Collective Rationality and Monotone Path Division Rules

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Collective Rationality and Monotone Path Division Rules*

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Abstract

We impose the axiom Independence of Irrelevant Alternatives on division rules for the conflicting claims problem. With the addition of Consistency and Resource Monotonicity, this characterizes a family of rules which can be described in three different but intuitive ways. First, a rule is identified with a fixed monotone path in the space of awards, and for a given claims vector, the path of awards for that claims vector is simply the monotone path truncated by the claims vector. Second, a rule is identified with a set of parametric functions indexed by the claimants, and for a given claims problem, each claimant receives the value of his parametric function at a common parameter value, but truncated by his claim. Third, a rule is identified with an additively separable, strictly concave social welfare function, and for a given claims problem, the amount awarded is the maximizer of the social welfare function subject to the constraint of choosing a feasible award. This third way of describing the family of rules is similar to Lensberg's (1987) solution for bargaining problems applied to conflicting claims problems.

1 Introduction

A conflicting claims problem is a situation in which a divisible homogeneous good must be distributed among a group, each individual in the group having an objective claim on the good, but where the amount of the good is insufficient to satisfy all the claims. An example is dividing the liquidated value of a bankrupt firm among its creditors. How should the good be divided among the claimants? We seek a rule which chooses, for any problem, a feasible allocation or award. (A feasible award is one in which every individual receives an amount between 0 and his claim, and which completely exhausts the good to be divided.)

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1.1 Overview of Results

We impose the axiom Independence of Irrelevant Alternatives (henceforth IIA) on rules. This axiom states that if the chosen award for a problem is also feasible for a second problem whose feasible set is a subset of the original problem, then that award is also chosen for the second problem. This is the same axiom originally used by Nash (1950) in the domain of bargaining problems. In the context of individual choice, this axiom is sometimes known as Chernoff's condition (Chernoff, 1954) or Sen's α (Sen, 1969).

We also impose two axioms that are common in the literature: Consistency and Resource Monotonicity. Consistency states that if a division rule chooses an award for a group of claimants, then it should not choose to reallocate the awards of any subgroup when considered as a separate problem. Resource Monotonicity states that if the amount to be divided increases, then no claimant's award should decrease.

Theorem 1 shows that IIA, Consistency, and Resource Monotonicity characterize a family of rules which can be described in three different but intuitive ways:

- Consider a fixed, weakly monotone path in the space of awards. For any group of claimants and any vector of claims for that group, the path of awards is simply the fixed path truncated by the claims vector. We refer to all such rules as monotone path rules.

- Consider a set of parametric functions, one for each individual. Each parametric function depends only on a single parameter, in which it is weakly increasing. For any problem, each parametric function is truncated by the individual's claim, and a common parameter is found so that the sum of the truncated parametric functions evaluated at that parameter equals the amount to be divided. We refer to all such rules as claims independent parametric rules.

- Consider an additively separable, strictly concave social welfare function. For any problem, the amount awarded is the maximizer of the social welfare function subject to the constraint of choosing a feasible award. We refer to all such rules as collectively rational additively separable (henceforth CRAS) rules.

We also consider a property which is dual to IIA. Rather than taking the awards as what matters to the individuals, as IIA does, this dual property takes the losses (the difference between an individuals claim and his award) as what matters. Theorem 2 shows that IIA and its dual are effectively incompatible: the queueing rule is the only rule to satisfy Consistency, Resource Monotonicity, IIA, and the dual of IIA.

If there is no a priori reason to treat the claimants differently, then one would want the rule to give the same award to individuals with the same claim, a property known as Symmetry. Theorem 3 shows that the constrained equal awards rule is the only rule in our family which satisfies Symmetry. Additionally, we consider the case where each individual has not just a claim on the good to be divided, but also some endowment which cannot be taken from him. In such a case, it may be reasonable to treat individuals with the same claim differently because of differing endowments.

We characterize the rule which, when not bound by feasibility constraints, equalizes the sum of awards and endowments for all individuals.

1.2 Related Literature

To our knowledge, the only other work to consider IIA in the domain of claims problems are a pair of papers by Kıbrıs (2012, 2013). From these papers, the result closest to ours (Kıbrıs, 2012, Theorem 3) is one which characterizes the family of rules that maximize some social welfare function (not necessarily additively separable). The axioms imposed by Kıbrıs are IIA, Continuity, and an axiom called Others-oriented Claims Monotonicity. In general, these rules are not Resource Monotonic. Another way in which this result differs from ours is that the population of claimants is fixed, and thus Consistency does not apply.

Recently, Stovall (2013) characterized the family of (asymmetric) parametric rules, of which the claims independent parametric rules are a special case. The family of symmetric parametric rules (Young, 1987) is also a special case of the family of parametric rules. The only overlap between the family of symmetric parametric rules and the family of claims independent parametric rules is the constrained equal awards rule (see Theorem 3). Additionally, Stovall shows that a parametric rule maximizes an additively separable and claims dependent social welfare function. This differs from our result as Theorem 1 characterizes a rule which maximizes a social welfare function which does not depend on the claims.

Considering other domains, the monotone path rules and the CRAS rules each have analogues in the literature on the bargaining problem. Monotone path rules are similar to the solutions given by Thomson and Myerson (1980), though they consider only strictly monotone paths and a fixed population. CRAS rules are analogous to the family of rules characterized by Lensberg (1987). Since the domain of bargaining problems is much richer than the domain of claims problems, it should come as no surprise that these families of rules are *not* equivalent in the domain of bargaining problems. The main axiom imposed by Thomson and Myerson is a strong monotonicity axiom, which would be equivalent to a strict resource monotonicity axiom here. The main axiom imposed by Lensberg is a consistency axiom similar to the one we impose. Interestingly, in the domain of bargaining problems and in conjunction with Lensberg's other axioms, this implies IIA.

In the literature on fair division under single-peaked preferences, Moulin (1999) characterizes the family of monotone path rules. The key axioms are a consistency axiom and resource monotonicity axiom (similar to the ones we impose), and, because of strategic considerations for an individual reporting his peak, a strategy-proofness axiom.

This work joins a growing literature studying asymmetric rules for the claims problem. In addition to the work by Stovall and Kıbrıs discussed above, Moulin (2000), Naumova (2002), Chambers (2006), and Hokari and Thomson (2003) all consider rules which are not symmetric.

For a recent survey of the literature on the claims problem, see Thomson (2013).

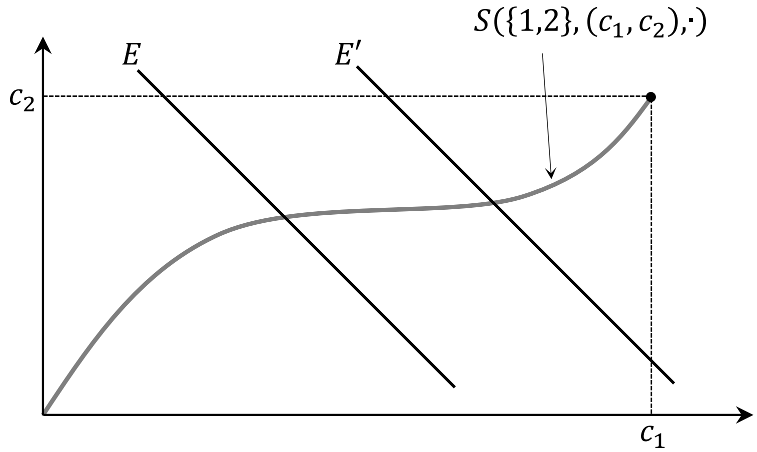

Figure 1: ***Path of awards.*** *The path of awards is the set of all awards as the endowment varies from zero to the sum of claims.*

2 Definitions

We adopt the following notation. Let $\mathcal{N}$ denote the set of finite subsets of the natural numbers, $\mathbb{N}$. Let $\mathbb{R}_+$ denote the non-negative real numbers, $\mathbb{R}_{++}$ the positive real numbers, and $\overline{\mathbb{R}}$ the extended real numbers. Let $\mathbf{0}$ denote a vector of zeros and Ω a vector in which all coordinates are $+\infty$. For $x, y \in \mathbb{R}^N$, we use the vector inequalities $x \geqq y$ if $x_i \geq y_i$ for all $i \in N$, $x \geq y$ if $x \geqq y$ and $x \neq y$, and $x > y$ if $x_i > y_i$ for every $i \in N$. Also, let $x \wedge y$ denote the meet of x and y, i.e. $x \wedge y = (\min\{x_i, y_i\})_{i \in N}$. For $N' \subset N$, let $x_{N'}$ denote the projection of x onto the subspace $\mathbb{R}^{N'}$.

A *claims problem* is a tuple (N, c, E), where $N \in \mathcal{N}$, $c \in \mathbb{R}^N_{++}$, and $E \in \mathbb{R}_+$, all satisfying $\sum_{i \in N} c_i \leq E$. Let $X(N, c, E)$ denote the set of efficient feasible awards vectors for the problem (N, c, E), i.e.

$$X(N, c, E) \equiv \left\{ x \in \mathbb{R}^N_+ : \mathbf{0} \leqq x \leqq c \text{ and } \sum_{i \in N} x_i = E \right\}.$$

A *division rule* is a function S such that for every problem (N, c, E), we have $S(N, c, E) \in X(N, c, E)$.

Some well-known rules include the proportional rule, the constrained equal awards rule, the constrained equal losses rule, the Talmud rule (Aumann and Maschler, 1985), and the queueing rule. The constrained equal awards and queueing rules are members of the family which we characterize, while the others are not.

A convenient way of graphically representing a rule is by the path of awards it generates. For a fixed $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, the path of awards generated by S is the set of all awards $S(N, c, E)$ as E varies from 0 to the sum of claims $\sum_N c_i$. See Figure 1. Thus a rule can be identified with a collection of paths, one for every $N \in \mathcal{N}$ and every $c \in \mathbb{R}^N_{++}$.

3 Main Results

In this section we introduce the axioms imposed and the family of rules which are characterized by those axioms.

3.1 Axioms

We impose three axioms. The first states that increasing the amount to divide should not cause any claimant's award to decrease.

Resource Monotonicity. *For every (N, c, E) and $E' < E$, we have $S(N, c, E') \leq S(N, c, E)$.*

The next axiom states that how a rule divides between two claimants does not change if all other claimants are removed from the problem.

Bilateral Consistency. *For every (N, c, E) and $\{i, j\} \subset N$, if $x = S(N, c, E)$, then*

$$(x_i, x_j) = S(\{i, j\}, (c_i, c_j), x_i + x_j).$$

A more general version of this axiom called Consistency holds for any $N' \subset N$.

Our final axiom axiom is well-known and has been used in many other contexts. It states that if the solution to a problem is feasible for a different problem with a smaller feasible set, then it will be the solution to this second problem.

Independence of Irrelevant Alternatives. *For every (N, c, E) and (N', c', E'), if $X(N', c', E') \subset X(N, c, E)$ and $S(N, c, E) \in X(N', c', E')$, then $S(N', c', E') = S(N, c, E)$.*

We abbreviate the name to IIA. Given the structure here, it is easy to more precisely characterize the conditions of IIA. Given S and (N, c, E), the set of all (N', c', E') such that $X(N', c', E') \subset X(N, c, E)$ and $S(N, c, E) \in X(N', c', E')$ is precisely the set of all (N', c', E') such that $N' = N$, $E' = E$, and $S(N, c, E) \leqq c' \leqq c$. Thus IIA can alternatively be written as:

> For every (N, c, E) and (N, c', E) satisfying $S(N, c, E) \leqq c' \leqq c$, we have $S(N, c', E) = S(N, c, E)$.

IIA is undoubtedly a strong assumption. In essence, it states that the division rule should assign allocations independently of the claims problem at hand. For example, suppose (N, c, E) and c' satisfy the conditions of IIA. Suppose further that $c'_j = c_j$ for all $j \neq i$ and that $c'_i < c_i$. IIA says that since the division rule deemed $x = S(N, c, E)$ a just allocation for the problem (N, c, E), then it must deem x a just allocation for (N, c', E). The fact that the claim of individual i decreased does not alter this assessment since x is still feasible for the problem (N, c', E).

Note that this argument takes for granted the idea that it is the awards that matter. Thus IIA would not be as compelling in applications where the award is actually a bad, e.g. fair taxation. (Taxation problems are formally identical to claims

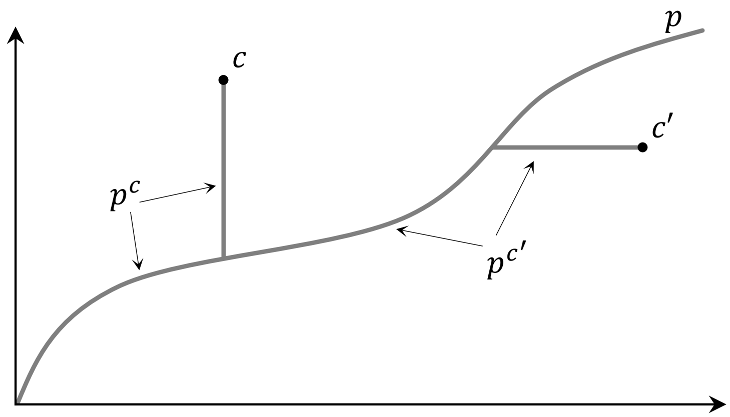

Figure 2: ***Monotone Path Rule.*** *For two claimants, a monotone path p, as well as the truncated paths for claims c and c'.*

problems: c is interpreted as the vector of incomes, E is interpreted as the revenue that must be raised via taxation, and $S(N,c,E)$ is the assignment of taxes.) We will return to this point in Section 4.

In the context of individual choice, IIA is considered to be a standard rationality assumption. Thus one can think of IIA as imposing some amount of "rationality" on the social choice function S. One may then wonder what the implications would be of assuming the stronger axiom WARP. Given the structure here, it is not difficult to show that IIA and WARP are in fact equivalent.

3.2 Monotone Path Rules

We describe here a family of rules defined from a monotone path.

For $M \subset \mathbb{N}$ and for $x, y \in \overline{\mathbb{R}}^M$, a path from x to y is a continuous function $p : [0,1] \to \overline{\mathbb{R}}^M$ such that $p(0) = x$ and $p(1) = y$. A path is weakly monotone if $t > t'$ implies $p(t) \geqq p(t')$. Let $\mathcal{P}$ denote the family of weakly monotone paths in $\overline{\mathbb{R}}_+^{\mathbb{N}}$ from $\mathbf{0}$ to Ω. Call $p \in \mathcal{P}$ a *monotone path*.

For a given $p \in \mathcal{P}$, $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, let p^c denote the path from $\mathbf{0}$ to c obtained by taking, for every t, the meet of c and the projection of $p(t)$ onto $\overline{\mathbb{R}}_+^N$, i.e.

$$p^c(t) \equiv p_N(t) \wedge c.$$

See Figure 2.

Note that for any $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, the function $\sum_{i\in N} p^c_i(t)$ is continuous, weakly increasing, and satisfies $\sum_{i\in N} p^c_i(0) = 0$ and $\sum_{i\in N} p^c_i(1) = \sum_{i\in N} c_i$. Hence, for any problem (N,c,E), there exists $t \in [0,1]$ such that $\sum_{i\in N} p^c_i(t) = E$. Moreover, if t and t' are such that $\sum_{i\in N} p^c_i(t) = \sum_{i\in N} p^c_i(t') = E$, then $p^c_i(t) = p^c_i(t')$ for every $i \in N$. Hence for any $p \in \mathcal{P}$, we can define a division rule S^p as:

$$S^p(N,c,E) = p^c(t),$$

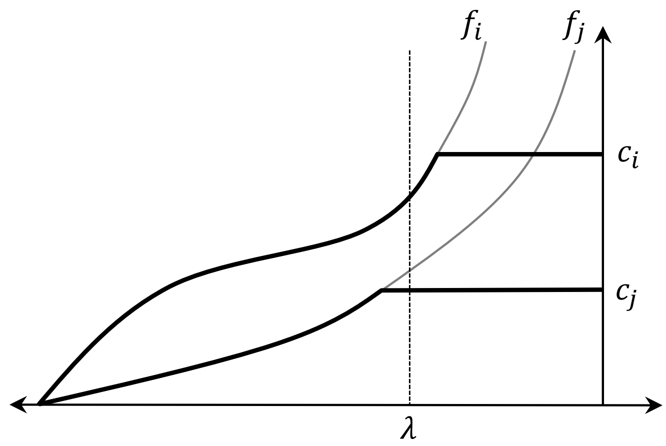

Figure 3: ***Claims Independent Parametric Rule.*** *The parametric functions for two claimants.*

where t is chosen such that $\sum_{i \in N} p_i^c(t) = E$. We say a rule S is a *monotone path rule* if there exists $p \in \mathcal{P}$ such that $S = S^p$.

3.3 Claims Independent Parametric Rules

Let $\mathcal{F}$ denote the family of functions $f : \mathbb{N} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}_+$ such that, for any $i \in \mathbb{N}$, $f(i, \cdot)$ is continuous, weakly increasing, and satisfies $f(i, -\infty) = 0$ and $f(i, +\infty) = +\infty$. From now on we write $f(i, \cdot)$ as f_i.

Note that for any $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, the function $\sum_{i \in N} \min\{f_i(\lambda), c_i\}$ is continuous and weakly monotonic in λ, and that $\sum_{i \in N} \min\{f_i(-\infty), c_i\} = 0$ and $\sum_{i \in N} \min\{f_i(+\infty), c_i\} = \sum_{i \in N} c_i$. Hence, for any claims problem (N, c, E), there exists $\lambda \in \overline{\mathbb{R}}$ such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$. Furthermore, if λ and λ' are such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = \sum_{i \in N} \min\{f_i(\lambda'), c_i\} = E$, then it must be that $\min\{f_i(\lambda), c_i\} = \min\{f_i(\lambda'), c_i\}$ for every $i \in N$. Hence for any $f \in \mathcal{F}$, we can define a division rule S^f as:

$$S^f(N, c, E) = \left(\min\{f_i(\lambda), c_i\}\right)_{i \in N},$$

where λ is chosen such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$. We say a rule S is a *claims independent parametric rule* if there exists $f \in \mathcal{F}$ such that $S = S^f$. See Figure 3.

The claims independent parametric rules are a special case of the (asymmetric) parametric rules characterized by Stovall (2013). In that paper, a parametric function was a continuous function $g : \mathbb{N} \times \mathbb{R}_{++} \times \overline{\mathbb{R}}$ which was weakly increasing in the third argument and satisfying $g_i(c_i, -\infty) = 0$, $g_i(c_i, +\infty) = c_i$. Thus the more general parametric functions depend not only on the parameter but also on the individual's claim. A division rule could be defined from such a function as was done above. Thus the parametric function g is a claims independent parametric function if for every $i \in \mathbb{N}$, $c_i > 0$, and $\lambda \in \overline{\mathbb{R}}$, there exists $f \in \mathcal{F}$ such that $g_i(c_i, \lambda) = \min\{f_i(\lambda), c_i\}$.

As we show later in Theorem 1, monotone path rules and claims independent parametric rules are in fact the same family of rules. Indeed, this is easy to see now. For $f \in \mathcal{F}$, define the monotone path

$$p(t) = (f_i(g(t)))_{i \in \mathbb{N}}$$

where g is any strictly increasing bijection from $[0,1]$ to $\overline{\mathbb{R}}$. Showing the converse is similar.

3.4 Collectively Rational Additively Separable Rules

A social welfare function (SWF) is a real-valued function of awards vectors. We say a SWF is additively separable if there exists $U : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}$ such that for any $N \in \mathcal{N}$ and $x \in \mathbb{R}^N_+$, the SWF can be written in the form $\sum_{i \in N} U(i, x_i)$. Let $\mathcal{U}$ denote the family of functions $U : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}$ that are continuous and strictly concave. From now on we write $U(i, \cdot)$ as U_i.

Note that for any $U \in \mathcal{U}$, $\arg\max_{x \in X(N,c,E)} \sum_{i \in N} U_i(x_i)$ is single-valued. Hence for any $U \in \mathcal{U}$, we can define a division rule S^U as:

$$S^U(N, c, E) = \arg\max_{x \in X(c,E)} \sum_{i \in N} U_i(x_i).$$

We say a rule S is a *collectively rational additively separable (CRAS) rule* if there exists $U \in \mathcal{U}$ such that $S = S^U$.

The family of CRAS rules are similar to the family of rules characterized by Lensberg (1987) for the Nash bargaining problem.[1] Indeed, it is easy to see that any claims problem is also a Nash bargaining problem. However Lensberg's result does not imply ours as the class of Nash bargaining problems is much larger than the class of claims problems.

3.5 Theorem

Our main theorem states that these three axioms characterize each of these families of rules, and thus these families are in fact one and the same.

Theorem 1 *The following are equivalent:*

1. *S satisfies Resource Monotonicity, Bilateral Consistency, and Independence of Irrelevant Alternatives.*

2. *S is a monotone path rule.*

3. *S is a claims independent parametric rule.*

4. *S is a CRAS rule.*

[1] Lensberg requires the SWF to be only strictly quasi-concave. Strict concavity is needed here to guarantee Resource Monotonicity.

The proof is in the appendix (with all subsequent proofs).

We note a few things concerning this theorem. First, it shows the appeal of this family of rules as it can be described in three different but intuitive ways. Monotone path rules are geometrically appealing. Claims independent parametric rules are easy to relate to the family of parametric rules. CRAS rules are a natural method of division, one that has been proposed in many other social choice problems.

A second thing to note is that continuity of the rule is not assumed, though it is implied by the axioms. It is a well-known result that Resource Monotonicity implies continuity in the endowment, and this fact is used in the proof. The other part of continuity, that of continuity in the claims vector, is not needed in the proof (though obviously IIA would play a key role in establishing this property).

The following examples demonstrate that the axioms in Theorem 1 are independent.

- **Bilateral Consistency.** Let $p, p' \in \mathcal{P}$ be two different monotone paths. Consider the rule which divides according to S^p for all two-person claims problems and $S^{p'}$ for all claims problems with more than two claimants. Such a rule would satisfy Resource Monotonicity and IIA but not Bilateral Consistency.

- **IIA.** The proportional rule

$$P(N, c, E) = \frac{E}{\sum_N c_i} c$$

  satisfies Bilateral Consistency and Resource Monotonicity, but not IIA.

- **Resource Monotonicity.** We sketch an example here, but a complete example is given in the appendix. Consider a rule for only two claimants. This rule divides according to a fixed path, which is not weakly monotone. Specifically, this path is always increasing in the first claimant's coordinate, but does sometimes decrease in the second claimant's coordinate. However it never has a slope less than -1, thus guaranteeing that it intersects with any endowment line only once. Given this fixed path, we now describe the path of awards for a given claims vector. For $(c_1, c_2) \in \mathbb{R}^2_{++}$, the path of awards is the intersection of the the fixed path with the rectangle defined by the origin and (c_1, c_2). If the fixed path exits and then subsequently re-enters the rectangle, then the path of awards travels along the border of the rectangle from the point of exit to the point of entry. Once the fixed path leaves the rectangle permanently, the path of awards travels along the border of the rectangle from the point of exit to the point (c_1, c_2). See Figure 4. We have thus described the path of awards for any claims vector, and thus completely described the rule for two claimants. Such a rule satisfies IIA but obviously not Resource Monotonicity. In the appendix, we describe how this rule can be extended to any problem so as to satisfy Bilateral Consistency.

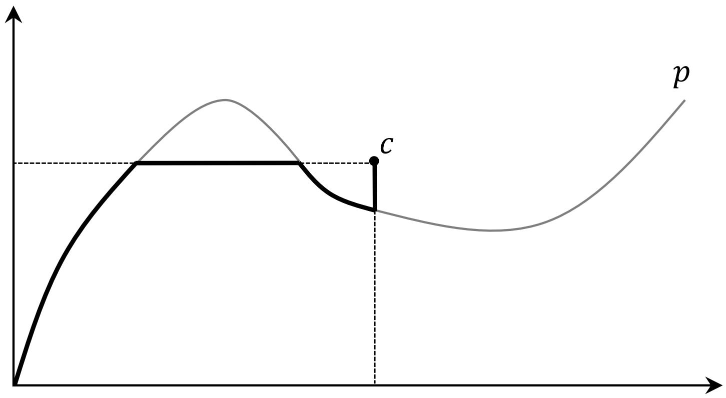

Figure 4: ***Violation of Resource Monotonicity.*** *The path is not monotone, and so Resource Monotonicity is violated. A path of awards is shown in bold for the claims vector c.*

4 Duality

An alternative way of viewing a claims problem is how to divide up the losses. Since the losses for the problem (N, c, E) is $E - \sum_N c_i$, we define the *dual of* (N, c, E) as the problem $(N, c, \sum_N c_i - E)$. Let $L(N, c, E)$ denote the set of efficient feasible loss vectors for the problem (N, c, E), i.e.

$$L(N,c,E) \equiv \left\{ x \in \mathbb{R}^N_+ : \mathbf{0} \leqq x \leqq c \text{ and } \sum_{i \in N} x_i = \sum_{i \in N} c_i - E \right\}.$$

Note that for every (N, c, E), we have $X(N, c, E) = L(N, c, \sum_N c_i - E)$, which is to say that the set of feasible awards for a problem is equal to the set of feasible losses for the dual of that problem. The *dual of the rule* S is

$$S^d(N, c, E) \equiv c - S(N, c, \sum_{i \in N} c_i - E).$$

Thus S^d allocates losses the same way that S allocates gains, and vice versa. The *dual of the axiom* A is the axiom A^d such that

$$S \text{ satisfies } A \text{ if and only if } S^d \text{ satisfies } A^d.$$

One can show that the dual of Consistency is Consistency itself. Similarly, the dual of Resource Monotonicity is itself. This is not true of IIA.

IIA-Dual. *For every (N, c, E) and (N', c', E'), if $L(N', c', E') \subset L(N, c, E)$ and $c - S(N, c, E) \in L(N', c', E')$, then $c' - S(N', c', E') = c - S(N, c, E)$.*

Similar to IIA, we can more precisely characterize the conditions of this axiom. Namely, the conditions are met if $N = N'$, $\sum c_i - E = \sum c'_i - E'$, and $S(N, c, E) \leqq c' \leqq c$.

Thus IIA-Dual treats losses as what matters, and so would be more normatively appealing in applications in which the resource to be divided is actually a bad.

It is easy to see the implications of replacing IIA with IIA-Dual in Theorem 1. Namely, the result would be a rule which divides losses according to the monotone path rule. However what would be the result if IIA-Dual was added to the axioms in Theorem 1? Since the intuition behind IIA and IIA-Dual are somewhat incompatible, the result is a rule which is usually considered to be normatively unappealing.

The queueing rule divides the endowment by lining up the claimants in a queue and then awarding the first person in the queue his full claim, then the second person his full claim, and continuing until the endowment is exhausted. Let $\mathcal{L}$ denote the set of strict linear orders over $\mathcal{N}$. For $\succ \in \mathcal{L}$, define the rule $S^{\succ}$ as:

$$S^{\succ}(N,c,E) = \left(\min\left\{c_i, \max\left\{0, E - \sum_{j\in N: j\succ i} c_j\right\}\right\}\right)_{i\in N}$$

We say a rule S is a queueing rule if there exists $\succ \in \mathcal{L}$ such that $S = S^{\succ}$.

Theorem 2 *The rule S satisfies Resource Monotonicity, Bilateral Consistency, IIA, and IIA-Dual if and only if S is a queueing rule.*

5 Symmetry

If there is no a priori reason to treat the claimants differently, then one would want the rule to give the same award to individuals with the same claim. This is captured in the following axiom.

Symmetry. *For every problem (N,c,E) and $\{i,j\} \in N$, if $c_i = c_j$, then $S_i(N,c,E) = S_j(N,c,E)$.*

The constrained equal awards rule is the rule that gives everyone the same award, unless that award is more than a claimant's claim in which case that claimant gets his full claim:

$$CEA(N,c,E) = (\min\{c_i, \lambda\})_{i\in N}$$

where λ is chosen such that $\sum_N CEA_i(N,c,E) = E$. It should be obvious that the constrained equal awards rule is a claims independent parametric rule and that it satisfies Symmetry. In fact it is the only claims independent parametric rule to satisfy Symmetry.

Theorem 3 *The constrained equal awards rule is the only rule to satisfy Resource Monotonicity, Bilateral Consistency, IIA, and Symmetry.*

The proof is straightforward given Theorem 1, and so is omitted. We note that adding IIA to the set of axioms in Young (1987, Theorem 1) would give the same result. Thus we could replace Resource Monotonicity with the assumption that S is continuous in Theorem 3 and the result would still hold.

Even if one thought the claimants should be treated equally, it may be the case that there is other information known about the claimants (other than their claims) that would cause one not to want to impose Symmetry. For example, suppose that each claimant had an endowment w_i of the good which could not be taken from the claimant. Then one may want the rule to sometimes allocate less to individuals who have larger endowments.

We formalize this as follows. A problem is now a tuple (N, c, w, E), where $N \in \mathcal{N}$, $c \in \mathbb{R}^N_{++}$, $w \in \mathbb{R}^N_+$, and $E \in \mathbb{R}_+$, all satisfying $\sum_{i \in N} c_i \leq E$. The variables N, c, and E are as before, but here w represents the vector of endowments, which doesn't put any additional constraint on the problem. Let $X(N, c, w, E)$ denote the set of efficient awards vectors for the problem (N, c, w, E), i.e.

$$X(N, c, w, E) \equiv \left\{ x \in \mathbb{R}^N_+ : \mathbf{0} \leqq x \leqq c \text{ and } \sum_{i \in N} x_i = E \right\}.$$

A *division rule* is a function S such that for every problem (N, c, w, E), we have $S(N, c, w, E) \in X(N, c, w, E)$.

Outcome Symmetry. *For every problem (N, c, w, E) and $\{i, j\} \in N$, if $c_i + w_i = c_j + w_j$, $w_j \leq w_i$, and $S_j(N, c, w, E) \geq w_i - w_j$, then $S_i(N, c, w, E) + w_i = S_j(N, c, w, E) + w_j$.*

Here, we think of the sum $S_i(N, c, w, E) + w_i$ as being the outcome for individual i. Appropriate modifications of Bilateral Consistency, Resource Monotonicity, and IIA do not involve w, and are thus omitted.

The constrained equal outcomes rule gives everyone the same outcome, unless that outcome implies a claimant receives either a negative award or more than his claim, in which case that claimant receives either zero or his full claim respectively:

$$CEO(N, c, w, E) = (\min\{c_i, \max\{0, \lambda - w_i\}\})_{i \in N}$$

where λ is chosen such that $\sum_N CEO_i(N, c, w, E) = E$. See Figure 5

Theorem 4 *The constrained equal outcome rule is the only rule to satisfy Resource Monotonicity, Bilateral Consistency, IIA, and Outcome Symmetry.*

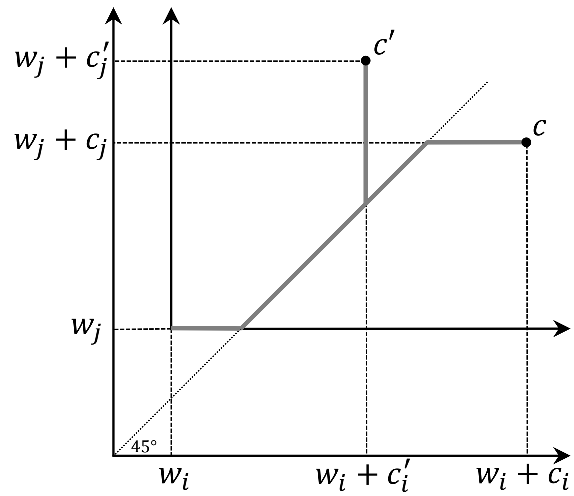

Figure 5: ***Constrained Equal Outcome Rule.*** *The paths of awards for the constrained equal outcomes rules for two different claims vectors.*

Appendix

Throughout the appendix, we shorten notation by writing the problem (N, c, E) as (c, E), as the group of claimants N is implicit in the claims vector c.

A Proof of Theorem 1

It is a straightforward exercise to show that each of these families of rules satisfies the axioms.

So we show that the axioms are sufficient to get each family of rules, i.e. we show that statement 1 implies 2, 3, and 4. Let S satisfy Bilateral Consistency, Resource Monotonicity, and IIA. It is straightforward to show that Resource Monotonicity implies the following continuity axiom.

Resource Continuity. *For every (c, E), for every sequence $(E^k)_{k\in\mathbb{N}}$ where $E^k \to E$, we have $S(c, E^k) \to S(c, E)$.*

We follow the general proof strategy of Kaminski (2000, 2006) and Stovall (2013) by defining a binary relation over a suitable outcome space. The key step in the proof is showing that this binary relation has a numerical representation.

For $i \neq j$ and $x_i, x_j > 0$, define:

$$G(i, x_i, j, x_j) \equiv \inf\{E : x_i = S_i((x_i, x_j), E)\}$$

Resource Continuity implies that we can replace inf with min. Note that we always have $G(i, x_i, j, x_j) \leq x_i + x_j$. Set

$$Y \equiv \mathbb{N} \times \mathbb{R}_+$$

and define the binary relation R_1 over Y as follows:

$$(i, x_i)R_1(j, x_j) \text{ if } i \neq j \text{ and either } x_i = 0 \text{ or } G(i, x_i, j, x_j) \leq G(j, x_j, i, x_i).$$

Note that if $(i, x_i)R_1(j, x_j)$ and $x_j = 0$, then it must be that $x_i = 0$. Define the binary relation R_2 over Y as follows:

$$(i, x_i)R_2(j, x_j) \text{ if } i = j \text{ and } x_i \leq x_j.$$

Set $R \equiv R_1 \cup R_2$.

A.1 R Has a Numerical Representation

We show that R is complete, transitive, and that there exists a countable R-dense subset of Y. Thus, by Cantor's classic result, R has a numerical representation.

By definition, it is obvious that R is complete. The following series of lemmas prove the other properties.

Lemma 1 *Suppose $i \neq j$ and $x_i, x_j > 0$. Then $(i, x_i)R_1(j, x_j)$ if and only if $G(j, x_j, i, x_i) = x_i + x_j$.*

Proof. Let $E^i = G(i, x_i, j, x_j)$ and $E^j = G(j, x_j, i, x_i)$. Note that $S_i((x_i, x_j), E^i) = x_i$ and $S_j((x_i, x_j), E^j) = x_j$. Suppose $E^i \leq E^j$. Then Resource Monotonicity implies $S_i((x_i, x_j), E^j) = x_i$, which implies $x_i + x_j = E^j$. Going the other direction, if $E^i > E^j$, then Resource Monotonicity implies $S_j((x_i, x_j), E^i) = x_j$. Hence $E^i = x_i + x_j > E^j$. ■

Thus for $i \neq j$ and $x_i, x_j > 0$, $(i, x_i)P_1(j, x_j)$ if and only if $G(i, x_i, j, x_j) < x_i + x_j$.

Lemma 2 *For every $(i, x_i), (j, x_j), (k, x_k) \in Y$ where i, j, k are distinct and $x_i, x_j, x_k > 0$, there exists E such that*

$$S((x_i, x_j, x_k), E) = (G(k, x_k, i, x_i) - x_k, G(k, x_k, j, x_j) - x_k, x_k).$$

Proof. Set

$$E \equiv G(k, x_k, i, x_i) + G(k, x_k, j, x_j) - x_k$$

and

$$(y_i, y_j, y_k) \equiv S((x_i, x_j, x_k), E).$$

We show that we must have $y_i + y_k = G(k, x_k, i, x_i)$ and $y_j + y_k = G(k, x_k, j, x_j)$. By way of contradiction, assume without loss of generality that $y_i + y_k < G(k, x_k, i, x_i)$. Then since $y_i + y_j + y_k = E$ it must be that $y_j + y_k > G(k, x_k, j, x_j)$. But then Bilateral Consistency and Resource Monotonicity would imply both $y_k < x_k$ and $y_k = x_k$.

Since $y_i + y_k = G(k, x_k, i, x_i)$, Bilateral Consistency implies that $y_k = x_k$. This implies $y_i = G(k, x_k, i, x_i) - x_k$. Similarly, we have $y_j = G(k, x_k, j, x_j) - x_k$. ■

Lemma 3 *Suppose $i \neq j$, $x_i, x_j > 0$, and $(j, x_j)R_1(i, x_i)$. Set $E^* \equiv G(j, x_j, i, x_i)$ and $x_i^* \equiv E^* - x_j$. Then for any $(c_i, c_j) \geq (x_i^*, x_j)$, we have $S((c_i, c_j), E^*) = (x_i^*, x_j)$.*

Proof. First consider the case when $(c_i, c_j) \geq (x_i, x_j)$. Then $X((x_i, x_j), E) \subset X((c_i, c_j), E)$ for every $E \leq x_i + x_j$. Also note that Lemma 1 implies $E^* \leq x_i + x_j$, which implies $x_i^* \leq x_i$.

Note that for every $E' \leq \min\{x_i, x_j\}$, we have $X((x_i, x_j), E') = X((c_i, c_j), E')$, and so IIA implies $S((x_i, x_j), E') = S((c_i, c_j), E')$. Thus if $E^* \leq \min\{x_i, x_j\}$, then we are done since $S((x_i, x_j), E^*) = (x_i^*, x_j)$. So assume $E^* > \min\{x_i, x_j\}$. Note also that for every $E'' < E^*$, we have $S_j((x_i, x_j), E'') < x_j$ (by definition of E^*) and $S_i((x_i, x_j), E'') < x_i$ (since $(j, x_j)R_1(i, x_i)$). Thus for every $E' \leq \min\{x_i, x_j\} < E^*$, we have $S_j((c_i, c_j), E') < x_j$ and $S_i((c_i, c_j), E') < x_i$.

So set $E^1 \equiv \min\{x_i, x_j\}$ and $(y_i^1, y_j^1) \equiv S((c_i, c_j), E^1) = S((x_i, x_j), E^1)$. Thus $y_i^1 < x_i$ and $y_j^1 < x_j$.

Set $E^2 \equiv \min\{E^*, E^1 + \min\{x_i - y_i^1, x_j - y_j^1\}\}$ and $(y_i^2, y_j^2) \equiv S((c_i, c_j), E^2)$. Resource Monotonicity implies that $(y_i^2, y_j^2) \geq (y_i^1, y_j^1)$. Since $y_i^2 \geq y_i^1$, we have $y_j^2 = E^2 - y_i^2 \leq E^2 - y_i^1$. But by definition, $E^2 \leq E^1 + x_j - y_j^1$, and thus $y_j^2 \leq E^1 + x_j - y_j^1 - y_i^1$. But since $y_j^1 + y_i^1 = E^1$, this implies $y_j^2 \leq x_j$. Similarly, we can show $y_i^2 \leq x_i$. Thus we have $(0, 0) \leqq (y_i^2, y_j^2) \leqq (x_i, x_j)$ and $y_i^2 + y_j^2 = E^2$, which means $(y_i^2, y_j^2) \in X((x_i, x_j), E^2)$. IIA then implies $(y_i^2, y_j^2) = S((c_i, c_j), E^2) = S((x_i, x_j), E^2)$. If $E^2 = E^*$, then we have proved the result since $S((x_i, x_j), E^*) = (x_i^*, x_j)$. So assume $E^2 < E^*$. Thus we have $y_i^2 < x_i$ and $y_j^2 < x_j$.

Similarly, for $n = 3, 4, \ldots$, set $E^n \equiv \min\{E^*, E^{n-1} + \min\{x_i - y_i^{n-1}, x_j - y_j^{n-1}\}\}$ and $(y_i^n, y_j^n) \equiv S((c_i, c_j), E^n)$. As before, we can show $(y_i^n, y_j^n) = S((c_i, c_j), E^n) = S((x_i, x_j), E^n)$. Again, if $E^n = E^*$ for any n, then we are done. So we assume $E^n < E^*$ for every n. Thus $y_i^n < x_i$ and $y_j^n < x_j$.

Now we show $E^n \to E^*$. So suppose not, i.e. $E^n \to \hat{E} < E^*$. Then since $E^n < E^*$ by assumption, we must have $\min\{x_i - y_i^n, x_j - y_j^n\} \to 0$. So either $y_i^n \to x_i$ or $y_j^n \to x_j$. If the former, then $S((x_i, x_j), E^n) = (y_i^n, y_j^n) \to (x_i, \hat{E} - x_i)$. Resource Continuity implies then that $S((x_i, x_j), \hat{E}) = (x_i, \hat{E} - x_i)$. But then by definition of E^* and the fact that $(j, x_j)R_1(i, x_i)$, we must have $\hat{E} \geq E^*$, which is a contradiction. Similarly, we get a contradiction if $y_j^n \to x_j$.

Since $E^n \to E^*$, Resource Continuity implies $S((c_i, c_j), E^n) \to S((c_i, c_j), E^*)$ and $S((x_i, x_j), E^n) \to S((x_i, x_j), E^*) = (x_i^*, x_j)$. But since $S((c_i, c_j), E^n) = S((x_i, x_j), E^n)$ for every n, we must have $S((c_i, c_j), E^*) = (x_i^*, x_j)$. This proves the result for $(c_i, c_j) \geq (x_i, x_j)$.

Now take any $(c_i, c_j) \geq (x_i^*, x_j)$. Set $(c_i', c_j') = (\max\{c_i, x_i\}, c_j)$. Then by the above result, we have $S((c_i', c_j'), E^*) = (x_i^*, x_j)$. But $(x_i^*, x_j) \in X((c_i, c_j), E^*)$. Hence IIA implies $S((c_i, c_j), E^*) = (x_i^*, x_j)$. ■

Lemma 4 *R is transitive.*

Proof. Let $(i, x_i)R(j, x_j)R(k, x_k)$. The proof is straightforward if either x_i, x_j, or x_k is 0. Hence assume $x_i, x_j, x_k > 0$.

Case 1: i, j, k distinct.

By way of contradiction, suppose $(k, x_k)P_1(i, x_i)$. Then Lemma 1 implies

$$G(k, x_k, i, x_i) < x_i + x_k. \tag{1}$$

By Lemma 2, there exists E such that

$$S((x_1, x_2, x_3), E) = (G(k, x_k, i, x_i) - x_k, G(k, x_k, j, x_j) - x_k, x_k).$$

For brevity, set $y_i \equiv G(k, x_k, i, x_i) - x_k$ and $y_j \equiv G(k, x_k, j, x_j) - x_k$. Lemma 1 implies $G(k, x_k, j, x_j) = x_j + x_k$. Hence $y_j = x_j$. Bilateral Consistency implies $S((x_i, x_j), y_i + x_j) = (y_i, x_j)$. However since $G(i, x_i, j, x_j) \leq G(j, x_j, i, x_i)$, Resource Monotonicity implies $y_i = x_i$. Hence $G(k, x_k, i, x_i) = x_i + x_k$, which contradicts (1).

Case 2: $i = k \neq j$.

So let $x'_i = x_k$. We need to show that $x_i \leq x'_i$. By way of contradiction, suppose $x_i > x'_i$. Let $E^1 = x'_i + x_j$ and $E^2 = x_i + x_j$. Then $E^1 < E^2$. Set $E^* = G(j, x_j, i, x'_i)$. By Lemma 1 we have $E^1 = G(i, x'_i, j, x_j)$. Hence $(j, x_j)R_1(i, x'_i)$ implies $E^* \leq E^1$. Lemma 1 also implies $E^2 = G(j, x_j, i, x_i)$. But Lemma 3 implies $S((x_i, x_j), E^*) = (x^*_i, x_j)$, which means we must have $E^2 \leq E^*$. But then $E^2 \leq E^1$, which is a contradiction.

Case 3: $i = j \neq k$.

So let $x'_i = x_j$. Then $x_i \leq x'_i$ by definition of R_2. By way of contradiction, suppose $(k, x_k)P_1(i, x_i)$. Since $(i, x'_i)R_1(k, x_k)$, Case 2 above implies $x'_i \leq x_i$. Hence $x'_i = x_i$. But then we have $(k, x_k)P_1(i, x_i)$ and $(i, x_i)R_1(k, x_k)$, a contradiction.

Case 4: $i \neq j = k$.

Similar to Case 3.

Case 5: $i = j = k$.

Then $x_i \leq x_j \leq x_k$. ■

Lemma 5 $\mathbb{N} \times \mathbb{Q}_{++}$ *is an R-dense subset of Y.*

Proof. Let $(i, x_i)P(j, x_j)$. Then we must have $x_j > 0$. (If $x_j = 0$, then $(j, x_j)R(i, x_i)$ which contradicts $(i, x_i)P(j, x_j)$.)

Case 1: $i = j$.

Then $x_i < x_j$, so the result follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Case 2: $i \neq j$ and $x_i = 0$.

Then for any $x'_j \in (0, x_j)$, we must have $(i, x_i)P(j, x'_j)P(j, x_j)$. The result follows from Case 1 and transitivity.

Case 3: $i \neq j$ and $x_i > 0$.

Set $E^* \equiv G(i, x_i, j, x_j)$ and $x^*_j \equiv E^* - x_i$. Obviously $S((x_i, x_j), E^*) = (x_i, x^*_j)$. Lemma 1 implies $G(i, x_i, j, x_j) < x_i + x_j$, which implies $x_j > x^*_j$. Choose $x'_j \in (x^*_j, x_j) \cap \mathbb{Q}$. Then $(j, x'_j)P(j, x_j)$ by definition. By way of contradiction, suppose $(j, x'_j)P(i, x_i)$. Set $E' \equiv G(j, x'_j, i, x_i)$ and $x'_i \equiv E' - x'_j$. Lemma 1 implies $G(j, x'_j, i, x_i) < x_i + x'_j$, which implies $x_i > x'_i$. Lemma 3 implies that $S((x_i, x_j), E') = (x'_i, x'_j)$. But as we already noted above $S((x_i, x_j), E^*) = (x_i, x^*_j)$, and since $x_i > x'_i$, Resource Monotonicity implies that $x^*_j \geq x'_j$, which is a contradiction. Hence it must be that $(i, x_i)R(j, x'_j)$. ■

Hence R is complete, transitive, and there exists a countable R-dense subset of Y. Thus there exists $r : Y \to \mathbb{R}$ such that

$$(i, x_i)R(j, x_j) \Leftrightarrow r(i, x_i) \leq r(j, x_j).$$

Note that for every i, $r(i,\cdot)$ is strictly increasing.

Before continuing, we prove the following lemma, which will be useful in what follows.

Lemma 6 *Fix* (c,E) *and set* $x \equiv S(c,E)$. *Set* $\lambda^* \equiv \max_{i\in N}\{r(i,x_i)\}$. *Then for every* $i \in N$ *where* $x_i < c_i$,

$$r(i,x_i) \leq \lambda^* \leq \lim_{x_i' \downarrow x_i} r(i,x_i').$$

Proof. Obviously $r(i,x_i) \leq \lambda^*$. If $r(i,x_i) = \lambda^*$, then the result obviously holds since $r(i,\cdot)$ is strictly increasing. So suppose $r(i,x_i) < \lambda^* = r(j,x_j)$ for some $j \in N$ such that $j \neq i$.

We show that for every $x_i' \in (x_i,c_i)$, we have $\lambda^* < r(i,x_i')$. By way of contradiction, suppose $\lambda^* = r(j,x_j) \geq r(i,x_i')$. Since $r(i,\cdot)$ is strictly increasing, we can assume $r(j,x_j) > r(i,x_i')$ without loss of generality. Note that $x_i' > x_i \geq 0$. Thus $\lambda^* \geq r(i,x_i')$ implies that $x_j > 0$. So set $E^* \equiv G(i,x_i',j,x_j)$ and $x_j^* \equiv E^* - x_i'$. Lemma 1 implies $E^* = x_i' + x_j^* < x_i' + x_j$, which implies $x_j^* < x_j$. By Lemma 3, $S((c_i,c_j),E^*) = (x_i',x_j^*)$. However Bilateral Consistency implies $S((c_i,c_j),x_i+x_j) = (x_i,x_j)$. Since $x_i < x_i'$ and $x_j > x_j^*$, this must violate Resource Monotonicity.

Thus for $x_i' \in (x_i,c_i)$, we have $\lambda^* < r(i,x_i')$. Hence we must have $\lambda^* \leq \lim_{x_i' \downarrow x_i} r(i,x_i')$. ■

A.2 Claims Independent Parametric Rule

Here we show that S must be a claims independent parametric rule.

For $\lambda \in \overline{\mathbb{R}}$, define

$$f_i(\lambda) \equiv \begin{cases} 0 & \text{if } \lambda < r(i,0) \\ \sup\{x_i : r(i,x_i) \leq \lambda\} & \text{if } \lambda \geq r(i,0) \end{cases}.$$

Lemma 7 *For every* $(i,x_i) \in Y$, *we have* $f_i(r(i,x_i)) = x_i$.

Proof. Fix $(i,x_i) \in Y$. Since $r(i,\cdot)$ is increasing, we have

$$\{x_i' : r(i,x_i') \leq r(i,x_i)\} = \{x_i' : x_i' \leq x_i\}.$$

Hence $f_i(r(i,x_i)) = \sup\{x_i' : x_i' \leq x_i\} = x_i$. ■

Lemma 8 $f \equiv \{f_i\}_{\mathbb{N}} \in \mathcal{F}$.

Proof. Fix $i \in \mathcal{N}$.

First we show f_i is weakly increasing. Let $\lambda,\lambda' \in \overline{\mathbb{R}}$ be given where $\lambda < \lambda'$. If $\lambda < r(i,0)$, then $f_i(\lambda) = 0$. Since the definition of f_i implies $f_i(\lambda') \geq 0$, we have $f_i(\lambda) \leq f_i(\lambda')$. If $\lambda \geq r(i,0)$, then observe that $\emptyset \neq \{x_i : r(i,x_i) \leq \lambda\} \subset \{x_i : r(i,x_i) \leq \lambda'\}$. Hence $f_i(\lambda) \leq f_i(\lambda')$.

Next we show f_i is continuous. Suppose f_i were discontinuous. Then since f_i is weakly increasing, there must exist $x_0 \in \mathbb{R}_+$ such that x_0 is not in the image of f_i. But this is a contradiction since $\lambda = r(i, x_0) \in \mathbb{R}$ and $f_i(\lambda) = x_0$ by Lemma 7.

Finally we show $f_i(-\infty) = 0$ and $f_i(+\infty) = +\infty$. Note that $-\infty < r(i, 0)$. Hence $f_i(-\infty) = 0$ by definition of f_i. Note that $+\infty \geq r(i, x_i)$ for every $x_i \in \mathbb{R}_+$. Hence $f_i(+\infty) = \sup \mathbb{R}_+ = +\infty$. ■

Lemma 9 *$S = S^f$.*

Proof. Fix (c, E) and set $x \equiv S(c, E)$. Set $\lambda^* \equiv \max_{i \in N}\{r(i, x_i)\}$. Fix $i \in N$.

Note that $\lambda^* \geq r(i, x_i)$, which means $f_i(\lambda^*) \geq f_i(r(i, x_i))$ since f_i is weakly increasing. Lemma 7 then implies $f_i(\lambda^*) \geq x_i$.

If $x_i = c_i$, then $\min\{f_i(\lambda^*), c_i\} = c_i$, which proves the result. So assume $x_i < c_i$. By Lemma 6, $\lambda^* \leq \lim_{x'_i \downarrow x_i} r(i, x'_i)$. Thus $f_i(\lambda^*) \leq f_i(\lim_{x'_i \downarrow x_i} r(i, x'_i))$ since f_i is weakly increasing. But then Lemma 7 and continuity of f_i imply $f_i(\lambda^*) \leq x_i$. We have thus established that for the case where $x_i < c_i$, we must have $f_i(\lambda^*) = x_i$. Thus $\min\{f_i(\lambda^*), c_i\} = x_i$. ■

A.3 Monotone Path Rule

Here we show that S must be a monotone path rule.

Set

$$p(t) = (f_i(g(t)))_{i \in \mathbb{N}}$$

where g is any strictly increasing bijection from $[0, 1]$ to $\overline{\mathbb{R}}$. For example: $g(t) = 1 - \frac{1}{2t}$ for $0 \leq t \leq \frac{1}{2}$ and $g(t) = \frac{t}{1-t} - 1$ for $\frac{1}{2} < t \leq 1$.

Lemma 10 *$p \in \mathcal{P}$.*

Proof. Since g is a strictly increasing bijection from $[0, 1]$ to $\overline{\mathbb{R}}$, we must have $g(0) = -\infty$ and $g(1) = +\infty$. Also, Lemma 8 establishes that $f_i(-\infty) = 0$ and $f_i(+\infty) = +\infty$ for every $i \in \mathbb{N}$. Hence $p(0) = \mathbf{0}$ and $p(1) = \Omega$. Also, p must be weakly monotone and continuous since g is increasing and continuous and each f_i is weakly increasing and continuous. ■

Lemma 11 *$S = S^p$.*

Proof. Follows from Lemma 9. ■

A.4 CRAS Rule

Here we show that S must be a CRAS rule.

Since $r(i, a)$ is strictly increasing, it is Riemann integrable. Define

$$U_i(x_0) \equiv -\int_0^{x_0} r(i, a)\, \mathrm{d}a.$$

Lemma 12 $U \equiv \{U_i\}_{\mathbb{N}} \in \mathcal{U}$.

Proof. Fix $i \in \mathbb{N}$. First we show U_i is strictly concave. Fix $x, x' \in \mathbb{R}_+$ such that $x < x'$. Fix $\alpha \in (0,1)$. Set $\hat{x} \equiv \alpha x + (1-\alpha)x'$. Note that

$$U_i(\hat{x}) = U_i(x) - \int_x^{\hat{x}} r(i,a)\,\mathrm{d}a$$

and

$$\begin{aligned}\alpha U_i(x) + (1-\alpha)U_i(x') &= U_i(x) - (1-\alpha)\int_x^{x'} r(i,a)\,\mathrm{d}a \\ &= U_i(x) - \int_x^{\hat{x}} r(i,a)\,\mathrm{d}a + \alpha\int_x^{\hat{x}} r(i,a)\,\mathrm{d}a - (1-\alpha)\int_{\hat{x}}^{x'} r(i,a)\,\mathrm{d}a.\end{aligned}$$

Hence, U_i is strictly concave if and only if

$$\alpha\int_x^{\hat{x}} r(i,a)\,\mathrm{d}a - (1-\alpha)\int_{\hat{x}}^{x'} r(i,a)\,\mathrm{d}a < 0.$$

But since $x < \hat{x} < x'$ and since $r(i,a)$ is strictly increasing, we have

$$\int_x^{\hat{x}} r(i,a)\,\mathrm{d}a < \int_x^{\hat{x}} r(i,\hat{x})\,\mathrm{d}a = (1-\alpha)(x'-x)r(i,\hat{x})$$

and

$$\int_{\hat{x}}^{x'} r(i,a)\,\mathrm{d}a > \int_{\hat{x}}^{x'} r(i,\hat{x})\,\mathrm{d}a = \alpha(x'-x)r(i,\hat{x}),$$

which, inputting into the above equation, gives the result.

The continuity of U_i follows from the Fundamental Theorem of Calculus. ■

Lemma 13 *Suppose $i \neq j$, $x_i, x_j > 0$ and $(j,x_j)P_1(i,x_i)$. Set $x_i^* \equiv G(j,x_j,i,x_i) - x_j$. Then for any $x_i' \in (x_i^*, x_i)$, we have $(j,x_j)P_1(i,x_i')$.*

Proof. Lemma 1 implies $G(j,x_j,i,x_i) < x_i + x_j = G(i,x_i,j,x_j)$. Thus $x_i^* < x_i$ so (x_i^*, x_i) is not empty. Fix $x_i' \in (x_i^*, x_i)$. Lemma 3 implies $S((x_i', x_j), G(j,x_j,i,x_i)) = (x_i^*, x_j)$. Thus $G(j,x_j,i,x_i') \leq G(j,x_j,i,x_i)$. Since $x_i' > x_i^* = G(j,x_j,i,x_i) - x_j$, we have $G(j,x_j,i,x_i') < x_i' + x_j$. Lemma 1 then implies $(j,x_j)P_1(i,x_i')$. ■

Lemma 14 *For every $i \in \mathbb{N}$ and $x_i \in \mathbb{R}_+$, it is without loss of generality to assume $\lim_{x_i' \uparrow x_i} r(i,x_i') = r(i,x_i)$.*

Proof. If $x_i = 0$, then the only sequence in $\mathbb{R}_+$ below 0 is the sequence of zeros. Hence $\lim_{x_i' \uparrow 0} r(i,x_i') = r(i,0)$.

So assume $x_i > 0$. Since $r(i,\cdot)$ is strictly increasing, we obviously have $\lim_{x_i' \uparrow x_i} r(i,x_i') \leq r(i,x_i)$. By way of contradiction, suppose $a \equiv \lim_{x_i' \uparrow x_i} r(i,x_i') < r(i,x_i)$. Note that without loss of generality, we can assume that there exists $(j,x_j) \in Y$ where $j \neq i$ such that $r(j,x_j) \in (a, r(i,x_i))$. (If no such (j,x_j) existed, then we could "cut out" the discontinuity at (i,x_i) without changing the ordering of r, and thus have $a = r(i,x_i)$.) Then for every $x_i' \in (0,x_i)$, we have $r(i,x_i') < r(j,x_j) < r(i,x_i)$. But then $(j,x_j)P_1(i,x_i)$, so Lemma 13 implies that $r(j,x_j) < r(i,x_i')$ for every $x_i' \in (G(j,x_j,i,x_i) - x_j, x_i)$, which is a contradiction. ■

Lemma 15 *Fix (c, E) and set $x \equiv S^U(c, E)$. Suppose $0 < E < \sum_N c_i$. Then there exists λ^* such that for every $i \in N$ where $x_i < c_i$,*

$$r(i, x_i) \leq \lambda^* \leq \lim_{x_i' \downarrow x_i} r(i, x_i').$$

Proof. Set $C \equiv \{y \in \mathbb{R}^N : 0 \leqq y \leqq c\}$. Note that x is the unique solution to the problem

$$\min_{y \in C} \sum_{i \in N} -U_i(y_i) \text{ subject to } E - \sum_{i \in N} y_i = 0.$$

Also note that there exists y in the interior of C such that $E = \sum_N y_i$ and $\sum_N -U_i(y_i)$ is finite. Hence (Rockafellar, 1970, Corollary 28.2.2), there exists λ^* such that x is the solution to

$$\min_{y \in C} \left[\sum_{i \in N} -U_i(y_i) + \lambda^* \left(E - \sum_{i \in N} y_i \right) \right] = \lambda^* E + \sum_{i \in N} \min_{y_i \in [0, c_i]} \left[-U_i(y_i) - \lambda^* y_i \right].$$

Thus for every i and $y_i \in [0, c_i]$, we have $-U_i(y_i) - \lambda^* y_i \geq -U_i(x_i) - \lambda^* x_i$. So if $x_i \in [0, c_i)$, then $-U_i(y_i) - \lambda^* y_i \geq -U_i(x_i) - \lambda^* x_i$ for every $y_i \in \mathbb{R}_+$ since $-U_i(y_i)$ is strictly convex (by Lemma 12). Thus for every i where $x_i < c_i$, λ^* is a subderivative of $-U_i(\cdot)$ at x_i. Thus by Lemma 14 and (Rockafellar, 1970, Theorem 24.2), we have $r(i, x_i) \leq \lambda^* \leq \lim_{x_i' \downarrow x_i} r(i, x_i')$. ■

Lemma 16 $S = S^U$.

Proof. Fix (c, E) and set $x \equiv S(c, E)$ and $x^U \equiv S^U(c, E)$. Obviously if $E = 0$ then $x = x^U = \mathbf{0}$. Also, if $E = \sum_N c_i$ then $x = x^U = c$. So assume $0 < E < \sum_N c_i$. By Lemma 6, we have (setting $\lambda \equiv \max_{i \in N}\{r(i, x_i)\}$)

$$r(i, x_i) \leq \lambda \leq \lim_{x_i' \downarrow x_i} r(i, x_i')$$

for every $i \in N$ where $x_i < c_i$. By Lemma 15, there exists λ^U such that

$$r(i, x_i^U) \leq \lambda^U \leq \lim_{x_i' \downarrow x_i^U} r(i, x_i')$$

for every $i \in N$ where $x_i^U < c_i$.

By way of contradiction, suppose $x \neq x^U$. Then there exists $i \in N$ such that $x_i \neq x_i^U$. If $x_i^U < x_i \leq c_i$, then $\lim_{x_i' \downarrow x_i^U} r(i, x_i') < r(i, x_i)$ since $r(i, \cdot)$ is strictly increasing. Note that $r(i, x_i) \leq \lambda$ (even if $x_i = c_i$) by definition of λ. Hence $\lambda^U < \lambda$. But this implies $x_j^U \leq x_j$ for every $j \in N$, which implies $\sum_N x_j^U < \sum_N x_j$. But this is a contradiction since both equal E. If $x_i < x_i^U \leq c_i$, then by similar reasoning we get a contradiction. Hence we must have $x = x^U$. ■

B Example Violating Resource Monotonicity

Instead of describing the paths of awards for the rule as we did in Section 3, we will define a function that is almost parametric. This function will be parametric for every $i \in \mathbb{N}$ (meaning f_i satisfies the conditions of a parametric function) except for $i = 2$, which will have f_2 *not* weakly increasing.

Let $\hat{f} \in \mathcal{F}$ satisfy $\hat{f}_1(0) = +\infty$ and $\hat{f}_i(0) = 0$ for all $i \neq 1$. Thus $\hat{f}$ gives priority to claimant 1 over all the other claimants. Also assume $\hat{f}_1$ is strictly increasing and concave on the interval $(-\infty, 0)$ (i.e. $\hat{f}_1'(\lambda) > 0$ and $\hat{f}_1''(\lambda) \geq 0$) and that $\hat{f}_i$ is strictly increasing on the interval $(0, +\infty)$ for every $i \neq 1$. Define the function f from $\hat{f}$ as follows: $f_i = \hat{f}_i$ for every $i \neq 2$. For $i = 2$, let

$$f_2(\lambda) = \begin{cases} f_1(\lambda) & \lambda < a \\ \frac{-\epsilon}{b-a}(\lambda - a) + f_1(a) & a \leq \lambda < b \\ f_1(\lambda) - f_1(b) + f_1(a) - \epsilon & \lambda \geq b \end{cases}$$

for some fixed a, b, and ϵ satisfying $-\infty < a < b < 0$, $0 < \epsilon < f_1(a)$, and $\frac{\epsilon}{b-a} < f_1'(a)$. Thus we have the following: $f_2(-\infty) = 0$, $f_2(0) = +\infty$, and $f_1 + f_2$ is strictly increasing on the interval $(-\infty, 0)$.

For $N \in \mathcal{N}$ where either $1, 2 \in N$ or $2 \notin N$, and for any $c \in \mathbb{R}^N_{++}$, the function $\sum_{i \in N} \min\{f_i(\lambda), c_i\}$ is continuous and increasing in λ, and that $\sum_{i \in N} \min\{f_i(-\infty), c_i\} = 0$ and $\sum_{i \in N} \min\{f_i(+\infty), c_i\} = \sum_{i \in N} c_i$. Hence, for any E satisfying $0 \leq E \leq \sum c_i$, there exists $\lambda \in \overline{\mathbb{R}}$ such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$. Furthermore, if λ and λ' are such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = \sum_{i \in N} \min\{f_i(\lambda'), c_i\} = E$, then it must be that $\min\{f_i(\lambda), c_i\} = \min\{f_i(\lambda'), c_i\}$ for every $i \in N$.

For $N \in \mathcal{N}$ where $2 \in N$ and $1 \notin N$, and for $c \in \mathbb{R}^N_{++}$, the function $\sum_{i \in N} \min\{f_i(\lambda), c_i\}$ is continuous and satisfies $\sum_{i \in N} \min\{f_i(-\infty), c_i\} = 0$ and $\sum_{i \in N} \min\{f_i(+\infty), c_i\} = \sum_{i \in N} c_i$. Hence, for any E satisfying $0 \leq E \leq \sum c_i$, there exists $\lambda \in \overline{\mathbb{R}}$ such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$. However note that $\sum_{i \in N} \min\{f_i(\lambda), c_i\}$ may not be increasing. However here it does not matter because individual 2 is given priority over every individual $i \neq 1$. Thus as before, if λ and λ' are such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = \sum_{i \in N} \min\{f_i(\lambda'), c_i\} = E$, then it must be that $\min\{f_i(\lambda), c_i\} = \min\{f_i(\lambda'), c_i\}$ for every $i \in N$.

So we can define a division rule S^f as:

$$S^f(N, c, E) = \left(\min\{f_i(\lambda), c_i\}\right)_{i \in N},$$

where λ is chosen such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$.

Obviously S^f does not satisfy Resource Monotonicity.

To see that S^f satisfies Consistency, let (N, c, E) and $N' \subset N$ be given. Set $x = S(N, c, E)$. Then there exists λ such that $x_i = \min\{f_i(\lambda), c_i\}$ for every $i \in N$. But then λ also satisfies $\sum_{N'} \min\{f_i(\lambda), c_i\} = \sum_{N'} x_i$. Thus by the definition of S^f, we have $S(N', c_{N'}, \sum_{N'} x_i) = x_{N'}$.

To see that S^f satisfies IIA, let (N, c, E) and c' satisfy $S(N, c, E) \leqq c' \leqq c$. Thus there exists λ such that for every $i \in N$, we have $\min\{f_i(\lambda), c_i\} \leq c_i' \leq c_i$. If

$\min\{f_i(\lambda), c_i\} = f_i(\lambda)$, then we have $f_i(\lambda) \leq c'_i$. If $\min\{f_i(\lambda), c_i\} = c_i$, then we have $c_i = c'_i$. In either case, we must have $\min\{f_i(\lambda), c'_i\} = \min\{f_i(\lambda), c_i\}$. Note then that $\sum_N \min\{f_i(\lambda), c'_i\} = \sum_N \min\{f_i(\lambda), c_i\} = E$. Thus by definition of S^f, we must have $S(N, c', E) = (\min\{f_i(\lambda), c'_i\})_N$, which means $S(N, c', E) = S(N, c, E)$.

C Proof of Theorem 2

Let S satisfy Bilateral Consistency, Resource Monotonicity, IIA, and IIA-Dual.

Define the binary relation $\succ$ over $\mathbb{N}$ as follows: $i \succ j$ if $i \neq j$ and there exists $(c_i, c_j) \in \mathbb{R}^{\{i,j\}}_{++}$ such that $S_i((c_i, c_j), c_i) = c_i$. The following lemmas demonstrate that $\succ$ is a strict linear order over $\mathbb{N}$.

Lemma 17 *It is never the case that for $N = \{i, j\} \subset \mathbb{N}$, $c \in \mathbb{R}^{\{i,j\}}_{++}$, and $E \in (0, c_i + c_j)$, that we have $0 < S(c, E) < c$.*

Proof. By way of contradiction, suppose there exists $N = \{i, j\} \subset \mathbb{N}$, $c \in \mathbb{R}^{\{i,j\}}_{++}$, and $E \in (0, c_i + c_j)$ such that $0 < S(c, E) < c$. Without loss of generality, assume $c_i \leq c_j$. There exists $\epsilon > 0$ such that $2\epsilon < S_i(c, E) < c_i - 2\epsilon$. Set $c' \equiv (c_i - \epsilon, c_j + \epsilon)$. Note that for $E \leq c_j$, we have $X(c', E) \subset X(c, E)$ and $S(c, E) \in X(c', E)$. Thus IIA implies $S(c, E) = S(c', E)$. Similarly, for $E \geq c_i$, we have $L(c', E) \subset L(c, E)$ and $c - S(c, E) \in L(c', E)$. Thus IIA implies $c - S(c, E) = c' - S(c', E)$.

Suppose $E \in [c_i, c_j]$. Then we have $S(c, E) = S(c', E)$ and $c - S(c, E) = c' - S(c', E)$. But this implies $c = c'$, which is a contradiction.

Suppose $E > c_j$. Then we have $c - S(c, E) = c' - S(c', E)$, which implies $S(c, E) = c - c' + S(c', E) = S(c', E) + (\epsilon, -\epsilon)$. Thus $S(c, E) \in X(c', E)$ and $S(c', E) \in X(c, E)$. Consider the claim $c'' = (c_i, c_j + \epsilon)$. Note that $X(c, E) \subset X(c'', E)$ and $X(c', E) \subset X(c'', E)$ and either $S(c'', E) \in X(c, E)$ or $S(c'', E) \in X(c', E)$. If $S(c'', E) \in X(c, E)$, then IIA implies $S(c'', E) = S(c, E)$. But since $S(c, E) \in X(c', E)$, IIA implies $S(c'', E) = S(c', E)$. Thus $S(c, E) = S(c', E)$. But then we get $c = c'$ which is a contradiction. Similarly, we get a contradiction if $S(c'', E) \in X(c', E)$.

Similarly, if $E < c_i$, we get a contradiction. But that exhausts all possible values of E. ■

Lemma 18 *For every $\{i, j\} \subset \mathbb{N}$, either $i \succ j$ or $j \succ i$, but not both.*

Proof. Fix $\{i, j\} \subset \mathbb{N}$. Lemma 17 implies that either $i \succ j$ or $j \succ i$. We now show that both cannot be true. By way of contradiction, suppose both $i \succ j$ and $j \succ i$. Thus there exists $(c_i, c_j) \in \mathbb{R}^{\{i,j\}}_{++}$ and $(c'_i, c'_j) \in \mathbb{R}^{\{i,j\}}_{++}$ such that $S((c_i, c_j), c_i) = (c_i, 0)$ and $S((c'_i, c'_j), c'_j) = (0, c'_j)$. But then Lemma 3 implies that both $S((c_i, c'_j), c_i) = (c_i, 0)$ and $S((c_i, c'_j), c'_j) = (0, c'_j)$, which must contradict Resource Monotonicity. ■

Lemma 19 *$\succ$ is transitive.*

Proof. Let $i \succ j$ and $j \succ k$. By Lemma 18, for any $c_i, c_j, c_k > 0$, we have $S_i((c_i, c_j), c_i) = c_i$ and $S_j((c_j, c_k), c_j) = c_j$. Note then that we must have $G(j, c_j, i, c_i) = c_i + c_j$ and $G(j, c_j, k, c_k) = c_j$. Lemma 2 implies there exists E such that

$$S((c_i, c_j, c_k), E) = (G(j, c_j, i, c_i) - c_j, c_j, G(j, c_j, k, c_k) - c_j) = (c_i, c_j, 0).$$

Bilateral Consistency then implies $S((c_i, c_k), c_i) = (c_i, 0)$. Thus $i \succ k$ by definition. ■

We now complete the proof by showing that S must be the queueing rule.

Lemma 20 $S = S^{\succ}$.

Proof. Fix (c, E) and set $x \equiv S(c, E)$. Let $\{i, j\} \in N$. We show $i \succ j$ if and only if $x_j = c_j$ implies $x_i = c_i$. Bilateral Consistency implies $(x_i, x_j) = S((c_i, c_j), x_i + x_j)$. Suppose $i \succ j$ and $x_j = c_j$. Then Lemmas 17 and 18 imply $S((c_i, c_j), c_i) = (c_i, 0)$. Resource Monotonicity then implies $x_i = c_i$. Going the other direction, suppose that $x_j = c_j$ implies $x_i = c_i$. Then it cannot be that $S((c_i, c_j), c_j) = (0, c_j)$. Hence $j \not\succ i$. By Lemma 18 we must have $i \succ j$. ■

D Proof of Theorem 4

Let S satisfy Bilateral Consistency, Resource Monotonicity, IIA, and Outcome Symmetry. For $i \neq j$ and $x_i, x_j > 0$, define:

$$G'(i, x_i, w_i, j, x_j, w_j) \equiv \inf\{E : x_i = S_i((x_i, x_j), (w_i, w_j), E)\}$$

Resource Continuity implies that we can replace inf with min.

Lemma 21 *Suppose $i \neq j$, $x_i, x_j > 0$, and $G(j, x_j, w_j, i, x_i, w_i) \leq G(i, x_i, w_i, j, x_j, w_j)$. Set $E^* \equiv G(j, x_j, w_j, i, x_i, w_i)$ and $x_i^* \equiv E^* - x_j$. Then for any $(c_i, c_j) \geq (x_i^*, x_j)$, we have $S((c_i, c_j), (w_i, w_j), E^*) = (x_i^*, x_j)$.*

The proof is similar to the proof for Lemma 3, and so is omitted.

Lemma 22 *Suppose $i \neq j$ and $x_i, x_j > 0$. Then $G(j, x_j, w_j, i, x_i, w_i) \leq G(i, x_i, w_i, j, x_j, w_j)$ if and only if $x_j + w_j \leq x_i + w_i$.*

Proof. ($\Rightarrow$) Set $E^* \equiv G(j, x_j, w_j, i, x_i, w_i)$ and $x_i^* \equiv E^* - x_j$. Note that $x_i^* \leq x_i$.

Case 1. $w_i \geq w_j$ and $x_j < w_i - w_j$.

Then $x_j + w_j < w_i < w_i + x_i$.

Case 2. $w_i \geq w_j$ and $x_j \geq w_i - w_j$.

Choose any $(c_i, c_j) \geq (x_i^*, x_j)$ such that $c_i + w_i = c_j + w_j$. By Lemma 21, $S((c_i, c_j), (w_i, w_j), E^*) = (x_i^*, x_j)$. By Outcome Symmetry, $x_j + w_j = x_i^* + w_i \leq x_i + w_i$.

Case 3. $w_i < w_j$.

First we show that we must have $x_i^* \geq w_j - w_i$. If $x_i^* < w_j - w_i$, then set $x_i' \equiv w_j - w_i + x_j > x_i^*$. Then by Lemma 21, we have $S((x_i', x_j), (w_i, w_j), E^*) = (x_i^*, x_j)$. But then Outcome Symmetry implies $x_i^* + w_i = x_j + w_j$. Hence $x_i^* = x_i'$, a contradiction. Hence $x_i^* \geq w_j - w_i$. So similar to Case 2, we can show using Lemma 21 and Outcome Symmetry that $x_j + w_j = x_i^* + w_i \leq x_i + w_i$.

($\Leftarrow$) We prove the contrapositive. So suppose $G(j, x_j, w_j, i, x_i, w_i) < G(i, x_i, w_i, j, x_j, w_j)$. Set $E^* \equiv G(j, x_j, w_j, i, x_i, w_i)$ and $x_i^* \equiv E^* - x_j$. Note that $x_i^* < x_i$. Following the three cases above, one can show $x_j + w_j < x_i + w_i$. ■

Lemma 23 *Fix (c, w, E) and set $x \equiv S(c, w, E)$. For every $i, j \in N$ where $0 < x_i < c_i$ and $0 < x_j < c_j$, we have $x_i + w_i = x_j + w_j$.*

Proof. Suppose $i, j \in N$ satisfy $0 < x_i < c_i$, $0 < x_j < c_j$, and $x_i + w_i < x_j + w_j$. Then choose $x_i' \in (x_i, \min\{c_i, x_j + w_j - w_i\})$. Thus $x_i' + w_i < x_j + w_j$ and $0 < x_i < x_i' < c_i$. Lemma 22 implies $G(i, x_i', w_i, j, x_j, w_j) < G(j, x_j, w_j, i, x_i', w_i)$. Set $E^* \equiv G(i, x_i', w_i, j, x_j, w_j)$ and $x_j^* \equiv E^* - x_i'$. Note $x_j^* < x_j$. Then Lemma 21 implies $S((c_i, c_j), (w_i, w_j), E^*) = (x_i', x_j^*)$. However Bilateral Consistency implies $S((c_i, c_j), (w_i, w_j), x_i + x_j) = (x_i, x_j)$. Since $x_i < x_i'$ and $x_j > x_j^*$, this must violate Resource Consistency. ■

Set

$$f_i(\lambda) = \begin{cases} 0 & \text{if } \lambda < w_i \\ \lambda - w_i & \text{if } \lambda \geq w_i \end{cases} = \max\{0, \lambda - w_i\}.$$

Using Lemma 23, it is straightforward to show that $S = S^f$.

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