ECONOMICS

Warwick Economics Research Paper Series

Sovereign debt and incentives to default with uninsurable risks

Gaetano Bloise, Herakles Polemarchakis and Yiannis Vailakis

This paper also appears as *CRETA Discussion Paper No: 15*

It has been published in *Theoretical Economics, Volume 12 (3). pp. 1121-1154*.

January 2016
Series Number: 1107
ISSN 2059-4283 (online)
ISSN 0083-7350 (print)

SOVEREIGN DEBT AND INCENTIVES TO DEFAULT WITH UNINSURABLE RISKS

GAETANO BLOISE, HERAKLES M. POLEMARCHAKIS, AND YIANNIS VAILAKIS

Current version: January 12, 2016

Abstract

Sovereign debt is not sustainable even in the presence of uninsurable risks; which extends the result of Bulow and Rogoff (1989). But the argument is not as general. Indeed, examples show that positive borrowing may be enforced even though the sovereign's natural debt limits, corresponding to the most pessimistic evaluation of future endowment, are finite. Unsustainable sovereign debt in incomplete asset markets requires a strong version of high implied interest rates: the value of the most optimistic evaluation of future endowment is finite

Keywords: Sovereign risk, Ponzi games, Reputational debt, Incomplete markets.

JEL Classification: F34, H63.

1. Introduction

The impossibility result of Bulow and Rogoff (1989) asserts that sovereign debt is unsustainable if debt contracts are not supported by direct sanctions and default carries only a ban from ever borrowing in financial markets.[1] The intuition is that, when solvency constraints preclude the roll-over of debt, a country can always improve upon contractual arrangements that involve repayments (*i.e.*, positive net transfers from the country to foreign investors) by defaulting at the date-event associated with the maximum expansion of debt. When markets are complete, debt unsustainability obtains as long as effective debt limits are as tight as, or tighter than, the natural debt limits (that is, dominated by a finite present

We would like to thank Alessandro Citanna, Martin Kaae Jensen and Filipe Martins-da-Rocha for comments and suggestions. Gaetano Bloise acknowledges the financial support of the Italian Ministry of Education (PRIN 2010-2012). Yiannis Vailakis acknowledges the financial support of an ERC starting grant (FP7, Ideas specific program, Project 240983 DCFM) and of an ANR research grant (Project Novo Tempus). Yiannis Vailakis acknowledges the hospitality of Kobe University (RIBE) and Université de Dauphine at early stages of the project.

[1] Bulow and Rogoff (1989) led to a vast literature studying alternative mechanisms enforcing debt repayment in the absence of sanctions. We refer to Aguiar and Amador (2014) and Wright (2011) for a thorough discussion of the literature.

value of the sovereign's future endowment), for otherwise debt could be rolled-over indefinitely and default would not be profitable.[2]

The logic underlying the unsustainability of sovereign debt is deeper than intuition suggests, and, as a matter of fact, its proof is subtle.[3] Even though solvency constraints impose debt redemption, repayments might be extremely dispersed over time under uncertainty. During the repayment phase, trade in financial instruments is still necessary for protection against adverse shocks, and a temporary expansion of liabilities might be indispensable for consumption smoothing purposes. For default to be profitable because of saved repayments, financial markets must allow for a similar risk diversification when borrowing is prohibited. Under incomplete markets, it is not clear that such a replication is feasible. This casts doubts on the relevance of the result in Bulow and Rogoff (1989).

We provide examples of sovereign debt sustained by reputation under incomplete markets when natural debt limits are finite. Incentives for debt repayment are stronger, because, as liabilities are not allowed after default, some insurance opportunities cannot be replicated. Default involves a benefit (saved repayments) along with a cost (incomplete risk-diversification). The incentive to default depends on this trade-off and, for high risk-aversion, the cost may overcome the benefit. This suggests that, in order to restore the validity of the impossibility result, we need to identify conditions that allow for replication under incomplete asset markets.

When asset markets are incomplete, the evaluation of non-tradable claims is ambiguous (Santos and Woodford (1997)). At a competitive equilibrium, under full commitment, debt is restricted by feasible repayment, the natural debt limit, corresponding to the most pessimistic evaluation of future endowment. Our examples reveal that impossibility may fail even though natural debt limits are finite, a legitimate analog of high implied interest rates for incomplete markets. We argue that, in general, replication requires that the most optimistic evaluation of future endowment be also finite, a substantially more demanding condition. And this is because it guarantees the existence of a trading strategy that finances any budget-affordable net consumption plan through portfolios that involve no liabilities. We show that, under an additional mild technical assumption (related to the continuity of the underlying pricing functional), this is sufficient to restore the validity of the impossibility result unless the sovereign can run a Ponzi game.

[2] Hellwig and Lorenzoni (2009) (also Bidian and Bejan (2014)) show that low implied interest rates is a necessary and sufficient condition for the exact roll-overing of equilibrium self-enforcing debt limits.

[3] Martins-da-Rocha and Vailakis (2014) show that the original argument of Bulow and Rogoff (1989) does not go through when the output of the sovereign may vanish along a path of successive low productivity shocks or when it may grow unboundedly along a path of successive high productivity shocks.

Pesendorfer (1992) studies repayment incentives of small open economies trading with competitive risk-neutral foreign investors while having access to a limited set of financial assets. His analysis differs from ours in a crucial aspect. The punishment in Pesendorfer (1992) is that defaulters may not hold a negative position in any of the available assets. As long as debtors cannot form portfolios consistent with full insurance, such a punishment for default may support for positive borrowing. We instead do allow defaulters to sell assets short, but only insofar as their portfolio does not involve negative payoffs (future net obligations or net liabilities). In some circumstances, these two formulations are equivalent: under complete markets, when a full set of elementary Arrow securities is available and, under incomplete markets, when only a risk-free bond is traded. In general, our no liability constraint is a weaker punishment when all securities yield positive payoffs and, thus, default is more likely.

It is worth noticing that, following Pesendorfer (1992), one can construct examples in which a ban on short sales supports positive borrowing even if there is a complete set of assets. In fact, in order to establish Bulow and Rogoff (1989)'s claim of unsustainable sovereign debt, it is in general necessary that all positive contingent claims remain available to a defaulting country, because short sales might be essential to replicate insurance opportunities after default. In this sense, we extend Bulow and Rogoff (1989) to incomplete markets under comparable conditions.

Evidently, the no short sale constraint can be implemented without observing trades in other securities of a defaulting country. However, this is also true for the no liability constraint when all redundant securities are traded in the market. Thus, ultimately, the separating hypothesis is whether redundant securities can be issued. Permitting such trades upon default seems, besides necessary for their claim, more in the spirit of Bulow and Rogoff (1989)'s cash-in-advance contract (an up-front payment in exchange of positive contingent deliveries in the future). Market access is preserved after default to the extent that the other parties are not exposed to the insolvency of the country. Moreover, the no liability constraint seems also consistent with competitive markets, as the creation of redundant securities does not raise any pricing issue. We can imagine this process of (redundant) financial innovation as a sort of competition among intermediaries providing packaged securities at their market price (with the addition of a negligible fee).

Our paper is also related to the recent work of Auclert and Rognlie (2015). They obtain an incomplete markets Bulow-Rogoff counterpart under Markovian uncertainty when a risk-free bond is the only available asset. Under risk-neutral pricing, when the interest rate is constant, it is easy to exhibit a simple replication strategy. However, these restrictive assumptions on pricing do not allow for drawing general implications about debt sustainability at a competitive equilibrium.

The paper is organised as follows. In section 2, we present the intuition for our analysis. In section 3, we lay out the fundamentals of the economy. In section 4, we discuss the meaning of, and the link between, various restrictions on the asset pricing kernel. In section 5, we present examples that bring out the insight underlying the failure of Bulow and Rogoff (1989)'s unsustainable debt result in incomplete markets. In section 6, we show which conditions on arbitrage-free prices restore the validity of the impossibility result of Bulow and Rogoff (1989). It is worth noticing that we introduce a different approach that applies independently of the extension to incomplete markets. For completeness, we gather some technical properties of incomplete-markets pricing in Appendices A and B.

2. INTUITION

We develop the intuition for unsustainable sovereign debt in a simple deterministic economy. Our approach is different from the original argument in Bulow and Rogoff (1989). This alternative method brings out the logic that underlies the incentives to default in a way that can then be immediately extended to uncertainty even under incomplete markets.

Time, t, is discrete, and the initial date is $t = 0$. The sovereign is entitled to an endowment $e = (\ldots, e_t, \ldots) > 0$, and consumes $c = (\ldots, c_t, \ldots) > 0$. The flow budget constraint requires that

$$p_{t+1}v_{t+1} + p_t (c_t - e_t) \leq p_t v_t,$$

where $p = (\ldots, p_t, \ldots) > 0$ is a process for present value prices and $v = (\ldots, v_t, \ldots)$ is the evolution of sovereign wealth. Prices are obtained by compounding interest rates over time or, equivalently, their ratios are determined by interest rates; v_t is a claim if positive and a liability if negative. Both prices and the endowment might be highly non-stationary processes; if not, the analysis is straightforward.

Bulow and Rogoff (1989) assume that sovereign debt never exceeds the market value of a claim on the country's future income stream, that is,

$$-g_t = -\frac{1}{p_t} \sum_{r \geq 0} p_{t+r} e_{t+r} \leq v_t.$$

Clearly, this is restrictive only if the claim has finite value or, according to the terminology of Alvarez and Jermann (2000), only if the hypothesis of high implied interest rates is satisfied. Bulow and Rogoff (1989) argue that, when the plan involves liabilities over time, the country will have an incentive to default and to revert to *cash-in-advance* contracts. These are budget-balanced plans involving no debt over time, that is, in this simple deterministic economy, budget-balanced plans fulfilling an additional no borrowing constraint. In this sense, reputational debt is unsustainable.

In general, a default incentive for the sovereign balances a benefit (no repayments according to the previous debt obligations) with a cost (reduced trade opportunities, due to the subsequent no borrowing constraint). The intuition of Bulow and Rogoff (1989)'s paradox is instead grounded on a pure arbitrage principle: the sovereign is able to replicate the previous consumption plan without borrowing and, hence, incurs no substantial cost upon default. The key step is the construction of this replication policy.

Define $b = (\ldots, b_t, \ldots)$ as the process satisfying, at every t,

$$b_t = \sup_{r \geq 0} \frac{1}{p_t} p_{t+r} \left(-v_{t+r}\right).$$

In economic terms, b_t is the minimum amount of resources that enables the sovereign to extinguish the debt in any arbitrary date beginning from t. This value is finite since debt is bounded by the present value of the country's income stream. In particular, $-v_t \leq b_t \leq g_t$.

It is immediate to verify that the process b is deflating, since the cost of solvency cannot increase over time in present value. Thus,

$$p_{t+1} b_{t+1} \leq p_t b_t.$$

This crucial observation reveals that replication is feasible. Indeed, the alternative financial plan $w = v + b \geq 0$ satisfies the budget constraint with no liabilities, that is,

$$p_{t+1} \left(v_{t+1} + b_{t+1}\right) + p_t \left(c_t - e_t\right) \leq p_t \left(v_t + b_t\right).$$

In other terms, at every point in time, b_t is the minimum amount of resources that would allow the sovereign to dispense with liabilities at no cost in terms of future trade opportunities. Furthermore, if the inequality is slack,

$$p_{t+1} b_{t+1} < p_t b_t,$$

then $w_t = v_t + b_t = 0$, which uncovers a strict benefit from defaulting (and restarting with $w_t = 0$). In the spirit of Bulow and Rogoff (1989), in this situation, sovereign debt has reached its maximum expansion and, thus, the country begins a repayment policy. Defaulting allows the country to save on these repayments and enjoy higher consumption.

For default not to be profitable, the adapted process b must be rolling-over exactly, that is,

$$p_{t+1} b_{t+1} = p_t b_t,$$

which implies that sovereign debt is not deflating over time and, therefore, a repayment policy never begins. No default incentive emerges, but this contradicts the assumption that debt is bounded by the present value of the future income stream, because, in this case,

$$0 < p_0 b_0 \leq \lim_{t\to\infty} p_t b_t \leq \lim_{t\to\infty} p_t g_t \leq 0.$$

Notice that, under high implied interest rates, when debt exceeds the present value of future income, the sovereign is necessarily running a Ponzi game, as it is rolling over the portion of its liabilities exceeding the value of its endowment.

We develop this line of reasoning in order to provide a proof under incomplete markets. An analogously constructed process reveals the cost for extinguishing the debt under uncertainty at any contingent truncation. The fact that truncations are contingent is an essential step in the argument, for otherwise the process would not be deflating and would not be suitable for replication. When markets are incomplete, the adapted process might not be in the space of tradable claims. Here we exploit the basic principles for the transfer of resources under incomplete markets. The current cost of meeting a future debt obligation, using available securities, can be computed as its largest evaluation at state prices consistent with no arbitrage opportunities. Thus, the largest evaluation of future liabilities identifies a plan that eliminates the debt at the predefined contingent truncation, which allows us to extend the argument for default to incomplete markets.

3. FUNDAMENTALS

3.1. **Uncertainty.** Trading occurs at each date-event in the set $\mathcal{S}$ along an infinite horizon. Time is indexed by t in $\mathbb{T} = \{0, 1, 2, \ldots\}$. We use the common notation s^t to denote one of the date-events in $\mathcal{S}$ that may be reached in period t in $\mathbb{T}$. Date-events in $\mathcal{S}$ are endowed with a partial ordering $\succeq$, that is, whenever date-event s^{t+r} (s^{t-r}) in $\mathcal{S}$ succeeds (precedes) date-event s^t in $\mathcal{S}$, we write $s^{t+r} \succeq s^t$ ($s^t \succeq s^{t-r}$). Thus, $\left\{s^{t+1} \in \mathcal{S} : s^{t+1} \succ s^t\right\}$ is the *finite* set of immediate successors of date-event s^t in $\mathcal{S}$, whereas $\left\{s^t \in \mathcal{S} : s^{t+1} \succ s^t\right\}$ is the *unique* predecessor of date-event s^{t+1} in $\mathcal{S}$. There is a unique initial date-event s^0 in $\mathcal{S}$. The set $\mathcal{S}$, endowed with the partial order $\succeq$, is the event-tree.

The *continuation tree* at date-event s^t in $\mathcal{S}$ is $\mathcal{S}\left(s^t\right) = \left\{s^{t+r} \in \mathcal{S} : s^{t+r} \succeq s^t\right\}$. A *finite contingent truncation* $\mathcal{F}\left(s^t\right)$ of the continuation tree $\mathcal{S}\left(s^t\right)$ is a finite set of $\succeq$-unordered elements in $\mathcal{S}\left(s^t\right)$ such that any date-event in $\mathcal{S}\left(s^t\right)$ admits either a (weak) successor or a (weak) predecessor in $\mathcal{F}\left(s^t\right)$. Figure 1 presents an example of a contingent truncation in a simple binomial tree: the continuation tree is initiated at the date-event corresponding to the first solid circle, while a contingent truncation is identified by the thick circles.

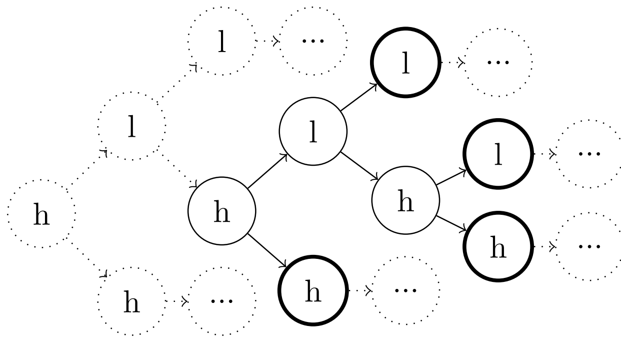

FIGURE 1. Finite contingent truncation

3.2. **Basic notation.** Denote by L the linear space of all maps $x : \mathcal{S} \to \mathbb{R}$. A map x in L is positive, $x \geq 0$, when $x\left(s^t\right) \geq 0$ for every date-event s^t in $\mathcal{S}$; non-trivially positive, $x > 0$, when positive and $x\left(s^t\right) > 0$ for some date-event s^t in $\mathcal{S}$; strictly positive, $x \gg 0$, when $x\left(s^t\right) > 0$ for every date-event s^t in $\mathcal{S}$. The positive cone is $L^+ = \{x \in L : x \geq 0\}$. For an element x on L, $\left(x\left(s^t\right)\right)_{s^t \in \mathcal{D}}$ is also regarded as an element of the linear space L for any subset $\mathcal{D}$ of $\mathcal{S}$, that is, as an element of L vanishing out of the subset $\mathcal{D}$ of $\mathcal{S}$.

3.3. **Consumption and preferences.** We let e be the element of L^+ representing the sovereign's endowment process of the single commodity, where $e\left(s^t\right)$ in $\mathbb{R}_+$ is the available value at date-event s^t in $\mathcal{S}$. The sovereign's preferences on consumption plans c in L^+ are defined by a *contingent utility function* $U : L^+ \to L$, where $U(c)\left(s^t\right)$ is the utility value beginning from date-event s^t in $\mathcal{S}$. It is assumed that $U(\hat{c})\left(s^t\right) > U(\tilde{c})\left(s^t\right)$ whenever $\left(\hat{c}\left(s^{t+r}\right)\right)_{s^{t+r} \in \mathcal{S}(s^t)} > \left(\tilde{c}\left(s^{t+r}\right)\right)_{s^{t+r} \in \mathcal{S}(s^t)}$. Strict monotonicity is the only restriction on preferences.

3.4. **Markets.** To simplify notation, at no loss of generality, incomplete markets are represented by a linear subspace V of L such that v is in V if and only if $\left(v\left(s^{t+1}\right)\right)_{s^{t+1} \succ s^t}$ is also in V for every s^t in $\mathcal{S}$. In other terms, the space of tradable claims decomposes sequentially in a collection of components for every date-event, each with deliveries only at subsequent date-events. For a tradable claim v in V, we use the canonical decomposition $v = v^+ - v^-$, separating *claims* v^+ in L^+ from *liabilities* v^- in L^+. These are interpreted as *net* positions, since the portfolio composition is not explicit. We maintain the assumption that some strictly positive element u on L is also in V, that is, available financial instruments allow for a (possibly risky) strictly positive transfer. The presence of a risk-free bond would be sufficient to ensure this, though it is more demanding than necessary.

For concreteness, as an example, consider the case where a finite set J of short-lived (one-period) securities is traded at every date-event s^t in $\mathcal{S}$. Each security j in J is represented as a return process y_j in L, interpreted as promises to deliver at successor nodes. A portfolio z in L^J specifies holdings of available securities at all contingencies, with $z_j\left(s^t\right)$ in $\mathbb{R}$ being the holding of security j in J at date-event s^t in $\mathcal{S}$. In this case, the space of tradable contingent claims V consists of all v in L such that, for some portfolio process z in L^J, at every date-event s^t in $\mathcal{S}$,

$$v\left(s^{t+1}\right)=\sum_{j\in J} z_j\left(s^t\right) y_j\left(s^{t+1}\right),$$

where it is understood that s^{t+1} in $\mathcal{S}$ is the successor of s^t in $\mathcal{S}$.

To maintain the analogy with complete markets, the market pricing of securities is represented as an element φ on V. This is at no loss of generality, as we assume that the Law of One Price is satisfied (see Ch. 2 and Ch. 17 in LeRoy and Werner (2001)). Thus, at every date-event s^t in $\mathcal{S}$, the market value of any portfolio with deliveries $\left(v\left(s^{t+1}\right)\right)_{s^{t+1}\succ s^t}$ in V is given by

$$\sum_{s^{t+1}\succ s^t} \varphi\left(s^{t+1}\right) v\left(s^{t+1}\right).$$

No arbitrage implies that, whenever $\left(v\left(s^{t+1}\right)\right)_{s^{t+1}\succ s^t}$ in V is a (non-trivial) positive claim, then its market value must be strictly positive. In other terms, any (non-trivially) positive claim is costly on the market.

A *price* p in P is a strictly positive element of L satisfying, for every tradable claim v in V, at every date-event s^t in $\mathcal{S}$,

$$\sum_{s^{t+1}\succ s^t} \varphi\left(s^{t+1}\right) v\left(s^{t+1}\right)=\frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) v\left(s^{t+1}\right).$$

By the assumption of no arbitrage, implicit prices exist and form a (non-empty) convex cone P. Observe that only the ratios are relevant for the determination of such implicit prices. This provides an equivalent representation of the asset pricing kernel. Indeed, as prices are invariant on the space of tradable claims V, at every date-event s^t in $\mathcal{S}$, the market value of claims v in V is given by

$$\inf_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) v\left(s^{t+1}\right)=\sup_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) v\left(s^{t+1}\right).$$

Here we take these basic facts as a primitive framework. They are well-established results in the literature and need no justification (see, for instance, Magill and Quinzii (1996), Santos and Woodford (1997) and

LeRoy and Werner (2001)). To make the paper self-contained, the relevant theorems are collected in Appendix A.

3.5. **Ponzi games.** A country is running a *Ponzi scheme*, or *game*, whenever it is persistently financing liabilities by means of other liabilities. Precisely, a financial plan v in V involves a Ponzi scheme if there exists a process b in V, with non-trivial part b^{+} in L^{+}, such that, at every date-event s^{t} in $\mathcal{S}$,

$$b\left(s^{t}\right) \leq v^{-}\left(s^{t}\right), \tag{PG-1}$$

and

$$b\left(s^{t}\right) \leq \sum_{s^{t+1} \succ s^{t}} \varphi\left(s^{t+1}\right) b\left(s^{t+1}\right). \tag{PG-2}$$

These conditions capture the fact that some outstanding liabilities are served by new liabilities perpetually and, thus, are rolled over indefinitely. Notice that, because of incomplete markets, assuming that b^{-} in L^{+} is zero would be unnecessarily restrictive. For later use, we provide an operational reformulation based on the dual representation of incomplete markets.

Lemma 3.1 (Ponzi schemes)**.** *A financial plan v in V involves a Ponzi scheme if and only if there exists a non-trivial process b^{*} in L^{+} such that, at every date-event s^{t} in $\mathcal{S}$,*

$$b^{*}\left(s^{t}\right) \leq v^{-}\left(s^{t}\right)$$

and

$$b^{*}\left(s^{t}\right) \leq \inf_{p \in P} \frac{1}{p\left(s^{t}\right)} \sum_{s^{t+1} \succ s^{t}} p\left(s^{t+1}\right) b^{*}\left(s^{t+1}\right).$$

Proof. Assuming that conditions (PG-1)-(PG-2) are satisfied, define $b^{*}=b^{+}$ in L^{+}. When $b^{+}\left(s^{t}\right)=0$ at a date-event s^{t} in $\mathcal{S}$, the claim is true. In the other case, (PG-2) implies, for every price p in P,

$$b^{+}\left(s^{t}\right) \leq \frac{1}{p\left(s^{t}\right)} \sum_{s^{t+1} \succ s^{t}} p\left(s^{t+1}\right) b\left(s^{t+1}\right) \leq \frac{1}{p\left(s^{t}\right)} \sum_{s^{t+1} \succ s^{t}} p\left(s^{t+1}\right) b^{+}\left(s^{t+1}\right),$$

so proving necessity. For sufficiency, by the Theorem of Duality (see Appendix A), there exists a tradable plan b in V such that, at every date-event s^{t} in $\mathcal{S}$,

$$b\left(s^{t}\right) \leq b^{*}\left(s^{t}\right)$$

and

$$\sum_{s^{t+1} \succ s^{t}} \varphi\left(s^{t+1}\right) b\left(s^{t+1}\right)=\inf_{p \in P} \frac{1}{p\left(s^{t}\right)} \sum_{s^{t+1} \succ s^{t}} p\left(s^{t+1}\right) b^{*}\left(s^{t+1}\right).$$

It is simple to verify that conditions (PG-1)-(PG-2) are satisfied. □

3.6. **Self-enforcing contracts.** A *contract* c in L^+ is sustained by a financial plan v in V if, at every date-event s^t in $\mathcal{S}$,

$$\sum_{s^{t+1}\succ s^t} \varphi\left(s^{t+1}\right) v\left(s^{t+1}\right) + \left(c\left(s^t\right) - e\left(s^t\right)\right) = v\left(s^t\right).$$

Notice that budget exhaustion is unrestrictive because of the existence of a strictly positive tradable claim u in V. Obviously, without any solvency requirement, the budget constraint is vacuous when Ponzi games are not ruled out. Thus, remaining agnostic on the nature of debt limits, we only say that a contract c in L^+ is *budget-feasible* if it is sustained by a financial plan v in V which does not involve any Ponzi scheme. Budget restrictions are compatible with liabilities, though debt cannot be rolled over indefinitely.

The sovereign can default at any date-event. Upon default financial instruments remain available subject to a no liability restriction. Thus, after default, financial plans w are restricted to $V \cap L^+$. A budget-feasible contract c in L^+ is *immune to default* if, at every date-event s^t in $\mathcal{S}$, for every alternative contract $\hat{c}$ in L^+ which is sustained by a restricted financial plan w in $V \cap L^+$ with $w\left(s^t\right) = 0$, $U(c)\left(s^t\right) \geq U(\hat{c})\left(s^t\right)$. In other terms, a contract is immune to default whenever, at every date-event, a country would not benefit from defaulting and trading subject to the no liability restriction thereafter.

3.7. **Replication.** For the understanding of default incentives, it is important to identify the role of liabilities in providing insurance opportunities. To this purpose, we say that a contract c in L^+ is *replicable* whenever it is sustained by a financial plan w in $V \cap L^+$. When not all contracts are replicable, default involves the implicit cost of restricting insurance opportunities.

We preliminarily observe that, when a contract can be replicated, it is sustainable by conducting previously planned trades in securities along with a rolling-over policy. When the risk-free bond is available in the market, the rolling-over component can consist of this only security, provided that the interest rate is uniformly positive and the endowment process is bounded. Otherwise, the risk-free bond might be unsuitable to replicate insurance opportunities. We provide below an example to support this claim (see Example 5.3). In general, under incomplete markets, the portfolio for the rolling-over component can only be identified abstractly.

Proposition 3.1 (Rolling-over)**.** *A contract c in L^+, sustained by a financial plan v in V, is replicable if and only if there is a process b^* in V such that, at every date-event s^t in $\mathcal{S}$,*

$$v\left(s^t\right) + b^*\left(s^t\right) \geq 0,$$

and

$$b^*\left(s^t\right)=\sum_{s^{t+1}\succ s^t}\varphi\left(s^{t+1}\right)b^*\left(s^{t+1}\right).$$

Proof. Notice that $w=v+b^*$ is a financial plan in $V\cap L^+$. The proof is obvious because the budget is unaffected when the rolling over condition is satisfied. □

We here provide a more fundamental necessary condition for replication. Replication requires a finite present value of future debt at any finite contingent truncation. In other terms, according to the interpretation in §2, the cost of solvency is finite at every date-event. As we shall prove later on (Proposition 6.1), this condition is also sufficient for replication.

Proposition 3.2 (Necessary condition for replication)**.** *A contract c in L^+, sustained by a financial plan v in V, is replicable only if, at every date-event s^t in $\mathcal{S}$,*

$$b\left(s^t\right)=\sup_{p\in P}\sup_{\mathcal{F}(s^t)\subset\mathcal{S}(s^t)}\frac{1}{p\left(s^t\right)}\sum_{s^{t+r}\in\mathcal{F}(s^t)}p\left(s^{t+r}\right)\left(-v\left(s^{t+r}\right)\right)\text{ is finite,}$$

where $\mathcal{F}\left(s^t\right)$ is any finite contingent truncation of $\mathcal{S}\left(s^t\right)$.

Proof. Fix any p in P and notice that, by Proposition 3.1,

$$\begin{aligned}\frac{1}{p\left(s^t\right)}\sum_{s^{t+r}\in\mathcal{F}(s^t)}p\left(s^{t+r}\right)\left(-v\left(s^{t+r}\right)\right) &\leq \frac{1}{p\left(s^t\right)}\sum_{s^{t+r}\in\mathcal{F}(s^t)}p\left(s^{t+r}\right)b^*\left(s^{t+r}\right)\\ &\leq b^*\left(s^t\right),\end{aligned}$$

where $\mathcal{F}\left(s^t\right)$ is any finite contingent truncation of $\mathcal{S}\left(s^t\right)$. This is sufficient to establish the claim. □

4. High interest rates

Under full commitment, liabilities are traditionally bounded by natural debt limits. This is justified by the requirement that sustainable debt need be repayable in finite time out of the available endowment. As established by Hernández and Santos (1996), Levine and Zame (1996) and Santos and Woodford (1997), when markets are incomplete, the natural debt limit is given by the worst evaluation of future endowment. Thus, at a competitive equilibrium of this sort,

(E) $$\inf_{p\in P}\frac{1}{p\left(s^0\right)}\sum_{s^t\in\mathcal{S}}p\left(s^t\right)e\left(s^t\right)\ \text{ is finite,}$$

for otherwise solvency would not be enforced. In general, this is the only restriction on state prices at the competitive equilibrium (see Magill and Quinzii (1994) and Hernández and Santos (1996) for the existence of a competitive equilibrium under incomplete markets).

We provide examples in §5 showing that, in general, the impossibility result of Bulow and Rogoff (1989) fails under condition (E), because insurance opportunities cannot be replicated without liabilities after default. We need to further restrict prices by assuming that

$$\text{(F)} \qquad \sup_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}} p\left(s^t\right) e\left(s^t\right) \text{ is finite.}$$

That is, the value of the endowment is (uniformly) finite for all prices consistent with the absence of arbitrage opportunities. In analogy with the terminology used in complete markets (see Alvarez and Jermann (2000)), we refer to this property as *high implied interest rates*.

Under complete markets, high implied interest rates deliver the continuity of the pricing kernel in a topology which is coherent with impatience: the value of residual claims in the remote future vanishes. We need a similar property under incomplete markets, namely,

$$\text{(H)} \qquad \lim_{t \to \infty} \sup_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right) = 0,$$

where $\mathcal{S}^t$ contains all date-events in $\mathcal{S}$ at date t in $\mathbb{T}$. When the pricing kernel satisfies condition (H), we say that it exhibits *uniformly high implied interest rates*.

Remark 4.1. As the terminology suggests, condition (H) implies condition (F). Indeed, assuming that (F) is violated, we can show that, for any $\epsilon > 0$, at every date t in $\mathbb{T}$,

$$\sup_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right) \geq \epsilon,$$

thus violating restriction (H). To this purpose, it suffices to argue that, for every t in $\mathbb{T}$,

$$\sup_{p \in P} \frac{1}{p(s^0)} \sum_{s^{t-r} \in \mathcal{S}^{0,t}} p\left(s^{t-r}\right) e\left(s^{t-r}\right) \text{ is finite,}$$

where $\mathcal{S}^{0,t}$ contains all contingencies in $\mathcal{S}$ from the initial date up to date t in $\mathbb{T}$. This is what we accomplish in the following, by exploiting that some strictly positive claim u is in the tradable space V.

Fixing any t in $\mathbb{T}$, suppose that $\left(w\left(s^{t+1}\right)\right)_{s^{t+1} \in \mathcal{S}^{t+1}}$ is a tradable claim in $V \cap L^+$. It is immediate to verify that there exists a tradable claim $\left(w\left(s^t\right)\right)_{s^t \in \mathcal{S}^t}$ in $V \cap L^+$ such that, for every price p in P, at every date-event s^t in $\mathcal{S}^t$

$$\sum_{s^{t+1} \succ s^t} p\left(s^{t+1}\right) w\left(s^{t+1}\right) + p\left(s^t\right) e\left(s^t\right) \leq p\left(s^t\right) w\left(s^t\right).$$

This is true because, for some sufficiently large $\lambda > 0$, the expansion $\lambda\left(u\left(s^t\right)\right)_{s^t \in \mathcal{S}^t}$ is an arbitrarily large strictly positive tradable claim in $V \cap L^+$, where u is the strictly positive claim in V. Therefore, by backward induction, beginning with $\left(w\left(s^{t+1}\right)\right)_{s^{t+1} \in \mathcal{S}^{t+1}} = 0$, there exists a sufficiently large $w\left(s^0\right)$ in $\mathbb{R}_+$ such that, for every p in P,

$$\sum_{s^{t-r} \in \mathcal{S}^{0,t}} p\left(s^{t-r}\right) e\left(s^{t-r}\right) \leq p\left(s^0\right) w\left(s^0\right),$$

thus proving the claim.

Remark 4.2. Condition (F) does not imply condition (H) when markets are incomplete. We provide an example in Appendix B, where we also present a complete (rather technical) characterisation of this lack of coincidence. The failure of condition (H), when condition (F) is satisfied, only happens in rather singular situations.

Remark 4.3. Here is an important case in which condition (H) is satisfied. Let us consider an economy with a tradable uncontingent bond in which there exists a sufficiently large $1 > \beta > 0$ such that, at every date-event s^t in $\mathcal{S}$,

$$\beta \geq \sup_{p \in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1} \succ s^t} p\left(s^{t+1}\right).$$

As the right-hand side is the price of the uncontingent bond (with unitary deliveries), this restriction imposes a sort of lower bound on the interest rates uniformly across all contingencies. When this uniform lower bound exists, the hypothesis of uniformly high implied interest rates is satisfied in an economy with *bounded* endowment.[4] Indeed, given any price p in P, at every t in $\mathbb{T}$, it follows that

$$\beta \sum_{s^t \in \mathcal{S}^t} p\left(s^t\right) \geq \sum_{s^{t+1} \in \mathcal{S}^{t+1}} p\left(s^{t+1}\right),$$

where $\mathcal{S}^t$ contains all date-events in $\mathcal{S}$ at date t in $\mathbb{T}$. Therefore, at every t in $\mathbb{T}$,

$$\frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right) \leq \eta \frac{\beta^t}{1-\beta}$$

where $\eta > 0$ is such that $e\left(s^t\right) \leq \eta$ for every s^t in $\mathcal{S}$.

[4] This is for instance the case in the environment studied in Auclert and Rognlie (2015).

Remark 4.4. We here present a relevant case in which *high implied interest rates are violated* when the uncontingent bond is the *only* tradable asset. Suppose that the price of this bond is constantly equal to $1 > \beta > 0$ and that the endowment evolves as a random walk, that is, at every date-event s^t in $\mathcal{S}$,

$$e\left(s^t\right) = \sum_{s^{t+1} \succ s^t} \mu_{t+1}\left(s^{t+1} | s^t\right) e\left(s^{t+1}\right),$$

where $\mu_{t+1}\left(s^{t+1} | s^t\right)$ is the probability conditional on date-event s^t in $\mathcal{S}$. In such an environment, a particular price p^* in P is given by $p^*\left(s^t\right) = \beta^t \mu_t\left(s^t\right) e\left(s^t\right)$ at every date-event s^t in $\mathcal{S}$, because

$$\frac{1}{p^*\left(s^t\right)} \sum_{s^{t+1} \succ s^t} p^*\left(s^{t+1}\right) = \beta,$$

where $\mu_t\left(s^t\right)$ is the unconditional probability of date-event s^t in $\mathcal{S}$. Notice that, given any date-event s^t in $\mathcal{S}$, simple computations deliver

$$p^*\left(s^t\right) e\left(s^t\right) \leq \sum_{s^{t+1} \succ s^t} p^*\left(s^{t+1}\right) e\left(s^{t+1}\right),$$

where we have used a mild hypothesis on the conditional variance of the endowment process, according to which

$$1 \leq \beta \sum_{s^{t+1} \succ s^t} \mu_{t+1}\left(s^{t+1} | s^t\right) \left(\frac{e\left(s^{t+1}\right)}{e\left(s^t\right)}\right)^2.$$

Thus, high implied interest rates are violated, as there is a price p^* in P such that the value of the endowment is inflating over time.

5. Examples

We here present some examples of failure of the impossibility of Bulow and Rogoff (1989) under incomplete markets. The cause of this failure is that the incompleteness of markets does not allow for replication when liabilities are prohibited after default. The evaluation of future endowment is finite for some prices, *i.e.*, condition (E) is satisfied. However, condition (F) of high implied interest rates is violated. We also provide an example to illustrate that, under high implied interest rates, a risk-free bond might be ineffective in replicating insurance opportunities after default, according to the rolling-over logic of Proposition 3.1, even when financial markets are complete.

Example 5.1. The first example is simple, but it delivers the basic intuition underlying the failure of Bulow and Rogoff (1989)'s unsustainable debt result. The economy is subject to binomial uncertainty over states $S = \{l, h\}$ occurring with equal probability. To fix ideas, also assume that preferences over consumption are additively separable, *i.e.*,

$$U(c)\left(s^0\right) = \sum_{s^t \in \mathcal{S}} \beta^t \mu_t\left(s^t\right) u\left(c\left(s^t\right)\right),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a Bernoulli function, $1 > \beta > 0$ is the discount factor and $\mu_t\left(s^t\right)$ is the unconditional probability of date-event s^t in $\mathcal{S}$. Here, as well as in the other examples, the set of date-events $\mathcal{S}$ consists of all partial histories of Markov states in S having strictly positive probability, given a predefined initial Markov state in S.

Markets are incomplete. Indeed, at every date-event s^t in $\mathcal{S}$, there is a single asset with payoffs $(y_l, y_h) = (1, -1)$ and price $q\left(s^t\right) = 0$. The endowment is $(e_l, e_h) = (0, 2)$. The economy begins at state h in S with an inherited liability $v\left(s^0\right) = -1$. Trivially, holding one unit of the security permits complete insurance at constant consumption $c\left(s^t\right) = 1$ at every date-event s^t in $\mathcal{S}$. However, whenever the economy is in state h in S, the country holds a liability. We verify whether default is profitable in such contingencies.

Upon default, liabilities are not allowed. Hence, in this simple economy, no asset can be traded and autarchy is the only budget-feasible consumption after default. In general, defaulting may produce no benefit, thus violating Bulow and Rogoff (1989)'s impossibility result. Indeed, a sufficient condition for this is that the instantaneous utility function satisfies the following inequality

$$(1-\beta)(u(2) - u(1)) < \beta\left(u(1) - \frac{u(2) + u(0)}{2}\right).$$

That is, the sovereign cannot benefit from defaulting if the current gain of not repaying the debt is compensated by the loss of smoothing future consumption.

Remark 5.1. It is immediate to see that for this example condition (F) and, hence, condition (H) is violated while condition (E) is satisfied. Indeed, the present value of the endowment is infinite (respectively, finite) for every implicit price process p^* in P such that, at every date-event s^t in $\mathcal{S}$, $p^*\left(s^t\right) = (1/2)^t \delta^t$ with $\delta > 1$ (respectively, $0 < \delta < 1$).

Example 5.2. The previous example does not satisfy our assumptions because markets do not permit a strictly positive transfer. We here develop a more complicated example under this additional restriction. Uncertainty is given by Markov states $S = \{l, m, h\}$, all occurring with the same probability. The economy begins in state h in S.

As in the previous example, preferences are assumed to be additively separable with a Bernoulli utility function of the form

$$u(c) = \frac{c^{1-(1/\gamma)} - 1}{1 - (1/\gamma)},$$

where $\gamma > 0$ is the elasticity of intertemporal substitution. Notice that, for any γ in the interval $(0, 1/2)$, this utility function is uniformly bounded from above, as

$$\text{(*)} \qquad u(c) \leq \frac{-1}{1 - (1/\gamma)} \leq 1.$$

Furthermore, as it can be verified by direct computation, for every $1 > \eta > 0$,

$$\text{(**)} \qquad \lim_{\gamma \to 0} u(1 - \eta) = \lim_{\gamma \to 0} \frac{(1 - \eta)^{1-(1/\gamma)} - 1}{1 - (1/\gamma)} = -\infty.$$

The important implication is that any small drop in consumption induces an arbitrarily large loss in utility when $\gamma > 0$ is sufficiently small.

At every date-event s^t in $\mathcal{S}$, there are only two securities paying off, for some sufficiently small $1 > \epsilon > 0$, $(y_l, y_m, y_h) = (1, \epsilon, 0)$ and $(y_l, y_m, y_h) = (0, \epsilon, 1)$. The price of each security is $q = (1/3)\,\beta(1 + \epsilon)$, where $1 > \beta > 0$ is the discount factor. The endowment is $(e_l, e_m, e_h) = (0, 1, 2)$. Notice that strictly positive holdings of both securities permit a strictly positive transfer, as long as $\epsilon > 0$, thus satisfying our general assumptions.

It is easy to construct a balanced portfolio with deliveries $(v_l, v_m, v_h) = (1, 0, -1)$. This sustains full-insurance with constant consumption $c\left(s^t\right) = 1$ at every date-event s^t in $\mathcal{S}$. Furthermore, it is optimal for all sufficiently large bounded debt limits. Does the country benefit from defaulting when holding a liability (and, hence, at the initial date-event s^0 in $\mathcal{S}$ when the economy is in state h in S)?

Preliminarily notice that, at every date-event s^t in $\mathcal{S}$, by budget feasibility without liabilities, consumption is bounded by a process ξ in L^+ such that, at every date-event s^t in $\mathcal{S}$,

$$\xi\left(s^{t+1}\right) \geq \frac{1}{q}\xi\left(s^t\right) + e\left(s^{t+1}\right),$$

for some sufficiently large initial value $\xi\left(s^0\right)$ in $\mathbb{R}_+$. Such bounds overestimate the payoffs of available assets (because $1 > \epsilon > 0$). Moreover, they hold true independently of any sufficiently small $\epsilon > 0$. We assume that the following inequality is satisfied:

$$\text{(***)} \qquad \frac{q}{\epsilon} > \frac{1}{3}\frac{\beta}{\epsilon} > 1.$$

Under this condition, we evaluate default incentives along a sequence of monotonically vanishing $\gamma > 0$.

Suppose that there exists a sequence of consumption plans $(c^{\gamma})_{\gamma>0}$ such that each plan is supported by a trading strategy involving no liabilities and guarantees an overall utility after defaulting at least equal to the overall utility from full insurance. That is, given any $\gamma > 0$, assume that

$$U(c^{\gamma})\left(s^0\right) \geq U(c)\left(s^0\right) = 0.$$

At no loss of generality, it can be assumed that the sequence of consumption plans converges, that is, at every date-event s^t in $\mathcal{S}$, $c^0\left(s^t\right) = \lim_{\gamma\to 0} c^{\gamma}\left(s^t\right) \leq \xi\left(s^t\right)$. We first argue that this limit guarantees at least the full-insurance consumption.

To verify this, assume that, at some date-event s^t in $\mathcal{S}$, $c^0\left(s^t\right) < 1 - \eta$ for some $1 > \eta > 0$. By condition (**), this implies an infinite loss, which cannot be compensated by bounded gains in other periods, because of (*). Hence, at every date-event s^t in $\mathcal{S}$, the consumption in the limit exceeds the full-insurance consumption, that is, $c^0\left(s^t\right) \geq 1$. We shall now argue by contradiction.

Consider a date-event s^t in $\mathcal{S}$ in which the economy is in state m in S. Suppose that the economy remains in state m in S for r in $\mathbb{N}$ consecutive dates and, after this phase, enters the averse state l in S. This happens at the date-event s^{t+r} in $\mathcal{S}$. A simple computation yields

$$c^0\left(s^{t+r}\right) \leq \left(\frac{1}{q}\right)\left(\frac{\epsilon}{q}\right)^{r-1}\xi\left(s^t\right).$$

For the computation of this bound, we use the fact, in the limit as $\gamma > 0$ vanishes, the endowment is completely consumed in state m in S. Initial resources are rolled over so as to overestimate their contribution to consumption when the economy enters in state l in S. For a sufficiently large r in $\mathbb{N}$, by condition (***), full-insurance consumption cannot be guaranteed.

Remark 5.2. We can verify that, for Example 5.2, condition (F) and, hence, condition (H) is violated, while condition (E) is satisfied. Indeed, as the economy is stationary, state prices (π_l, π_m, π_h) in $\mathbb{R}^S_+$ are only restricted by the two pricing equations

$$\pi_h + \epsilon\pi_m = q = \frac{1}{3}\beta(1+\epsilon) \text{ and } \pi_l + \epsilon\pi_m = q = \frac{1}{3}\beta(1+\epsilon).$$

State prices correspond to the ratios of implicit present value prices in our general analysis. For any choice of (stationary) state prices π in $\mathbb{R}^S_+$, the present value of the endowment, if finite, is determined by the system of equations

$$\begin{aligned}
Q_l(\pi) &= e_l + \pi_l Q_l(\pi) + \pi_m Q_m(\pi) + \pi_h Q_h(\pi), \\
Q_m(\pi) &= e_m + \pi_l Q_l(\pi) + \pi_m Q_m(\pi) + \pi_h Q_h(\pi), \\
Q_h(\pi) &= e_h + \pi_l Q_l(\pi) + \pi_m Q_m(\pi) + \pi_h Q_h(\pi).
\end{aligned}$$

We shall now argue that the value is finite for some state prices and infinite for other state prices.

Setting $\pi_h = \pi_l = 0$, we obtain $\pi_m = (q/\epsilon)$. For these state prices, the present value of the endowment satisfies

$$Q_m(\pi) = e_m + \pi_l Q_l(\pi) + \pi_m Q_m(\pi) + \pi_h Q_h(\pi) \geq e_m + \frac{q}{\epsilon} Q_m(\pi).$$

No positive solution exists whenever $\epsilon > 0$ fulfils condition (***) in Example 5.2, thus showing that condition (F) and, hence, condition (H) is violated.

To show that condition (E) holds true, it suffices to exhibit alternative state prices π in $\mathbb{R}_+^S$ for which the value of the endowment is finite. For instance, setting $\pi_h = \pi_m = \pi_l = (1/3)\beta$, a positive solution exists and is bounded by the value of receiving surely the largest endowment forever, $e_h/(1-\beta)$.

Example 5.3. The purpose of this example is to clarify that, under high implied interest rates, the risk-free bond might be unsuitable to replicate insurance opportunities after default. As this observation is independent of incomplete markets, we consider the case in which markets are indeed complete.

Uncertainty is given by a Markov chain on states $S = \{l, m, h\}$ with the following transitions: from state l in S (respectively, state h in S), states $\{m, h\}$ (respectively, $\{m, l\}$) are reached with equal probability; state m in S is absorbing; the initial state is h in S. Hence, the economy is fluctuating between states l and h, with a probability of entering the absorbing state m. The endowment depends only on the Markov state and it is given by $(e_l, e_m, e_h) = (1-\epsilon, \eta, 1+\epsilon)$ for some $1 > \epsilon > 0$ and $1 > \eta > 0$. Preferences are additively separable and the smooth Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and strictly concave with $\lim_{c \to 0} u'(c) = \infty$. Future utility is discounted at a rate $1 > \beta > 0$.

At every date-event s^t in $\mathcal{S}$, a risk-free bond is traded in the market. Its price depends on the current state in S. When the economy reaches one of the transient states $\{l, h\} \subset S$, the price of the bond is

$$q^f\left(s^t\right) = \frac{1}{2}\beta + \frac{1}{2}\beta \frac{u'(\eta)}{u'(1)} > 1.$$

When the economy is in the absorbing state m in S, the price is $q^f\left(s^t\right) = \beta$. At every date-event s^t in $\mathcal{S}$, when the economy is in the transient state l in S (respectively, h in S), the country can additionally trade an elementary Arrow security, delivering a unitary payoff conditional on state h in S (respectively, l in S) at the price $q\left(s^t\right) = (1/2)\,\beta$.

In this setting, the constant consumption $c\left(s^t\right) = 1$ is sustainable at every date-event s^t in $\mathcal{S}$ in which the economy reaches one of the transient states $\{l, h\} \subset S$. This can be achieved by trading in the elementary Arrow security only. Budget restrictions imply

$$\left(\frac{1}{2}\right)\beta z_l - \epsilon = z_h \text{ and } \left(\frac{1}{2}\right)\beta z_h + \epsilon = z_l,$$

where z_l and z_h are the holdings of the elementary Arrow security, respectively, in states l and h in S. These are solved by $z_l = z$ and $z_h = -z$, where $z = \epsilon/\left(1 + (1/2)\,\beta\right)$. Consumption at every date-event s^t in $\mathcal{S}$ in which the economy has reached the absorbing state m in S is given by the endowment, *i.e.*, $c\left(s^t\right) = \eta$. This plan is optimal provided that debt is bounded by some sufficiently large constant limit. Notice that trade in the risk-free bond cannot increase utility because its market price is determined at marginal utilities.

Observe that the safe interest rate is negative before the economy enters the absorbing state, though this is perfectly consistent with high implied interest rates. Indeed, the present value of the endowment is determined by

$$\begin{aligned} Q_l &= e_l + \pi_m Q_m + \pi_h Q_h, \\ Q_h &= e_h + \pi_m Q_m + \pi_l Q_l, \end{aligned}$$

where direct computation leads to

$$Q_m = \frac{\eta}{1-\beta}$$

and, by complete markets, state prices are given by $\pi_h = (1/2)\,\beta$ (respectively, $\pi_l = (1/2)\,\beta$) and

$$\pi_m = \frac{1}{2}\beta\frac{u'(\eta)}{u'(1)}.$$

The above pricing equations admits a finite positive solution.

The sovereign can benefit from default by repudiating the debt and replicating the same consumption plan by means of a translation of the holdings of the elementary Arrow security. However, replication is not feasible by rolling-over the risk-free bond and conducting the same trades in the elementary Arrow security as it was doing previously. Indeed, suppose that the process b^* in V given in Proposition 3.1 only involves trades in the risk-free bond. This requires holdings of the risk-free bond z^f in L^+ such that, at every date-event s^t in $\mathcal{S}$,

$$b^*\left(s^{t+1}\right) = z^f\left(s^t\right)$$

and

$$q^f\left(s^t\right)z^f\left(s^t\right)=b^*\left(s^t\right),$$

where s^{t+1} in $\mathcal{S}$ is an immediate successor of s^t in $\mathcal{S}$. Because the interest rate is negative, $q^f\left(s^t\right)>1$, as long as the economy remains in states $\{l,h\}\subset S$, this plan is contracting over time and, thus, it cannot serve to balance the recurrent liability implied by the previous trades in the elementary Arrow security.

6. UNSUSTAINABLE DEBT

We here show that, under uniformly high implied interest rates, sovereign debt is unsustainable. This extends Bulow and Rogoff (1989)'s impossibility result (see also Martins-da-Rocha and Vailakis (2014)) under more restrictive assumptions than those for complete markets. Importantly, we provide an alternative argument which applies independently of the extension of market incompleteness.

Recall that budget restrictions are compatible with liabilities, though debt cannot be rolled over indefinitely. We first show that ruling out Ponzi games is equivalent to bound liabilities by the most favourable evaluation of future endowment. Notice that, in general, this is more permissive than the bound given by the natural debt limit, that is, the most pessimistic evaluation of future endowment (Santos and Woodford (1997)).

Lemma 6.1 (Bounds to liabilities)**.** *Under high implied interest rates, a contract c in L^+ is budget-feasible only if it is sustained by a financial plan v in V satisfying, at every date-event s^t in $\mathcal{S}$,*

$$v\left(s^t\right)+g\left(s^t\right)\geq 0,$$

where the adapted process g in L^+ is given by

$$g\left(s^t\right)=\sup_{p\in P}\frac{1}{p\left(s^t\right)}\sum_{s^{t+r}\in\mathcal{S}(s^t)}p\left(s^{t+r}\right)e\left(s^{t+r}\right).$$

Proof. By construction of the adapted process g in L^+, at every date-event s^t in $\mathcal{S}$, for every price p in P,

$$\sum_{s^{t+1}\succ s^t}p\left(s^{t+1}\right)g\left(s^{t+1}\right)+p\left(s^t\right)e\left(s^t\right)\leq p\left(s^t\right)g\left(s^t\right).$$

Hence, adding up with the budget constraint, for every price p in P,

$$\sum_{s^{t+1}\succ s^t}p\left(s^{t+1}\right)\left(v\left(s^{t+1}\right)+g\left(s^{t+1}\right)\right)\leq p\left(s^t\right)\left(v\left(s^t\right)+g\left(s^t\right)\right).$$

In turn, this implies that, for every price p in P,

$$p\left(s^t\right)\left(v\left(s^t\right)+g\left(s^t\right)\right)^- \leq \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right)\left(v\left(s^{t+1}\right)+g\left(s^{t+1}\right)\right)^- .$$

Finally, at every date-event s^t in $\mathcal{S}$,

$$\left(v\left(s^t\right)+g\left(s^t\right)\right)^- \leq v^-\left(s^t\right).$$

Therefore, as Ponzi schemes are ruled out, we conclude that $v+g\geq 0$, for otherwise the conditions in Proposition 3.1 would be satisfied by $b^*=(v+g)^-$ in L^+. $\square$

We now restore the impossibility result under uniformly high implied interest rates. The crucial step in the argument is the construction of a feasible replication policy after default. The necessary condition for replication, provided in Proposition 3.2, is also sufficient. When replication is feasible, the default option is profitable unless the sovereign's debt is not deflating over time. The role of strengthening condition (F) by assuming uniformity (*i.e.*, that condition (H) be satisfied) is precisely to rule out such a situation. In complete markets, the uniformity follows by assumption since prices are supposed to have a sequential representation. In incomplete markets, a non-sequential price, as the limit of admissible sequential prices, cannot be ruled out, unless some uniformity of evaluation is imposed on the pricing kernel (see Appendix B for a thorough discussion).

Proposition 6.1 (Sovereign debt paradox)**.** *Under uniformly high implied interest rates, a budget-feasible contract c in L^+ is immune to default only if it involves no liabilities. That is, any financial plan v in V sustaining this contract with no Ponzi schemes must be positive.*

Proof. Consider a budget-feasible contract c in L^+ which is sustained by a financial plan v in V (not involving Ponzi schemes). Given a date-event s^t in $\mathcal{S}$, define

$$b\left(s^t\right)=\sup_{p\in P}\sup_{\mathcal{F}(s^t)\subset\mathcal{S}(s^t)}\frac{1}{p\left(s^t\right)}\sum_{s^{t+r}\in\mathcal{F}(s^t)} p\left(s^{t+r}\right)\left(-v\left(s^{t+r}\right)\right)\geq -v\left(s^t\right),$$

where $\mathcal{F}\left(s^t\right)$ is any finite contingent truncation of $\mathcal{S}\left(s^t\right)$ (for the definition, see §3.1). Notice that, for fixed price p in P, given any contingent truncation $\mathcal{F}\left(s^t\right)$ of $\mathcal{S}\left(s^t\right)$,

$$\frac{1}{p\left(s^t\right)}\sum_{s^{t+r}\in\mathcal{F}(s^t)} p\left(s^{t+r}\right)\left(-v\left(s^{t+r}\right)\right)\leq\frac{1}{p\left(s^t\right)}\sum_{s^{t+r}\in\mathcal{F}(s^t)} p\left(s^{t+r}\right)g\left(s^{t+r}\right)\leq g\left(s^t\right),$$

where we use Lemma 6.1 and the fact that the process g in L^+ satisfies the weak roll-over condition

$$\sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) g\left(s^{t+1}\right) + p\left(s^t\right) e\left(s^t\right) \leq p\left(s^t\right) g\left(s^t\right).$$

Therefore, the process b in L obeys $-v \leq b \leq g$. Furthermore, the fact that truncations are contingent implies that b in L fulfils the following property:

$$b(s^t) = \max\left\{-v(s^t), \sup_{p\in P} \frac{1}{p(s^t)} \sum_{s^{t+1}\succ s^t} p(s^{t+1}) b(s^{t+1})\right\}.$$

This happens because $\mathcal{F}\left(s^t\right)$ is a non-trivial finite contingent truncation of $\mathcal{S}\left(s^t\right)$ if and only if

$$\mathcal{F}\left(s^t\right) = \bigcup_{s^{t+1}\succ s^t} \mathcal{F}\left(s^{t+1}\right),$$

where $\mathcal{F}\left(s^{t+1}\right)$ is a finite contingent truncation of $\mathcal{S}\left(s^{t+1}\right)$ for every $s^{t+1} \succ s^t$ in $\mathcal{S}$.

By the Theorem of Duality (see Appendix A), there exists a tradable claim b^* in V such that, at every date-event s^t in $\mathcal{S}$,

$$b\left(s^t\right) \leq b^*\left(s^t\right)$$

and

$$\sum_{s^{t+1}\succ s^t} \varphi\left(s^{t+1}\right) b^*\left(s^{t+1}\right) = \sup_{p\in P} \frac{1}{p(s^t)} \sum_{s^{t+1}\succ s^t} p(s^{t+1}) b(s^{t+1}).$$

Therefore, at every date-event s^t in $\mathcal{S}$,

$$\begin{aligned}
\sum_{s^{t+1}\succ s^t} \varphi\left(s^{t+1}\right)\left(v\left(s^{t+1}\right) + b^*\left(s^{t+1}\right)\right) + \left(c\left(s^t\right) - e\left(s^t\right)\right) &\leq \\
\sup_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right)\left(v\left(s^{t+1}\right) + b\left(s^{t+1}\right)\right) + \left(c\left(s^t\right) - e\left(s^t\right)\right) &\leq \left(v\left(s^t\right) + b\left(s^t\right)\right) \\
&\leq \left(v\left(s^t\right) + b^*\left(s^t\right)\right).
\end{aligned}$$

If there is a date-event s^t in $\mathcal{S}$ such that

$$b\left(s^t\right) > \sup_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) b\left(s^{t+1}\right),$$

then $v\left(s^t\right) + b\left(s^t\right) = 0$. Define the process w in $V \cap L^+$ as $w\left(s^t\right) = v\left(s^t\right) + b\left(s^t\right) = 0$ and $w\left(s^{t+r}\right) = v\left(s^{t+r}\right) + b^*\left(s^{t+r}\right) \geq 0$ at any strict successor s^{t+r} in $\mathcal{S}\left(s^t\right)$. This sustains the given consumption plan without exhausting the budget at the date event s^t in $\mathcal{S}$. That is, the consumption plan c in L^+ is not immune to default, a contradiction.

It follows that, at every date-event s^t in $\mathcal{S}$, we must have

$$b\left(s^t\right) = \sup_{p \in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1} \succ s^t} p\left(s^{t+1}\right) b\left(s^{t+1}\right).$$

The last condition implies that, for every arbitrary $\eta > 0$, there exists a price process p in P such that, at every date-event s^t in $\mathcal{S}$,

$$b\left(s^t\right) \leq \frac{1}{p\left(s^t\right)} \sum_{s^{t+1} \succ s^t} p\left(s^{t+1}\right) b\left(s^{t+1}\right) + \eta e\left(s^t\right).$$

Hence, recalling that $b \leq g$, we get

$$\begin{aligned} b\left(s^0\right) &\leq \frac{1}{p\left(s^0\right)} \sum_{s^{t+1} \in \mathcal{S}^{t+1}} p\left(s^{t+1}\right) b\left(s^{t+1}\right) + \eta \frac{1}{p\left(s^0\right)} \sum_{s^{t-r} \in \mathcal{S}^{0,t}} p\left(s^{t-r}\right) e\left(s^{t-r}\right) \\ &\leq \frac{1}{p\left(s^0\right)} \sum_{s^{t+1} \in \mathcal{S}^{t+1}} p\left(s^{t+1}\right) g\left(s^{t+1}\right) + \eta \frac{1}{p\left(s^0\right)} \sum_{s^{t-r} \in \mathcal{S}^{0,t}} p\left(s^{t-r}\right) e\left(s^{t-r}\right), \end{aligned}$$

where $\mathcal{S}^t$ (respectively, $\mathcal{S}^{0,t}$) contains all date-events in $\mathcal{S}$ at date t in $\mathbb{T}$ (respectively, from the initial date up to date t in $\mathbb{T}$). Taking the limit, by uniformly high implied interest rates (condition (H)), this implies that, for any arbitrary $\eta > 0$,

$$b\left(s^0\right) \leq \eta \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}} p\left(s^t\right) e\left(s^t\right),$$

which shows that $b\left(s^0\right) \leq 0$. Reproducing the argument beginning from any date-event proves that $b \leq 0$ and, as $v + b \geq 0$, $v \geq 0$, thus establishing the claim. □

7. Conclusion

We have shown, by means of examples, that market incompleteness may induce incentives for repayment when liabilities are prohibited after default. A sovereign may not benefit from defaulting on its debt and positive borrowing can be sustainable by reputation. But the Bulow and Rogoff (1989)'s impossibility result does extend to economies with uninsurable risks if the pricing functional satisfies stronger restrictions. In particular, replication is obtained under a continuity property implying that the value of the most optimistic evaluation of future endowment eventually vanishes in the long-run.

Appendix A. Arbitrage-free pricing

We here collect some basic facts about arbitrage-free asset pricing which are used in the body of the text. These are well-known implications of duality. We provide simple proofs for convenience, independently of their applications in this paper.

The space of tradable claims Y is a linear subspace of some (finite-dimensional) linear space X, endowed with its canonical ordering. The pricing of tradable claims is given by a linear map $\varphi : Y \to \mathbb{R}$. This map is arbitrage free, in the sense that, for any claim y in Y, $y > 0$ only if $\varphi(y) > 0$. We assume that there exists a strictly positive tradable claim u in Y with $\varphi(u) = 1$. This needs not be the safe asset, though a safe asset would be sufficient for this property to be satisfied. The internal product on X is denoted by $x \cdot y$.

Let Π be the convex set of positive linear functionals π in X such that, for every y in Y,

$$\varphi(y) = \pi \cdot y.$$

Here is the Fundamental Theorem of Finance.

Fundamental Theorem of Finance. *The set Π is compact and contains a strictly positive linear functional π on X.*

Proof. Notice that the convex set $K = \{x \in X_+ : x \cdot u = 1\}$ does not interest the linear subspace $Z = \{y \in Y : \varphi(y) = 0\}$. By the Strong Separation Theorem (see for instance Aliprantis and Border (1999), Theorem 5.58), there exists a non-null π in X such that, for every k in K and for every z in Z,

$$\pi \cdot k > \pi \cdot z.$$

As Z is a linear space, $\pi \cdot z = 0$ for every z in Z. If $\pi \cdot x \leq 0$ for some non-null x in X_+, then

$$0 \geq \frac{1}{x \cdot u} \pi \cdot x \geq \pi \cdot \left(\frac{1}{x \cdot u} x \right) > 0,$$

a contradiction. Hence, π is a strictly positive positive linear functional on X. We next show that π is in Π.

At no loss of generality, it can be assumed that $\pi \cdot \bar{y} = \varphi(\bar{y}) > 0$ for some $\bar{y}$ in Y. Given any y in Y, suppose that $\varphi(y) > \pi \cdot y$. Hence,

$$\varphi \left(y - \frac{\varphi(y)}{\varphi(\bar{y})} \bar{y} \right) = 0 \text{ and } \pi \cdot \left(y - \frac{\varphi(y)}{\varphi(\bar{y})} \bar{y} \right) < 0,$$

a contradiction. The set Π is compact as it is contained in $\{\pi \in X_+ : \pi \cdot u = \varphi(u)\}$. $\square$

When markets are incomplete, Π contains multiple value kernels. Nevertheless, values are restricted by upper and lower bounds.

Theorem of Duality. *For every x in X,*

$$\max_{\pi\in\Pi}\pi\cdot x=\min_{y\in Y}\left\{\varphi\left(y\right):x\leq y\right\}$$

and

$$\min_{\pi\in\Pi}\pi\cdot x=\max_{y\in Y}\left\{\varphi\left(y\right):y\leq x\right\}.$$

Proof. We prove the first statement only, as the argument is specular for the other statement. We first show that there exists $\bar{y}$ in Y such that $x\leq y$ and

$$\varphi\left(\bar{y}\right)=\min_{y\in Y}\left\{\varphi\left(y\right):x\leq y\right\}.$$

Observe that, for some sufficiently large $\lambda>0$,

$$-\lambda u\leq x\leq\lambda u,$$

where u is the strictly positive claim in Y. Thus, by no arbitrage,

$$-\lambda\varphi\left(u\right)\leq\inf_{y\in Y}\left\{\varphi\left(y\right):x\leq y\right\}\leq\lambda\varphi\left(u\right).$$

This shows that the infimum is finite. For every n in $\mathbb{N}$, there exists a claim y^n in $\{y\in Y:x\leq y\}$ such that

$$\varphi\left(y^n\right)\leq\inf_{y\in Y}\left\{\varphi\left(y\right):x\leq y\right\}+\frac{1}{n}.$$

If the sequence $\left(y^n\right)_{n\in\mathbb{N}}$ is bounded, then the claim follows. Otherwise, observe that $\hat{y}^n=y^n/\left\|y^n\right\|$ is also a tradable claim in Y satisfying

$$\frac{x}{\left\|y^n\right\|}\leq\hat{y}^n$$

and

$$\varphi\left(\hat{y}^n\right)\leq\frac{\inf_{y\in Y}\left\{\varphi\left(y\right):x\leq y\right\}}{\left\|y^n\right\|}+\frac{1}{n\left\|y^n\right\|}.$$

Taking a subsequence of $\left(\hat{y}^n\right)_{n\in\mathbb{N}}$ in Y converging to $\hat{y}$ in Y, we obtain that $\hat{y}>0$ and $\varphi\left(\hat{y}\right)\leq 0$, contradicting no arbitrage.

Clearly, $\pi\cdot\left(x-\bar{y}\right)\leq 0$ for every π in Π. To prove that the opposite inequality is satisfied by some π in Π, consider the convex set C in $\mathbb{R}\times X$ defined by

$$\left\{\left(\varphi\left(\bar{y}-y\right),y-x\right)\in\mathbb{R}\times X:y\in Y\right\}.$$

This set does not intersect $\mathbb{R}_+ \times X_{++}$. Hence, by the Separating Hyperplane Theorem, there exists a non-null (μ, π) in $\mathbb{R}_+ \times X_+$ such that, for every y in Y,

$$\mu\varphi(\bar{y} - y) \leq \pi \cdot (x - y).$$

It can be verified that $\mu > 0$ and, hence, $\mu = 1$ at no loss of generality. Also,

$$0 \leq \varphi(\bar{y} - \bar{y}) \leq \pi \cdot (x - \bar{y}) \leq 0,$$

thus proving that $\pi \cdot (x - \bar{y}) = 0$. Finally, notice that, when y lies in Y, also $(\bar{y} - y)$ is in Y. It follows that

$$\varphi(y) \leq \varphi(\bar{y} - (\bar{y} - y)) \leq \pi \cdot (x - (\bar{y} - y)) \leq \pi \cdot y.$$

As Y is a linear space, it is also true that $\varphi(-y) \leq \pi \cdot (-y)$. We conclude that, for every y in Y,

$$\varphi(y) = \pi \cdot y,$$

which reveals that π is an element of Π. □

Appendix B. Continuous (sub-linear) pricing

This is a rather technical appendix which clarifies under which conditions uniformly high implied interest rates fail when high implied interest rates are instead satisfied. To this purpose, consider the linear space

$$L(e) = \{x \in L : |x| \leq \lambda e \text{ for some } \lambda > 0\},$$

which is a Banach lattice when endowed with the norm $\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda e\}$. As usual, let $L^*(e)$ be its norm dual. Notice that, by Alaoglu's Theorem (see Aliprantis and Border (1999), Theorem 6.21), the closed unit ball in $L^*(e)$ is weak-$*$ compact.

Under high implied interest rates, the space of (normalized) state prices $P_0 = \left\{p \in P : p\left(s^0\right) = 1\right\}$ can be regarded as a set in the positive cone of $L^*(e)$, where the duality operation is given by

$$p(x) = \sum_{s^t \in \mathcal{S}} p\left(s^t\right) x\left(s^t\right).$$

Notice that, in general, the set P_0 is not weak-$*$ closed in $L^*(e)$, though, by Alaoglu's Theorem, it is contained in a weak-$*$ compact set.

We can now provide a full characterisation of uniformly high implied interest rates. The violation of uniformly high implied interest rates occurs if and only if some price in the closure of admissible prices contains a bubble, that is, a non-negligible value at infinity. Under complete markets, such a circumstance is ruled out by assumption, as prices are supposed to have a sequential representation. When markets are incomplete, a non-sequential price, as the limit of admissible prices, cannot be ruled out, unless some uniformity of evaluation is imposed on the pricing kernel.

Proposition B.1 (Value at infinity)**.** *Under high implied interest rates, the condition of uniformly high implied interest rates is satisfied if and only if, for every p in the weak-$*$ closure of P_0 in $L^*(e)$,*

$$\lim_{t \in \mathbb{T}} p\left(e^t\right) = p(e),$$

where e^t in L is the truncation of e in L at date t in $\mathbb{T}$.

Proof. For necessity, suppose that the equivalent condition is violated. By positivity, this means that there exists $\epsilon > 0$ such that, for some p^* in the weak-$*$ closure of P_0, at every t in $\mathbb{T}$,

$$p^*\left(e - e^t\right) = p^*(e) - p^*\left(e^t\right) > \epsilon.$$

As p^* lies in the weak-$*$ closure of P_0, for every t in $\mathbb{T}$, it can be approximated by some p in P_0 such that

$$p\left(e - e^t\right) > \epsilon.$$

Thus, for every t in $\mathbb{T}$,

$$\sup_{p \in P_0} p\left(e - e^t\right) > \epsilon,$$

which contradicts the fact that condition (H) is satisfied.

For sufficiency, suppose that the equivalent condition is satisfied and assume a violation of uniformly high implied interest rates. By monotonicity, this implies that there exists $\epsilon > 0$ such that, for every t in $\mathbb{T}$,

$$\sup_{p \in P_0} p\left(e - e^t\right) \geq \epsilon.$$

At every t in $\mathbb{T}$, consider the restricted set

$$\bar{P}_{0,t} = \left(\left\{p \in \bar{P}_0 : p\left(e - e^t\right) \geq \epsilon\right\}\right),$$

where $\bar{P}_0$ is the weak-$*$ closure of P_0 in the positive cone of $L^*(e)$. Observe that $\bar{P}_{0,t}$ is a non-empty closed subset of the compact set $\bar{P}_0$ and that, by monotonicity, $\bar{P}_{0,t+1} \subset \bar{P}_{0,t}$. By the Finite Intersection Property, as $\bar{P}_0$ is compact,

$$p^* \in \bigcap_{t \in \mathbb{T}} \bar{P}_{0,t} \subset \bar{P}_0.$$

For such an element p^* of $\bar{P}_0$, $p^*(e) \geq p^*\left(e^t\right) + \epsilon$ for every t in $\mathbb{T}$, thus delivering a contradiction. $\square$

Example B.1. As a complement to the previous characterisation, we provide an example in which condition (F) is satisfied and condition (H) is violated. The Markov states are $S = \left\{(l_t)_{t\in\mathbb{T}}, (h_t)_{t\in\mathbb{T}}\right\}$, with l_0 being the initial state and each h_t being an absorbing state. In period t in $\mathbb{T}$, when the economy is in state l_t, with equal probability, it will move to the absorbing state h_{t+1} or to state l_{t+1}. The endowment is $e = (1-\beta)$ in state l_t and $e = \beta^{-t}(1-\beta)$ in state h_t. The only asset is an uncontingent bond, delivering a unitary payoff, with constant price $1 > \beta > 0$.

To verify that condition (F) holds true, notice the most optimistic evaluation of the endowment satisfies the recursive equation

$$Q(l_t) = (1-\beta) + \sup_{(\pi_l,\pi_h)\in\mathbb{R}_+\times\mathbb{R}_+} \pi_l Q(l_{t+1}) + \pi_h Q(h_{t+1})$$

subject to

$$\pi_l + \pi_h = \beta.$$

Furthermore, by direct computation, $Q(h_t) = \beta^{-t}$. It also immediate to prove that this recursive equation is solved by

$$Q(l_t) = 1 + \beta^{-t}.$$

We now shall argue that condition (H) is instead violated.

For fixed non-initial t in $\mathbb{T}$, consider the feasible sequence of Markov states $(l_0, \ldots, l_{t-1}, h_t, h_t, \ldots)$ in S. Let $\mathcal{D}$ be the path of date-events in $\mathcal{S}$ corresponding to the selected sequence of Markov states and construct a price p in the closure of P by means of $p\left(s^{\hat{t}}\right) = \beta^{\hat{t}}$, if $s^{\hat{t}}$ lies in $\mathcal{D}$, and $p\left(s^{\hat{t}}\right) = 0$, otherwise. For such a price p in P, direct computation shows that

$$\frac{1}{p(s^0)} \sum_{s^t\in\mathcal{S}^t} \sum_{s^{t+r}\in\mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right) = \sum_{r\in\mathbb{T}} \beta^{t+r}\beta^{-t}(1-\beta) = 1.$$

Thus, the residual does not vanish, which shows that condition (H) cannot be satisfied.

References

Aguiar, M. and Amador, M.: 2014, Sovereign debt, *Handbook of International Economics Vol 4*, North-Holland, pp. 647–687.

Aliprantis, C. D. and Border, K. C.: 1999, *Infinite Dimensional Analysis*, 3rd edn, Berlin: Springer.

Alvarez, F. and Jermann, U. J.: 2000, Efficiency, equilibrium, and asset pricing with risk of default, *Econometrica* **68**(4), 775–797.

Auclert, A. and Rognlie, M.: 2015, Unique equilibrium in the eato-gersovitz model of sovereign debt. Electronic copy available at http://www.princeton.edu/~aauclert/uniqueness_EG.pdf.

Bidian, F. and Bejan, C.: 2014, Martingale properties of self-enforcing debt. Economic Theory, DOI:10.1007/s00199-014-0832-0.

Bulow, J. and Rogoff, K.: 1989, Sovereign debt: Is to forgive to forget?, *American Economic Review* **79**(1), 43–50.

Hellwig, C. and Lorenzoni, G.: 2009, Bubbles and self-enforcing debt, *Econometrica* **77**(4), 1137–1164.

Hernández, A. D. and Santos, M. S.: 1996, Competitive equilibria for infinite-horizon economies with incomplete markets, *Journal of Economic Theory* **71**(4), 102–130.

LeRoy, S. F. and Werner, J.: 2001, *Principles of Financial Economics*, 1st edn, Cambridge University Press.

Levine, D. K. and Zame, W. R.: 1996, Debt constraints and equilibrium in infinite horizon economies with incomplete markets, *Journal of Mathematical Economics* **26**(1), 103–131.

Magill, M. and Quinzii, M.: 1994, Infinite horizon incomplete markets, *Econometrica* **62**(4), 853–880.

Magill, M. and Quinzii, M.: 1996, *Theory of Incomplete Markets*, 1st edn, MIT Press.

Martins-da-Rocha, V. F. and Vailakis, Y.: 2014, On the sovereign debt paradox. Electronic copy available at https://sites.google.com/site/filipeecon/research.

Pesendorfer, W.: 1992, Sovereign debt: Forgiving and forgetting reconsidered. Mimeo, Northwestern University.

Santos, M. and Woodford, M.: 1997, Rational asset pricing bubbles, *Econometrica* **65**(1), 19–57.

Wright, M. L. J.: 2011, The theory of sovereign debt and default. Forthcoming, Encyclopedia of Financial Globalization, edited by G. Caprio, published by Elsevier.

G. Bloise, Yeshiva University, New York; University of Rome III, Rome
E-mail address: bloise@yu.edu

H.M. Polemarchakis, University of Warwick, UK
E-mail address: h.polemarchakis@warwick.ac.uk

Y. Vailakis, University of Glasgow, UK
E-mail address: yiannis.vailakis@glasgow.ac.uk