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The identification of beliefs from asset demand

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Abstract

The demand for assets as prices and initial wealth vary identifies beliefs and attitudes towards risk.

We derive conditions that guarantee identification with no knowledge either of the cardinal utility index or of the distribution of future endowments or payoffs of assets; the argument applies even if the asset market is incomplete and demand is observed only locally.

Key words: asset prices; beliefs; attitudes towards risk.

JEL classification: D80; G10.

1 Introduction

We consider an individual who trades in financial assets to maximise his time-separable (subjective) expected utility over life-time consumption; and we assume that we observe how his initial period demand for consumption and assets varies with prices and wealth, while the the investor's beliefs over the stochastic path of future prices, asset payoffs and endowments remain fixed and have finite support. We investigate conditions under which one can identify the investor's beliefs from his demand for assets.

The identification of fundamentals is of intrinsic theoretical interest; also, it serves to formulate policy: here, imperfections, like market incompleteness, are of interest, since it is imperfections that render interventions desirable. Identification is essential in order to understand better paradoxes that arise in classical consumption based asset pricing. In financial markets, prices are thought to be determined by the joint probability distribution of payoffs and idiosyncratic shocks to investors, as well as their risk-preferences. Deviations of the prices of assets from these "fundamentals" are often attributed to the beliefs of investors. Perhaps most famously, in Shiller (2015) unusual run-ups in asset prices are described as "irrational exuberance". In order to investigate the extent to which asset prices are determined by fundamentals or the beliefs of investors, it is necessary to identify these beliefs from market data. It is an open question to what extent the beliefs of investors can actually be deduced from their demand for consumption and assets. We investigate this question under the strong assumption that demand is observable, but we make few assumptions on the beliefs of the investor over asset payoffs and his endowments or the structure of the asset market.

It is clear that, even in a two-period setting, beliefs cannot always be identified. For example, if the investor has quadratic utility, his demand for assets only depends on the first and the second moments of asset payoffs – higher moments are irrelevant and, as a consequence, beliefs about higher moments of the distribution cannot possibly be identified, while subjective expected utility theory requires the individual to have beliefs over the joint distribution of asset payoffs and endowments. More interestingly, if the investor has log-utility, the entire distribution matters for his utility; however, if the investor does not have (labor) endowments beyond the first period, and if there is a single, risky asset available for trade, the demand for this asset only depends on the wealth of the investor and his discount factor, and beliefs over the payoffs of the assets do not matter and cannot be identified. The identification of beliefs may not be possible even when financial markets are complete. Since the observation of the demand for assets is equivalent to the observation of *excess demand* (but not necessarily consumption), the identification of beliefs turns out to be impossible if the cardinal utility exhibits constant absolute risk aversion.

In this paper, we derive conditions on fundamentals that ensure that beliefs can be identified. If cardinal utility is analytic, it can always be identified from demand for period zero consumption. Our main result is that identification is possible if the value functions across realisations of uncertainty are linearly independent. In a two-period model, and in the presence of a risk-free asset, beliefs over endowments and payoffs of assets can be identified if marginal utility, $u'(\cdot)$ is not *finitely mean-periodic*: that is, if for any $K > 1$, and any distinct $(e_k)_{k=1}^{K}$, the functions $(u'(e_k + x))_{k=1}^{K}$ are linearly independent. We characterise classes of utility functions with this property. Moreover, we show that in this framework, if the payoffs of risky assets *separate uncertainty* in the sense that for any two states some portfolio of assets has different payoffs across these states, beliefs can be identified whenever cardinal utility is not the product of the exponential function and some periodic function. In conclusion, we argue that the analysis extends to the case in which only aggregate demand or the equilibrium correspondence are observable.

The identification of fundamentals from observable data can be posed, most simply, in the context of certainty; there Mas-Colell (1977) showed that the demand function identifies the preferences of the consumer, while Chiappori, Ekeland, Kubler, and Polemarchakis (2004) extended the argument to show that aggregate demand or the equilibrium correspondence, as endowments vary, also allow for identification. Importantly, the argument for identification is local: if prices, in the case of demand, or endowments, in the case of equilibrium, are restricted to an open neighbourhood, they identify fundamentals in an associated neighbourhood. Evidently, the arguments extend to economies under uncertainty, but with a complete system of markets in elementary securities.

Identification becomes problematic, and more interesting, when the set of observations is restricted. Under uncertainty, this arises when the asset market is incomplete and the payoffs to investors are restricted to a subspace of possible payoffs. Nevertheless, Green, Lau, and Polemarchakis (1979), Dybvig and Polemarchakis (1981), Geanakoplos and Polemarchakis (1990) and Kubler, Chiappori, Ekeland, and Polemarchakis (2002) demonstrated that identification is possible as long as the utility function has an expected utility representation with a state-independent cardinal utility index, and the distribution of asset payoffs is known. Polemarchakis (1983) extended the argument to the joint identification of tastes and beliefs; but, the argument relied crucially on the presence of a risk-free asset and, more importantly, did not allow uncertainty due to future endowments.

It is interesting to note that the identification of preferences from the excess demand for commodities, that corresponds to the demand for elementary securities in a complete asset market, is, in general, not possible, as shown in Chiappori and Ekeland (2004) and Polemarchakis (1979). Here, restrictions on preferences, additive separability and stationarity or state-independence, allow for identification even in an asset market that is incomplete.

With a finite set of observation, Varian (1983) provided conditions necessary and sufficient for portfolio choices to be generated by expected utility maximisation with a known distribution of assets; which extends the characterisation of Afriat (1967). For the case of complete financial markets, Kubler, Selden, and Wei (2014) refined the argument to eliminate quantifiers and obtain an operational characterisation. In the same vane, Echenique and Saito (2015) extended the argument to case of subjective expected utility where beliefs are unknown. Importantly, the identification that we derive here is necessary for the convergence of preferences and beliefs constructed in Varian (1983) or Echenique and Saito (2015) as the number of observations increases.

A strand of literature in finance, inspired by Lucas (1978), most recently Ross (2015) and earlier work by He and Leland (1993), Wang (1993), Dybvig and Rogers (1997), Cuoco and Zapatero (2000) and Carr and Yu (2012) focuses on supporting prices and observations for a single realisation of the path of endowments or equivalently on equilibrium in an economy with a representative investor. In particular, Ross (2015) provides a simple framework where beliefs can be identified from asset prices. However, to obtain his results he needs to assume that there is a single (representative) agent, markets are complete and, importantly, the economy is stationary in levels – that Borovicka, Hansen, and Scheinkman (2014) pointed out.

First, we consider a two-period version of the problem and give conditions on fundamentals that ensure that beliefs can be identified; subsequently, we indicate how the argument can be extended to a multi-period setting.

2 Identification

Dates are $t = 0, 1$, and, at each date-event, there is a single perishable good. At date 0, assets, $a = 1, \ldots, A$, are traded and they pay off at $t = 1$.

An individual has subjective beliefs over the joint distribution of asset-payoffs and his endowments at $t = 1$ that, we assume, has finite support, S.

Consumption at date 0 is x_0, and it is x_s at state of the world $s = 1, \ldots, S$

at date 1. The individual maximizes time-separable expected utility

$$U(x_0, \ldots, x_s, \ldots) = u(x_0) + \beta \sum_{s=1}^{S} \pi_s u(x_s),$$

with the cardinal utility index, $u : (0, \infty) \to \mathbb{R}$, concave and strictly monotonically incresing.

Payoffs of an asset are $r_a = (r_{a,1}, \ldots, r_{a,s}, \ldots, r_{a,S})^\top$, and payoffs of assets at a state of the world are $R_s = (r_{1,s}, \ldots, r_{a,s}, \ldots, r_{A,s})$. Holdings of assets are $y = (\ldots, y_a, \ldots)^\top$.

At date 0, the endowment of the individual that, importantly, is observable is e_0, consumption is numéraire and prices of assets are $q = (\ldots, q_a, \ldots)$; at state of the world $s = 1, \ldots, S$, at date 1, consumption is, again, numéraire, and the endowment is e_s; across states of the world, $e = (e_1, \ldots, e_S)$.

The optimisation problem of the individual is

$$\begin{array}{ll} \max_{x \geq 0, y} & u(x_0) + \beta \sum_{s=1}^{S} \pi_s u(x_s) \\ \\ \text{s.t.} & x_0 + qy \leq e_0, \\ \\ & x_s - R_s y \leq e_s, \;\; s = 1, ..., S. \end{array}$$

The demand function for consumption and assets is $(x_0, y)(q, e_0)$; it defines the inverse demand function $(q, e_0)(x_0, y)$, and the marginal rate of substitution function

$$m(x_0, y) = q(x_0, y)$$

that, as a consequence, is observable.

Unobservable characteristics of an individual are the cardinal utility index, $u : (0, \infty) \to \mathbb{R}$, the discount factor, β, and beliefs over the distribution of future endowments and and payoffs of assets, $S \in \mathbb{N}$, $(\pi, R, e) \in \mathbb{R}^S_+ \times \mathbb{R}^{AS} \times \mathbb{R}^S_+$ with $\pi = (\ldots, \pi_s, \ldots) \in \Delta^{S-1}$ a probability measure.

Given any $(\bar{q}, \bar{e}_0)$ and a demand function $(x_0, y)(q, e_0)$ on an open neighbourhood of $(\bar{q}, \bar{e}_0)$, suppose $(x_0, y)(\cdot)$ solves the individual's maximisation problem, with $(x_0, \ldots, x_s, \ldots) \gg 0$.

Does the demand function identify the unobservable characteristics of the individual? This is the question we address in this paper.

We are mainly concerned with the identification of beliefs. The following result establishes a simple sufficient condition for the identification of the cardinal utility index.

Proposition 1. *If the cardinal utility index, $u : (0, \infty) \to \mathbb{R}$, is analytic, then, the demand function for consumption and assets identifies it.*

Proof. The demand for consumption and assets is defined by the the first order conditions

$$\beta \mathrm{E}_\pi u'(R_s y) = u'(x_0) m(x_0, y).$$

Successive differentiation with respect to consumption, x, yields [1]

$$0 = \sum_{k=0}^{n} b(n,k) u^{n+1-k}(x_0) m^k_{x_0}(x_0, y), \quad n = 1, \ldots,$$

with $b(n,k)$ binomial coefficients.

Since $b(n,0) = 1$, the system of equations identifies, sequentially,

$$\frac{u^{n+1}(x_0)}{u^1(x_0)}, \quad n = 1, \ldots.$$

If the cardinal utility index, u, is analytic, $u^{n+1}(\bar{x}_0)/u^1(\bar{x}_0))$, $n = 1, \ldots$ identifies u up to an affine transformation.

Alternatively, $u^2(x_0)/u^1(x_0)$, over its domain of definition, also suffices for identification. □

With $u(\cdot)$ given, the unknown characteristics are $\xi = (S, \beta, \pi, R, e)$. The question of identification is whether, given some ξ that generates the observed demand function, there is a different $\tilde{\xi}$ that would generate the same demand for assets on the specified neighbourhood of prices and wealth.

While we do not provide a complete answer to this question, we can find conditions on admissible beliefs and/or on cardinal utility that ensure identification. First note that we must assume that there is a portfolio of assets that has a positive payoff in all subsequent nodes. Probabilities of nodes at which no asset pays cannot possibly be identified. Without loss of generality, we take this to be $a = 1$; that is, we assume $r_{1s} > 0, s = 1, ..., S$.

It turns out that the key to the identification of ξ lies in the assumption that, for different states of the world, the functions $u(e_s + R_s y)$ together with the constant function[2], are linearly independent for $y \in \mathbb{R}^A$. Recall that functions, $f_1, ..., f_n$, are linearly independent on an open set $B \subset \mathbb{R}^m$ if there are no $\alpha_1, ..., \alpha_n$ so that $\sum_{k=1}^n \alpha_k f_k(x) = 0$, for all $x \in B$. For the discussion that follows it is useful to introduce the Wronskian matrix. For a family of functions $f : \mathbb{R}^m \to \mathbb{R}^n$, following the notation in Bostan and Dumas (2010), define differential operators

$$\Delta_k = (\frac{\partial}{\partial x_1})^{j_1} \ldots (\frac{\partial}{\partial x_m})^{j_m}, \quad j_1 + \ldots + j_m \leq k, \quad k = 0, \ldots, (n-1);$$

[1] We use $u'(\cdot)$ and $u^1(\cdot)$ interchangeably, and u^k for derivatives ofhigher order.

[2] Note that this property is, at it should be, true for all possible affine transformations of $u(\cdot)$.

note that, for $f : \mathbb{R} \to \mathbb{R}$, the operators are, simply, derivatives of order k : $\Delta_k = d^k f/dx^k$. The (generalized) Wronskian matrix associated to $\Delta_0, ..., \Delta_{n-1}$ is defined as

$$W = \begin{pmatrix} \Delta_0(f_1) & \dots & \Delta_0(f_i) & \dots & \Delta_0(f_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_k(f_1) & \dots & \Delta_k(f_i) & \dots & \Delta_k(f_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{n-1}(f_1) & \dots & \Delta_{n-1}(f_i) & \dots & \Delta_{n-1}(f_n) \end{pmatrix};$$

Bostan and Dumas (2010) show that, if the functions $f_1, ..., f_n$ are analytic, then, they and linearly independent if and only if the determinant of at least one of the Wronsikians of $f_1, ..., f_n$ is not identically equal to zero.

It is now possible to give necessary and sufficient conditions for identification.

Theorem. *Under the assumption that the cardinal utility index, $u(\cdot)$, is analytic and that $u(\cdot)$ has non-vanishing derivatives of any order, the demand function for consumption and assets identifies the unobservable characteristics ξ if and only if, for any $K \in \mathbb{N}$ and any $(e_s, R_s)_{s=1}^K$ with $(e_s, R_s) \neq (e_{s'}, R_{s'})$ for all s, s', the functions $f_s(y) = u(e_s + R_s y)$, $s = 1, ..., K$, along with the constant function $f_0(y) = 1$, are linearly independent.*

Proof. To prove sufficiency suppose that, with the cardinal utility index $u(\cdot)$ identified by Proposition 1, the characteristics $\xi = (S, \beta, e_1, R_1, \pi_1, \dots e_S, R_S, \pi_S)$ rationalize observed asset demand. Identification of beliefs is possible if there are no other characteristics, $\tilde{\xi} = (\tilde{S}, \tilde{\beta}, \tilde{e}_1, \tilde{R}_1, \tilde{\pi}_1, \dots \tilde{e}_{\tilde{S}}, \tilde{R}_{\tilde{S}}, \tilde{\pi}_{\tilde{S}})$, that rationalize the same demand.

Take $K \geq S$ to be the number of distinct (e_s, R_s) and $(\tilde{e}_s, \tilde{R}_s)$, and collect them in a vector $(e_1, R_1, \dots e_K, R_K)$. Consider asset demand in a fictitious problem of an investor with K states in the second period. The first order condition with respect to the demand for any asset a can be written as

$$\beta \sum_{s=1}^{K} \pi_s r_{as} u'(e_s + R_s y) = u'(x_0) q_a.$$

Since the functions $f_0, \dots, f_K$ are linearly independent, there must exist Δ_0, $\dots, \Delta_{K-1}$ such that the resulting generalised Wronskian matrix is invertible.

Note that since the first column of the Wronskian, that is, the derivatives of the constant function, are given by $(1, 0, \cdots, 0)^\top$, the invertibility of the Wronskian implies that there must be some asset a such that the matrix

$$\widetilde{W} = \begin{pmatrix} \Delta_0(r_{a1}u'(e_1 + R_1 y)) & \dots & \Delta_0(r_{aK}u'(e_K + R_K y)) \\ \vdots & \vdots & \vdots \\ \Delta_{K-1}(r_{a1}u'(e_1 + R_1 y)) & \dots & \Delta_{K-1}(r_{aK}u'(e_K + R_K y)) \end{pmatrix}$$

is invertible. Successive differentiation of the first order conditions with respect to this asset yields

$$\beta \sum_{s=1}^{K} \pi_s \Delta_n(r_{as}u'(e_s + R_s y)) = \Delta_n(u'(x_0)q_a), \quad n = 0, \dots, k-1.$$

Since $\widetilde{W}$ is invertible there is a unique solution for $\pi_1, \dots, \pi_K$ and β with $\sum_k \pi_k = 1$. But, if both the characteristics ξ and the characteristics $\tilde{\xi}$ generate the observed asset demand, there must be at least two distinct solutions for $\pi_1, \dots, \pi_K$ and β – one corresponding to ξ and the other one corresponding to $\tilde{\xi}$. Therefore, there cannot be a second set of characteristics that generates the same demand.

To prove necessity, note that linear dependence of $f_0, \dots, f_K$ implies that there exist $\alpha_1, \dots, \alpha_K$ such that $\sum_k \alpha_k r_{ak} u'(e_k + R_k y) = 0$, for all $a = 1, ..., A$. If $S = K$ and observed asset demand is rationalized for some β and probabilities $\pi_1, \dots, \pi_S > 0$ then, for any ϵ, the first order conditions can be written as follows

$$-\frac{q_a}{\beta}u'(x_0) + \sum_{s=1}^{S}(\pi_s + \epsilon\alpha_s) r_{as} u'(e_s + R_s y) = 0, \quad \text{for all} \quad a = 1, \dots, A.$$

For sufficiently small $\epsilon > 0$ we have that $\pi_s + \epsilon\alpha_s > 0$ for all s and we can define alternative probabilities $\tilde{\pi}_s = (\pi_s + \epsilon\alpha_s)/(1 + \epsilon \sum_k \alpha_k)$ and appropriately adjusted $\tilde{\beta} = \beta(1 + \epsilon \sum_k \alpha_k)$ that would rationalise the same demand function. $\square$

In the Theorem, we require that for any distinct $(e_s, R_s)_{s=1}^K$ the functions $u(e_s + R_s y)$ and f_0 are linearly dependent. If one poses restrictions on possible (e_s, R_s) the Theorem obviously goes through if one requires linear independence only for functions for which the e_s, R_s satisfy these restrictions.

The identification Theorem obviously raises the question whether there are assumptions on fundamentals, either assumptions on utility or restrictions on $(e_s, R_s)_{s=1}^K$ that guarantee independence as required.

3 Assumptions on fundamentals

First, examples show that, even when the support of beliefs, (S, R, e), is known, identification may not be possible. Note, that we are concerned with the seemingly much more demanding case, where nothing is known about the beliefs of the individual; nevertheless, it shall turn out that understanding these simple examples provides the key to our general identification results.

1. Suppose there is a single risky asset, second period endowments are zero, $e = 0$, and cardinal utility is logarithmic: $u(x) = \ln(x)$. A simple computation shows that $qy = (\beta e_0)/(1+\beta)$: the individual invests a fixed fraction of his wealth in the risky asset, and the demand for asset is identical for all π; beliefs are not identified.

2. Suppose there is a single risk-free asset, there is uncertainty about second period endowments, $e \neq 0$, and utility is CARA: it exhibits constant absolute risk aversion, and $u(x) = -\exp(-x)$. Direct computation shows that the demand for the risk-free asset is
$$y = \frac{1}{1+q}\left(e_0 - \ln(q) + \ln(\beta \sum_{s=1}^{S} \pi_s \exp(-e_s)\right);$$
beliefs are not identified.

There are two obvious ways to solve the problem. One could make assumptions on utility that rule out these cases; or, one could assume that there are several assets available for trade; we shall consider both in detail.

It is useful to note that, with two risky assets, with log-utility, identification might still be impossible.

3. Suppose there are two risky assets, there are no endowments, $e = 0$, and $u(x) = \ln(x)$. Recall that r_{as} is the payoff of asset a in state s. If, for states $s = 1, 2$,
$$\frac{r_{11}}{r_{12}} = \frac{r_{21}}{r_{22}},$$
then $r_{21}/r_{11} = r_{22}/r_{12}$, and the first order conditions that characterise asset demand can be written as
$$\begin{aligned}\frac{q_1}{\beta}u'(x_0) &= (\pi_1+\pi_2)\frac{1}{\theta_1 + \theta_2\frac{r_{21}}{r_{11}}} + \sum_{s=3}^{S}\pi_s r_{1s} u'(x_s),\\ \frac{q_2}{\beta}u'(x_0) &= (\pi_1+\pi_2)\frac{1}{\theta_1\frac{r_{11}}{r_{21}} + \theta_2} + \sum_{s=3}^{S}\pi_s r_{2s} u'(x_s);\end{aligned}$$
clearly, π_1 and π_2 cannot be identified separately.

Motivated by this example and to simplify the exposition, we will from now on focus on the case where a risk-free asset is available for trade. Unfortunately, the following example shows that identification might still be impossible, case even if there is a risky and a risk-less asset.

4. Suppose there is a risk-free asset (asset 1) and a risky asset (asset 2). Suppose $e \neq 0$, $u(x) = -\exp(-x)$ and

$$r_{21} = r_{22}, \quad e_1 \neq e_2.$$

The first order conditions that characterise asset demand can be written as

$$\begin{aligned}
\frac{q_1}{\beta}u'(x_0) &= \sum_{s=1}^{S} \exp(-(\theta_1 + \theta_2 r_{2s}))\pi_s \exp(-e_s), \\
\frac{q_2}{\beta}u'(x_0) &= \sum_{s=1}^{S} \exp(-(\theta_1 + \theta_2 r_{2s}))r_{2s}\pi_s \exp(-e_s);
\end{aligned}$$

beliefs, e_s and π_s cannot be identified separately.

In fact, in the example, identification is impossible even if markets are complete. As we mentioned earlier, existing results on the identification of preferences from demand do not apply when only excess demand is observable, which is the case here: since endowments are unknown, consumption is not observable.

Building on these examples, we now consider two cases. First we assume that there is both a risky and a risk-free asset available for trade and we give conditions on admissible beliefs and cardinal utility that ensure identification. We then consider the case where the is only a risk-free asset, and we give conditions on cardinal utility.

We now present assumptions on fundamentals that guarantee that the marginal utilities are linearly independent. First, assumptions on the asset structure and, then, assumptions on the cardinal utility index.

3.1 A risky asset separates all uncertainty

In this subsection, we assume that there are two assets available for trade, a risk-free asset (a= 1) and a risky asset (a = 2). It is without loss of generality to focus on the case of a single risky asset. If there are several risky assets, since we observe asset demand in a neighborhood, different portfolios of these risky assets can be used to apply the following arguments and to identify beliefs over the payoffs of all risky assets.

It is useful to first consider the situation where the risky asset defines all uncertainty: that is, all admissible beliefs can be described as beliefs over asset payoffs, and that there is a function from asset payoffs to individual endowments[3] In this case, the beliefs over the asset payoffs can be identified without additional assumptions on cardinal utility.

We then relax this assumption, and merely require that the payoff of the risky asset separates uncertainty: for all possible beliefs, $R_s \neq R_{s'}$, if $s \neq s'$. This assumption is strictly weaker since it could be the case that different agents assign the same asset payoffs to a particular state but different consumptions. Example 4 above implies that if the individual has CARA utility identification is no longer possible. It turns out that ruling out CARA utility (and a little more) ensures that the result is restored.

In what follows, and unless we indicate otherwise, we restrict attention to cardinal utility indices that are analytic and have non-vanishing derivatives of any order.

Proposition 2. *Suppose there exist a risk-free asset, $r_{1s} = 1$, for all s, and asset 2 separates all uncertainty; that is, for all possible beliefs,*

$$r_{2s} \neq r_{2s'} \quad \textit{for all} \quad s \neq s'.$$

The demand function for consumption and assets identifies the unobservable characteristics $(u, \beta, S, (\pi, e, R))$ if one of the two following assumption holds:

1. *There is some function $f : \mathbb{R} \to \mathbb{R}_+$, such that, for any possible individual characteristics (S, β, π, e, R),*

$$e_s = f(r_{2s}), \quad s = 1, ..., S.$$

2. *The cardinal utility cannot be written as*

$$u'(x) = s(x) \exp(-\alpha x), \tag{1}$$

for some periodic function $s(\cdot)$ and some $\alpha > 0$.

Proof. As in the proof of the the Theorem, suppose that characteristics ξ and $\tilde{\xi}$ rationalize observed asset demand, take $K \geq S$ to be number of distinct

[3]The case of zero individual endowments is obviously a special case.

(e_s, R_s) and $(\tilde{e}_s, \tilde{R}_s)$, and collect them in a vector $(e_1, R_1, \ldots e_K, R_K)$. Suppose the K functions $u'(e_s + y_1 + y_2 r_{2s})$, $s = 1, ..., K$ are linearly dependent; that is, there are $\alpha \in \mathbb{R}^K$, $\alpha \neq 0$, such that

$$\sum_{s=1}^{K} \alpha_s u'(e_s + y_1 + y_2 r_{2s}) = 0, \qquad \text{for all} \qquad y_1, y_2.$$

Repeated differentiation of this identity, n times with respect to y_1 and $k - 1 - n$ times with respect to y_2 gives the following system of k equations

$$\begin{aligned}
\sum_{s=1}^{K} r_{2s}^{K-1} \alpha_s u^{(K)}(e_s + y_1 + y_2 r_{2s}) &= 0, \\
\sum_{s=1}^{K} r_{2s}^{K-2} \alpha_s u^{(K)}(e_s + y_1 + y_2 r_{2s}) &= 0, \\
&\vdots \\
\sum_{s=1}^{K} \alpha_s u^{(K)}(e_s + y_1 + y_2 r_{2s}) &= 0.
\end{aligned}$$

Since the Vandermonde matrix

$$\begin{pmatrix} 1 & r_{21} & \ldots & r_{21}^{K-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & r_{2k} & \ldots & r_{2k}^{K-1} \end{pmatrix}$$

has full rank if $r_{2s} \neq r_{2s'}$ for all s, s' there exist a solution for non-zero α if and only if there are s, s' with $r_{2s} = r_{2s'}$. Therefore, condition 1 of Proposition2 ensures identification by construction.

Since under both beliefs the payoff of asset 2 separates uncertainty, r_{2s} can be identical across at most two states. But if for all s and s' with $r_{2s} = r_{2s'}$ we have that $\alpha_s u'(e_s + y_1 + y_2 r_{2s}) + \alpha_{s'} u'(e_{s'} + y_1 + y_2 r_{2s'}) \neq 0$ then we again obtain a Vandermonde matrix and there cannot be an $\alpha \neq 0$ with $\sum_{s=1}^{K} \alpha_s u'(e_s + y_1 + y_2 r_{2s}) = 0$. Therefore there must be s and s' with $r_2 = r_{2s} = r_{2s'}$, and an $\alpha \in \mathbb{R}^2$ such that $\alpha_1 u'(e_s + y_1 + y_2 r_2) + \alpha_2 u'(e_{s'} + y_1 + y_2 r_2) = 0$. But then, Theorem 1 in Laczkovich (1986) implies that u' must be of the form (1). □

The result in Polemarchakis (1983) was Part 1 of this proposition, with no future endowments, was

3.2 Unrestricted beliefs

In the spirit of the previous result, one could ask if, in a framework with a risky and a risk-free asset, we could make stronger assumptions on utility that would allow us to drop the assumption that the risky asset separates all uncertainty altogether.

We say that cardinal utility is *finitely mean-periodic* if there exist an S and distinct $e_k > 0$, $k = 1, ..., K$ and non-zero $\alpha \in \mathbb{R}^S$ such that

$$\sum_{k=1}^{K} \alpha_k u'(y + e_k) = 0 \text{ for all } \quad y.$$

A classic result in Schwartz (1947) characterises all complex-valued, mean-periodic functions; that is, functions that are continuous complex valued solutions to the integral equation

$$\int f(y + e) d\mu(e) = 0$$

for some non-zero measure with compact support, μ. Clearly ruling out all mean-periodic functions is sufficient for our result but more than we need. Moreover, the characterisation of all mean-periodic functions is not too insightful.

With our definition, the following result follows immediately from the identificationTheorem.

Proposition 3. *If the cardinal utility index, $u(\cdot)$, is not finitely mean-periodic, then, asset beliefs can be identified if there is a risk-free asset .*

Note that, in addition to the risk-free asset, there could be risky assets available for trade. Just considering the first order condition for the risk-free asset identifies $(e(s) + y_2 r_2(s))$, for all s, and, then, variations in y_2 identify $r_2(s)$ independently of $e(s)$.

As Laczkovich (1986) (remark iii) points out, there is no simple characterisation known for finitely mean-periodic functions. The negative exponential function is one example, but not the only one. In fact, it is easy to see that any real solution of a linear homogeneous differential equation with fixed coefficients

$$\alpha_1 u'(x) + \alpha_2 u^{(2)}(x) + ... + \alpha_k u^{(k)}(x) = 0$$

is finitely mean-periodic. However, this does not provide a full characterisation.

Remark 1. A simple sufficient condition for an analytic function not to be mean-periodic is that the function satisfy an Inada condition: $u'(c) \to \infty$, as $c \to 0$. If it was, then, there would exist $e_1 < e_2 < \ldots < e_K$ and $\alpha_1, \alpha_2, \ldots, \alpha_K$, all different from 0, such that $\sum_k \alpha_k u'(e_k + y) = 0$ for all y. But, as $y \to -e_1$, $u'(e_1 + y) \to \infty$, while all other $u(\cdot)$ remain finite. There cannot be a linear combination that stays equal to 0 and puts positive weight on $u'(e_1 + y)$.

Remark 2. As we show in the appendix, if the cardinal utility index is a polynomial, of degree n, there cannot exist distinct $e_1, \ldots, e_k, \ldots, e_K$, with $K < n$, such that $\sum_{k=1}^{K} \alpha_k u'(y + e_k) = 0, \quad \text{for all} \quad y$. As a consequence, identification is possible if $2S < n$.

Remark 3. Similarly, let $\mathcal{A}$ be a finite dimensional family of cardinal utility functions sufficiently rich in perturbations: if $u' = \sum_0^\infty a_k x^k \in \mathcal{A}$, then $u'(x) = \sum_0^n a'_k x^k + \sum_{k+1}^\infty a_k x^k \in \mathcal{A}$, for $a'_n \in (a_n - \varepsilon, a_n + \varepsilon)$, and any finite k. Then, for any $e_1, \ldots, e_k, \ldots, e_K$, with $K < n$, for a generic $u' \in \mathcal{A}$, there is no $\alpha \in \mathbb{R}^K$, $\alpha \neq 0$ such that $\sum_{k=1}^{K} \alpha_k u'(y + e_k) = 0, \quad \text{for all} \quad y$.

For simplicity of exposition, let $S = 2$ and consider the function[4] F : $\mathcal{S}^{S-1} \times \mathcal{A}$ defined by

$$F(\theta_1, \theta_2, \ldots, a_k, \ldots) = (\theta_1, \theta_2)W = (\theta_1, \theta_2)AB,$$

where

$$A = \begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & \ldots \\ 2a_2 & 6a_3 & 12a_4 & \ldots & \ldots \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 1 \\ (e_1 + x) & (e_2 + x) \\ (e_1 + x)^2 & (e_2 + x)^2 \\ \vdots & \vdots \end{pmatrix};$$

evidently, $W = AB$ is the Wronskian matrix.

By a direct computation,

$$D_{a_1,a_2,a_3} F = \begin{pmatrix} \theta_1 & 2\theta_2 + 2\theta_1 e_1 & \theta_2 6 e_1 + 3\theta_1 1 e_1^2 \\ \theta_1 & 2\theta_2 + 2\theta_1 e_2 & \theta_2 6 e_1 + 3\theta_1 1 e_2^2 \end{pmatrix}.$$

[4] $\mathcal{S}^k$ denotes the sphere of dimension k.

Since $(\theta_1, \theta_2) \in \mathcal{S}^1$, while $e_1 \neq e_2$, the matrix $D_{a_1,a_2,a_3}F$ has full row rank, which extends to the matrix DF. It follows that $F \pitchfork 0$, and, by the transversal density Theorem, $F_u : \mathcal{S}^1 \pitchfork 0$, for u in an subset of $\mathcal{A}$ of full Lebesgue measure. Then, since $\dim \mathcal{S}^1 < 2$, there is no (θ_1, θ_2) such that $F(\theta_1, \theta_2, \ldots, a_k, \ldots) = (\theta_1, \theta_2)AB = 0$.

4 Extensions

In conclusion, we indicate extensions of the results to the case in which only aggregate demand is observable or, even more generally, only the equilibrium correspondence as the distribution of endowments varies; and, lastly, to a dynamic setting.

4.1 Aggregation and equilibrium

For a finite collection of individuals, i, with distribution of initial wealth $\vec{e}_0 = (\ldots, e_0^i, \ldots)$ and characteristics $\vec{\xi}^{\,i} = (u^i, \beta^i, e^i, \pi^i, r^i)$, aggregate demand for consumption an assets is $(x_0^a, y^a)(q, \vec{e}_0) = \sum_i (x_0^i, y^i)(q, e_0^i)$[5].

The argument in Chiappori, Ekeland, Kubler, and Polemarchakis (2004), in an abstract context that applies here, is that the aggregate demand identifies individual demand as long as the latter satisfies a rank condition on wealth effects following in Lewbel (1991):

$$\left.\begin{array}{l} \frac{\partial^2 z_j^i}{\partial (e_0^i)^2} \neq 0, \\ \frac{\partial}{\partial e_0^i}(\ln \frac{\partial^2 z_k^i}{\partial (e_0^i)^2}) \neq \frac{\partial}{\partial e_0^i}(\ln \frac{\partial^2 z_l^i}{\partial (e_0^i)^2}), \end{array}\right. \quad j, k, l, \in \{0, \ldots, a, \ldots\}.$$

Indeed, the argument proceeds by observing, first, that aggregate demand identifies the wealth effects of individuals: $(\partial z_j^i / \partial e_0^i) = (\partial z_j^a / \partial (e_0^i))$; and, subsequently, by demonstrating that the rank condition allows the demands of individuals to be identified from higher order derivatives with respect to wealth of the identity $\partial z_k^i / \partial q_j - \partial z_j^i / \partial q_k = z_k(\partial z_j^i / \partial e_0^i) - z_j(\partial z_j^i / \partial e_0^i$ that follows from the Slutzky decomposition of the derivatives of demand into income and substitution effects and the symmetry of the latter.

An alternative formulation allows for equilibrium. Assets are productive (trees), and the, $r_{a,s}$ are output, wealth or consumption; and, individuals are endowed with assets, f_0^i. Equilibrium prices, then, satisfy $\sum_i y^i(q, e_0^i, f_0^i) =$

[5]To facilitate exposition, alternatively, we write $z^i = (z_0, \ldots, z_a, \ldots) = (x_0^i, \ldots, y_a^i, \ldots)$, and similarly, for z^a.

$\sum_i f_0^i$, and the set equilibria of equilibria as endowments vary is $\mathcal{W} = \{(q, \overrightarrow{(e_0^i, f_0^i)} : \sum_i y^i(q, e_0^i, f_0^i) = \sum_i f_0^i\}$. Under smoothness assumptions, the equilibrium set has a differentiable manifold structure, and it identifies aggregate demand locally; the previous argument identifies beliefs and preferences of individuals.

4.2 Multiple periods

We assume that an agent chooses consumption and assets over $T+1$ periods, $t = 0, 1, \ldots, T$. To simplify the setup we assume throughout that every period the agent's beliefs over endowments, prices and asset-payoffs has at most S points in its support. With this we can describe uncertainty by a finite event tree $\mathbf{\Sigma}$; the root node is 0, and $\widetilde{\mathbf{\Sigma}}$ is the tree excluding the root node. Each node, $s^t \in \widetilde{\mathbf{\Sigma}}$ can be identified by a history of shocks $s^t = (s_1, ..., s_t)$, with $s_t \in \mathbf{S} = \{1, ..., S\}$ for all t; we write $s^{t'} \succeq s^t$ if $s^{t'}$ is a successor of s^t . For what follows the S nodes at $t = 1$ will be of particular interest – in a slight abuse of notation we denote them by the realisation of the shock, $1, ..., S$.

An individual has beliefs over nodes, $\pi(s^t)$, for $s^t \in \widetilde{\mathbf{\Sigma}}$, that are consistent in the sense that for all $s^t \in \widetilde{\mathbf{\Sigma}}$

$$\sum_{s^t \succeq s^{t-1}} \pi(s^t) = \pi(s^{t-1}), \quad \text{with } \pi(0) = 1. \tag{2}$$

The individual maximises a time-separable subjective expected utility

$$U(x) = u(x(0)) + \sum_{t=1}^{T} \beta^t \sum_{s^t \in \mathbf{\Sigma}} \pi(s^t) u(x(s^t)), \quad x \geq 0,$$

where as before β is the discount factor, and $u : (0, \infty) \to \mathbb{R}$ satisfies standard assumptions.

One-period assets, $a = 1, \ldots, A$, are traded in periods $t = 0, \ldots (T - 1)$. Payoffs of an asset traded at s^{t-1} are $r_a(s^t)$ and payoffs across assets are $R(s^t) = (r_1(s^t), \ldots, r_a(s^t), \ldots, r_A(s^t))$. Holdings of assets are $y(s^t) = (\ldots, y_a(s^t), \ldots)^\top$.

To use our insights from the two period problem, it is useful to write the agent's problem recursively. Define

$$v_{s^T}(y) = u(e(s^T) + R(S^t)y) \text{ and } d_{s^T} = e(s^T),$$

and recursively for up to $t = 1$,

$$v_{s^t}(y) = \max_{y'} u(e(s^t) + R(s^t)y - q(s^t)y') + \beta \sum_{s^{t+1} \succeq s^t} \frac{\pi(s^{t+1})}{\pi(s^t)} v_{s^{t+1}}(y')$$

$$\text{s.t.} \qquad R(s^{t+1})y' + d_{s^{t+1}} \geq 0 \quad s^{t+1} \succeq s^t,$$

and

$$d_{s^t} = \max_{y'} e(s^t) + q(s^t)y' \text{ s.t. } R(s^{t+1})y' + d_{s^{t+1}} \geq 0 \quad s^{t+1} \succeq s^t.$$

While in a two period setting it is natural to assume that only asset demand in the first period is observable, in this multiple period setting one can consider various different cases. In the simplest case, one observes how demand at all nodes varies as asset prices vary at all nodes. We focus on the in a sense most demanding case and assume that one only observes how the demand for asset changes in the first period as prices and incomes change in the first period. We assume that all other prices are held fixed. We denote by $y_0(q_0, e_0)$ the demand for assets in period 0 as only prices in period 0 vary. We assume that this demand function is observed in an open neighbourhood of prices and endowments for which the resulting consumption is strictly positive at all nodes, that is, none of the inequality constraints on consumption are binding. Consistent with our earlier notation we define $v_s(y)$, for $s = 1, ..., S$, to be the possible value functions at $t = 1$. Since we assume that we observe asset demand in a neighbourhood where consumption at all nodes is strictly positive, by the analytic version of the implicit function theorem, Fritzsche and Grauert (2002), Theorem 7.6., the value functions v_s must be analytic on this neighbourhood whenever $u(\cdot)$ is analytic. Therefore, Proposition 1 above directly implies that the cardinal utility index can be identified whenever $u(\cdot)$ is analytic.

In order to find necessary conditions for identification one therefore needs to make additional assumption on fundamentals or on admissible beliefs (similar to the assumptions we made for Proposition 4 above). It is easy to see that without further assumptions identification is never possible: if along two paths prices, asset-payoffs and non-marketable endowments are identical it is impossible to identify their probabilities even if the agent believes that his marketable endowments are distributed differently over time.

It is useful to write fundamentals as $\xi = (\xi_1, ..., \xi_S) \in \Xi$ where each $\xi_s = (\pi(\sigma), e(\sigma), R(\sigma))_{\sigma \in \tilde{\Sigma}, \sigma \succeq s}$. With this we can write $v_s(y) = v(y|\xi_s)$ for some state invariant function $v(.)$. We denote Ξ to be the set of possible fundamentals along a path, that is, each $\xi_s \in \Xi$ Restrictions on beliefs mean that we impose that each $\xi_s \in \widehat{\Xi} \subset \Xi$.

The Theorem, then, translates to the following result:

Proposition 4. *The demand function for consumption and assets identifies the unobservable characteristics within a set $\widehat{\Xi}$ if for any $2S$ distinct $\xi_s \in \widehat{\Xi}$ the functions $f_s(y) = v(y|\xi_s))$, along with the constant function $f_0(y) = 1$, are linearly independent for all $s = 1, \ldots, 2S$.*

Without strong assumptions on $\widehat{\Xi}$, it is difficult to find assumptions on fundamentals that ensure that the sufficient condition in Proposition 4 hold. However, given our analysis in Section 3 above, there are two simple cases for which this is the case. They are summarised in the following result.

Proposition 5. *Suppose cardinal utility is analytic and has non-vanishing derivatives of any order. Under either of the following to assumptions on admissible beliefs identification is possible.*

1. *The payoff of risky assets separates all uncertainty in the following strong sense: For any $\xi, \xi' \in \widehat{\Xi}$ if $\xi \neq \xi'$ then $R(\xi) \neq R(\xi')$*

2. *The cardinal utility index, $u(\cdot)$ satisfies an Inada condition and different beliefs imply different implicit borrowing limits: For any $\xi, \xi' \in \widehat{\Xi}$ we have $d(\xi) \neq d(\xi')$.*

The proof follows directly from our discussion above.

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Appendix: Polynomials and power series

If the utility function is polynomial,

$$u(x) = a_0 + a_1 x + \dots, a_l x^l + \dots a^n x^n,$$

then [6]

$$u^{(l)}(x) = a_l l! + \dots a_k \frac{k!}{(k-l)!} x^{k-l} + \dots a_n \frac{n!}{(n-l)!} x^{n-l}, \quad l = 0, \dots n,$$

and, in particular,

$$u^n(x) = a_n n!.$$

[6] Here, we use $u^{(k)}$ for u^k.

Consider the matrix

$$A_n = \begin{pmatrix} a_1 & \dots & (1+k)a_k & \dots & \dots & \dots & \dots & na_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_l l! & \dots & a_{l+k}\frac{(l+k)!}{k!} & \dots & a_n \frac{n!}{(n-l)!}x^{n-l} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n n! & \dots & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

the submatrix

$$A_{S,n} = \begin{pmatrix} a_{n-S+1}(n-S+1)! & \dots & a_n \frac{n!}{(S-1)!}x^{(S-1)} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n n! & \dots & 0 & 0 & \dots \end{pmatrix},$$

and the matrix

$$B_n^S = \begin{pmatrix} \dots & 1 & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^k & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^{(S-1)} & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^{n-1} & \dots \end{pmatrix}.$$

The Wronskian of the family of functions $\{u^{(n-S+1)}(e_s+x)\}$, that is, of the derivatives of order $(n-S+1)$ of the functions $\{u(e_s+x)\}$, is

$$W_{(n-S+1)} = A_{S,n} B_n^S =$$

$$\begin{pmatrix} a_{n-S+1}(n-S+1)! & \dots & a_n \frac{n!}{(S-1)!} x^{(S-1)} \\ \vdots & \vdots & \vdots \\ a_n n! & \dots & 0 \end{pmatrix} \begin{pmatrix} \dots & 1 & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^k & \dots \\ \vdots & \vdots & \vdots \\ \dots & (e_s+x)^{(S-1)} & \dots \end{pmatrix},$$

a square matrix of dimension $S \times S$, that is invertible: it is the product of two square matrices, of which the first term is upper-diagonal, with non-vanishing terms on the diagonal, $a_n n! = u_{(n)}(x) \neq 0$, while the second is the Vandermonde matrix of the random variables $\{(e_s+x)\}$.

Since linear dependence of the functions $\{u^{(1)}(e_s+x)\}$ would imply linear dependence of the functions $\{u^{(n-S+1)}(e_s+x)\}$.

For power series, it is instructive to consider the case of CARA cardinal utility,

$$u(x) = -e^{-x} = -1 + x + \dots + (-1)^{(k+1)} \frac{1}{k!} x^k + \dots (-1)^{(n+1)} \frac{1}{n!} x^n, \dots;$$

In order to simplify the exposition, we restrict attention to the case $S=2$. The polynomial approximation of $u(x)$ of order n is

$$u_n(x) = -1 + x + \dots + (-1)^{(k+1)} \frac{1}{k!} x^k + \dots (-1)^{(n+1)} \frac{1}{n!} x^n;$$

evidently,

$$u_n^{(1)}(x) = 1 - x + \dots + (-1)^{(k+1)} \frac{1}{(k-1)!} x^{(k-1)} + \dots (-1)^{(n+1)} \frac{1}{(n-1)!} x^{(n-1)},$$

and

$$u_n^{(2)}(x) = -1 + x + \dots + (-1)^{(k+1)} \frac{1}{(k-2)!} x^{(k-2)} + \dots (-1)^{(n+1)} \frac{1}{(n-2)!} x^{(n-2)}.$$

It follows that, if

$$A^{2,n} = \begin{pmatrix} 1 & -1 & \dots & (-1)^{k+1}\frac{1}{(k-1)!} & \dots & \dots & (-1)^{(n+1)}\frac{1}{(n-1)!} \\ -1 & 1 & \dots & (-1)^{k+2}\frac{1}{(k-1)!} & \dots & (-1)^{n+1}\frac{1}{(n-2)!} & 0 \end{pmatrix},$$

and

$$B_n^2 = \begin{pmatrix} 1 & 1 \\ (e_1+x) & (e_2+x) \\ \vdots & \vdots \\ (e_1+x)^k & (e_2+x)^k \\ \vdots & \vdots \\ (e_1+x)^{(n-1)} & (e_2+x)^{(n-1)} \end{pmatrix},$$

the Wronskian of the family of functions $\{u^{(n-S+1)}(e_s+x)\}$, that is, of the derivatives of order $(n-S+1)$ of the functions $\{u(e_s+x)\}$, is

$$W_{2,n} = A_n^2 B_n^2 =$$

$$\begin{pmatrix} 1 & -1 & \dots & \dots & (-1)^{(n+1)}\frac{1}{(n-1)!} \\ -1 & 1 & \dots & (-1)^{n+1}\frac{1}{(n-2)!} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ (e_1+x) & (e_2+x) \\ \vdots & \vdots \\ (e_1+x)^k & (e_2+x)^k \\ \vdots & \vdots \\ (e_1+x)^{(n-1)} & (e_2+x)^{(n-1)} \end{pmatrix}.$$

For finite n, the rank of $W_{2,n} = A_n^2 B_n^2$ is not clear – evidently, the functions are independent as can be ascertained by considering the Wronskian of the derivatives of order $n-1$. But, as $n \to \infty$, the matrix A_n^2 converges to a matrix of row rank 1, which implies that the Wronskian is singular; this accounts for the failure of identification of CARA cardinal utility.

Alternatively, for CRRA cardinal utility, and, in particular,

$$u(x) = \ln x,$$

the power series expansion at $\bar{x} = 1$ is

$$u(x) = \ln x = 0 + (x-1) + \ldots + (-1)^{(k-1)}\frac{1}{k}(x-1)^k + \ldots (-1)^{(n-1)}\frac{1}{n}(x-1)^n, \ldots;$$

In order to simplify the exposition, we restrict attention to the case $S = 2$. The polynomial approximation of $u(x)$ of order n is

$$u_n(x) = 0 + (x-1) + \ldots + (-1)^{(k-1)}\frac{1}{k}(x-1)^k + \ldots (-1)^{(n-1)}\frac{1}{n}(x-1)^n;$$

evidently,

$$u_n^{(1)}(x) = 1 - x + \ldots + (-1)^k(x-1)^k + \ldots + (-1)^{(n-1)}x^{(n-1)},$$

and

$$u_n^{(2)}(x) = -1 + x + \ldots + (-1)^{(k+1)}(k+1)(x-1)^k + \ldots + (-1)^{(n-1)}(n-1)x^{(n-2)}.$$

It follows that

$$A_n^2 = \begin{pmatrix} 1 & -1 & \ldots & (-1)^k & \ldots & \ldots & (-1)^{(n-1)} \\ -1 & 2 & \ldots & (-1)^{k+1}(k+1) & \ldots & (-1)^{n-1} & 0 \end{pmatrix},$$

$$B_n^2 = \begin{pmatrix} 1 & 1 \\ (e_1 + x - 1) & (e_2 + x - 1) \\ \vdots & \vdots \\ (e_1 + x - 1)^k & (e_2 + x - 1)^k \\ \vdots & \vdots \\ (e_1 + x - 1)^{(n-1)} & (e_2 + x - 1)^{(n-1)} \end{pmatrix},$$

and the Wronskian of the family of functions $\{u^{(n-S+1)}(e_s + x)\}$ is

$$W_{2,n} = A_n^2 B_n^2 =$$

$$\begin{pmatrix} 1 & -1 & \dots & \dots & (-1)^{(n+1)}\frac{1}{(n-1)!} \\ -1 & 2 & \dots & (-1)^{n+1}\frac{1}{(n-2)!} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ (e_1 + x - 1) & (e_2 + x - 1) \\ \vdots & \vdots \\ (e_1 + x - 1)^k & (e_2 + x - 1)^k \\ \vdots & \vdots \\ (e_1 + x - 1)^{(n-1)} & (e_2 + x - 1)^{(n-1)} \end{pmatrix}.$$

For all n, and as $n \to \infty$, the matrix A_n^2 remains of rank 2; this is in contrast to the CARA case.