Psi(\bar{T}, n_1)$ at the two non-terminal nodes of the tree $\bar{T} = \bar{T}_{\lambda,\mu,\nu;\alpha}$ of Figure 3 are *dynamically consistent at node* n_0 provided that

$$\begin{aligned} \lambda \in \Psi(\bar{T}, n_1) &\iff \alpha\lambda + (1-\alpha)\nu \in \Psi(\bar{T}, n_0) \\ \text{and} \quad \mu \in \Psi(\bar{T}, n_1) &\iff \alpha\mu + (1-\alpha)\nu \in \Psi(\bar{T}, n_0) \end{aligned} \tag{40}$$

or equivalently, provided that $\Psi(\bar{T}, n_0) = \alpha\Psi(\bar{T}, n_1) + (1-\alpha)\{\nu\}$.

5.1.5. Separability after a Chance Node

As in Section 4.3.3, separability requires the "continuation subtree" $\bar{T}(n_1)$ of tree $\bar{T} = \bar{T}_{\lambda,\mu,\nu;\alpha}$ that starts at node n_1 to be treated as if it were a full decision tree. The difference from Section 4.3.3 is that there the preceding node was a decision node; here it is a chance node.

Here is a formal definition for an arbitrary finite tree T of separability at a chance node n of T.

Definition 17. Given any decision tree T, the planned consequence set $\Psi(T, n)$ at a chance node satisfies *separability after the chance node* n provided that it equals the planned consequence set $\Psi(T(n), n)$ at the initial node n of the continuation subtree $T(n)$.

5.1.6. Timing Invariance with Risk

Consider once again the two decision trees $\bar{T} = \bar{T}_{\lambda,\mu,\nu;\alpha}$ and $T' = T'_{\lambda,\mu,\nu;\alpha}$ shown in Figure 3. We have already noted in (38) that the two feasible sets $F(\bar{T}, n_0)$ and $F(T', n'_0)$ at the initial nodes must be equal. The only difference between these two trees is that in $\bar{T}$, the lottery that picks "up" with probability α and "down" with probability $1-\alpha$ precedes the decision node, whereas in T' this timing is reversed. Our last condition, which we state only for two trees like those in Figure 3, requires the planned consequence set to be invariant to this timing reversal. Formally:

Definition 18. Given the two decision trees $\bar{T} = \bar{T}_{\lambda,\mu,\nu;\alpha}$ and $T' = T'_{\lambda,\mu,\nu;\alpha}$ shown in Figure 3, which differ only in the timing of the decision and chance nodes, say that there is *timing invariance* provided that the two planned consequence sets $\Psi(\bar{T}, n_0)$ and $\Psi(T', n'_0)$ are identical.

5.2. An Independence Condition

The following Lemma establishes a useful condition precursor of the vNM independence axiom:

Lemma 8. *Given any three lotteries $\lambda, \mu, \nu \in \Delta(Y)$ (not necessarily distinct) and any scalar $\alpha \in (0,1)$, consider the four decision trees $\bar{T} = \bar{T}_{\lambda,\mu,\nu;\alpha}$, $T' = T'_{\lambda,\mu,\nu;\alpha}$, $\bar{T}(n_1) = \bar{T}_{\lambda,\mu,\nu;\alpha}(n_1)$, and $\hat{T} = \hat{T}_{\lambda,\mu,\nu;\alpha}$ as shown in Figures 3 and 4. Suppose that the planned consequence sets $\Psi(T, n)$ at the non-terminal nodes of these trees satisfy reduction of compound lotteries, dynamic consistency at a chance node, separability after a chance node, and timing invariance. Then the two planned consequence sets $\Psi(\hat{T}, \hat{n}_0)$ and $\Psi(\bar{T}(n_1), n_1)$ satisfy*

$$\Psi(\hat{T}, \hat{n}_0) = \alpha\Psi(\bar{T}(n_1), n_1) + (1-\alpha)\{\nu\} \tag{41}$$

Proof. Applying successively reduction of compound lotteries, timing invariance, dynamic consistency at a chance node, then separability after a chance node, we obtain the chain of equalities

$$\begin{array}{rcl} \Psi(\hat{T}, \hat{n}_0) & = & \Psi(T', n'_0) \;=\; \Psi(\bar{T}, n_0) \;=\; \alpha\Psi(\bar{T}, n_1) + (1-\alpha)\{\nu\} \\ & = & \alpha\Psi(\bar{T}(n_1), n_1) + (1-\alpha)\{\nu\} \end{array} \tag{42}$$

from which (41) follows trivially. □

The following Theorem establishes that our versions of Cubitt's (1996) axioms are sufficient for ordinality and independence.

Theorem 9. *Suppose that:*

1. *the hypotheses of Theorem 5 are satisfied whenever $F \subset \Delta(Y)$ is a feasible set consisting of at least 3 distinct lotteries;*

2. *the hypotheses of Lemma 8 are satisfied.*

Then there exists a preference ordering $\succsim$ on $\Delta(Y)$ satisfying the vNM independence axiom with the property that, in every reduced form finite decision tree $\hat{T}$ whose terminal nodes have consequences in $\Delta(Y)$, the planned consequence set satisfies

$$\Psi(\hat{T}, \hat{n}_0) = C^{\succsim}(F(\hat{T}, \hat{n}_0)) \tag{43}$$

where $\mathcal{F}(\Delta(Y)) \ni F \mapsto C^{\succsim}(F)$ is the ordinal choice function on non-empty finite subsets $F \subset \Delta(Y)$ that is generated by the ordering $\succsim$.

Proof. By Theorem 5 applied to the domain $\Delta(Y)$ instead of the domain Y, the ordering $\succsim$ exists on $\Delta(Y)$ and (43) is satisfied.

Given any three lotteries $\lambda, \mu, \nu \in \Delta(Y)$ and any scalar $\alpha \in (0,1)$, Lemma 8 implies that for the two decision trees $\bar{T}(n_1) = \bar{T}_{\lambda,\mu,\nu;\alpha}(n_1)$ and $\hat{T} = \hat{T}_{\lambda,\mu,\nu;\alpha}$ shown in Figure 4, one has

$$F(\bar{T}(n_1), n_1) = \{\lambda, \mu\} \quad \text{and} \quad F(\hat{T}, \hat{n}_0) = \alpha F(\bar{T}(n_1), n_1) + (1-\alpha)\{\nu\}$$

By definition of the ordering $\succsim$, it follows that

$$\begin{aligned} \lambda \in \Psi(\bar{T}(n_1), n_1) &\iff \lambda \succsim \mu \quad \text{and} \\ \alpha\lambda + (1-\alpha)\nu \in \Psi(\hat{T}, \hat{n}_0) &\iff \alpha\lambda + (1-\alpha)\nu \succsim \alpha\mu + (1-\alpha)\nu \end{aligned} \tag{44}$$

But equation (41) implies that

$$\lambda \in \Psi(T(n_1), n_1) \iff \alpha\lambda + (1-\alpha)\nu \in \Psi(\hat{T}, \hat{n}_0)$$

Combining this with (44) yields

$$\lambda \succsim \mu \iff \alpha\lambda + (1-\alpha)\nu \succsim \alpha\mu + (1-\alpha)\nu$$

which is precisely the vNM independence axiom. □

5.3. Behavior and Its Consequences

5.3.1. Folding Back at a Decision Node

We begin by recalling the notation of Section 4.5.1, where at any decision node n of any tree T, one has:

1. a set $N_{+1}(T, n)$ of immediately succeeding nodes in the decision tree T;
2. a non-empty behavior set $\beta(T, n) \subseteq N_{+1}(T, n)$;
3. a feasible consequence set $F(T, n) \subset \Delta(Y)$;
4. a consequence choice set $\Phi(T, n) \subseteq F(T, n)$ induced by behavior $\beta(T, n')$ at all nodes n' of the continuation subtree $T(n)$ whose initial node is n.

As in Section 4.5.1, we assume that at any decision node n of any decision tree T, the feasible set $F(T, n)$ and the consequence choice set $\Phi(T, n)$ satisfy the folding back rule (30), reproduced here for convenience:

$$F(T, n) = \bigcup\nolimits_{n' \in N_{+1}(T,n)} F(T, n') \text{ and } \Phi(T, n) = \bigcup\nolimits_{n' \in \beta(T(n),n)} \Phi(T, n')$$

5.3.2. Folding Back at a Chance Node

At the chance node n_0 of the first tree $\bar{T} = \bar{T}_{\lambda,\mu,\nu;\alpha}$ that is shown in Figure 3, we naturally assume that the random move to one of the two succeeding nodes in $N_{+1}(\bar{T}, n) = \{n_1, \nu\}$ is stochastically independent of both the two lotteries λ, ν at the terminal nodes succeeding n_1. This implies that a commitment to choose up at node n_1 results in the consequence lottery $\alpha\lambda + (1-\alpha)\nu$ back at n_0, whereas a commitment to choose down at node n_1 results in the consequence lottery $\alpha\mu + (1-\alpha)\nu$ back at n_0. Hence

$$F(\bar{T}, n_0) = \{\alpha\lambda + (1-\alpha)\nu, \alpha\mu + (1-\alpha)\nu\} = \alpha F(\bar{T}, n_1) + (1-\alpha)\{\nu\} \quad (45)$$

Furthermore, by a similar argument,

$$\begin{array}{rcccl} \lambda \in \beta(\bar{T}, n_1) & \Longleftrightarrow & \lambda \in \Phi(\bar{T}, n_1) & \Longleftrightarrow & \alpha\lambda + (1-\alpha)\nu \in \Phi(\bar{T}, n_0) \\ \mu \in \beta(\bar{T}, n_1) & \Longleftrightarrow & \mu \in \Phi(\bar{T}, n_1) & \Longleftrightarrow & \alpha\mu + (1-\alpha)\nu \in \Phi(\bar{T}, n_0) \end{array} \quad (46)$$

which obviously implies that

$$\Phi(\bar{T}, n_0) = \alpha\Phi(\bar{T}, n_1) + (1-\alpha)\{\nu\} \quad (47)$$

Equations (45) and (47) are special cases of the more general folding back rules

$$\begin{array}{rcl} F(T, n) & = & \sum_{n' \in N_{+1}(n)} \pi(n'|n)\, F(T, n') \\ \Phi(T, n) & = & \sum_{n' \in N_{+1}(n)} \pi(n'|n)\, \Phi(T, n') \end{array} \quad (48)$$

that apply in a general decision tree T to the feasible set and consequence choice set at any chance node n where the probabilities of reaching each immediately succeeding node $n' \in N_{+1}(n)$ are specified to be $\pi(n'|n)$.

5.3.3. Implications of Consequentialist Invariance

Consequentialist invariance was defined in Section 4.5.1. The following result demonstrates that it implies that the consequence choice sets $\Phi(T, n)$ satisfy the axioms for independence that were imposed in Section 5.1.

Theorem 10. *Given the four decision trees shown in Figures 3 and 4, suppose that behavior $\beta(T, n)$ and the consequence choice sets $\Phi(T, n)$ at all the nodes n of these trees satisfy consequentialist invariance. Then the consequence choice sets $\Phi(T, n)$ also satisfy reduction of compound lotteries, dynamic consistency at the chance node n_0, separability after the chance node n_0, and timing invariance.*

Proof. First, because $F(T', n_0') = F(\hat{T}, \hat{n}_0) = \alpha\{\lambda, \mu\} + (1-\alpha)\{\nu\}$, consequentialist invariance implies that $\Phi(T', n_0') = \Phi(\hat{T}, \hat{n}_0)$, so these two sets satisfy reduction of compound lotteries.

Second, the folding back rule (47) is identical to the equation $\Phi(\bar{T}, n_0) = \alpha\Phi(\bar{T}, n_1) + (1-\alpha)\{\nu\}$ in Definition 16, so dynamic consistency at the chance node n_0 is satisfied.

Third, because $F(\bar{T}, n_1) = F(\bar{T}(n_1), n_1) = \{\lambda, \mu\}$, consequentialist invariance implies that $\Phi(\bar{T}, n_1) = \Phi(\bar{T}(n_1), n_1)$, which is the condition for separability after the chance node n_0.

Finally, because $F(\bar{T}, n_0) = F(T', n_0') = \alpha\{\lambda, \mu\} + (1-\alpha)\{\nu\}$, consequentialist invariance implies that $\Phi(\bar{T}, n_0) = \Phi(T', n_0')$, which is the condition for timing invariance. □

5.4. Ordinal Dynamic Programming with Risky Consequences

The main result of this section extends the ordinal dynamic programming result in Theorem 7 of Section 4.6 to allow for chance as well as decision nodes.

Theorem 11. *Let $\succsim$ be any preference ordering on $\Delta(Y)$ that satisfies vNM independence. Let $C^{\succsim}$ denote the ordinal choice function on the domain $\mathcal{F}(\Delta(Y))$ of non-empty finite subsets of $\Delta(Y)$ that is induced by $\succsim$. Then there exists a behavior rule $(T, n) \mapsto \beta(T, n)$ defined on the domain $\mathcal{T}(\Delta(Y))$ of all finite decision trees with consequences in $\Delta(Y)$ having the property that the induced consequence choice sets, which are calculated by applying the folding back rules (30) and (48), satisfy consequentialist invariance with*

$$\Psi(T, n) = C^{\succsim}(F(T, n)) \tag{49}$$

at every node n of every tree T in $\mathcal{T}(\Delta(Y))$.

Proof. Given any decision node n of any tree T in $\mathcal{T}(\Delta(Y))$, construct the set

$$\beta(T, n) := \{n' \in N_{+1}(T, n) \mid C^{\succsim}(F(T, n)) \cap F(T, n') \neq \emptyset\} \tag{50}$$

as in the proof of Theorem 7. We now prove by backward induction that (49) holds at every node n of T.

First, at any terminal node n with a consequence $\lambda \in \Delta(Y)$ one has

$$\Psi(T, n) = F(T, n) = C^{\succsim}(F(T, n)) = \{\lambda\}$$

so (49) holds trivially.

As the induction hypothesis, suppose that $\Psi(T, n') = C^{\succsim}(F(T, n'))$ for every $n' \in N_{+1}(T, n)$. There are two cases to consider.

Case 1 occurs when n is a decision node of T. Then, as in the proof of Theorem 7, the construction (50) implies (33) which, when combined with the folding back rule (30) and the induction hypothesis, implies that

$$\Psi(T, n) = \bigcup\nolimits_{n' \in \beta(T,n)} \Psi(T, n') = \bigcup\nolimits_{n' \in \beta(T,n)} C^{\succsim}(F(T, n')) = C^{\succsim}(F(T, n))$$

This confirms the backward induction step at any decision node n in tree T.

Case 2 occurs when n is a chance node of T.

First, suppose that $\lambda \in \Psi(T, n)$, and consider any $\mu \in F(T, n)$. Then the rolling back rules imply that, given any $n' \in N_{+1}(n)$, there exist $\lambda(n') \in \Psi(T, n')$ and $\mu(n') \in F(T, n')$ satisfying

$$\lambda = \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n)\, \lambda(n') \quad \text{and} \quad \mu = \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n)\, \mu(n') \quad (51)$$

By the induction hypothesis, for all $n' \in N_{+1}(n)$ one has $\lambda(n') \in C^{\succsim}(F(T, n'))$, so $\lambda(n') \succsim \mu(n')$. But then, because $\succsim$ is transitive, repeated application of the vNM independence condition can be used to show that $\lambda \succsim \mu$. Because this is true for all $\mu \in F(T, n)$, it follows that $\lambda \in C^{\succsim}(F(T, n))$. We have therefore proved that $\Psi(T, n) \subseteq C^{\succsim}(F(T, n))$.

Second, suppose that $\lambda \in C^{\succsim}(F(T, n))$. Because $\lambda \in F(T, n)$, the folding back rule implies that for all $n' \in N_{+1}(n)$ there exists $\lambda(n') \in F(T, n')$ such that $\lambda = \sum_{n' \in N_{+1}(n)} \pi(n'|n)\, \lambda(n')$. Consider now any fixed $\bar{n} \in N_{+1}(n)$. Because $\lambda \in C^{\succsim}(F(T, n))$, for any $\mu \in F(T, \bar{n})$ it must be true that

$$\begin{aligned} \lambda &= \pi(\bar{n}|n)\, \lambda(\bar{n}) + \sum\nolimits_{n' \in N_{+1}(n) \setminus \{\bar{n}\}} \pi(n'|n)\, \lambda(n') \\ &\succsim \pi(\bar{n}|n)\, \mu + \sum\nolimits_{n' \in N_{+1}(n) \setminus \{\bar{n}\}} \pi(n'|n)\, \lambda(n') \end{aligned}$$

Because $\pi(\bar{n}|n) > 0$, the vNM independence condition then implies that $\lambda(\bar{n}) \succsim \mu$. This is true for all $\mu \in F(T, \bar{n})$, implying that $\lambda(\bar{n}) \in C^{\succsim}(F(T, \bar{n}))$, and so $\lambda(\bar{n}) \in \Phi(T, \bar{n})$ by the induction hypothesis. Since this is true for all $\bar{n} \in N_{+1}(n)$, one has

$$\lambda = \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n)\, \lambda(n') \in \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n)\, \Phi(T, n')$$

and so $\lambda \in \Phi(T,n)$ because of the rolling back rule. We have therefore proved that $C^{\succsim}(F(T,n)) \subseteq \Psi(T,n))$. Together with the result of the previous paragraph, this confirms that $\Psi(T,n) = C^{\succsim}(F(T,n))$ and so completes the final induction step. $\square$

5.5. Timing and Nonlinear Expectations

Halevy (2008) notices a parallel between observed preferences in examples like the Allais paradox we described in Section 3.5.2 and preferences for *timed consequences* $(t,y) \in \mathbb{R}_+ \times Y$ of the kind considered in Section 4.7.1 (see also Quiggin and Horowitz 1995; Ebert and Prelec, 2007). The connection seems to arise because, in contrast to sure consequences in the near future, delayed consequences are perceived as inherently risky (Dasgupta and Maskin, 2005). As such, non-linear probability transformations like those which some writers such as Kahneman and Tversky (1979) and Tversky and Kahneman (1992) applied to explain Allais-like behavior can also adjust discount factors in order to help explain choice of timed outcomes (Epper et al., 2011; Baucells and Heukamp, 2012).

Some recent experiments explicitly look at time preferences for *timed risky consequences* of the form $(t,\lambda) \in \mathbb{R}_+ \times \Delta(Y)$. The findings of Noussair and Wu (2006) and Abdellaoui et al. (2011) suggest that people become more risk tolerant as the risk they face is delayed — i.e., as the time when that risk is resolved recedes further into the future. This hints at the possibility that preferences for delayed risky consequences may conform better to the EU hypothesis than do the preferences for immediate risky consequences that most previous experiments have been designed to elicit. Investigating this further may shed light on the relationship between time preferences and standard preferences over risky consequences, as well as on the potential effecton choices over risky consequences of decreasing impatience and other timing related phenomena.

5.6. Experimental Tests

5.6.1. Potential Challenges for Dynamic Choice Experiments

Designing and implementing dynamic choice experiments is complex and challenging. It demands a lot of creativity to avoid introducing biases which may obscure the interpretation of the findings. We highlight a few difficulties here.

One major problem is that risk-taking behavior may change as a decision tree unfolds. Static choice may be influenced by numerous empirically well documented phenomena such as regret or disappointment, reference points and loss aversion, optimism and pessimism, or attitudes to ambiguity. All of these may arise in dynamic choice. These phenomena may themselves be affected in addition by real-time delays in resolving risk. Design of dynamic choice experiments to control for these features is therefore necessary.

Another potential difficulty results from the fact that real people make errors. Observing a single contradictory choice need not imply that the subject was violating a choice principle intentionally. To allow for errors like this requires runnning studies where some choice problems are repeated.

Yet such repetitions introduce new difficulties. The subject's preferences may change during the course of the experiment. With repeated or multiple decision tasks there are additional complications when financial incentives are provided. Indeed, if participants in a study are paid for each choice they make, income effects may lead to biases. On the other hand, if participants are paid at the end of the experiment based on just a small randomly chosen subset of all the decisions they make during the experiment, this means that participants are in fact facing one large dynamic choice problem instead of many small independent problems. For further discussion of the incentives one can provide to participants in dynamic choice experiments, see Cubitt et al. (1998).

Notwithstanding these and other significant difficulties, there have been sufficient advances in experimental design to allow a useful body of results to emerge during the last two decades. We summarize the main findings of the few studies we are aware of next focusing mainly on Cubitt (1996)'s factorized properties leading to the independence property.

5.6.2. Experimental Diagnosis of the Common Ratio Effect

One of the empirically most frequently documented descriptive findings in static choice is that people are extremely sensitive to probabilities near the two end points of the probability interval $[0, 1]$ (Wakker 1994, 2001, 2010). A typical experimental finding is that for good consequences small probabilities are overweighted, but for bad consequences they are underweighted. These observations hint at the possibility of finding subtle explanations for systematic violations of the independence axiom in particular choice problems. Naturally, such findings can inform the design of dynamic choice problems intended to test separately each of Cubitt's (1996) "factorized" dynamic choice principles.

Now, there is a large body of experimental evidence on static Allais paradoxes (see Camerer (1995) and Starmer (2000) for summaries), which also involve decision problems with small probabilities. So it is not surprising that a dynamic version of the choice problem involving the common ratio effect has often been used to study potential violations of those dynamic choice principles.

Cubitt et al. (1998) used an between-subject design to test for possible systematic variations in the frequency with which a consequence like λ is chosen in decision problems such as those shown in Figures 3 and 4. In their study, 451 subjects where divided into six groups, each subject facing a single decision problem which was played out for real. Cubitt et al. (1998) report violations of timing invariance. No violations were observed for the other dynamic choice principles (dynamic consistency after a chance node, separability after chance a node, and reduction of compound lotteries).

Busemeyer et al. (2000) also tested separability and dynamic consistency after chance nodes. They designed decision problems related to a tree with several successive decision nodes, at each of which there was a choice between: (i) ending the choice process at once and finishing with a deterministic payoff; (ii) continuing to the next decision node. At the final decision node the choice is between a safe consequence and a $50:50$ lottery. Busemeyer et al. (2000) used a within-subject design to test the dynamic consistency condition by comparing stated plans at the initial node with the choices actually made when the final decision node was reached. They also tested the separability condition by comparing: (i) the isolated choice in a "pruned off" problem that starts at the terminal decision node of the original problem; (ii) the actual choices of those subjects who had started at the initial node of the original problem and decided to follow the path to the final decision node. Overall, taking into account errors in choice (i.e., different choices when the same decision problem is presented again), Busemeyer et al. (2000) found statistically significant violations of dynamic consistency, but no statistically significant violation of separability.

Cubitt and Sugden (2001) also used multistage decision problems. In their experiment subjects who started with a monetary balance m had to choose between obtaining $3^k m$ if they were lucky enough to survive k chance events, or 0 otherwise. The survival probabilities for $k = 1, 2, \ldots, 6$ were set equal to $(7-k)/(7-k+1)$. In a between-subject design, subjects were allocated randomly to one of three conditions and paid according to a single decision that they make. In the first multi-stage condition subjects had to decide, prior to the first four chance events being resolved, whether they would like to continue or not with one (i.e., $k = 5$) or two (i.e., $k = 6$) subsequent chance events. By contrast, in the second condition, subjects had to take the same decision only after they knew they had survived the first four chance events. In the third condition, subjects where endowed with 3^4m and were then asked if they would choose none, one (i.e., $k = 5$) or two (i.e., $k = 6$) subsequent chance events (as if they had survived a series of four chance events). Comparing the choice frequencies across the first two different conditions revealed that neither of the two hypotheses of dynamic consistency or separability could be rejected. The third treatment, however, led to significantly different choice probabilities. One possible explanation is that participants may have perceived the endowment of 3^4m as "house money" (Thaler and Johnson 1990), making them more risk seeking than if they had earned this amount in previous lotteries, or otherwise. Without this house money effect, the data could be interpreted as supporting a reversed common ratio effect (see Cubitt and Sugden 2001, p. 121).

The previous studies provide some evidence that dynamic consistency is violated in dynamic choice problems. It seems that in multiple stage problems the number of chance nodes that subjects face matters because it affects their willingness to be exposed to additional chance events. Motivated by this observation, Johnson and Busemeyer (2001) replicated the study of Busemeyer *et al.* (2000) and added additional controls to analyze the data for the effect of the number of chance nodes (which they call the *length of planning horizon* in their study). Their data gives evidence against dynamic consistency, with the effect becoming stronger as the number of chance nodes increases.

In a recent study, Cubitt et al. (2010) tested separability by designing dynamic choice problems with a variable number of chance nodes preceding a decision node. They implemented a between-subject design and paid each participant the outcome of the single task that they answered. They collected data from three treatments. In one treatment subjects faced a decision problem similar to T' in Figure 3 that starts with a decision node. In a second treatment subjects faced a decision like that in $\bar{T}$ in Figure 3 where risk is resolved before a decision node like n_1 at which the subjects could either leave the experiment with nothing or else face the continuation subtree $\bar{T}(n_1)$ in Figure 4. In their third treatment the decision tree $\bar{T}$ was lengthened by adding five extra chance nodes before the decision node, at all of which the right chance move had to occur before subjects could reach $\bar{T}(n_1)$. According to their data, no evidence was found that separability is violated. This indicates that the history of how one arrives at the decision problem $\bar{T}(n_1)$, including the number of chance nodes one has passed through, appears not to affect subsequent choice behavior.

5.6.3. Evidence for Naïve, Sophisticated, or Resolute Choice

According to McClennen (1990) there are decision makers who, faced with a decision trees like the potential addict example shown in Figure 1 of Section 4.2, are neither naïve nor sophisticated. Instead, these individuals are *resolute* in planning b at n_0 and resisting the revised preference for a instead of b that becomes apparent at n_1. We point out that a version of this potential addict problem is something that can affect any experimental subject whose preferences violate the EU hypothesis; for EU maximisers, there is no such problem and in fact their behavior is the same whether they are resolute, naïve, or sophisticated (cf. Hammond, 1976).

With this possibility in mind, Hey and Paradiso (2006) conducted an experimental study in a dynamic choice setting whose aim was to find out how many subjects with non-EU preferences behaved in ways that could be most accurately described as naïve, resolute, or sophisticated. Their experiment used a within-subject design in order to obtain some information about how participants value strategically equivalent dynamic choice problems. They implemented a second-price sealed-bid auction mechanism combined with a random lottery incentive scheme. The advantage of this method is that it provides information about the strength of their preference for facing a particular problem. They study decision problems similar to those of Cubitt et al. (1998). They found that the majority, 56%, of people valued the three problems roughly equally, like EU maximizers. The residual fraction had non-EU preferences and violated a dynamic choice principle:

- 20% of the participants were resolute in that they both violated dynamic consistency and valued the opportunity to precommit to their decision prior before uncertainty was resolved;
- 18% were sophisticated planners who achieve dynamic consistency by using backward induction to reach their decision;
- only 6% were naïve in both violating dynamic consistency and valued the opportunity to change their original decision after uncertainty was resolved.

Hey and Lotito (2009) extended this earlier study by combining decisions in four dynamic choice problems with valuations. They added several other features to their design to account for errors in choice.[17] They found that 44% of the subjects have preferences that agree with EU theory. A further 30% of the subjects seem naïve and 24% seem resolute. Only 2% appear to be sophisticated decision makers.

A recent experimental study in which real time was explicitly added into the design of the study was provided in Hey and Panaccione (2011). The decision tree they designed involved allocating 40 euros on two successive occasions between two risky alternatives in the form of possible continuation subtrees. After the first stage there is a chance move to determine which subtree is encountered at the second stage, where subjects once more allocate the residual budget among two securities that pay off in two different states.

In total each subject had to respond to 27 dynamic decision problems that had different probabilities of continuation. One of these problems was randomly selected and played for real money. Subjects were assumed to have rank-dependent utility (RDU) preferences (Quiggin 1981, 1982) which combined a parametric inverse-S probability weighting function, as in Tversky and Kahneman (1992), with a power utility function consistent with constant relative risk aversion — preferences that Wakker and Zank (2002) had axiomatized.

This parametric specification allowed all their 71 participants to be classified, on the basis of their observed behavior, into four most likely types. The first three types were resolute, sophisticated, and naïve, as in previous studies. A fourth myopic type was added to describe a few agents whose first-stage choices seemed best explained by the hypothesis that they totally ignored the second stage choice problem. They found that only 28 subjects, that is 39.44%, had preferences that deviated significantly from the special case that fits the EU hypothesis, and so behaved in ways allowing them to be classified as resolute, sophisticated, or naïve. As a result, among all 71 subjects, a total of 25.35% were classified as resolute non-EU maximisers, 5.63% as sophisticated non-EU maximisers, 5.63% as naïve non-EU maximisers, and 2.82% as simply myopic.

[17]Hey and Lotito use deception in their incentive mechanism to a limited number of participants that end up being paid while extracting useful information from many participants (see their footnote 18, p. 13). Further, two of their decision trees include stochastically dominated alternatives, which may bias subjects towards particular choices.

The latter three studies suggest that a majority of participants either do not violate EU theory at all, or if they do, the violations are not very large. Those who are non-EU maximizers can be broken down further into resolute, naïve, and sophisticated subgroups. Of these a relatively large fraction are resolute, violating dynamic consistency because they prefer to stick to their originally planned action. Naïve planners, who also violate dynamic consistency, are a small group. Similarly, very few non-EU maximizers are sophisticated agents who anticipate the potential dynamic inconsistency of behavior, and apply backward induction in a way most game theorist might recommend.

5.6.4. Summary of Experimental Evidence

The few studies reviewed in the last two subsections indicate that dynamic consistency of planned behavior is frequently violated, but apparently not by the majority of subjects. Separability does not seem to be violated. The study by Barkan and Busemeyer (2003) discusses the potential role of reference points as causes for violations of dynamic consistency through changes in risk attitudes. A further review of experimental evidence and a discussion on future directions in modeling dynamic choice is provided in Cubitt et al. (2004). Nebout and Dubois (2011) indicate that the parameters used in the experiments seem to have significant influence on the frequency of violations of the vNM independence axiom and that these frequencies are related to the frequencies of dynamic consistency violations.

6. Dynamic Rationality with Uncertain Consequences

6.1. The Ellsberg Paradox

The following famous example is due to Ellsberg (1961). An urn conceals 90 balls, of which exactly 30 are known to be red, whereas the remaining 60 must be either black or yellow, but in unknown proportions. Apart from their colors, the balls are otherwise entirely identical, so that each ball has exactly the same 1/90 chance of being drawn at random from the urn. Before the ball is drawn, the decision maker is offered the choice between two different lottery pairs or "acts".

Act/Event	Red	Black	Yellow
L_1	\$100	\$0	\$0
L_2	\$0	\$100	\$0
L_3	\$100	\$0	\$100
L_4	\$0	\$100	\$100

Each of the four lotteries has a prize of either \$100 or \$0, depending on the color of the randomly drawn ball, as indicated in the table. Subjects were asked to make choices: (i) between lotteries L_1 and L_2; (ii) between lotteries L_3 and L_4. Typical reported preferences are $L_1 \succ L_2$ and $L_4 \succ L_3$. It would seem that the typical subject prefers the known probability $\frac{1}{3}$ of winning \$ 100 when L_1 is chosen to the unknown probability of winning \$100 when L_2 is chosen, but also prefers the known probability $\frac{1}{3}$ of not winning \$100 when L_4 is chosen to the unknown probability of not winning \$100 when L_3 is chosen.

Following Raiffa's (1961) suggestion, consider the two mixed lotteries

$$L' = \tfrac{1}{2}L_1 + \tfrac{1}{2}L_4 \quad \text{and} \quad L'' = \tfrac{1}{2}L_2 + \tfrac{1}{2}L_3$$

Whatever color ball is drawn from the urn, the results of both L' and L'' are exactly the same: there is a roulette lottery with a probability $\frac{1}{2}$ of winning \$100, and a probability $\frac{1}{2}$ of winning \$0. This suggests that L' and L'' should be indifferent. Yet the reported preferences $L_1 \succ L_2$ and $L_4 \succ L_3$ also suggests that L' offers a better lottery, either L_1 or L_4, with probability 1 compared to L'', which offers either L_2 or L_3.

Indeed, the preferences $L_1 \succ L_2$ and $L_4 \succ L_3$ not only contradict the subjective version of the EU hypothesis; they also exclude a form of "probabilistic sophistication" (cf. Machina and Schmeidler, 1992) whereby one attaches subjective probabilities p and q to the events of drawing a yellow and black respectively.

These subjective probabilities may come from a more basic hypothesis concerning b, the unknown number of black balls, with $60 - b$ as the number of yellow balls. Suppose that for each $b \in \{0, 1, 2, \ldots, 59, 60\}$ there is a subjective probability P_b that there will be b black balls in the urn. In this case the probability that the one ball drawn at random is black will be $p = \sum_{b=0}^{60} P_b b/90$, and that is yellow will be $q = \sum_{b=0}^{60} P_b(60 - b)/90$, with $p + q = \sum_{b=0}^{60} P_b(60/90) = \frac{2}{3}$ of course.

Assume that the decision maker prefers a higher to a lower probability of winning the \$100 prize. Given that the probability of red is supposed to be $\frac{1}{3}$, the preference $L_1 \succ L_2$ implies that $\frac{1}{3} > p$, whereas the preference $L_4 \succ L_3$ implies that $p + q > \frac{1}{3} + q$, or equivalently that $p > \frac{1}{3}$, which is an obvious contradiction. So the decision maker cannot be probabilistically sophisticated in this way.

6.2. States of the World, Events, Uncertain Consequences, and Preferences

Let S be a non-empty finite set of uncertain *states of the world* s, like the color of a ball drawn from an urn in Ellsberg's example. An *event* E is a non-empty subset of S.

For each $s \in S$, assume there is a state-contingent *consequence domain* Y_s. Then, for each event $E \subseteq S$, the Cartesian product space $Y^E := \prod_{s \in E} Y_s$, or equivalently the space of mappings $E \ni s \mapsto y_s \in Y_s$, is the domain of *contingent consequences* conditional upon the event E.[18] Each point $y^E := \langle y_s \rangle_{s \in E} \in Y^E$ then represents a pattern of uncertain consequences, given that E is known to occur. By definition, the event S is always known to occur.

In dynamic decision problems, it is important to keep track of what event is known to the decision maker at each stage. The relevant version of the ordinality hypothesis, in fact, is that for each event $E \subseteq S$, there is a *conditional preference ordering* $\succsim^E$ *given* E defined on the domain Y^E. Thus, we shall need to consider the whole family $\succsim^E$ $(\emptyset \neq E \subseteq S)$ of conditional preference orderings as E varies over all possible events in S.

6.3. The Sure Thing Principle

One of Savage's (1954) main axioms is the following:[19]

Definition 19. The family $\succsim^E$ $(\emptyset \neq E \subseteq S)$ of conditional preference orderings satisfies the *sure thing principle* (or STP) provided that whenever the two non-empty events $E, E' \subseteq S$ are disjoint, and whenever $a^E, b^E \in Y^E$,

[18] This choice reveals a preference for Debreu's (1959) terminology, in connection with contingent consumption vectors, over Savage's (1954), who uses the term *act* instead.

[19] Technically, we have simplified Savage's axiom by excluding "null events" E (relative to S) with the property that $(a^E, c^{S \setminus E}) \sim^S (b^E, c^{S \setminus E})$ for all $a^E, b^E \in Y^E$. One logical implication of the consequentialist approach is that an event E can be null only if $a^E \sim^E b^E$ for all $a^E, b^E \in Y^E$. In Hammond (1998b) the axiom we have stated here is called STP*, in contrast to Savage's which is called STP.

$c^{E'} \in Y^{E'}$, then the preference relation between $(a^E, c^{E' \setminus E})$ and $(b^E, c^{E' \setminus E})$ in $Y^{E \cup E'} = Y^E \times Y^{E' \setminus E}$ satisfies

$$(a^E, c^{E'}) \succsim^{E \cup E'} (b^E, c^{E'}) \Longleftrightarrow a^E \succsim^E b^E \tag{52}$$

The rest of this section is devoted to providing an axiomatic foundation for this principle that is similar to those we have already provided for both ordinality and vNM independence.

6.4. Axioms for the Sure Thing Principle

6.4.1. A Family of Simple Trees with Uncertain Consequences

As in Sections 4.3 and 5.1, we are again going to introduce a parametric family of decision trees. This time our goal is to develop variations of Cubitt's (1996) axioms that apply to choice under uncertainty, and which together imply Savage's sure thing principle.

The parameters we use are a pair of disjoint events $E, E' \subseteq S$, and a triple of contingent consequences $a^E, b^E \in Y^E$, $c^{E'} \in Y^{E'}$, as in the statement of STP. For each value of the parameter vector $(E, E'; a^E, b^E, c^{E'})$ there will again be four closely related trees

$$\bar{T}_{E,E';a^E,b^E,c^{E'}},\ T'_{E,E';a^E,b^E,c^{E'}},\ \bar{T}_{E,E';a^E,b^E,c^{E'}}(n_1),\ \text{and } \hat{T}_{E,E';a^E,b^E,c^{E'}}$$

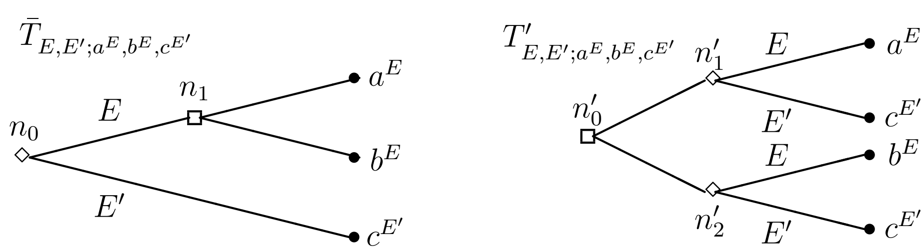

Figure 5: The Decision Tree $\bar{T}_{E,E';a^E,b^E,c^{E'}}$ and the Variation $T'_{E,E';a^E,b^E,c^{E'}}$

The four trees are illustrated in Figures 5 and 6. Rather than chance nodes, the three nodes n_0, n'_1 and n'_2 marked by diamonds are *event nodes* at which "Nature" makes a move, and the decision maker discovers which of the two disjoint events E or E' has occurred. Each terminal node of each tree has been marked by a contingent consequence in the appropriate domain, Y^E or $Y^{E'}$, depending on what happened at the previous event node.

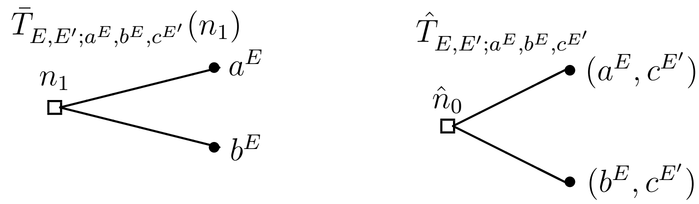

Figure 6: The Continuation $\bar{T}_{E,E';a^E,b^E,c^{E'}}(n_1)$ and the Reduction $\hat{T}_{E,E';a^E,b^E,c^{E'}}$

The trees can be used to illustrate Ellsberg's example for the special case when one ignores the information that 30 of the 90 balls in the urn are red. To do so, take the events $E := \{\text{red}, \text{black}\}$ and $E' := \{\text{yellow}\}$, along with contingent consequences $a^E = (100, 0)$, $b^E = (0, 100)$, and then $c^{E'}$ either 0 in the case when the choice is between L_1 and L_2, or 100 in the other case when the choice is between L_3 and L_4.

6.4.2. Preliminaries

Because of the need to keep track of what states $s \in S$ are possible, we must adapt the notation that was introduced in Section 4.3.1 for decision trees with deterministic consequences, and in Section 5.1.1 for decision trees with risky consequences.

Definition 20. In each of the four decision trees T of Figures 5 and 6, and for each of the nodes n of T, let:

1. $N_{+1}(T, n)$ denote the set of nodes that immediately succeed n in the tree T;

2. $E(T, n) \subset S$ denote the event containing those states $s \in S$ that are possible after reaching node n in the tree T;

3. $F(T, n) \subset Y^{E(T,n)}$ denote the *feasible set* of contingent consequences given that node n has been reached in decision tree T;

4. $\Psi(T, n) \subseteq F(T, n)$ denote the *planned set* contingent of consequences at node n.

Note that, in the four trees $\bar{T}$, T', $\bar{T}(n_1)$ and $\hat{T}$ of Figures 5 and 6, the relevant values of $E(T, n)$ and of $F(T, n)$ are obvious at the terminal nodes because they correspond to the contingent consequences; at the different non-terminal nodes of these trees, however, they are given:

1. at the node n_1 of trees $\bar{T}$ and $\bar{T}(n_1)$, by

$$E(\bar{T}, n_1) = E(\bar{T}(n_1), n_1) = E \text{ and } F(\bar{T}, n_1) = F(\bar{T}(n_1), n_1) = \{a^E, b^E\}$$

2. at all the other non-terminal nodes, by $E(T, n) = E \cup E'$, and

$$F(T', n'_1) = \{(a^E, c^{E'})\}; F(T', n'_2) = \{(b^E, c^{E'})\}$$

$$F(\bar{T}, n_0) = F(\bar{T}, n_1) \times \{c^{E'}\} = \{a^E, b^E\} \times \{c^{E'}\} = \{(a^E, c^{E'}), (b^E, c^{E'})\}$$
$$F(T', n'_0) = F(T', n'_1) \cup F(T', n'_2) = \{(a^E, c^{E'}), (b^E, c^{E'})\}$$

Note that at any event node, the feasible sets all satisfy the equality

$$F(T, n) = \prod\nolimits_{n' \in N_{+1}(n)} F(T, n') \tag{53}$$

whose right-hand side is the Cartesian product. This is the general rule for folding back the feasible sets at an event node. Of course, we still have the folding back rule $F(T, n) = \cup_{n' \in N_{+1}(n)} F(T, n')$ when n is a decision node.

6.4.3. Reduction of Compound Events

The two trees $T' = T'_{E,E';a^E,b^E,c^{E'}}$ in Figure 5 and $\hat{T} = \hat{T}_{E,E';a^E,b^E,c^{E'}}$ in Figure 6 both start with a decision node. In $\hat{T}$ a move at the initial decision node $\hat{n}_0$ leads directly to one of the two contingent consequences $(a^E, c^{E'})$ or $(b^E, c^{E'})$; in T' a move at the initial decision node n'_0 leads instead to one of the two immediately succeeding event nodes n'_1 or n'_2. But at both these event nodes Nature makes a move resulting in either the event E or the event E'. In either case, when the consequences of this move by Nature are folded back in the obvious way, the final result is again one of the two contingent consequences $(a^E, c^{E'})$ or $(b^E, c^{E'})$.

The following axiom requires that, because the differences between T' and $\hat{T}$ are essentially irrelevant, there should be no difference in the planned consequence sets $\Psi(T', n'_0)$ and $\Psi(\hat{T}, \hat{n}_0)$.

Definition 21. Given the two decision trees $T' = T'_{E,E';a^E,b^E,c^{E'}}$ in Figure 5 and $\hat{T} = \hat{T}_{E,E';a^E,b^E,c^{E'}}$ in Figure 6, say that *reduction of compound events* is satisfied provided that the two planned consequence sets $\Psi(T', n'_0)$ and $\Psi(\hat{T}, \hat{n}_0)$ coincide.

6.4.4. Dynamic Consistency at an Event Node

Consider again the decision tree $\bar{T}$ shown in Figure 5. Regardless of what is planned at node n_0, the decision maker will be taken to decision node n_1 in case the event E occurs, as opposed to E'. It should also be anticipated that a specific non-empty subset $\Psi(\bar{T}, n_1) \subseteq \{a^E, b^E\}$ will be selected at that node.

With $c^{E'}$ fixed, we now require the bijection $y^E \leftrightarrow (y^E, c^{E'})$ between the two sets $F(\bar{T}, n_1)$ and $F(\bar{T}, n_0)$ to induce a correspondence between the two planned consequence sets $\Psi(\bar{T}, n_1)$ and $\Psi(\bar{T}, n_0)$. Formally:

Definition 22. The planned consequence sets $\Psi(\bar{T}, n_0)$ and $\Psi(\bar{T}, n_1)$ at the two non-terminal nodes of the tree $\bar{T} = \bar{T}_{E,E';a^E,b^E,c^{E'}}$ of Figure 5 are *dynamically consistent at event node* n_0 provided that

$$\begin{aligned} a^E \in \Psi(\bar{T}, n_1) &\iff (a^E, c^{E'}) \in \Psi(\bar{T}, n_0) \\ \text{and} \quad b^E \in \Psi(\bar{T}, n_1) &\iff (b^E, c^{E'}) \in \Psi(\bar{T}, n_0) \end{aligned} \tag{54}$$

or equivalently, provided that $\Psi(\bar{T}, n_0) = \Psi(\bar{T}, n_1) \times \{c^{E'}\}$.

6.4.5. Separability After an Event Node

As in Sections 4.3.3 and 5.1.5, separability requires the "continuation subtree" $\bar{T}(n_1)$ of tree $\bar{T} = \bar{T}_{E,E';a^E,b^E,c^{E'}}$ that starts at node n_1 to be treated as if it were a full decision tree. The difference from before is that here the preceding node is an event node.

Here is a formal definition for an arbitrary finite tree T of separability at an event node n of T.

Definition 23. Given any decision tree T, the planned consequence set $\Psi(T, n)$ at a chance node satisfies *separability after the event node* n provided that it equals the planned consequence set $\Psi(T(n), n)$ at the initial node n of the continuation subtree $T(n)$.

6.4.6. Timing Invariance with Uncertainty

Consider once again the two decision trees $\bar{T} = \bar{T}_{E,E';a^E,b^E,c^{E'}}$ and $T' = T'_{E,E';a^E,b^E,c^{E'}}$ shown in Figure 5. The calculations in Section 6.4.2 already imply that the sets $F(\bar{T}, n_0)$ and $F(T', n'_0)$ must be equal. The only difference between these two trees is that in $\bar{T}$, the move by Nature that picks "up" in event E and "down" in event E' precedes the decision node, whereas this timing is reversed in T'. We introduce the following counterpart of the timing reversal condition stated in Section 5.1.6.

Definition 24. Given the two decision trees $\bar{T} = \bar{T}_{E,E';a^E,b^E,c^{E'}}$ and $T' = T'_{E,E';a^E,b^E,c^{E'}}$ shown in Figure 5, which differ only in the timing of the decision and event nodes, say that there is *timing invariance under uncertainty* provided that the two planned consequence sets $\Psi(\bar{T}, n_0)$ and $\Psi(T', n'_0)$ coincide.

6.5. A Precursor of the Sure Thing Principle

We now have the following useful result:

Lemma 12. *Given any pair of disjoint events $E, E' \subseteq S$, and any triple of contingent consequences $a^E, b^E \in Y^E$, $c^{E'} \in Y^{E'}$, consider the four decision trees $\bar{T}_{E,E';a^E,b^E,c^{E'}}$, $T'_{E,E';a^E,b^E,c^{E'}}$, $\bar{T}_{E,E';a^E,b^E,c^{E'}}(n_1)$, and $\hat{T}_{E,E';a^E,b^E,c^{E'}}$ as shown in Figures 5 and 6. Suppose that the planned consequence sets $\Psi(T, n)$ at all the non-terminal nodes n of each tree T among these four satisfy: (i) reduction of compound events; (ii) dynamic consistency at an event node; (iii) separability after an event node; (iv) timing invariance under uncertainty. Then the two planned consequence sets $\Psi(\hat{T}, \hat{n}_0)$ and $\Psi(\bar{T}(n_1), n_1)$ in the reduced trees $\hat{T}$ and $\bar{T}(n_1)$ of Figure 6 satisfy*

$$\Psi(\hat{T}, \hat{n}_0) = \Psi(\bar{T}(n_1), n_1) \times \{c^{E'}\} \tag{55}$$

Proof. Applying the four conditions (i)–(iv) one after another yields successively each equality in the chain

$$\begin{aligned} \Psi(\hat{T}, \hat{n}_0) &= \Psi(T', n'_0) = \Psi(\bar{T}, n_0) = \Psi(\bar{T}, n_1) \times \{c^{E'}\} \\ &= \Psi(\bar{T}(n_1), n_1) \times \{c^{E'}\} \end{aligned} \tag{56}$$

from which (55) follows trivially. □

We can now show that appropriate versions of the three axioms introduced in Sections 4.3.2–4.3.4 as well as here are sufficient for ordinality and a strengthened version of Savage's sure thing principle.

Theorem 13. *Suppose that:*

1. *for each event $E \subset S$, the hypotheses of Theorem 5 are satisfied whenever $F \subset Y^E$ is a feasible set consisting of at least 3 distinct contingent consequences;*

2. *the hypotheses of Lemma 12 are satisfied.*

Then there exists a family $\succsim^E$ ($\emptyset \neq E \subseteq S$) of conditional preference orderings on the respective domains Y^E which satisfy the sure thing principle, and also have the property that, in every reduced form finite decision tree $\hat{T}$ whose terminal nodes have consequences in Y^E, the planned consequence set satisfies

$$\Psi(\hat{T}, \hat{n}_0) = C^{\succsim^E}(F(\hat{T}, \hat{n}_0)) \tag{57}$$

where $\mathcal{F}(Y^E) \ni F \mapsto C^{\succsim^E}(F)$ is the ordinal choice function on non-empty finite subsets $F \subset Y^E$ that is generated by the ordering $\succsim^E$.

Proof. For each event $E \subset S$, applying Theorem 5 to the domain Y^E instead of Y implies that the ordering $\succsim^E$ exists on Y^E and (57) is satisfied.

Given any two disjoint events $E, E' \subseteq S$ and any $a^E, b^E \in Y^E$, $c^{E'} \in Y^{E'}$, applying Lemma 12 implies that for the two decision trees $\bar{T}(n_1) = \bar{T}_{\lambda,\mu,\nu;\alpha}(n_1)$ and $\hat{T} = \hat{T}_{\lambda,\mu,\nu;\alpha}$ shown in Figure 4, one has

$$F(\bar{T}(n_1), n_1) = \{a^E, b^E\} \quad \text{and} \quad F(\hat{T}, \hat{n}_0) = F(\bar{T}(n_1), n_1) \times \{c^{E'}\}$$

By definition of each ordering $\succsim^E$, it follows that

$$\begin{array}{rcl} a^E \in \Psi(\bar{T}(n_1), n_1) & \Longleftrightarrow & a^E \succsim^E b^E \quad \text{and} \\ (a^E, c^{E'}) \in \Psi(\hat{T}, \hat{n}_0) & \Longleftrightarrow & (a^E, c^{E'}) \succsim^{E \cup E'} (b^E, c^{E'}) \end{array} \tag{58}$$

But equation (55) implies that

$$a^E \in \Psi(T(n_1), n_1) \Longleftrightarrow (a^E, c^{E'}) \in \Psi(\hat{T}, \hat{n}_0)$$

Combining this with (58) yields

$$a^E \succsim^E b^E \Longleftrightarrow (a^E, c^{E'}) \succsim^{E \cup E'} (b^E, c^{E'})$$

which is precisely the STP axiom as stated in (52). □

6.6. Behavior and Its Uncertain Consequences

6.6.1. Folding Back at Decision and Event Nodes

We begin by recalling the notation of Section 4.5.1, and introducing some new notation to recognise the unfolding of events as one passes through an event node. So at any decision node n of any tree T, one has:

1. a set $N_{+1}(T, n)$ of immediately succeeding nodes in the decision tree T;

2. a non-empty behavior set $\beta(T,n) \subseteq N_{+1}(T,n)$;

3. a non-emptyset $E(T,n) \subset S$ of possible states;

4. a feasible consequence set $F(T,n) \subseteq Y^{E(T,n)}$;

5. a consequence choice set $\Phi(T,n) \subseteq F(T,n)$ induced by behavior $\beta(T,n')$ at all nodes n' of the continuation subtree $T(n)$ whose initial node is n.

As in Section 4.5.1, we assume that at any decision node n of any decision tree T, the feasible set $F(T,n)$ and the consequence choice set $\Phi(T,n)$ satisfy the folding back rule (30), reproduced below for convenience:

$$F(T,n) = \bigcup\nolimits_{n' \in N_{+1}(T,n)} F(T,n') \text{ and } \Phi(T,n) = \bigcup\nolimits_{n' \in \beta(T(n),n)} \Phi(T,n')$$

The folding back rules at an event node n of any decision tree T allowing uncertainty include an extra rule for the sets $E(T,n)$ of possible states. They take the form

$$\begin{array}{rrcl} & F(T,n) & = & \prod_{n' \in N_{+1}(T,n)} F(T,n') \\ & \Phi(T,n) & = & \prod_{n' \in \beta(T(n),n)} \Phi(T,n') \\ \text{and} & E(T,n) & = & \bigcup_{n' \in N_{+1}(T,n)} E(T,n') \end{array} \tag{59}$$

The third equation reflects the fact that Nature's move at the event node n generates a partition of the set of states $E(T,n)$ into the collection $\{E(T,n') \mid n' \in N_{+1}(T,n)\}$. In the first two equations, the left hand side is a subset of $Y^{E(T,n)} := \prod_{n' \in N_{+1}(T,n)} Y^{E(T,n')}$, so matches the right hand side as one would expect.

6.6.2. Implications of Consequentialist Invariance

The following result demonstrates that consequentialist invariance as defined in Section 4.5.1 implies that the consequence choice sets $\Phi(T,n)$ satisfy our axioms for the sure thing principle.

Theorem 14. *Given the four decision trees shown in Figures 5 and 6, suppose that behavior $\beta(T,n)$ and the consequence choice sets $\Phi(T,n)$ at all the nodes n of these trees satisfy consequentialist invariance. Then the consequence choice sets $\Phi(T,n)$ also satisfy the four properties: (i) reduction of compound events; (ii) dynamic consistency at the event node n_0; (iii) separability after the event node n_0; (iv) timing invariance under uncertainty.*

Proof. The analysis of Section 6.4.2 establishes that $F(T', n'_0) = F(\hat{T}, \hat{n}_0) = \{a^E, b^E\} \times \{c^{E'}\}$. Consequentialist invariance then implies that $\Phi(T', n'_0) = \Psi(\hat{T}, \hat{n}_0)$, which is the condition for the consequence choice sets to satisfy reduction of compound events.

The relevant second folding back rule in (59) is identical to the equation $\Phi(\bar{T}, n_0) = \Phi(\bar{T}, n_1) \times \{c^{E'}\}$ in Definition 22, so dynamic consistency at the event node n_0 is satisfied.

Next, because $F(\bar{T}, n_1) = F(\bar{T}(n_1), n_1) = \{a^E, b^E\}$, consequentialist invariance implies that $\Phi(\bar{T}, n_1) = \Phi(\bar{T}(n_1), n_1)$, which is the condition for separability after the event node n_0.

Finally, because $F(\bar{T}, n_0) = F(T', n'_0) = \{a^E, b^E\} \times \{c^{E'}\}$, consequentialist invariance implies that $\Phi(\bar{T}, n_0) = \Phi(T', n'_0)$, which is the condition for timing invariance under uncertainty. □

6.7. Ordinal Dynamic Programming under Uncertainty

Let $\mathcal{T}(S; Y^S)$ denote the domain of all finite decision trees T with decision and event nodes for which there exists some event $E \subset S$ such that the initial set of states is $E(T, n_0) = E$, and also $F(T, n) \subset Y^{E(T,n)}$ at every node of T.

The ordinal dynamic programming result stated as Theorem 11 of Section 5.4 for the case when decisions have risky consequences has the following counterpart for decisions under uncertainty, when the decision tree has event rather than chance nodes.

Theorem 15. *Let $\succsim^E$ ($\emptyset \neq E \subseteq S$) be any family of conditional preference orderings on the respective domains Y^E that satisfy the sure thing principle. For each event $E \subseteq S$, let $C^{\succsim^E}$ denote the ordinal choice function on the domain $\mathcal{F}(Y^E)$ of non-empty finite subsets of Y^E that is induced by $\succsim^E$. Then there exists a behavior rule $(T, n) \mapsto \beta(T, n)$ defined on the domain $\mathcal{T}(S; Y^S)$ having the property that the induced consequence choice sets, which are calculated by applying the folding back rules (30) and (59), satisfy consequentialist invariance with*

$$\Psi(T, n) = C^{\succsim^{E(T,n))}}(F(T, n)) \tag{60}$$

at every node n of every tree T in $\mathcal{T}(S; Y^S)$.

Proof. See Hammond (1998b) for a proof of a comparable result in a more complicated setting that combines risk with uncertainty. □

6.7.1. Experimental Tests: Ellsberg and Beyond

The vNM independence axiom has been tested experimentally very often; despite Ellsberg's early example, the sure-thing principle has been tested much more rarely, especially in a dynamic setting.

A summary of early empirical evidence for and against the sure thing principle appeared in Camerer and Weber (1992). They identify two kinds of experimental study. The first kind, represented by Dominiak et al. (2012), sticks quite closely to Ellsberg's (1961) original example. The second kind of study focuses on phenomena related to attitudes towards "ambiguity", following a more modern interpretation of the Ellsberg example. A prominent example is the work of Baillon et al. (2011) on Machina's (2009) extension of that famous example. Other works on ambiguity are too numerous to discuss here, and are anyway covered elsewhere in this *Handbook*.

The only study of separability and dynamic consistency at event nodes that we are aware of is Dominiak et al. (2012). They also report frequent violations of dynamic consistency and few of separability. We hope that our extension of Cubitt's factorised approach, with its distinction between dynamic choice principles at decision and event nodes, will be helpful for designing further experimental tests beyond the theoretical discussion provided here.

References

Abdellaoui, M., Diecidue, E., Öncüler, A. (2011). "Risk preferences at different time periods: an experimental investigation" . Management Science 57, 975–987.

Ainslie, G.W. (1992). Picoeconomics. Cambridge: Cambridge University Press.

Akerlof, G.A. (1991). "Procrastination and obedience" . American Economic Review, Papers and Proceedings 81, 1–19.

Allais, M. (1953). "Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Américaine" . Econometrica 21, 503–546.

Allais, M. (1979). "The so-called Allais paradox and rational decisions under uncertainty" . In: Allais M., Hagen, O. (Eds.), Expected Utility Hypotheses and the Allais Paradox. Reidel, Dordrecht.

Anscombe, F.J., Aumann, R.J. (1963). "A definition of subjective probability" . Annals of Mathematical Statistics 34, 199–205.

Anscombe, G.E.M. (1958). "Modern moral philosophy" . Philosophy 33, 1–19.

Arrow, K.J. (1951a). "Alternative approaches to the theory of choice in risk-taking situations" . Econometrica 19, 404–437.

Arrow, K.J. (1951b; 2nd edn 1963). Social Choice and Individual Values. John Wiley, New York.

Arrow, K.J. (1959). "Rational choice functions and orderings" . Economica 26, 121–127.

Arrow, K.J. (1965). "The theory of risk aversion" . In: Aspects of the Theory of Risk Bearing. Yrjo Jahnssonin Saatio, Helsinki. Reprinted in: Essays in the Theory of Risk Bearing (1971, pp. 90–109). Markham Publ. Co., Chicago.

Baillon, A., L'Haridon, O., Placido., L. (2011). "Ambiguity Models and the Machina Paradoxes" . American Economic Review 101, 1547–1560.

Barkan R., Busemeyer, J.R. (2003). "Modeling dynamic inconsistency with a changing reference point" . Journal of Behavioral Decision Making 16, 235–255.

Barro, R.J. (1999). "Ramsey meets Laibson in the neoclassical growth model" . Quarterly Journal of Economics 114, 1125–1152.

Baucells, M., Heukamp, F. (2012). "Probability and time tradeoff" . Management Science 58, 831–842.

Bernheim, B.D., Rangel, A. (2007). "Behavioral public economics: welfare and policy analysis with non-standard decision-makers" . In: Diamond, P., Vartiainen, H. (Eds.) Behavioral Economics and its Applications. Princeton, Princeton University Press.

Bernoulli, D. (1738). "Specimen theoriae novae de mensura sortis" . In: Commentarii Academiae Scientiarum Imperialis Petropolitanae; translated by Sommer, L. (1954) as "Exposition of a new theory on the measurement of risk" . Econometrica 22, 23–36.

Blackorby, C., Nissen, D., Primont, D., Russell, R.R. (1973). “Consistent intertemporal decision making” . Review of Economic Studies 40, 239–248.

Bleichrodt, H., Rohde, K.I.M., Wakker, P.P. (2009). “Non-hyperbolic time inconsistency” . Games and Economic Behavior 66, 27–38.

Bordes, G. (1976). “Consistency, rationality, and collective choice” . Review of Economic Studies 43, 451–457.

Busemeyer, J.R., Weg, E., Barkan, R., Li, X., Ma, Z. (2000). “Dynamic and consequential consistency of choices between paths of decision trees” . Journal of Experimental Psychology: General 129, 530–545.

Butler, D.J., Loomes, G.C. (2007). “Imprecision as an account of the preference reversal phenomenon” . American Economic Review 97, 277–297.

Camerer, C. (1995). “Individual decision making” . In: Kagel, J., Roth, A.E. (Eds.) Handbook of Experimental Economics. Princeton University Press, Princeton.

Camerer, C., Weber, M., (1992). “Recent developments in modelling preferences: uncertainty and ambiguity” . Journal of Risk and Uncertainty 5, 325370.

Chernoff, H. (1954). “Rational selection of decision functions” . Econometrica 22, 423–443.

Cooter, R., Rappoport, P. (1984). “Were the ordinalists wrong about welfare economics?” Journal of Economic Literature 22, 507–530.

Cramer, G. (1728). Letter to Nicholas Bernoulli; extracts printed in Bernoulli (1738).

Cubitt, R.P. (1996). “Rational dynamic choice and expected utility theory” . Oxford Economic Papers 48, 1–19.

Cubitt, R.P., Ruiz-Martos, M., Starmer, C. (2012). “Are bygones bygones?” Theory and Decision 73, 185–202.

Cubitt, R.P., Starmer, C., Sugden, R. (1998). "Dynamic choice and the common ratio effect: an experimental investigation" . Economic Journal 108, 1362–1380.

Cubitt, R.P., Starmer, C., Sugden, R. (2004). "Dynamic decisions under uncertainty: some recent evidence from economics and psychology" . In: Brocas, I., Carrillo, J.D. (Eds.) The Psychology of Economic Decisions, Volume II. Oxford University Press.

Cubitt, R.P., Sugden, R. (2001). "Dynamic decision making under uncertainty: an experimental investigation of choices between accumulator gambles" . Journal of Risk and Uncertainty 22, 103–128.

Dalkey, N.C. (1949). "A numerical scale for partially ordered utilities" . RM-296, RAND Corporation, Santa Monica, CA.

Dasgupta, P. (1974). "On some alternative criteria for justice between generations" . Journal of Public Economics 3, 405–423.

Dasgupta, P., Maskin, E. (2005). "Uncertainty and hyperbolic discounting" . American Economic Review 95, 1290–1299.

Debreu, G. (1959). Theory of Value: An Axiomatic Analysis of Economic Equilibrium. Yale University Press, New Haven and London.

Dominiak, A., Dürsch, P., Lefort, J.-P. (2012). "A dynamic Ellsberg urn experiment" . Games and Economic Behavior 75, 625–638.

Ebert, J., Prelec, D. (2007). "The fragility of time: time-insensitivity and valuation of the near and far future" . Management Science 53, 1423–1438.

Ellsberg, D. (1961). "Risk, ambiguity and the Savage axioms" . Quarterly Journal of Economics 75, 643–669.

Epper, T., Fehr-Duda, H., Bruhin, A. (2011). "Viewing the future through a warped lens: why uncertainty generates hyperbolic discounting" . Journal of Risk and Uncertainty 43, 169–203.

Farquharson, R. (1969). Theory of Voting. Blackwell, Oxford.

Fishburn, P.C. (1970). Utility Theory for Decision Making. John Wiley and Sons, New York.

Fishburn, P.C., Rubinstein, A. (1982). "Time preference" . International Economic Review 23, 677–694.

Fishburn, P.C., Wakker, P.P. (1995). "The invention of the independence condition for preferences" . Management Science 41, 1130–1144.

Frederick, S., Loewenstein, G., O'Donoghue, T. (2002). "Time discounting and time preference: a critical review" . Journal of Economic Literature 40, 351–401.

Gigerenzer, G., Todd, P.M., the ABC Research Group (1999). Simple Heuristics That Make Us Smart. Oxford University Press, New York.

Grether, D.M., Plott, C.R. (1979). "Economic theory of choice and the preference reversal phenomenon" . American Economic Review 69, 623–639.

Halevy, Y. (2008). "Strotz meets Allais: diminishing impatience and certainty effect" . American Economic Review 98, 1145–1162.

Hammond, P.J. (1973). Consistent Planning and Intertemporal Welfare Economics. Ph.D. dissertation, University of Cambridge.

Hammond, P.J. (1976). "Changing tastes and coherent dynamic choice" . Review of Economic Studies 43, 159–173.

Hammond, P.J. (1977). "Dynamic restrictions on metastatic choice" . Economica 44, 337–350.

Hammond, P.J. (1983). "Ex-post optimality as a dynamically consistent objective for collective choice under uncertainty" . In: Pattanaik, P.K., Salles, M. (Eds.) Social Choice and Welfare. North-Holland.

Hammond, P.J. (1988a). "Consequentialist foundations for expected utility" . Theory and Decision 25, 25–78.

Hammond, P.J. (1988b). "Orderly decision theory: a comment on Professor Seidenfeld" . Economics and Philosophy 4, 292–297.

Hammond, P.J. (1989). "Consistent plans, consequentialism, and expected utility" . Econometrica 57, 1445–1449.

Hammond, P.J. (1998a). "Objective expected utility: a consequentialist perspective" . In: Barberà, S., Hammond, P.J., Seidl, C. (Eds.) Handbook of Utility Theory, Volume I: Principles. Kluwer, Dordrecht, The Netherlands.

Hammond, P.J. (1998b). "Subjective expected utility" . In: Barberà, S., Hammond, P.J., Seidl, C. (Eds.) Handbook of Utility Theory, Volume I: Principles. Kluwer, Dordrecht, The Netherlands.

Harris, C.J., Laibson, D.I. (2001). "Dynamic choices of hyperbolic consumers" . Econometrica 69, 935–957.

Hayashi, T. (2003). "Quasi-stationary cardinal utility and present bias" . Journal of Economic Theory 112, 343–52.

Herstein, I.N., Milnor, J. (1953). "An axiomatic approach to measurable utility" . Econometrica 21, 291–297.

Hey, J.D., Lotito, G. (2009). "Naive, resolute or sophisticated? A study of dynamic decision making" . Journal of Risk and Uncertainty 38, 1–25.

Hey, J.D., Paradiso, M. (2006). "Preferences over temporal frames in dynamic decision problems: an experimental investigation" . The Manchester School 74, 123–137.

Huber, J., Payne, J. W., Puto, C. (1982). "Adding asymmetrically dominated alternatives: violations of regularity and the similarity hypothesis" . Journal of Consumer Research 9, 90–98.

Inagaki, M. (1970). Optimal Economic Growth: Shifting Finite versus Infinite Time Horizon. North-Holland. Amsterdam.

Isoni, A., Loomes, G., Sugden, R. (2011). "The willingness to pay—willingness to accept gap, the 'endowment effect,' subject misconceptions, and experimental procedures for eliciting valuations: comment" . American Economic Review 101, 991–1011.

Jensen, N.E. (1967). "An introduction to Bernoullian utility theory, I: utility functions," Swedish Journal of Economics 69, 163–183.

Johnson, J.G., Busemeyer, J.R. (2001). “Multiple-stage decision-making: the effect of planning horizon length on dynamic consistency.” Theory and Decision 51, 217–246.

Kahneman, D., Tversky, A. (1979). “Prospect theory: an analysis of decision under risk” . Econometrica 47, 263–291.

Karni, E., Schmeidler, D. (1991). “Atemporal dynamic consistency and expected utility” . Journal of Economic Theory 54, 401–408.

Klein, D.B. (1990). “The microfoundations of rules vs. discretion” . Constitutional Political Economy 1, 1–19.

Knight, F. H. (1921). Risk, Uncertainty, and Profit. Hart, Schaffner & Marx, Houghton Mifflin Company, Boston, MA.

Koopmans, T.C. (1960). “Stationary ordinal utility and impatience” . Econometrica 28, 287309.

Krusell, P., Smith Jr., A.A. (2003). “Consumption-savings decisions with quasi-geometric discounting” . Econometrica 71, 365–375.

Kydland, F.E., Prescott, E.C. (1977). “Rules rather than discretion: the inconsistency of optimal plans”. Journal of Political Economy 85, 473–492.

Laibson, D.I. (1997). “Golden eggs and hyperbolic discounting” . Quarterly Journal of Economics 112, 443–477.

Laibson, D.I. (1998). “Life-cycle consumption and hyperbolic discounting” . European Economic Review 42, 861–871.

LaValle, I.H., Wapman, K.R. (1986). “Rolling back decision trees requires the independence axiom!” Management Science 32, 382–385.

Leonard, R.J. (1995). “From parlor games to social science: von Neumann, Morgenstern, and the creation of game theory 1928–1944” . Journal of Economic Literature 33, 730–761.

Lichtenstein, S., Slovic, P. (1971). “Reversals of preference between bids and choices in gambling decisions” . Journal of Experimental Psychology 89, 46–55.

List, J.A. (2002). “Preference reversals of a different kind: the ”more is less” phenomenon” . American Economic Review 92, 1636–1643.

Loewenstein, G.F., Prelec, D. (1992). “Anomalies in intertemporal choice: evidence and an interpretation” . Quarterly Journal of Economics 107, 573597.

Loomes, G., Starmer, C., Sugden, R. (2010). “Preference reversals and disparities between willingness to pay and willingness to accept in repeated markets” . Journal of Economic Psychology 31, 374–387.

Machina, M.J. (1989). “Dynamic consistency and non-expected utility models of choice under uncertainty” . Journal of Economic Literature 27, 1622–1688.

Machina, M.J. (2009). “Risk, ambiguity, and the rank-dependence axioms” . American Economic Review 99, 385–92.

Machina, M.J., Schmeidler, D. (1992). “A more robust definition of subjective probability” . Econometrica 60, 745–780.

Malinvaud, E. (1952). “Note on von Neumann–Morgenstern’s strong independence axiom” . Econometrica 20, 679.

Manzini, P., Mariotti, M. (2007). “Sequentially rationalizable choice” . American Economic Review 97, 1824–1839.

Marschak, J.A. (1950). “Rational behavior, uncertain prospects, and measurable utility” . Econometrica 18, 111–141.

Mas-Colell, A., Whinston, M.W., Green, J.R. (1995). Microeconomic Theory. Oxford University Press.

McClennen, E.F. (1986). “Prisoner’s dilemma and resolute choice” . In: Campbell, R., Sowden, L. (Eds.) Paradoxes of Rationality and Cooperation. University of British Columbia Press, Vancouver.

McClennen, E.F. (1990). Rationality and Dynamic Choice: Foundational Explorations. Cambridge University Press, Cambridge.

Moulin, H. (1979). “Dominance solvable voting schemes” . Econometrica 47, 1337–1351.

Nash, J.F. (1950). "The bargaining problem" . Econometrica 18, 155–162.

Nebout, A., Dubois, D. (2011). "When Allais meets Ulysses: dynamic axioms and the common ratio effect" . Research Paper 2011-02, LAMETA Working Paper Series, University of Montpellier. http://www.asfee.fr/sites/default/files/201102_asfee.pdf.

Noussair, C, Wu, P., 2006. "Risk tolerance in the present and the future: An experimental study. Managerial and Decision Economics 27, 401–412.

O'Donoghue, T., Rabin, M. (1999a). "Doing it now or later" . American Economic Review 89, 103–124.

O'Donoghue, T., Rabin, M. (1999b). "Incentives for procrastinators" . Quarterly Journal of Economics 114, 769–816.

O'Donoghue, T., Rabin, M. (2001). "Choice and procrastination" . Quarterly Journal of Economics 116, 121–160.

Peleg, B., Yaari, M.E., 1973. "On the existence of a consistent course of action when tastes are changing" . Review of Economic Studies 40, 391–401.

Phelps, E.S., Pollak, R.A. (1968). "On second-best national saving and game-equilibrium growth" . Review of Economic Studies 35, 185–199.

Plott, C.R., Zeiler, K. (2005). "The willingness to pay–willingness to accept gap, the 'endowment effect,' subject misconceptions, and experimental procedures for eliciting valuations" . American Economic Review 95, 530–545.

Pollak, R.A. (1968). "Consistent planning" . Review of Economic Studies 35, 201–208.

Pratt, J.W. (1964). "Risk aversion in the small and in the large" . Econometrica 32, 122–136.

Quiggin, J., Horowitz, J. (1995). "Time and risk" . Journal of Risk And Uncertainty 10, 37–55.

Raiffa, H. (1961). "Risk, ambiguity, and the Savage axioms: comment" . Quarterly Journal of Economics 75, 690–694.

Raiffa, H. (1968). Decision Analysis: Introductory Lectures on Choice Under Uncertainty. Addison-Wesley, Reading, MA, USA.

Samuelson, P.A. (1937). "A note on measurement of utility" . Review of Economic Studies 4, 155–161.

Samuelson, P.A. (1952). "Probability, utility, and the independence axiom" . Econometrica 20, 670–678.

Samuelson, P.A. (1983). Foundations of Economic Analysis. (Enlarged Edition) Harvard University Press.

Sarin R., Wakker, P.P. (1994). "Folding back in decision trees" . Management Science 40, 625–628.

Savage, L.J. (1954). The Foundations of Statistics. Wiley, New York, USA.

Seidl, C. (2002). "Preference reversal" . Journal of Economic Surveys 16, 621–655.

Seidenfeld, T. (1988). "Decision theory without 'independence' or without 'ordering': what is the difference?" Economics and Philosophy 4, 267–290.

Selten, R. (1965). "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit" Zeitschrift für die gesamte Staatswissenschaft 12, 201–324."

Sen, A.K. (1977). "Social choice theory: a re-examination" . Econometrica 45, 53–89.

Simonson, I. (1989). "Choice based on reasons: the case of attraction and compromise effects" . Journal of Consumer Research 16, 158–174.

Sobel, M.J. (1975) "Ordinal dynamic programming" . Management Science 21, 967–975.

Starmer, C. (2000). "Developments in non-expected utility theory: the hunt for a descriptive theory of choice under risk" . Journal of Economic Literature 38, 332–382.

Steele, K.S. (2010). "What are the minimal requirements of rational choice? Arguments from the sequential-decision setting" . Theory and Decision 68, 463–487.

Strotz, R.H. (1956). "Myopia and inconsistency in dynamic utility maximization" . Review of Economic Studies 23, 165–180.

Thaler, R.H. (1981). "Some empirical evidence on dynamic inconsistency" . Economics Letters 8, 201-207.

Thaler, R.H., Johnson, E.J. (1990). "Gambling with the house money and trying to break even: the effects of prior outcomes on risky choice" . Management Science 36, 643–660.

Tsetsos, K., Usher, M., Chater, N. (2010). "Preference reversals in multiattribute choice" . Psychological Review 117, 1275–1293.

Tversky, A. (1972). "Choice by elimination" . Journal of Mathematical Psychology 9, 341–367.

Tversky, A., Kahneman, D. (1992). "Advances in prospect theory: cumulative representation of uncertainty" . Journal of Risk and Uncertainty 5, 297–323.

Tversky, A., Sattath, S., Slovic, P. (1988). "Contingent weighting in judgment and choice" . Psychological Review 95, 371–384.

Tversky A., Slovic, P., Kahneman, D. (1990). "The causes of preference reversal" . American Economic Review 80, 204–217.

Volij, O. (1994). "Dynamic consistency, consequentialism and reduction of compound lotteries" . Economics Letters 46, 121–129.

von Neumann, J. (1928). "Zur Theorie der Gesellschaftsspiele" . Mathematische Annalen 100, 295–320; translated as "On the theory of games of strategy" in: Luce, R.D., Tucker, A.W. (Eds., 1959) Contributions to the Theory of Games, Vol. IV. Princeton University Press, Princeton.

von Neumann, J., Morgenstern, O. (1944; 3rd edn. 1953). Theory of Games and Economic Behavior. Princeton University Press, Princeton.

Wakker, P.P. (1994). "Separating marginal utility and probabilistic risk aversion" . Theory and Decision 36, 1–44.

Wakker, P.P. (2001). "Testing and characterizing properties of nonadditive measures through violations of the sure-thing principle" . Econometrica 69, 1039–1059.

Wakker, P.P. (2010). Prospect Theory: For Risk and Ambiguity. Cambridge University Press, Cambridge, UK.