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The social value of information in economies with mandatory savings *

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Abstract

We study the value of public information in a stochastic exchange economy where agents trade assets to reallocate risk and mandatory (retirement) savings imposes a lower bound on the market value of some agents' holdings of a financial asset. Since equilibrium prices depend on the agents' beliefs about the states of nature, the arrival of information shifts the agents' mandatory savings constraints. We show that the arrival of public information can generate an ex-ante Pareto improvement relative to an uninformative equilibrium even when ex-post improvements are not possible.

1 Introduction

Nicholas Barr and Peter Diamond [2] write about the "aging crisis", discussing how in recent decades reductions in mortality and fertility, and lower labor force participation by older men, have increased pension costs to unsustainable levels in many countries. Several countries have responded to this crisis by changing their pension systems, including the famous reform in Chile in 1981 which introduced mandatory fully funded individual savings accounts.

In economies with mandatory individual savings, the government may give consumers a choice about where to invest their savings. In the literature on the design of such systems, information is a key input into the consumer's decision making process. In this paper, we study the value of information in a stochastic pure exchange economy where all agents trade assets in financial markets to reallocate risk, and some of the agents face a mandatory savings constraint.

These systems involve complicated rules and poor financial literacy is widespread in many countries, so "more" information is normally viewed as desirable. However information can affect welfare through another channel, by affecting prices. Surprisingly, to the best of our knowledge, this channel has been largely ignored in the literature on pensions, even though the theoretical literature suggests that the welfare effects of public information might be ambiguous, depending on the asset structure. Cuevas, Bernhardt and Sanclemente [5], and Da, Larrain, Sialm and Tesada [6] show how the release of information has led to massive coordinated movements between the available social security portfolios in Chile,

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affecting the prices of the social security portfolios and the domestic stock market. Thus the release of new information appears to affect those who did not act upon the new information, as well as those outside the pension system. Therefore, it is important to understand the welfare effect of information-driven price changes when some agents face mandatory savings constraints.

We consider a two-period exchange economy with two agents, a single consumption good and uncertainty about the state of nature in period 1. There are two states of nature that realise in period 1 but no aggregate uncertainty. In period zero, agents trade two Arrow securities and a risk-free bond.[1] One of the agents faces a mandatory savings constraint modelled as a lower bound on the value of the holdings of the risk-free bond, and short-sale constraints on the other assets. We call her the constrained agent. The other agent is unconstrained.[2] We call her the unconstrained agent. We model an information structure as signals that lead agents to update their common prior belief over the states of nature via Bayes' rule.

We study the welfare effects caused by information that arrives before agents trade in financial markets. In particular, we look for an information structure that allows a benevolent planner to obtain a Pareto improvement relative to the equilibrium in which agents do not have access to information before trading, hereafter the *uninformative equilibrium*. Throughout our analysis we assume the Planner faces the same constraints as the agents in terms of the available assets, information, and savings constraints. We consider two welfare notions depending on whether one evaluates allocations ex-ante or ex-post, i.e. before or after observing the signal.

In the absence of the savings and short-sell constraints, the existing assets would allow the agents to generate any desirable future consumption vector. However, the savings constraints and short-sale constraints prevent the equilibrium from being fully Pareto efficient when they are binding. Since the savings constraint is modeled as a lower bound on the value of the holdings of the risk-free bond, it depends on the equilibrium (gross) interest rate. Different information structures lead to different posterior beliefs and thus result in different equilibrium prices and, possibly, a different interest rate. Therefore changes in information can shift the lower bound on the holdings of the risk-free bond. If the savings constraint is relaxed, the Planner can access allocations that are Pareto superior, but that were not feasible at the uninformative equilibrium interest rate.

When studying the existence of ex-post improvements, i.e. Pareto improvements under each signal, we show that an ex-post improvement exists if and only if the savings constraint is relaxed under *every* signal. We begin our analysis by asking if there exists an alternative allocation and a belief different to the common prior such that the alternative allocation satisfies the savings constraint at the initial equilibrium prices, and that Pareto dominates the uninformative equilibrium allocation when utilities are computed using the new belief. The answer is no. For every alternative belief, the allocations that Pareto dominate the uninformative equilibrium allocation are outside the constrained feasible set of the uninformative equilibrium. *All* the Pareto superior allocations violate the savings constraint at initial equilibrium prices. Thus ex-post constrained Pareto improvements exist if and only if the constraint set

[1] We consider mandatory savings in a redundant assets to mimic how these types of social security systems work in practice.

[2] We work with an unconstrained agent because that is what we observe in the Chilean case, where only formal workers are forced to save. If we were to assume that both agents have mandatory savings constraints, but the unconstrained agent were completely free to choose the Arrow-Debreu securities, then the bond would need to be in positive net supply. We expect all our results to hold with this modification.

of the Planner is enlarged for *every* signal.

To study ex-ante improvements, i.e. Pareto improvements in expected value, we define the Pareto frontier as the maximum utility the constrained agent can attain in the constrained feasible set as a function of her posterior belief and the utility of the unconstrained agent. We relate the existence of ex-ante improvements to the concavity of the Pareto frontier. We show that there exist ex-ante improvements if and only if the concavification of the Pareto frontier evaluated at the prior belief and the uninformative equilibrium utility level for the unconstrained agent lies above the uninformative equilibrium utility for constrained agent, i.e. the Pareto frontier evaluated at that point.

The Pareto frontier fails to be concave in beliefs and the utility of the unconstrained agent under general conditions. In the simple case where the savings constraint does not prevent full consumption smoothing across states, the objective function defining the Pareto frontier is independent of the belief. Therefore the posterior belief only affects the Pareto frontier through its effect on the equilibrium interest rate and, therefore, on the constrained feasible set. Thus if prices do not change with changes in information it is not possible to obtain ex-ante improvements. Consequently, changes in prices are a necessary condition for the existence of ex-ante improvements.

We show there exists a threshold on the posterior beliefs such that equilibria are first best if and only if the belief is weakly above that threshold. Since there is no aggregate uncertainty, the gross interest rate is equal to one in *every* first best equilibria. Fixing the utility of the unconstrained agent, this means that the constrained feasible set, and thus the Pareto frontier, is constant for beliefs above the threshold. For beliefs below the threshold, the savings constraint is binding in equilibrium and the equilibrium interest rate is strictly less than one, as equilibrium prices must induce the unconstrained agent to increase her consumption in period zero. If the common prior is below the threshold, then the constrained feasible set in the uninformative equilibrium contains the constrained feasible set for beliefs above the threshold. This implies that the Pareto frontier at the uninformative equilibrium attains a higher value relative to the constant value to the right of the threshold.

Following Kamenica and Gentzkow [14], we say a distribution of beliefs is Bayes' plausible if the expected posterior belief is equal to the prior. If the common prior is below the threshold, then for any Bayes' plausible distribution of beliefs with support equal to the common prior and a belief to the right of the threshold, the expected value of the Pareto frontier is greater than the Pareto frontier at the threshold. Consequently, the frontier fails to be concave in the posterior belief.

Finally, we show that there are information structures that allow the Planner to obtain ex-ante improvements, i.e. we show that the concavification of the Pareto frontier lies above the Pareto frontier when evaluated at the prior and the initial utility for the unconstrained agent. To show the existence of ex-ante improvements we fix the utility of the unconstrained agent at the uninformative equilibrium level for every posterior belief. We fix a support for the beliefs and we look for the probabilities of the beliefs that satisfy Bayes' plausibility. Then we ask what are the probabilities (of the beliefs) that keep the constrained agent indifferent with respect to the initial equilibrium. We show that if the prior is close to the aforementioned threshold, the Bayes' plausible probabilities differ from the probabilities that keep the constrained agent indifferent. In particular, the constrained agent is better off relative to the uninformative equilibrium, and as the unconstrained agent is indifferent by construction, we obtain an ex-ante improvement.

Our work is related to a large literature that analyzes the effect of public information in two-period competitive economies with homogeneous beliefs and complete markets to share risk.[3] In a seminal paper, Jack Hirshleifer [13] considers a situation where initially uninformed agents are revealed the true state of the world before trading and, therefore, no risk sharing trade that benefits all agents is possible. Initially uninformed traders cannot all be made better off even if the new information that is revealed before trading is only partially revealing. Marshall [16] showed that the contract curve is independent of the posteriors when beliefs are homogeneous (and markets are complete), therefore if the equilibrium without information lies in the contract curve, then it also belongs to the contract curve for every vector of posterior beliefs. Thus there are no ex-post improvements. In the special case where initial endowments are an equilibrium in the economy without information, he showed that changes in information cannot reduce the ex-ante utility of any agent, as they can always stand pat and not act upon the new information, nor increase agents' ex-ante utility. When endowments are not an equilibrium without information, he argued that public information cannot obtain Pareto improvements. This result, closely related to the Sunspot theorem by Cass and Shell [4], was formally proved by Hakansson, Kunkel and Ohlson [11] who gave a set of sufficient and necessary conditions for public information to have social value in pure exchange economies under uncertainty. Full Pareto efficiency of the initial equilibrium is a sufficient condition for public information to be of no social value.

When prior beliefs are heterogeneous, Marshall [16] gave an early example of public information having social value. Ng [19] showed that when initial endowments are an equilibrium in the economy without information, then the arrival of information makes some individual better off and no individual worse off. This follows directly from the fact that agents can always decide not to trade. If beliefs are heterogeneous, new information will make individuals willing to trade, hence they cannot be worse off. He also showed that when initial endowments are not an equilibrium in the economy without information, if prices are the same in the economies with and without information, then the equilibrium with information represents an ex-ante constrained Pareto improvement when beliefs coincide after the release of information. The results follow from a standard revealed preference argument.[4]

Recently, Maurer and Tran [18] study the value of public information in an economy with multiple consumption and trading dates. They show that when beliefs are heterogeneous, the Hirshleifer effect is reversed if information arrives before the first round of trading. This result holds when agents anticipate small benefits from risk sharing and large benefits from intertemporal consumption smoothing.

Gottardi and Rahi [10] consider the case of a pure exchange economy with one good, two periods and incomplete markets. They show that ex-post Pareto improvements can be attained for any change in information, by adjusting agents' asset holdings to the new information.[5] However when comparing different equilibria, they show that the overall effect on welfare can go in any direction. The difference lies in an additional welfare effect that arises due to the adjustment in equilibrium prices. They conclude that competitive markets typically do not deal with changes in information in a way that is welfare-

[3] With the exception of Maurer and Tran [18], this literature does not consider markets that are open before signals arrive.

[4] In his analysis, Ng [19] uses the value of optimization to measure changes in welfare. That is, he evaluates different allocations at the new beliefs induced by the new information, but he fails to integrate across all the signals.

[5] Since the condition for constrained Pareto optimally of equilibria can be characterized in terms of the equality of the marginal rate of substitution between assets and present consumption for all agents (Diamond [9]), it typically depends on the beliefs of the agents. If one keeps the allocation fixed and changes the beliefs, nothing ensures that the condition for constrained Pareto optimality holds for the new beliefs.

improving even though it is feasible to do so.[6] In contrast, in our model ex-post improvements are only possible under special conditions. There in no ex-post constrained Pareto improvements with respect to the uninformative equilibrium if and only if the uninformative equilibrium interest rate is greater than the equilibrium interest rate of *all* the signals of the informative information structure.

From a methodological point of view, this paper is also related to the literature on Bayesian persuasion. Kamenica and Gentzkow [14] study a symmetric information model where two players, a sender and a receiver, interact. The sender can send signals to the receiver, who upon observing the signal takes an action that affects the payoff of both agents. The authors show that the sender can send signals that result in the receiver taking an action that gives the sender higher expected utility, relative to the action taken based on the prior, if and only if the concavification of the sender's expected utility lies above the expected utility function when evaluated at the action the receiver takes under his prior. Thus we can relate our analysis for a competitive economy to the literature on Bayesian persuasion. We can think of the Planner as the sender, whose payoff function is given by the Pareto frontier. The agents play the role of the receiver. They observe the signal sent by the Planner and take an action, their excess demand, to maximize their payoff, taking prices as given. These actions affect the Planner's payoff as they determine equilibrium prices and the constrained feasible set.

In the next section we formally introduce our model. In section three we define an equilibrium in our economy. In section four we relate changes in information and welfare, and in the next two sections we study ex-post and ex-ante improvements. We finish with conclusions in section seven. All proofs are relegated to the appendix.

2 The model

There are two periods, 0 and 1, and a single consumption good. At period 1, $s = 1, 2$ states of the world realize.

The economy is populated by two agents indexed by h. A consumption plan for agent h is given by $x^h = \left(x_0^h, x_1^h, x_2^h\right)$. Agent h has endowments $w^h = (w_0^h, w_1^h, w_2^h)$ where $w_0^h > 0$ is agent h's endowment in period 0 and $w_s^h > 0$ is agent h's endowment in state $s = 1, 2$. There is no aggregate uncertainty, $\sum_h w_s^h = w$ for $s = 0, 1, 2$.

Agents' common belief about the probability of state one is denoted by $\pi \in [0, 1]$. Both agents have utility functions $V(x^h, \pi) : \mathbb{R}_+^3 \times [0, 1] \mapsto \mathbb{R}$ that can be represented as a time-separable expected utility function with Bernoulli function $v : \mathbb{R}_+ \mapsto \mathbb{R}$. That is, for every consumption plan x^h,

$$V(x^h, \pi) = v(x_0^h) + \pi v(x_1^h) + (1 - \pi) v(x_2^h).$$

We assume that v is twice continuously differentiable, strictly increasing, strictly concave, and that $\lim_{x_s^h \to 0} v'(x_s^h) = \infty$, where v' denotes the first derivative of v with respect to its argument.

[6]They relate their analysis to the *Hishleifer effect* and the *Blackwell effect*. The former effect follows from the example by Hirshleifer [13] and it is related to how information affects welfare through the change in equilibrium prices. The latter effect makes reference to Blackwell [3], who showed that agents can hedge risk more efficiently by adjusting their portfolios to the new information.

2.1 Financial markets

There are three securities, indexed by l, that are traded in a competitive market at period 0. The payoffs of these assets are in units of the consumption good of period 1. Assets 1 and 2 are Arrow-Debreu securities while asset 3 is a risk free bond paying one unit of the consumption good in all states. Let

$$\Phi = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

be the payoff matrix, where the element (s, l) of Φ is the payoff of asset l in state s.

Let $q = (q_0, \ldots, q_3) \in \mathbb{R}^4_{++}$ be the vector of period 0 prices, where q_0 is the price of the consumption good and q_l is the price of asset $l = 1, 2, 3$, and let $z^h = (z^h_1, z^h_2, z^h_3) \in \mathbb{R}^3$ be agent h's vector of asset holdings. We say a consumption plan x^h can by financed at (q, w^h) by a portfolio z^h if:

$$q_0 x^h_0 + \sum_s q_s z^h_s = q_0 w^h_0,$$

$$x^h_s = w^h_s + z^h_s + z^h_3, \forall s = 1, 2.$$

One of the two agents living in this economy faces a mandatory savings constraint. Let's call her agent c (for *constrained*). Agent c's mandatory savings constraint is such that the value of her holdings of asset three have to satisfy:

$$q_3 z^c_3 \geq q_0 \theta_3, \tag{1}$$

where $\theta_3 > 0$ is an exogenous parameter. The savings constraint on asset three implies that she faces a lower bound on the value of her holdings of the risk free bond, and this lower bound depends on equilibrium prices.

In addition to the mandatory savings constraint, agent c faces short-sale constraints on assets one and two. That is, her holdings of assets one and two have to satisfy:

$$z^c_l \geq \theta_l, \tag{2}$$

with $\theta_l \leq 0$ an exogenous constant for $l = 1, 2$. Short-sale constraints are needed in order to have equilibria where the mandatory savings constraint is binding. If there were no short-sale constraints, then agent c could undo the savings constraint (1) as she could generate any date 1 consumption plan in $\mathbb{R}^2_+$ with the existing assets. In that case any equilibrium would be fully Pareto optimal.[7] The other agent, agent u (for *unconstrained*), faces no constraints besides the usual budget constraint.

2.2 Information

Prior to trading, agents observe a public signal possibly correlated with the state of the world s. This signal does not directly affects utility functions, endowments, or asset payoffs. We fix a finite set of signal realizations $\mathcal{Y} = (y_1, y_2, y_3)$.[8] Once signal y_k is observed agents update their common prior belief about

[7] An alternative approach, used in a previous version of this paper, is to assume that the government can target agents' savings directly, i.e. we could assume the government imposes a lower bound on total savings. All the result in the current paper hold if we use the alternative approach.

[8] The number of signal realization is set equal to the number of agents plus the number of states of the world minus one. We need at least three signals to use Carathéodory's theorem in the proof of Corollary 4.

the state of nature, π^0, to the posterior belief $\pi(y_k)$ via Bayes' rule. Let $\text{pr}(y_k|s)$ denote the conditional probability of signal y_k given state s. The 2×3 matrix of conditional probabilities:

$$Y = \begin{pmatrix} \text{pr}(y_1|1) & \text{pr}(y_2|1) & \text{pr}(y_3|1) \\ \text{pr}(y_1|2) & \text{pr}(y_2|2) & \text{pr}(y_3|2) \end{pmatrix},$$

is called an information structure. Let $\text{pr}(y_k) = \sum_s \text{pr}(y_k|s)\pi_s^0$. Then beliefs are given by:

$$\pi_s(y_k) = \frac{\text{pr}(y_k|s)\pi_s^0}{\text{pr}(y_k)}.$$

If Y is such that $\text{pr}(y_k|1) = \text{pr}(y_k|2)$ for all k then the beliefs coincide with the prior for all signals. We call such an information structure *uninformative*. We call the information structure *informative* otherwise.

3 Equilibrium

Both agents take the price vector q as given. Agent c's budget set, denoted $\mathbb{B}^c(q, w^c)$, is the set of consumption plans that can be financed at (q, w^c) by a portfolio z^c that satisfies constraints (1) - (2). That is,

$$\mathbb{B}^c(q, w^c) \equiv \Bigg\{ x^c \in \mathbb{R}^3_+ \;\Bigg|\; \exists z^c \in \mathbb{R}^3 \text{ s.t. } q_0 x_0^c + \sum_s q_s z_s^c = q_0 w_0^c,\ x_s^c = w_s^c + z_s^c + z_3^c \ \forall s,$$
$$z_s^c \geq \theta_s \ \forall s,\ q_3 z_3^c \geq q_0 \theta_3 \Bigg\}.$$

Since agent u neither faces a savings nor short-sale constraints, her budget set, denoted $\mathbb{B}^u(q, w^u)$, is simply defined as the set of consumption plans that can be financed at (q, w^u) by a portfolio z^u. That is,

$$\mathbb{B}^u(q, w^u) \equiv \Bigg\{ x^u \in \mathbb{R}^3_+ \;\Bigg|\; \exists z^u \in \mathbb{R}^3 \text{ s.t. } q_0 x_0^u + \sum_s q_s z_s^u = q_0 w_0^u,\ x_s^u = w_s^u + z_s^u + z_3^u \ \forall s \Bigg\}.$$

A financial market equilibrium is defined as follows:

Definition 1. Given π, a financial market equilibrium is a collection of prices $\overline{q} \in \mathbb{R}^4_{++}$, consumption plans $\overline{x} = (\overline{x}^u, \overline{x}^c) \in \mathbb{R}^6_+$ and portfolios $\overline{z} = (\overline{z}^u, \overline{z}^c) \in \mathbb{R}^6$, where $\overline{z}^h$ finances the consumption plans $\overline{x}^h$ at (q, w^h), such that:

1. $\overline{x}^h \in \text{argmax } \{V^h(x^h, \pi)\, |x^h \in \mathbb{B}^h(\overline{q}, w^h)\} \ \forall h \in \{c, u\}$,
2. $\sum_h \overline{z}^h = 0$.

Hereafter we refer to any consumption plan satisfying Definition 1 as an *equilibrium allocation*.

To prove the existence of a financial market equilibrium for all beliefs $\pi \in [0, 1]$, we need the following assumptions on $c's$ endowment vector:

$$\begin{aligned} w_0^c &\geq \theta_3, \\ w_s^c &\geq -\theta_s \text{ for } s = 1, 2. \end{aligned} \tag{A0}$$

Proposition 1. *If (A0) is satisfied, then a financial market equilibrium exist for all $\pi \in [0, 1]$.*

Proposition 1 is proved in several steps. First we define the concept of a non-arbitrage equilibrium following Magill and Quinzii [15]. Then we show the equivalence between both types of equilibria in the absence of arbitrage. Working with a non-arbitrage equilibrium is useful because it allows us to use the standard existence proof in models with contingent consumption.

Given a vector of prices q, we say there is no arbitrage if there does not exist $z \in \mathbb{R}^3$ such that:

$$\begin{bmatrix} -q \\ \Phi \end{bmatrix} z > 0.$$

It is not difficult to show that with our payoff matrix there is no arbitrage if and only if $q_1 + q_2 = q_3$.[9] It is direct to see that u's optimization problem has no solution if there are arbitrage opportunities. Thus every equilibrium satisfies absence of arbitrage. Since the gross interest rate is given by $R = (q_1 + q_2)^{-1}$, then in equilibrium $R = q_3^{-1}$.

Assumption (A0) is needed to ensure that c's optimization problem has a solution for every $\pi \in [0, 1]$. The savings and short-sale constraints imply that there is a lower bound on x_s^c for $s = 1, 2$ given by:[10]

$$x_s^c \geq w_s^c + \theta_s + \theta_3 R \text{ for } s = 1, 2. \tag{3}$$

The second condition in assumption (A0) implies that the lower bound on x_s^c is positive when prices are strictly positive. When both lower bounds are binding, c's period 0 consumption is given by: $x_0^c = w_0^c - \theta_3 - \sum p_s \theta_s$. If prices are positive, then the first condition in (A0) ensures $x_0^c \geq 0$.

From here onwards we normalize $q_0 = 1$, i.e. the equilibrium price of the consumption good at period 0 is set equal to one. Before discussing the effect of changes in information on welfare, we show that, given a belief, if the Planner is constrained to satisfy c's mandatory savings and short-sale constrains he cannot obtain a Pareto improvement. We follow the literature on constrained Pareto optimality and assume that the Planner can freely allocate period 0 consumption, but period 1 consumption can only be allocated using the existing assets. The allocation assigned by the Planner to agent c is required to satisfy the short-sale constraints and the savings constraint, that is $q_3 z_3^c \geq \theta_3$ where q_3 is the period zero equilibrium price of the bond *before* the Planner redistribute assets and period 0 consumption.

Definition 2. The consumption allocation $x \in \mathbb{R}_+^6$ is constrained feasible if there exists an asset allocation $z \in \mathbb{R}^6$ such that:

1. $x_s^h = w_s^h + z_s^h + z_3^h$, for $s = 1, 2$. $\forall h$,
2. $\sum_h z^h = 0$,
3. $z_s^c \geq \theta_s$, for $s = 1, 2$,
4. $z_3^c \geq \theta_3 R$.

We define the constrained feasible set under belief π, CFS(R), as the set of all constrained feasible allocations. Formally:

[9] If $q_1 + q_2 > q_3$, use the vector $z = (-1, -1, 1)$. If $q_1 + q_2 < q_3$, use the vector $z = (1, 1, -1)$.

[10] To see this just replace the lower bounds on z_l^c for all l in the expressions for x_s^c for $s = 1, 2$.

Definition 3. The constrained feasible set, CFS(R), is the set of all constrained feasible allocations:

$$\text{CFS}(R) = \Big\{ x \in \mathbb{R}^6_+ \;\Big|\; \exists z \in \mathbb{R}^6 \text{ s.t. } x^h_s = w^h_s + z^h_s + z^h_3 \; \forall (h,s),$$
$$z^c_s \geq \theta_s \; \forall s, \; z^c_3 \geq \theta_3 R \text{ and } \sum_h z^h = 0 \Big\}.$$

It is straightforward to redefine the constrained feasible set independently of the portfolio of agent c by replacing the lower bounds on the components of this vector straight into the expressions for x^c_s:

Definition 4. Given a belief π, the constrained feasible set, CFS′(R), is the set of all constrained feasible allocations:

$$\text{CFS}'(R) = \left\{ x \in \mathbb{R}^6_+ \mid x^c_s \geq w^c_s + \theta_s + \theta_3 R \; \forall s, \text{ and } x^u = w - x^c \right\}.$$

It is trivial to show that CFS(R) ⊂ CFS′(R). It is also true that CFS′(R) ⊂ CFS(R).[11][12]

Proposition 2. *There is no constrained feasible allocation that Pareto dominates the financial market equilibrium allocation.*

To prove Proposition 2 we follow the standard proof of constrained Pareto optimality in economies with two period and a single consumption good, for our definition for the constrained feasible set.

4 Changes in information and welfare

In this section we define the welfare notions we use in the rest of the paper, and we relate changes in the information agents have before trading to the agents' welfare. Proposition 2 tells us that changing the beliefs, and hence the information structure, is a necessary condition for obtaining Pareto improvements if the Planner must satisfy c's constraints at equilibrium prices.

The mandatory savings constraint for agent c depends on equilibrium prices and, therefore, on the agents' beliefs. By changing the information structure, the Planner can change the posterior belief of the agents and, as R is a function of π, shift the mandatory savings constraints to reach allocations that were not feasible under the original information structure. Thus it may be the case that changing the information structure allows the Planner to reach allocations that are Pareto superior with respect to the starting information structure, but that weren't feasible before the change in information. For example if R is increasing in π, by inducing a belief below the prior the Planner can enlarge the constrained feasible set. For this reason from now on we make explicit the dependence of the constrained feasible set on the belief, i.e. we write CFS($R(\pi)$).

We wish to study the welfare consequences of changes in information. Our reference point will always be what we call the *uninformative equilibrium*. Under an *uninformative* information structure, by definition $\pi(y_k) = \pi^0$ for $k = 1, 2, 3$. We define an uninformative equilibrium, as an equilibrium of the economy where agents make decisions on the basis of the prior π^0.

[11] To prove that CFS′(R) ⊂ CFS(R), fix $z^c_3 = \theta_3 R$ and notice that with the existing asset structure, and ignoring c's constraints, the agents can generate any period 1 consumption in $\mathbb{R}^2_+$. Hence there exist a z^c_s such that $w^c_s + z^c_s + z^c_3 = x^c_s \geq w^c_s + \theta_s + \theta_3 R$ for $s = 1, 2$. Our choice of z^c_3 implies: $z^c_s \geq \theta_s$ for $s = 1, 2$. Finally $x^u = w - x^c$ implies $\sum_h z^h = 0$.

[12] Notice that if we were to write Definition 2 in terms of contingent consumption instead, we would have to rewrite the conditions for feasibility, and replace conditions 3. and 4. by equation (3).

By Proposition 1 there exist at least one vector of prices that clears the market at π^0. If the equilibrium is unique, it is clear that this unique equilibrium is the uninformative equilibrium. If there are multiple equilibria, we pick one of them, and set the uninformative equilibrium equal to it for every signal realisation. Therefore, by definition, the uninformative equilibrium is signal invariant.

Definition 5. Consider an information structure Y. Let $x(y_k)$ be an (equilibrium) allocation when signal y_k is observed in the economy with information structure Y. Then, $x(Y) = (x(y_1), x(y_2), x(y_3))$ is an (equilibrium) allocation under information structure Y. If Y is the uninformative information structure, then we define an uninformative equilibrium allocation as the equilibrium allocation under information structure Y, say $x(Y)$, that satisfies $x(y_1) = x(y_2) = x(y_3)$.

Starting from the uninformative equilibrium, we will look for an *informative* information structure that allows the Planner to obtain welfare improvements in a sense that will be made precise below. We consider two welfare notions depending on whether one evaluate allocations ex-ante or ex-post, i.e. before or after observing the signal. Formally:

Definition 6. Let $\overline{Y}$ and $\widehat{Y}$ be an uninformative and an informative information structure respectively. Let $\pi(y_k)$ and $\text{pr}(y_k)$ for $k = 1, 2, 3$, be the beliefs and the probabilities of the beliefs under $\widehat{Y}$, respectively. We say $x(\overline{Y})$ is ex-ante constrained Pareto efficient under $\widehat{Y}$ if there is no allocation under information structure $\widehat{Y}$, call it $x(\widehat{Y}) = (\widehat{x}(y_1), \widehat{x}(y_2), \widehat{x}(y_3))$, such that $\widehat{x}(y_k) \in \text{CFS}(R(\pi(y_k)))$ for all k and:

$$\sum_k \text{pr}(y_k) V(\widehat{x}^h(y_k), \pi(y_k)) \geq \sum_k \text{pr}(y_k) V(\overline{x}^h(y_k), \pi(y_k)),$$

for all $h \in H$ and with strict inequality for some h.

Definition 7. Let $\overline{Y}$ and $\widehat{Y}$ be an uninformative and an informative information structure respectively. Let $\pi(y_k)$ and $\text{pr}(y_k)$ for $k = 1, 2, 3$, be the beliefs and the probabilities of the beliefs under $\widehat{Y}$, respectively. We say $x(\overline{Y})$ is ex-post constrained Pareto efficient under information structure $\widehat{Y}$ if there is no allocation under information structure $\widehat{Y}$, call it $x(\widehat{Y}) = (\widehat{x}(y_1), \widehat{x}(y_2), \widehat{x}(y_3))$, such that $\widehat{x}(y_k) \in \text{CFS}(R(\pi(y_k)))$ for all k and:

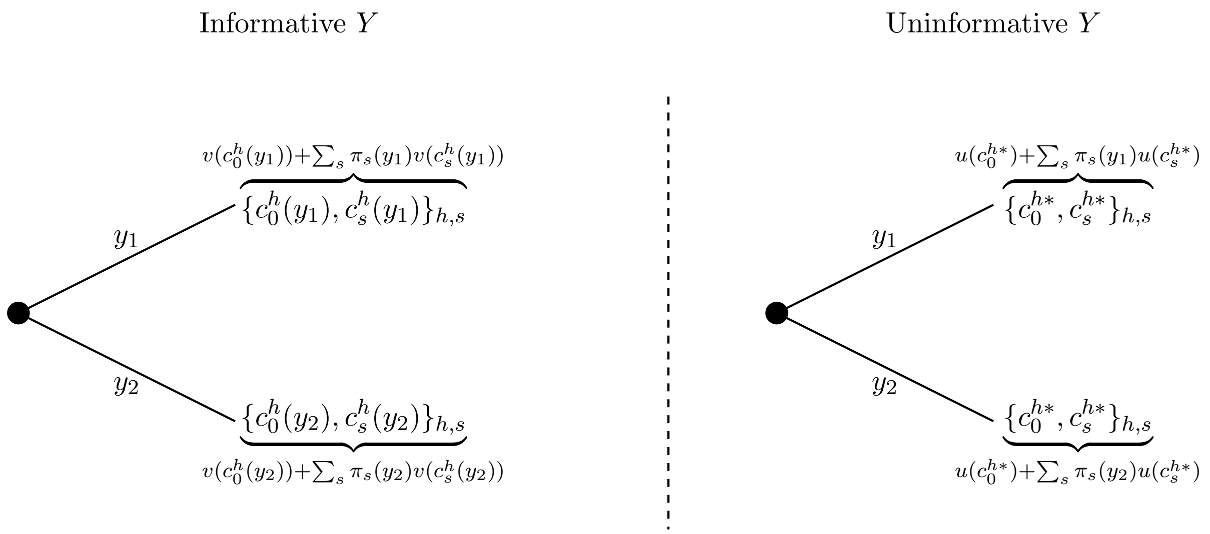
$$V(\widehat{x}^h(y_k), \pi(y_k)) \geq V(\overline{x}^h(y_k), \pi(y_k)),$$

for all $h \in H$ and with strict inequality for some h, for all k.

From Definition 6 we see that the concept of an ex-ante constrained Pareto improvement is related to an (constrained) improvement in expected value. Definition 7 provides a stronger concept of improvement: a (constrained) Pareto improvement for every signal at the new beliefs. Clearly an ex-post improvement is a sufficient condition for an ex-ante improvement.

To clarify the difference between these two concepts, consider the right diagram in Figure 1. There we show the uninformative equilibrium under each signal, denoted by c^*, for a two dimensional information structure. Remember that this equilibrium is signal invariant by definition. In the left diagram we show some constrained feasible allocation under every signal of an informative information structure. Notice that $c(y_1)$ and $c(y_2)$ may differ. When looking for ex-ante improvements, we take each utility level under the informative information structure, and compute the expected value for every agent using the probability of the signals as weights. Then we compare this expected value with the utility that the agents obtain in the uninformative equilibrium. We look if they are better off before receiving information. When looking for ex-post improvements, we compare the $c(y_1)$ allocation with c^*, and $c(y_2)$ with c^* separately. If there is a constrained Pareto improvement for both cases, then we have an ex-post constrained Pareto improvement. We look if they are better off after receiving each signal.

Figure 1: Ex-ante and Ex-post constrained Pareto improvements.



To prove the existence of ex-ante improvements one could work with the ex-ante utility function used in Definition 6. Noticing that the existence of ex-ante improvement is closely related to the shape of the boundary of the utility possibility set, the Pareto frontier, helps us to simplify the problem. Formally, we define the Pareto frontier as a function $F : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$ given by:

$$\begin{aligned} F(\pi, V) = \{ \underset{x^c}{\text{Max}} \quad & v(x_0^c) + \sum \pi_s v(x_s^c) \\ \text{s. t.} \quad & x_s^c \geq w_s^c + \theta_s + \theta_3 R(\pi) \text{ for } s = 1, 2, \\ & v(w - x_0^c) + \sum \pi_s v(w - x_s^c) \geq V \; \} . \end{aligned} \tag{4}$$

Since we assume v is strictly increasing, the last constraint in the problem defining (4) will always bind at any solution. Furthermore, Bayes' rule implies that beliefs have to satisfy: $\sum_k \text{pr}(y_k)\pi(y_k) = \pi^0$ for any information structure. This condition is what Kamenica and Gentzkow [14] call *Bayes-plausibility* of beliefs. Since by definition $V(x^c, \pi) = v(x_0^c) + \sum \pi_s v(x_s^c)$, we can relate the existence of an ex-ante constrained Pareto improvement to the properties of function F.

Proposition 3. *Let π^0 and V_0^h be the common prior and the equilibrium utility level of agent h in the uninformative equilibrium, respectively. The uninformative equilibrium allocation is not ex-ante constrained Pareto efficient if and only if there exist vectors $(\tau_1, \tau_2, \tau_3) \in \Delta^2$, $(\pi_1, \pi_2, \pi_3) \in [0,1]^3$ and $(V_1, V_2, V_3) \in \mathbb{R}^3$ such that:*

1. $\sum_k \tau_k \pi_k = \pi^0$,
2. $\sum_k \tau_k V_k = V_0^u$,
3. $\sum_k \tau_k F(\pi_k, V_k) > F(\pi^0, V_0^u)$.

The proof of Proposition 3 is direct from the definitions of an ex-ante improvement and the Pareto frontier, using constrained Pareto optimality of the uninformative equilibrium, and the fact that the uninformative equilibrium is signal invariant. Proposition 3 shows that we can work with the dsistribution of posterior beliefs and the Pareto frontier to study the existence of an ex-ante improvement. In Corollary 4 we relate this to the concavification of F.

Definition 8. Let $g : [0,1]\times\mathbb{R} \mapsto \mathbb{R}$. The concavification of g is given by: cav $g(x) \equiv \sup\{y|(x,y) \in co(g)\}$, where $co(g)$ denotes the convex hull of the graph of g.

From Definition 8 it is direct to notice that cav g is concave, and everywhere weakly greater than g. While there are alternative definitions for the concavification of a function, the one we use is useful when applying Carathéodory's theorem when proving the "if" part in Corollary 4.

Corollary 4. *There exist an ex-ante constrained Pareto improvement over the uninformative equilibrium if and only if the concavification of F at (π^0, V_0^u) is greater than $F(\pi^0, V_0^u)$. There cannot be an ex-ante constrained Pareto improvement if F is concave.*

In Kamenica and Gentzkow [14] the authors study a symmetric information model where a sender can choose a signal she reveals to a receiver, who takes a (contractable) action that affects the payoff of both agents. They ask whether there exist a signal that leads the receiver to take an action that benefits the sender, relative to the equilibrium where actions are taken based on the prior beliefs. They show that there exist such a signal if and only if the concavification of the expected utility of the Sender evaluated at the action based on the prior beliefs lies above the expected utility of the Sender evaluated at the same point.

Similar to their analysis we can relate the existence of an information structure that makes the Planner better off with the concavification of the Pareto frontier. We can think of the Planner as the sender, who can choose the information structure. Agents c and u play the role of the receiver. They observe the signal and their actions (excess demands) determine the equilibrium interest rate, which affect the Planner's payoff through the constrained feasible set.

A difference between our model and those considered in Kamenica and Gentzkow [14], is that in their models Bayes plausibility is the only constraint that the sender has to satisfy. Our sender also has to make sure that the levels of utility he assigns to agent u in every signal are such that she is indifferent with respect to the uninformative equilibrium in expected value. In a sense this makes the problem of the sender more flexible as his expected payoff function can fail to be concave in beliefs, utility levels, or both. However it makes the analysis more complicated.

The *concavification* of a function has also been used to analyze the value of knowledge in a game theoretical context. Aumann and Maschler [1] use it to analyze whether a player benefits from using his knowledge of chance's choice in a infinite 2-person game. To our knowledge there are no papers taking this approach when studying the value of public information in a market economy. The "usual" approach is to work with information structure defined as we did in section 2 (or by joint probability distributions as in Gottardi and Rahi [10]). From now on we think of information structures as a vector of beliefs and a vector of probabilities of signals such that *Bayes-plausibility* is satisfied.

5 Ex-post improvements

In this section we study the existence of ex-post constrained Pareto improvements. In a similar model, but without c's savings and short-sale constraints, Hakansson, Kunkel and Ohlson [11] showed that if prior beliefs and information structures are homogeneous, utility functions are time-additive, and the uninformative equilibrium is fully Pareto efficient, then there cannot be an ex-ante improvement over the uninformative equilibrium. Thus no ex-post improvements exist either.

Consider an uninformative equilibrium where c's savings constraint is not binding. Since with the existent assets agent h can generate any consumption vector in $\mathbb{R}^2_+$, this implies that such an equilibrium belongs to the Pareto set and hence is fully Pareto efficient. Therefore, a necessary condition for the existence of an ex-post improvement is for c's savings constraint to be binding in the uninformative equilibrium. To ensure this is the case, we need to introduce some assumptions in the parameters of the model.

First, endowments in state one and state two have to differ, otherwise there is no uncertainty. We assume that c is relatively rich in state one:

$$w_1^c > w_2^c. \tag{A1}$$

Second, a priori we do not know which of the two lower bounds on c's consumption in period one is higher, but to know their relative size simplifies the analysis. We assume that the lower bound on x_1^c is the biggest of the two:

$$w_1^c + \theta_1 \geq w_2^c + \theta_2. \tag{A2}$$

Define the beliefs $\underline{\pi}_1$, $\underline{\pi}_2$ and $\underline{\pi}$ as:

$$\begin{aligned} \underline{\pi}_1 &\equiv \frac{2\theta_3 + 2(\theta_1 + w_1^c) - w_0^c - w_2^c}{w_1^c - w_2^c}, \\ \underline{\pi}_2 &\equiv \frac{2\theta_3 + 2(\theta_2 + w_2^c) - w_0^c - w_2^c}{w_1^c - w_2^c}, \\ \underline{\pi} &\equiv \max\{\underline{\pi}_1, \underline{\pi}_2\} = \underline{\pi}_1. \end{aligned}$$

Notice that the denominator of $\underline{\pi}_1$ and $\underline{\pi}_2$ is positive by (A1), and (A2) implies that the numerator of $\underline{\pi}_1$ is weakly greater than that of $\underline{\pi}_2$.

Proposition 5. *Assume (A1) and (A2) hold. Equilibria are Pareto efficient if and only if $\pi \geq \underline{\pi}$. If an equilibrium is not Pareto efficient, then the lower bound on x_1^c is binding at that equilibrium, i.e. $x_1^c = w_1^c + \theta_1 + \theta_3 R(\pi)$.*

Proposition 5 show that $\underline{\pi}$ is a threshold such that if the prior is below $\underline{\pi}$, then the uninformative equilibrium is not fully Pareto efficient. For $\underline{\pi}$ to be in the interval $(0, 1)$ we have to assume:

$$(w_0^c - w_1^c) > 2(\theta_1 + \theta_3) > (w_0^c - w_1^c) - (w_1^c - w_2^c). \tag{A3}$$

Assumptions (A1) and (A2) are without loss of generality, we just specify the relative magnitudes to simplify our analysis.[13] Assumption (A3) is important. This condition ensures that we can partition $[0, 1]$ in two non-empty sets such that equilibria are first best if and only if agents' belief is in one of the

[13] If we write (A1) with the reverse inequality, then the condition in Proposition 5 would also hold with the reverse inequality. If (A2) were to hold with the reverse inequality then $\underline{\pi} = \underline{\pi}_2$, but we would have to adjust assumption (A3) to make sure that $\underline{\pi}_2 \in (0, 1)$.

partitions, and if agents' belief is in the other partition at least one of the lower bounds on c's period one consumption is binding. We exploit this difference in the section on ex-ante improvements.[14]

From now on, assume $\pi^0 \leq \underline{\pi}$. We show at the end of section 3 that equilibria are constrained Pareto efficient. The following proposition shows that if the constrained feasible set does not vary with changes in the information structure, the Planner cannot obtain an ex-post constrained Pareto improvement, that is even if he uses a belief different from the common prior to compute expected utilities.

Proposition 6. *Let x be the equilibrium allocation under π^0. There is no $\widehat{x} \in CFS(R(\pi^0))$ and $\pi^1 \in [0,1]$ different from π^0, such that $V(\widehat{x}^h, \pi^1) \geq V(x^h, \pi^1)$ for all h and with strict inequality for some h.*

By constrained Pareto optimality, it follows that the allocations that make both agents better off at the prior are not constrained feasible. Proposition 6 shows that this remains true even if we use a different belief to compute the utility that both agents get from consuming their allocation.

To understand the idea behind Proposition 6 see Figure 2. For simplicity, assume there is no consumption in period 0, and that c faces some lower bound on period 1 consumption, depicted by the dashed straight lines in the figure. The shaded area represents the constrained feasible set. Assume both constraints are binding at equilibrium and that this equilibrium is not in the Pareto set, which is depicted by the diagonal of the box. The solid indifference curves represent this equilibrium. We need to understand how the indifference curves that pass trough the initial equilibrium allocation change when we increase the belief of state one. By definition the new indifference curves, depicted by the dashed lines, pass through the initial equilibrium. To keep u indifferent we need to increase consumption in one state and reduce consumption in the other state, but this implies reducing one of c's consumptions. By doing that we are choosing an allocation outside of the constrained feasible set.

If only one constraint is binding at equilibrium,[15] then the argument we just gave is not enough. If only the constraint on x_1^c is binding, then $x_1^c > x_2^c$ and $x_2^u > x_1^u$.[16] Thus we could reduce x_1^u and increase x_2^u to make u indifferent without leaving the constrained feasible set. But as $\text{MRS}_{1,2}^c < \text{MRS}_{1,2}^u$ this change cannot make c better off, where $MRS_{1,2}^h = \frac{\partial v(x_1^h)}{\partial x_1^h} / \frac{\partial v(x_2^h)}{x_2^h}$.

Proposition 6 makes clear that the only way to obtain an ex-post improvement is to find beliefs such that the equilibrium interest rate under each signal is below the interest rate of the uninformative equilibrium. In that case the constrained feasible set in the uninformative equilibrium is contained in the constrained feasible set of each signal of the *informative* information structure. The next theorem is a direct consequence of Proposition 6.

Theorem 7. *Let π^k for $k = 1, 2, 3$ be the beliefs under some informative information structure. The uninformative equilibrium is ex-post constrained Pareto efficient if and only if $R(\pi^k) \geq R(\pi^0)$ for some k.*

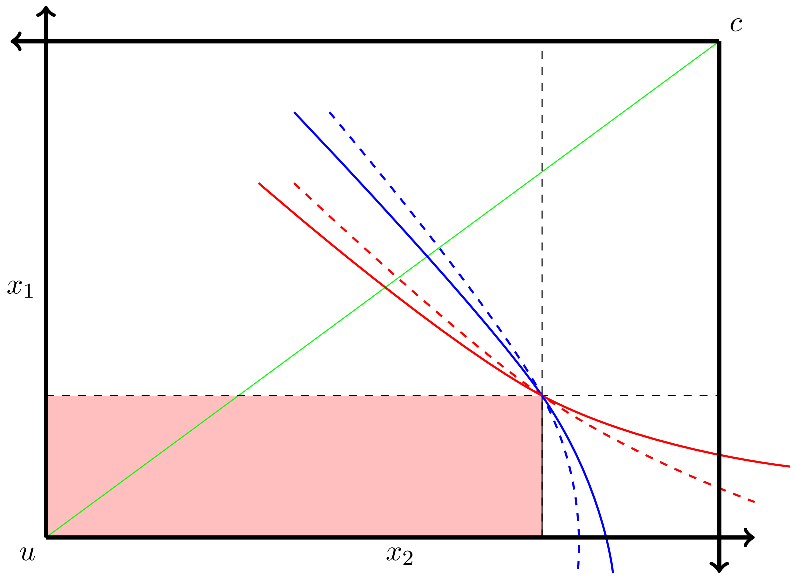
Theorem 7 shows that ex-post improvements are possible if and only if the constrained feasible set under π^0 is contained in the constrained feasible set of the beliefs under *all* signals of the *informative*

[14] Notice that if the conditions in (A0) are satisfied with equality, then for (A3) to hold we need $\theta_2 > -\theta_3 > \theta_1$. Assuming this last condition is true and fixing the endowments of the agents, then (A0) and (A3) are still satisfied if we reduce θ_3 and increase θ_1 in the same magnitude and any θ_2 satisfying (A0), i.e. if we work with $\theta_3' = \theta_3 - \Delta$, and $\theta_1' = \theta_1 + \Delta$, for a small and positive Δ.

[15] By Proposition 5 there are no equilibria that are not fully Pareto efficient and the lower bound on x_1^c is not binding.

[16] In this case the lower bound on x_2^c is somewhere to the right of the vertical dashed line depicted in the figure.

Figure 2: Indifference curves rotate at the equilibrium allocation when the belief of state one is increased.



information structure. This is the case when $R(\pi^0) > R(\pi(y_k))$ for all k. When considering marginal changes in the belief, then an ex-post improvement exists if and only if R attains a strict local maximum at π^0. This differs with Gottardi and Rahi [10] who show that when markets are incomplete ex-post improvements are always feasible, and in no way this depends on how prices react to the new information.

The next corollary is implied directly by Theorem 7:

Corollary 8. *If R is monotonic, then the uninformative equilibrium is ex-post constrained Pareto efficient for every informative information structure.*

The shape of the equilibrium interest rate as a function of the belief is key for the existence of ex-post improvements. In appendix H we show that Corollary 8 is not an empty statement. In Lemma 3 we use the implicit function theorem to prove that when (A2) holds with equality, R is a monotone increasing function below $\underline{\pi}$ for *any* v strictly increasing and strictly concave.

6 Ex-ante improvements

In this section we study the existence of ex-ante constrained Pareto improvements. In the first subsection we study the shape of the function F, and in particular we investigate when it is concave. In the second subsection we look for sufficient conditions for ex-ante improvements to exist.

For the discussion in the main text we assume that assumption (A2) is satisfied with equality, i.e $w_1^c + \theta_1 = w_2^c + \theta_2$. The proofs in the general case are relegated to the appendix. In what follows we assume $\pi^0 < \underline{\pi}$ to ensure the uninformative equilibrium is not Pareto efficient.

6.1 Non-concavity of the Pareto frontier

In Corollary 4 we have shown that a necessary condition for the existence of ex-ante improvements is for F to be non-concave. To prove the non-concavity of $F(\pi, V)$, we show it is not concave in π when V is restricted to be u's utility level at the uninformative equilibrium, $V^u(\pi^0)$, that is we define a new function $f(\pi,\pi^0) \equiv F(\pi, V^u(\pi^0))$ and show that $f(\pi,\pi^0)$ is not concave in π. Formally,

$$\begin{aligned} f(\pi,\pi^0) \equiv F(\pi, V^u(\pi^0)) = \{ \underset{x^c}{\text{Max}} \quad & v(x_0^c) + \sum \pi_s v(x_s^c) \\ \text{s. t.} \quad & x_s^c \geq w_s^c + \theta_s + \theta_3 R(\pi) \text{ for } s = 1, 2, \\ & v(w - x_0^c) + \sum \pi_s v(w - x_s^c) \geq V^u(\pi^0) \, \} \, . \end{aligned} \tag{5}$$

When $w_1^c + \theta_1 = w_2^c + \theta_2$, there is *always* full smoothing in period 1 consumption in equilibrium, as both lower bounds on period 1 consumption coincide. The solution to the maximization problem in (5) is characterized by $x_1^c = x_2^c$ for all π.[17] Therefore the function f coincides with the function $\widehat{f}$ defined below:

$$\begin{aligned} \widehat{f}(\pi,\pi^0) = \{ \underset{\{x_0^c, x_1^c\}}{\text{Max}} \quad & v(x_0^c) + v(x_1^c) \\ \text{s. t.} \quad & x_1^c \geq w_1^c + \theta_1 + \theta_3 R(\pi) \, , \\ & v(w - x_0^c) + v(w - x_1^c) \geq V^u(\pi^0) \, \} \, . \end{aligned}$$

Proposition 9. *Assume f is defined at $(\underline{\pi},\pi^0)$. If (A2) is satisfied with equality, or if (A2) is satisfied with strict inequality and f is not differentiable at $\underline{\pi}$, then f is not concave.*

Notice that the function $\widehat{f}(\pi,\pi^0)$ depends on π only through $R(\cdot)$ as the objective function is independent of π. If $\widehat{f}(\pi,\pi^0)$ is defined at $(\underline{\pi},\pi^0)$, then $\widehat{f}(\pi,\pi^0)$ is equal to some constant, say $\underline{f}(\pi^0)$ for all $\pi \geq \underline{\pi}$, as $R(\pi) = 1$ for all $\pi \geq \underline{\pi}$, i.e. the constrained feasible set is constant to the right of $\underline{\pi}$.[18] Also notice that $\widehat{f}(\pi^0,\pi^0) > \widehat{f}(\pi,\pi^0)$ for all $\pi \geq \underline{\pi}$, this follows from the fact that the constrained feasible set under $\pi \geq \underline{\pi}$ is a proper subset of the constrained feasible set under π^0, as $R(\pi^0) < 1 = R(\underline{\pi})$.[19] Therefore $\widehat{f}(\pi^0,\pi^0) \geq \widehat{f}(\underline{\pi},\pi^0)$. In fact, as the constrained feasible set for a given belief is a convex set and the maximiser of the problem defining $\widehat{f}(\pi,\pi^0)$ belongs to CFS$(R(\pi^0))$ for all $\pi \geq \underline{\pi}$, strict concavity of v implies that $\widehat{f}(\pi^0,\pi^0) > \widehat{f}(\pi,\pi^0)$ for all $\pi \geq \underline{\pi}$.

Consider the beliefs $\pi^1 = \pi^0$ and $\pi^2 = 1$. There exist a $\tau \in (0,1)$ such that $\tau\pi^1 + (1-\tau)\pi^2 = \underline{\pi}$. But $\tau\widehat{f}(\pi^1,\pi^0) + (1-\tau)\widehat{f}(\pi^2,\pi^0) > \widehat{f}(\underline{\pi},\pi^0)$. Hence the function $\widehat{f}(\pi,\pi^0)$, and therefore $f(\pi,\pi^0)$, is not concave in π.

Notice that the analysis above helps us explain why there cannot be an improvement if R is independent of π, or in the standard pure exchange economy with complete markets and no aggregate uncertainty. In both cases the objective function and the constraints are independent of π. Strict concavity of the objective function and convexity of the constrained set imply concavity of F.

In the general case, when (A2) is satisfied with strict inequality, it is not longer true that the objective function in the problem defining f is independent of the belief, as the solution may not display full smoothing in period 1. This implies that we cannot use the approach explained above to prove non-concavity.

Notice that when (A2) holds with strict inequality, it is still true that $R(\pi) = 1$ for all $\pi \geq \underline{\pi}$. Lemma 4 in appendix H shows that $R(\pi) < 1$ for all $\pi < \underline{\pi}$. Thus the right derivative of R with respect to π at

[17] If the constraints on x_1^c and x_2^c are both binding at the solution, then $x_1^c = x_2^c$ at the solution. If both constraints are not binding, then $x_0^c = x_1^c = x_2^c$ at the solution. If only the constraint on x_1^c is binding, then at the solution $x_0^c = x_2^c > x_1^c$, but then if we assign to c the constrained feasible allocation $(x_0^c, \widetilde{x}^c, \widetilde{x}^c)$, where $\widetilde{x}^c = \pi x_1^c + (1-\pi)x_2^c$, we make both agents better off as v is strictly concave. Therefore we cannot have $x_0^c = x_2^c > x_1^c$ at a solution. To discard the case when $x_0^c = x_1^c > x_2^c$ at equilibrium, we use the same logic.

[18] The same argument implies that if $\widehat{f}$ is defined at $(\underline{\pi},\pi^0)$, then it is defined at (π,π^0) for all $\pi \in [\underline{\pi}, 1]$.

[19] See appendix H.

$\underline{\pi}$ is zero. If the left derivative of R with respect to π at $\underline{\pi}$ is different from zero, the functions F and f are not differentiable. In the appendix we show that a sufficient condition for non-differentiability of R is:

$$w_0^c - w_2^c \neq 2(\theta_1 + \theta_3). \tag{A4}$$

Numerical results confirm that the intersection between the subset of parameter values that satisfy assumptions (A0), (A1), (A2) with strict inequality and (A3), and the subset of parameter values that satisfy (A4) is non-empty.

To prove the non-concavity of f in the general case we do an analysis around $(\underline{\pi}, \pi^0)$ and we exploit the non-differentiability of f at this point. We show that when f is non-differentiable at $(\underline{\pi}, \pi^0)$ its slope marginally to the right of $\underline{\pi}$ is bigger than its slope marginally to the left of $\underline{\pi}$. This condition allows us to prove non-concavity independently of the actual sign of these slopes.

So far we have assumed that f is defined at $(\underline{\pi}, \pi^0)$. In appendix I, Lemma 8 provides a sufficient condition for f to be defined at $(\underline{\pi}, \pi^0)$. We show that if π^0 is sufficiently close to $\underline{\pi}$, then f is indeed defined at $(\pi^0, \underline{\pi})$.

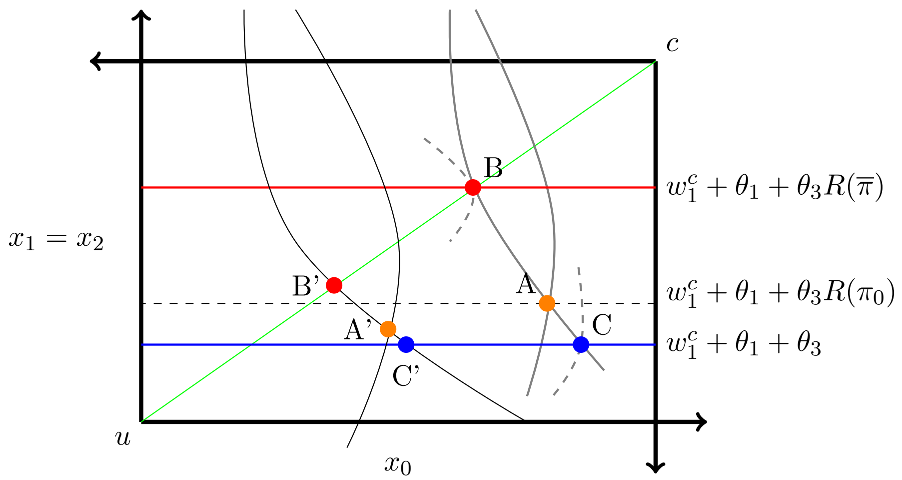
6.2 Sufficient conditions for an ex-ante improvement

In the previous subsection we argued that f, and therefore F, is not concave. In this subsection we first determine the actual shape of f and that of its concavification. We complete the analysis with a sufficient condition for the existence of ex-ante constrained Pareto improvements.

By Corollary 4 and the definition of f, to see if ex-ante improvements are possible we need to compare cav f and f at the point (π^0, π^0). For simplicity, in the main text we assume that $f(\pi, \pi^0)$ is defined for all $\pi \in [0, 1]$.

Remember that when (A2) is satisfied with equality we always have $x_1^c = x_2^c$ in equilibrium. Therefore we can depict the equilibrium in an Edgeworth box. In Figure 3, where we have period 0 and period 1 consumption in the horizontal and vertical axes respectively, the dashed horizontal line represents the lower bound on period one consumption and the uninformative equilibrium is given by a point like A for any $\pi_0 < \underline{\pi}$. The Pareto set, characterized by full consumption smoothing across periods (and states), is depicted by the diagonal line connecting the bottom left and upper right corners of the box.

Figure 3: Uninformative equilibrium and allocations $\overline{x}^h(\pi^0)$ and $\underline{x}^h(\pi^0)$.



Let $\mathbb{1} \in \mathbb{R}^3$ be the vector of ones. Consider the allocation

$$\overline{x}^c(\pi^0) = \mathbb{1}\overline{\alpha}(\pi^0) \quad \text{and} \quad \overline{x}^u(\pi^0) = \mathbb{1}w - \overline{x}^c(\pi^0),$$

where $\overline{\alpha}(\pi^0)$ is the solution to

$$2v(w - \alpha) = V^u(\pi^0).$$

That is, $\overline{x}^c(\pi^0)$ is a constant consumption plan for c that gives u her uninformative equilibrium utility, $V^u(\pi^0)$. Allocation $(\overline{x}^c(\pi^0), \overline{x}^u(\pi^0))$ is depicted as point B in Figure 3, the point where u's indifference curve that passes through the uninformative equilibrium allocation (point A) cuts the Pareto set. Notice that this allocation gives c the highest possible utility conditional on u being indifferent with respect to the uninformative equilibrium. Thus, if $(\overline{x}^c(\pi^0), \overline{x}^u(\pi^0))$ is constrained feasible for π, $f(\pi, \pi^0) = V(\overline{x}^c(\pi^0), \pi)$.

Consider also the allocation

$$\underline{x}^c(\pi^0) = \left(\underline{\alpha}(\pi^0), w_1^c + \theta_1 + \theta_3, w_1^c + \theta_1 + \theta_3\right) \quad \text{and} \quad \underline{x}^u(\pi^0) = \mathbb{1}w - \underline{x}^c(\pi^0),$$

where $\underline{\alpha}(\pi^0)$ is the solution to:

$$v(w - \alpha) + v(w_1^u - \theta_1 - \theta_3) = V^u(\pi^0).$$

That is, $\underline{x}(\pi^0)$ is the period 0 consumption level for c such that if she consumes $w_1^c + \theta_1 + \theta_3$ in both states in period 1, then u is indifferent with respect to the uninformative equilibrium. Allocation $(\underline{x}^c(\pi^0), \underline{x}^u(\pi^0))$ is depicted as point C in Figure 3. Notice that $\underline{x}^c(\pi^0)$ is constrained feasible for every belief as $R(\pi) \leq 1$ for every $\pi \in [0, 1]$.

By construction u is indifferent between the uninformative equilibrium and allocations $\underline{x}^u(\pi^0)$ and $\overline{x}^u(\pi^0)$. We have drawn Figure 3 in such a way that these allocations are well defined, however it may be the case that one or both of them are not feasible. Lemma 9 in appendix J gives us a sufficient condition for both allocations to be feasible. Assume for now that both $\overline{\alpha}(\pi^0)$ and $\underline{\alpha}(\pi^0)$ are in $[0, w]$.

Define $\overline{\pi}(\pi^0)$ as a belief such that $\overline{\alpha}(\pi^0)$ is c's equilibrium consumption in state one. That is, $\overline{\pi}(\pi^0)$ is the value of π that solves:

$$\overline{\alpha}(\pi^0) = w_1^c + \theta_1 + \theta_3 R(\pi).$$

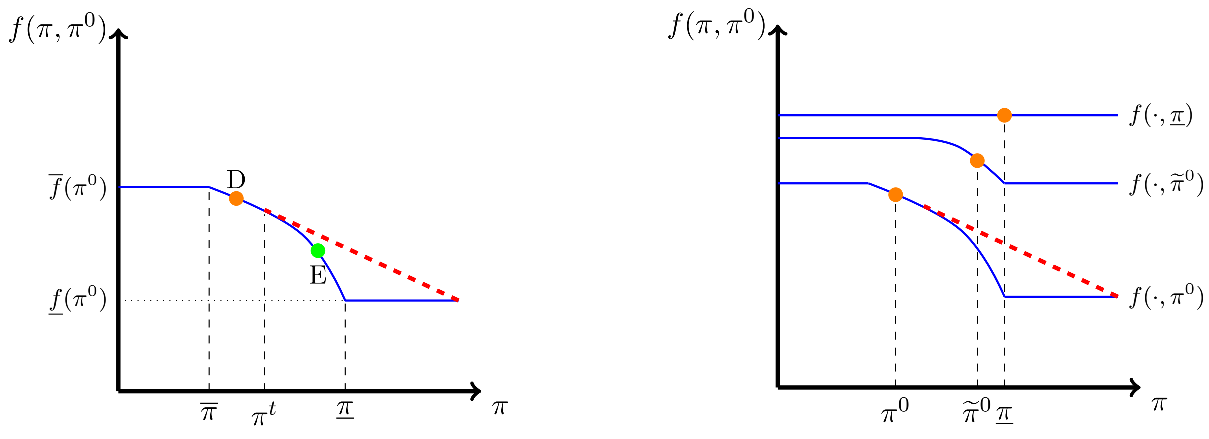
Graphically, $\overline{\pi}(\pi^0)$ is the belief that makes the lower bound on state one consumption to pass through point B in Figure 3. Assume for now that $\overline{\pi}(\pi^0) \in [0, 1]$. Note that $\overline{\pi}(\pi^0) < \pi^0$ as R is monotone increasing below $\underline{\pi}$ whenever (A2) holds with equality and $w_1^j + \theta_1 + \theta_3 R(\pi^0) > w_1^j + \theta_1 + \theta_3 R(\overline{\pi}(\pi^0))$ (See Figure 3).

As $R(\pi)$ is increasing in π for all $\pi < \underline{\pi}$, then $\overline{x}^c(\pi^0)$ is constrained feasible for all $\pi \leq \overline{\pi}(\pi^0)$. Consequently, f is constant, say equal to $\overline{f}(\pi^0)$, to the left of $\overline{\pi}(\pi^0)$. Note that $f(\pi^0, \pi^0)$ is c's utility associated with the indifference curve through point A, and $f(\overline{\pi}(\pi^0), \pi^0)$ is c's utility associated with the indifference curve through point B. Since point A is not in the Pareto set by Proposition 5, then $f(\pi, \pi^0) > f(\pi^0, \pi^0)$ for all $\pi \leq \overline{\pi}(\pi^0)$.

The function f is constant to the left of $\overline{\pi}(\pi^0)$ and in the previous subsection we argue it is also constant to the right of $\underline{\pi}$, i.e. $f(\pi, \pi^0) = \underline{f}(\pi^0)$ for all $\pi \geq \underline{\pi}$. A priori we do not know the shape of $f(\pi, \pi^0)$ for $\pi \in [\overline{\pi}(\pi^0), \underline{\pi}]$. If f is convex in this interval, then we have ex-ante improvements, as the concavification of f in this case is equal to f to the left of $\overline{\pi}(\pi^0)$ and to the right of $\underline{\pi}$, and the straight line joining $f(\underline{\pi}, \pi^0)$ and $f(\overline{\pi}(\pi^0), \pi^0)$ elsewhere.

In Figure 4a we have drawn $f(\pi, \pi^0)$ assuming it is strictly concave for $\pi \in [\overline{\pi}, \underline{\pi}]$. In this case the concavification of f is given by the straight dashed line starting from $f(1, \pi^0)$ that is tangent to f, and cav f coincides with f for beliefs below the tangency point. We denote the tangency point by π^t. Therefore, we have ex-ante improvements if and only if $\pi^0 > \pi^t$.

Figure 4: f and its concavification when (A2) holds with equality.



(a) $\pi^0 \neq \underline{\pi}$ (b) $\pi^0 \to \underline{\pi}$

Suppose we have a common prior π^0 below π^t so that cav $f(\pi^0, \pi^0)$ coincides with $f(\pi^0, \pi^0)$ as in point D in Figure 4a. A hasty conjecture would be that for a common prior π^0 to the right of π^t, cav $f(\pi^0, \pi^0)$ would lie strictly above $f(\pi^0, \pi^0)$, like in point E in the figure. However, notice that $\overline{\pi}$, π^t, $\overline{f}$ and $\underline{f}$ are all functions of π^0. So by increasing π^0 we change $\overline{\pi}$ and π^t and also the value that f attains at those points and at the point $\underline{\pi}$, as by changing the common prior we are changing the expected wealth of the agents in the uninformative equilibrium and the position of u's indifference curve at this equilibrium.

To understand how f changes as we increase the prior it is useful to go back to Figure 3. In Figure 3 we show the effect of an increase in the prior on allocations $\overline{x}^h$ and $\underline{x}^h$. If we change π^0 to $\widetilde{\pi}^0 > \pi^0$, as u is relatively poor in period 1 (assumption (A1)), and as utilities are independent of the probability of state one (since (A2) holds with equality), u's indifference curve at the uninformative equilibrium is shifted down. The same argument gives us that c's indifference curve is shifted upwards and the uninformative equilibrium moves from point A to point A', i.e. c attains a higher utility at the uninformative equilibrium. Thus $f(\widetilde{\pi}^0, \widetilde{\pi}^0) > f(\pi^0, \pi^0)$. By definition, $\overline{f}(\widetilde{\pi}^0)$ is the utility of agent c at the point where u's uninformative equilibrium cuts the Pareto set. As u's indifference curve is shifted down, it now cuts the Pareto set in a point that gives c more consumption in every period, thus $\overline{f}(\widetilde{\pi}^0) > \overline{f}(\pi^0)$ (see points B' and B). As R is monotone increasing below $\underline{\pi}$ and $w_1^c + \theta_1 + R(\overline{\pi}(\widetilde{\pi}^0)) > w_1^c + \theta_1 + R(\overline{\pi}(\pi^0))$, then $\overline{\pi}(\widetilde{\pi}^0) > \overline{\pi}(\pi^0)$. Finally, as u's indifference curve is shifted down and $\underline{x}_1^c = w_1 + \theta_1 + \theta_3$ is unchanged the point where it cuts the horizontal line at $\underline{x}_1^c = w_1 + \theta_1 + \theta_3$ gives more period zero consumption to c, i.e. $\underline{x}_0^c$ is increased. Thus $\underline{f}(\widetilde{\pi}^0) > \underline{f}(\pi^0)$ (see point C' and C).

The analysis above explains why f is shifted upwards when we increase agents' prior, as shown in Figure 4b. The extreme case when $\pi^0 = \underline{\pi}$ is also shown in Figure 4b. In that case the uninformative equilibrium is in the Pareto set, and it is constrained feasible for every belief, hence f is flat, and it lies above $f(\pi, \pi^0)$ for all $\pi \in [0, 1]$ and all $\pi^0 < \underline{\pi}$. See point A in Figure 5.

Thus we see that even though f fails to be concave, it is not direct to see if cav $f > f$ at the point (π^0, π^0). If for a given prior we have $\pi^0 < \pi^t(\pi^0)$, it is also not clear if by changing the prior we can move the economy to an uninformative equilibrium that is not ex-ante constrained Pareto efficient.

We will show that ex-ante improvements exist if π^0 is close to $\underline{\pi}$. Let's redefine the prior on state one as $\pi^0(\epsilon) \equiv \underline{\pi} - \epsilon$, with $\epsilon \in [0, \underline{\pi}]$. Therefore now all $\overline{x}$, $\underline{x}$ and $\overline{\pi}$ are functions of ϵ.

The uninformative equilibrium displays full smoothing if and only if $\pi^0 \geq \underline{\pi}$, hence $\overline{\pi}(\epsilon) = \pi^0(\epsilon)$ if and only if $\epsilon = 0$. Above we argue that $\overline{\pi}(\epsilon) < \pi^0(\epsilon)$ for all $\epsilon > 0$.[20] Consider the beliefs $\pi^1 = \overline{\pi}(\epsilon)$, $\pi^2 = 1$ and $\pi^3 = \pi^0(\epsilon)$. Notice that $\overline{x}^c(\epsilon)$ is constrained feasible under π^1, and $\underline{x}^c(\epsilon)$ is constrained feasible under π^2. For an ex-ante improvement to exist, it is *sufficient* to find $\tau^1 \in (0,1)$ and $\tau^3 \in [0,1)$ such that $\tau^1 + \tau^3 < 1$ and:

$$\pi^0(\epsilon) = \tau^1\pi^1 + (1 - \tau^1 - \tau^3)\pi^2 + \tau^3\pi^3,$$
$$V_0^c < \tau^1 V(\overline{x}^c(\epsilon), \pi^1) + (1 - \tau^1 - \tau^3)V(\underline{x}^c(\epsilon), \pi^2) + \tau^3 V_0^c,$$

since u is indifferent with respect to the uninformative equilibrium by construction when allocated $\overline{x}^u$ and $\underline{x}^u$.

Bayes plausibility implies that τ^1 has to satisfy:

$$\tau^1 = (1 - \tau^3)\frac{\pi^0(\epsilon) - \pi^2}{\pi^1 - \pi^2} = (1 - \tau^3)\frac{\pi^0(\epsilon) - 1}{\overline{\pi}(\epsilon) - 1} \equiv \widetilde{\tau}(\epsilon).$$

Let

$$\overline{V}(\epsilon) \equiv V(\overline{x}^c(\epsilon), \overline{\pi}(\epsilon)),$$
$$\underline{V}(\epsilon) \equiv V(\underline{x}^c(\epsilon), 1),$$
$$V_0(\epsilon) \equiv V_0^c.$$

Agent c is ex-ante indifferent between mixing $\overline{V}(\epsilon)$, $\underline{V}(\epsilon)$ and $V_0(\epsilon)$, and the uninformative equilibrium if she gets $\overline{V}(\epsilon)$ with probability:

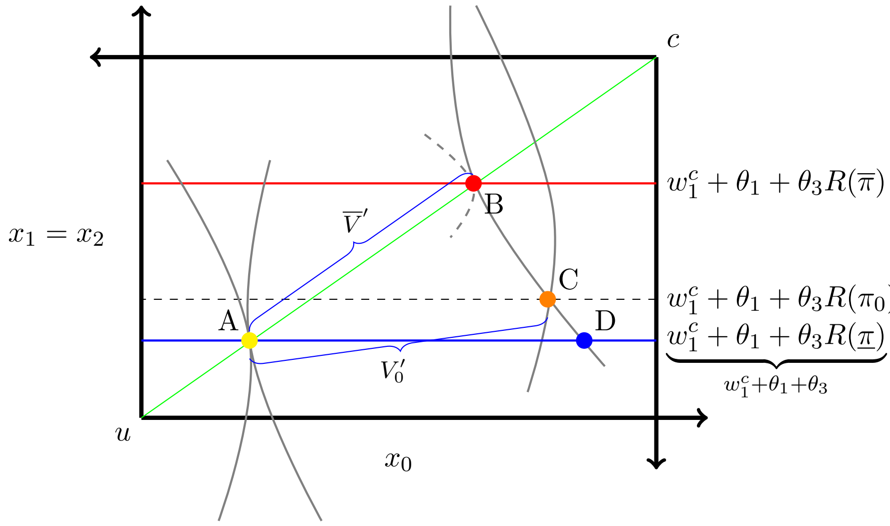
$$\tau^1 = (1 - \tau^3)\frac{V_0(\epsilon) - \underline{V}(\epsilon)}{\overline{V}(\epsilon) - \underline{V}(\epsilon)} \equiv \widehat{\tau}(\epsilon).$$

If $\widehat{\tau}(\epsilon) \leq \widetilde{\tau}(\epsilon)$, then there exist an ex-ante constrained Pareto improvement as $\overline{V}(\epsilon) > V_0(\epsilon) > \underline{V}(\epsilon)$ (see points B, A and C in Figure 3). Equality between $\widehat{\tau}$ and $\widetilde{\tau}$ is sufficient as the Planner could still smooth period 0 consumption between signals.

The limit as ϵ goes to 0 of $\widetilde{\tau}(\epsilon)$ is equal to $1 - \tau_3$, as $\overline{\pi}(\epsilon)$ converges to $\pi^0(\epsilon)$ as ϵ tends to zero. If we can show that the limit as ϵ tends to 0 of $\widehat{\tau}(\epsilon)$ is equal to $\overline{a}(1 - \tau_3)$ for some constant $\overline{a} < 1$, then we would have proved the existence of an ex-ante improvement.

[20] At a first glance there is no reason to think that $\overline{\pi}(\epsilon) \in [0,1]$. For example if $\overline{x}(\pi^0) < \min_{\{\pi\}} \{w_1^c + \theta_1 + \theta_3 R(\pi)\}$ then $\overline{x}(\pi^0)$ is never constrained feasible, or if R is strictly increasing below $\underline{\pi}$, we may have $\overline{x}(\pi^0) < w_1^c + \theta_1 + \theta_3 R(0)$ in which case the same conclusion applies. In appendix J we formally show that $\overline{\pi}$ is well defined in an interval around $\epsilon = 0$.

Figure 5: Indifference curves at $\epsilon = 0$.



In Figure 5 we show the picture that led us to think that the limit of $\widehat{\tau}$ is strictly lower than $1 - \tau_3$. For this, we need the limit of $\frac{V_0(\epsilon)-\underline{V}(\epsilon)}{\overline{V}(\epsilon)-\underline{V}(\epsilon)}$ to be strictly lower than one. From the figure we see that the ratio is strictly lower than one for every $\epsilon > 0$. Agent c's indifference curve at point C represents $V_0(\epsilon)$, her indifference curve at point B represents $\overline{V}(\epsilon)$, and her indifference curve at point D represents $\underline{V}(\epsilon)$. Thus it is direct to see that $\frac{V_0(\epsilon)-\underline{V}(\epsilon)}{\overline{V}(\epsilon)-\underline{V}(\epsilon)} < 1$ for all $\epsilon > 0$. However this is not sufficient to prove that the limit of the ratio is strictly below one, as the ratio is not defined at $\epsilon = 0$. Point A depicts the equilibrium when $\epsilon = 0$. In that point $\overline{V}$, $\underline{V}$ and V_0 take the same value.

If we increase ϵ marginally starting from $\epsilon = 0$, the indifference curves of the agents are shifted from point A to those in the right of the figure. From c's perspective, point B is associated with higher utility relative to point C, i.e. $\overline{V}(\epsilon) > V_0(\epsilon)$. As we can think of the (right) derivatives of $\overline{V}$ and V_0 at $\epsilon = 0$ as the distance between c's indifference curves at points A and B, or points A and C respectively, this suggest that if we apply L'Hôpital's rule to compute the limit of $\frac{V_0(\epsilon)-\underline{V}(\epsilon)}{\overline{V}(\epsilon)-\underline{V}(\epsilon)}$, we obtain a limit strictly lower than one.

Theorem 10. *Suppose either (A2) is satisfied with equality or (A4) holds. Then there exist a $\delta > 0$ such that if $\epsilon \in (0, \delta)$ then the uninformative equilibrium is not ex-ante constrained Pareto efficient.*

However when using L'Hôpital's rule to compute the limit of $\widehat{\tau}(\epsilon)$, we need to apply it twice as in the limit when the increment of ϵ goes to zero, V_0', $\overline{V}'$, and $\underline{V}'$ coincide. We use an alternative approach to compute the limit of $\widehat{\tau}(\epsilon)$. First we define the function $g(a, \epsilon) = a\overline{V}(\epsilon) + (1 - a)\underline{V}(\epsilon) - V_0(\epsilon)$, mapping $\mathbb{R} \times [0, \underline{\pi}]$ into $\mathbb{R}$. Then we show that there exist $\overline{a} < 1$ such that $g(\overline{a}, \epsilon)$, as a function of ϵ, attains a *strict* local minimum at $(\overline{a}, 0)$, and that $g(\overline{a}, 0) = 0$. Thus $g(\overline{a}, \epsilon) > 0$ for all ϵ in a neighborhood of zero. Using the definition of g, $g(\overline{a}, \epsilon) > 0$ is equivalent to $\overline{a}\overline{V}(\epsilon) + (1 - \overline{a})\underline{V}(\epsilon) - V_0(\epsilon) > 0$, or $\frac{V_0(\epsilon)-\underline{V}(\epsilon)}{\overline{V}(\epsilon)-\underline{V}(\epsilon)} < \overline{a} < 1$ for all ϵ in the neighborhood. This last expression tells us that the limit of $\widehat{\tau}$ is strictly below $1 - \tau_3$.

When assumption (A2) holds with strict inequality, the difference is that we do not know a priori if $\overline{\pi}(\epsilon)$ is above or below $\pi^0(\epsilon)$, as we do not know how R changes when we vary the belief. In appendix J we show that $\overline{\pi}(\epsilon)$ is a continuous function in an interval around $\epsilon = 0$. As $\overline{\pi} = \pi^0$ if and only if $\epsilon = 0$, continuity implies that $\overline{\pi}(\epsilon)$ is always above or below $\pi^0(\epsilon)$. If $\overline{\pi}(\epsilon) < \pi^0(\epsilon)$, then the analysis explained above is still valid. If $\overline{\pi}(\epsilon) > \pi^0(\epsilon)$, then the only difference is that we need to set $\pi^2 = \pi^0(\epsilon) - \gamma$ for some fixed and small $\gamma > 0$.

7 Conclusion

We have shown that in economies where savings and short-sale constraints may prevent equilibrium from being fully Pareto efficient, public information may have positive social value. Information-driven price changes may allow a benevolent social planner facing the same information and asset constraints, to obtain *ex-ante* constrained Pareto improvements under quite general conditions. Unlike Gottardi and Rahi [10], ex-post improvements are attainable only under special conditions for the equilibrium interest rate. The reaction of prices due to the arrival of new information is a necessary condition for information to have social value. Thus we need to be careful when judging the welfare implications of such price changes. Cuevas, Bernhardt and Sanclemente [5] documents that the Chilean authorities viewed the arrival of new information as bad for the economy because it affected asset prices. Our analysis showed that a Planner could take advantage of such a situation and improve welfare. However, our result on ex-post improvements suggest that the authorities are right to be worried if they are concerned with welfare from an ex-post point of view.

Our results provide new insights on the value of public information in exchange economies where equilibria are not necessarily fully Pareto efficient. Gottardi and Rahi [10] show that with incomplete markets a Planner can obtain ex-post improvements for any initial information structure, by locally changing the information agents receive before trading. In our setting, constrained ex-post improvements are possible only under special circumstances: the uninformative equilibrium interest rate needs to be above the equilibrium interest rate for every signal of the informative information structure. In the main result of the paper we show that ex-ante improvements exist if the common prior is sufficiently close to the threshold dividing first best equilibria from equilibria where the savings constraint and short-sale constraint are binding. These ex-ante improvements are not marginal in nature. We consider a situation where posterior beliefs can be far away from the prior, in fact one of the posterior beliefs is equal to one.

Finally, we have shown that the study of the value of information in exchange economies can be simplified by adopting the techniques used in the literature on Bayesian persuasion. The simplification lies in realizing that information structures can be defined as a vector of beliefs and a vector of probabilities of the beliefs such that Bayes plausibility is satisfied. By taking the Planner as the sender and the agents as the receivers, whose actions affect the Planner's payoff by changing equilibrium prices and the constrained feasible set, we can relate the value of information to the concavity of the Planner's utility function, the Pareto frontier. The difference between our model and the standard problem in the Bayesian persuasion literature is that on top of Bayes plausibility, the sender has to make sure that the utility levels he assigns to the unconstrained agent, leave her indifferent with respect to the uninformative equilibrium.

Our analysis used the condition that the total endowment is constant in every period and state of the world. Possible extension of this work may be to relax this assumption and see if our results extend to this more general setting. Also, in our current model there is no reason why agents should face mandatory constraints, therefore the best thing the Planner could do is to remove the mandatory savings altogether. We plan to extend our model to the case of preferences involving hyperbolic discounting or temptation, where having mandatory savings constraints can be optimal. Finally, we have been silent about the existence of ex-ante improvement through marginal changes in beliefs, in future work we plan to relate this to Radner and Stiglitz [20].

Appendix A Proof of Proposition 1

To prove the existence of an equilibrium, instead of working with the financial market equilibrium defined in Definition 1, it is easier to write the model in terms of contingent consumption. For this, we define a non-arbitrage equilibrium following Magil and Quinzii [15]. We then prove the equivalence between both types of equilibria. First, let's define alternative budget sets for both agents:

$$
\begin{aligned}
\mathcal{B}^c(p, w^c) &= \left\{ x^c \in \mathbb{R}^3_+ \left| \sum_{s=0}^{2} p_s x_s^c = \sum_{s=0}^{2} p_s w_s^c,\ x_s^c \geq w_s^c + \theta_s + \frac{p_0 \theta_3}{p_1 + p_2}\ \forall s \in \{1,2\} \right. \right\}, \\
\mathcal{B}^u(p, w^u) &= \left\{ x^u \in \mathbb{R}^3_+ \left| \sum_{s=0}^{2} p_s x_s^u = \sum_{s=0}^{2} p_s w_s^u \right. \right\}.
\end{aligned}
\tag{6}
$$

where $p = (p_0, p_1, p_2)$ is the vector of contingent consumption prices. Using the budget sets in (6) we can now define a non-arbitrage equilibrium:

Definition 9. Given π, a non-arbitrage equilibrium is a collection of prices $\overline{p} \in \mathbb{R}^3_{++}$ and consumption plans $\overline{x} = (\overline{x}^u, \overline{x}^c) \in \mathbb{R}^6_+$, such that:

1. $\overline{x}^h \in \ \operatorname{argmax} \ \left\{ V^h(x^h, \pi) \left| x^h \in \mathcal{B}^h(\overline{p}, w^h) \right. \right\} \ \forall h \in \{c, u\}$,
2. $\sum_h \left(\overline{x}^h - w^h \right) = 0$.

Having defined both types of equilibria, we can now prove their equivalence. First we show the equivalence of both budget sets in the absence of arbitrage.

Lemma 1. *Let $q = (1, q^1) \in \mathbb{R}^4_{++}$, and $p = (1, p^1) \in \mathbb{R}^3_{++}$. If $q^1 = p^1 \Phi$, then $\mathbb{B}^h(q, w^h) = \mathcal{B}^h(p, w^h)$ for all h.*

Proof. Notice that the non-arbitrage condition $q^1 = p^1 \Phi$ implies: $q_1^1 = p_1^1$, $q_2^1 = p_2^1$, and $q_3^1 = p_1^1 + p_2^1$. Assume $x^h \in \mathbb{B}^h(q, w^h)$ for all h, then:

$$
x_0^h - w_0^h = -q^1 z^h = -p^1 \Phi z^h = -\left(\sum_s p_s^1 \Phi_s \right) z^h, \tag{7}
$$

where Φ_s represents row s of matrix Φ. The budget constraint in period 1 implies:

$$
x_s^h - w_s^h = \Phi_s z^h,
$$

thus we can rewrite (7) as:

$$
x_0^h - w_0^h = -\sum_s p_s^1 (x_s^h - w_s^h). \tag{8}
$$

Furthermore as $z_s^c \geq \theta_s$ for $s = 1, 2$, and $q_3^1 z_3^c \geq \theta_3$:

$$
x_s^c = w_s^c + z_s^c + z_3^c \geq w_s^c + \theta_s^c + \frac{\theta_3}{q_3^1}. \tag{9}
$$

Equations (8) and (9) imply that $x^h \in \mathcal{B}^h(p, w^h)$ for all h.

Assume now $x^h \in \mathcal{B}^h(p, w^h)$ for all h. Then $x_s^c - w_s^c \geq \theta_s + \theta_3/(p_1^1 + p_2^1)$. Fix $z_3^c = \theta_3/q_3^1$ and $z_S^c = \theta_s$ for $s = 1, 2$, and let consumption in period s be given by:

$$
x_s^c - w_s^c = z_s^c + z_3^c.
$$

As $q_3^1 = p_1^1 + p_2^1$, replacing $x_s^c = w_s^c + z_s^c + z_3^c$ into $\sum_{s=0}^{2} p_s x_s^c = \sum_{s=0}^{2} p_s w_s^c$ we obtain $x_0^c + \sum_k q_k^1 z_k^c = w_0^c$. These two results imply that $x^c \in \mathbb{B}^h(q, w^h)$. For u the result follows from noticing that she can freely choose the $z^u \in \mathbb{R}^3$. Then for any $x_s^u - w_s^u$ there exist $z^u \in \mathbb{R}^3$ such that $x_s^u - w_s^u = z_s^u + z_3^u$. ☐

From the equivalence between the two budget sets under no arbitrage, when can prove the equivalence between the two types of equilibria.

Lemma 2.

1. *If $(\overline{x}, \overline{z}, \overline{q})$ is a financial market equilibrium with $\overline{q} = (1, \overline{q}^1)$, then $(\overline{x}, \overline{p})$, with $\overline{p} = (1, \overline{p}^1)$, and $\overline{p}^1$ satisfying $\overline{q}^1 = \overline{p}^1 \Phi$, is a non-arbitrage equilibrium.*

2. *If $(\overline{x}, \overline{p})$ is a non-arbitrage equilibrium with $\overline{p} = (1, \overline{p}^1)$, then there exist portfolios $\overline{z}^u$ and $\overline{z}^c$ and asset prices $\overline{q}^1 = \overline{p}^1 \Phi$ such that $(\overline{x}, \overline{z}, (1, \overline{q}^1))$ is a financial market equilibrium.*

Proof.

1. By Lemma 1, $\overline{x}^h \in \text{argmax}\{V(x^h, \pi) \,|x^h \in \mathbb{B}^h(\overline{q}, w^h)\}$ imply$\overline{x}^h \in \text{argmax}\{V(x^h, \pi) \,|x^h \in \mathcal{B}^h(\overline{p}, w^h)\}$ for all h. As $\sum_h z_l^h = 0$ for $l = 1, 2, 3$, then $\sum_h (x_s^h - w_s^h) = 0$ for $s = 0, 1, 2$.

2. By Lemma 1, $\overline{x}^h \in \text{argmax}\{V(x^h, \pi) \,|x^h \in \mathcal{B}^h(\overline{p}, w^h)\}$ imply$\overline{x}^h \in \text{argmax}\{V(x^h, \pi) \,|x^h \in \mathbb{B}^h(\overline{q}, w^h)\}$ for all h. As $\sum_h (x_s^h - w_s^h) = 0$ for $s = 0, 1, 2$, then $\sum_h z_l^h = 0$ for $l = 1, 2, 3$.

□

Using the equivalence between financial market equilibria and non-arbitrage equilibria, we can now prove the existence of a financial market equilibrium following the standard proof involving contingent consumption.

Proof of Proposition 1:

We will prove the existence of a non-arbitrage equilibrium, and then invoke Lemma 2.

It is well-known that agent u's optimal demand function for contingent consumption is continuous, homogeneous of degree zero, satisfies Walras' law, satisfies non-negativity, and has the appropriate boundary behavior. The proof can be found, for example, in Hildenbrand and Kirman [12]. Below we will argue that c's demand function has the same properties. Let's start with continuity: As $V(\cdot)$ is continuous in consumption, if we can show that c's budget correspondence, defined in (6), is compact-valued and continuous, continuity of c's demand function follows from the maximum theorem. That $\mathcal{B}^c(p, w)$ is compact-valued is direct when prices are strictly positive. Let's study its continuity.

Upper hemi continuity: Take a sequence $(p_n, w_n) \in \mathbb{R}^4_{++} \times \mathbb{R}^3_+$ converging to $(p, w) \in \mathbb{R}^4_{++} \times \mathbb{R}^3_+$. Let $(x_n) \in \mathbb{R}^3_+$ be a sequence such that $(x_n) \in \mathcal{B}^c(p_n, w_n)$ $\forall n$. As we explained after Proposition 1 in page 7, our assumptions on θ_l for $l = 1, 2, 3$, guarantee that the budget correspondence is never the empty set. Clearly the sequence (x_n) is bounded below by the zero vector. Let $\overline{p} = \max_s (\sup_n p_{s,n}) > 0$, where $s \in \{0, 1, 2\}$; $w^* = \max_s (\sup_n w_{s,n}) > 0$, and $\underline{p} = \min_s (\inf_n p_{s,n}) > 0$.[21] Then $x_{s,n} \leq \frac{\overline{p} w^*}{\underline{p}}$ for all s and n. Hence the sequence (x_n) is bounded, and by the Bolzano-Weierstrass Theorem it has a convergent subsequence: $x_{n_k} \to x$. Since $x_{n_k} \in \mathcal{B}^c(p_{n_k}, w_{n_k})$: $\sum_{s=0}^2 p_{s,n_k} x_{s,n_k} \leq \sum_{s=0}^2 p_{s,n_k} w_{s,n_k}$, and $x_{s,n_k} \geq w_{s,n_k} + \theta_s + \frac{p_{0,n_k}\theta_3}{p_{1,n_k} + p_{2,n_k}}$ for $s = 1, 2$. Taking limits it's direct to see that $x \in \mathcal{B}^c(p, w)$, as weak inequalities hold at the limit.

[21] Let ϵ be such that $p_s - \epsilon > 0$, where p_s is the limit of the convergent sequence $(p_{s,n})$. Then there exist N such that for all $n \geq N$: $p_s + \epsilon > p_{s,n} > p_s - \epsilon > 0$. Therefore $p_{s,n} > \min\{p_s - \epsilon, \min\{p_{s,1}, ..., p_{s,N-1}\}\} > 0$ for all n, as the sequence only takes strictly positive values. This implies $\inf_n p_{s,n} > 0$. A similar argument gives us that w^* and $\overline{p}$ are strictly positive and do not diverge to infinity.

Lower hemi continuity: Fix $(p, w) \in \mathbb{R}^4_{++} \times \mathbb{R}^3_+$. Let O be an open subset of $\mathbb{R}^3_+$ such that $\mathcal{B}^c(p,w) \cap O \neq \emptyset$. Suppose $\mathcal{B}^c(p,w)$ is not lower hemi continuous at (p,w), then for every $n \in \mathbb{N}$ there exist a (p_n, w_n) within a $\frac{1}{n}$-neighborhood of (p,w) such that $\mathcal{B}^c(p_n, w_n) \cap O = \emptyset$. Take any $x \in \mathcal{B}^c(p,w) \cap O$ such that x is in the interior of $\mathcal{B}^c(p,w)$. Then $\lambda x \in \mathcal{B}^c(p,w) \cap O$ for $\lambda \in (0,1)$ sufficiently close to 1. But as (p_n, w_n) converges to (p,w) and $px - pw < 0$ and $x_s > w^x_s + \theta_s + p_0\theta_3/(p_1+p_2)$ for $s = 1,2$, continuity of $\lambda p_n x - p_n w_n$ and $x_s - w^x_s - \theta_s - p_0\theta_3/(p_1+p_2)$ implies that $\lambda p_n x - p_n w_n < 0$, and $x_s > w^c_{s,n} + \theta_s + p_{0,n}\theta_3/(p_{1,n} + p_{2,n})$, for $s = 1,2$; for n large enough. But then $x \in \mathcal{B}^c(p_n, w_n)$ for such n. This contradicts O and $\mathcal{B}^c(p_n, w_n)$ being disjoint. Thus the budget correspondence is continuous, and we obtain continuity of demand using the maximum theorem.

Walras' law follows from strong monotonicity of preferences, and homogeneity of degree zero follows from the fact that the budget set does not change if we multiply all prices by the same constant.

Define the excess demand function of agent h as:

$$\phi^h(p) = x^h(p, w^h) - w^h,$$

where $x^h(p, w^h)$ is h's Walrasian demand function. The aggregate excess demand function of the economy is:

$$\phi(p) = \sum_h \phi^h(p).$$

As $x^h(\cdot)$ is continuous, homogeneous of degree zero and satisfy Walras' law, these properties are directly inherited by $\phi(p)$. As $x^h(\cdot) \geq 0$, this implies there exist an $m > 0$ such that $\phi_s(p) > -m$ for every s and all p. Finally we have to prove that if $p^n \to p$, where $p \neq 0$ and $p_s = 0$ for some s, then $\max\{\phi_0(p^n), \phi_1(p^n), \phi_2(p^n)\} \to \infty$. Suppose this is not true. Then the sequences $\max\left\{\phi^h_0(p^n), \phi^h_1(p^n), \phi^h_2(p^n)\right\}$ does not diverge to infinity for any h, and so each of the $\phi^h_s(p^n)$ for $s = 0,1,2$ does not diverge to infinity for any h. Assume the value of c's endowment is different from zero at the limit. Then, there is a bounded set $B \subset \mathbb{R}^3_+$ such that $\phi^c(p^n) \cap B \neq \emptyset$ for infinitely many n. Then the sequence $(\phi^c(p^n)) \in B$ has a convergent subsequence. Let ϕ^{c*} be the limit of this subsequence, and define $x^{c*} = \phi^{c*} + w^c$. Then $x^{c*} \in \mathbb{R}^3_+$ and $px^{c*} = pw^c$. Take any other $x^c \in \mathbb{R}^3_+$ such that $px^c \leq pw^c$. If $px^c < pw^c$, then for n large enough $p^n x^c < p^n w^c$. Let $x^c_n = \phi^c(p^n) + w^c$, then $x^c_n \succsim x^c$. By continuity of preferences: $x^{*c} \succsim x^c$. If $px^c = pw^c$ we can find a sequence $(\widehat{x}^c_n)$ converging to x^c with $p\widehat{x}^c_n < pw^c$, but then $x^{*c} \succsim \widehat{x}^c_n$, and by continuity of preferences: $x^{*c} \succsim x^c$. But this is a contradiction since by strong monotonicity the demand of c at p is not well defined , because by consuming more of the good with price equal to zero, she can increase her utility at no cost. If the value of c's endowment at the limit is equal to zero, then the result follows from doing the sames analysis for u as total endowment w is assumed to be strictly positive.

As the excess demand function is defined for all strictly positive price vectors, and satisfy all the properties explained above, existence of equilibrium follows from Proposition 17.C.1 in Mas-Colell, Whinston and Green [17]. □

Appendix B Proof of Proposition 2

Let $x = (x^u, x^c)$ be the competitive equilibrium allocation and normalize $q_0 = 1$. Assume x is not constrained Pareto optimal, then there exist a constrained feasible allocation $\overline{x}$ and a supporting portfolio $\overline{z}$ such that $v(\overline{x}^h) \geq v(x^h)$ for all h and, say, $v(\overline{x}^c) > v(x^c)$. As $\overline{x}$ is feasible, $\sum_h \overline{x}^h_s - w^h_s \leq 0$ for

$s = 1, 2$. This implies $\Phi \sum_h \bar{z}^h \leq 0$. As equilibrium prices q satisfy no arbitrage:[22] $q \sum_h \bar{z}^h \leq 0$. Local non satiation of preferences imply that: $\bar{x}_0^u + \sum_{s=1}^3 q_s \bar{z}_s^u \geq w_0^u$. Similarly, as $v(\bar{x}^c) > v(x^c)$: $\bar{x}_0^c + \sum_{s=1}^3 q_s \bar{z}_s^c > w_0^c$. Adding across consumers: $\sum_h \bar{x}_0^h + q \sum_h \bar{z}^h > w$. Thus $\sum_h \bar{x}_0^h > w$ which contradicts constrained feasibility of $\bar{x}$. □

Appendix C Proof of Proposition 3

If there exist an ex-ante Pareto improvement, then there exist an informative information structure $\widehat{Y}$, and a constrained feasible allocation under $\widehat{Y}$, $(\widehat{x}(y_1), \widehat{x}(y_2), \widehat{x}(y_3))$ with:

$$\begin{aligned} \sum_k \text{pr}(y_k) V^u(\widehat{x}^u(y_k), \pi(y_k)) &= \sum_k \text{pr}(y_k) V^u(\bar{x}^u(y_k), \pi(y_k)), \\ \sum_k \text{pr}(y_k) V^c(\widehat{x}^c(y_k), \pi(y_k)) &> \sum_k \text{pr}(y_k) V^c(\bar{x}^c(y_k), \pi(y_k)). \end{aligned} \tag{10}$$

where $\bar{x}$ is the uninformative equilibrium allocation. Set $\pi_k = \pi(y_k)$ and $\tau_k = \text{pr}(y_k)$ for $k = 1, 2, 3$, where $\pi(y_k)$ and $\text{pr}(y_k)$ are the beliefs and signal probabilities implied by $\widehat{Y}$. Bayes' rule imply that the beliefs under $\widehat{Y}$, $\pi(y_k)$, have to satisfy condition 1 in the proposition. Let $V_k = V^u(\widehat{x}(y_k), \pi(y_k))$ for $k = 1, 2, 3$, then condition 2 is also satisfied. The allocation $(\widehat{x}^c(y_1), \widehat{x}^c(y_2), \widehat{x}^c(y_3))$ satisfy all the constraint in (4) when F is evaluated at $(\pi(y_1), V_1)$, $(\pi(y_2), V_2)$, and $(\pi(y_3), V_3)$ respectively. Hence $F(\pi(y_k), V_k) \geq V^c(\widehat{x}^c(y_k), \pi(y_k))$, and using (10), condition 3 is satisfied, as the uninformative equilibrium is constrained Pareto efficient by Proposition 2, i.e. $F(\pi^0, V_0^u) = V^c(\bar{x}^c, \pi^0)$ and $\sum_k \text{pr}(y_k) V^c(\bar{x}^c(y_k), \pi(y_k)) = V_0^c$.

Assume there exist vectors $(\tau_1, \tau_2, \tau_3) \in \Delta^2$, $(\pi_1, \pi_2, \pi_3) \in [0, 1]^3$ and a vector of utility levels $(V_1, V_2, V_3) \in \mathbb{R}^3$ such that conditions 1, 2 and 3 are satisfied. Let $\widehat{x}^c(y_k)$ be the argmax of the optimization problem defining $F(\pi_k, V_k)$, and $\widehat{x}^u(y_k) = w - \widehat{x}^c(y_k)$ for $k = 1, 2, 3$. Then $(\widehat{x}(y_1), \widehat{x}(y_2), \widehat{x}(y_3))$ is constrained feasible. Furthermore $V^u(\widehat{x}^u(y_k), \pi_k) = V_k$, and $V^c(\widehat{x}^c(y_k), \pi_k) = F(\pi_k, V_k)$ for $k = 1, 2, 3$. Then conditions 2, and 3 imply that the allocation $(\widehat{x}(y_1), \widehat{x}(y_2), \widehat{x}(y_3))$ obtains an ex-ante Pareto improvement over the uninformative equilibrium, as $F(\pi^0, V_0^u) = V_0^c = \sum_k \tau_k V^c(\bar{x}^c(y_k), \pi_k)$. □

Appendix D Proof of Corollary 4

We will start from the last part of the corollary. If F is concave, then $\sum_k \text{pr}(y_k)\pi(y_k) = \pi^0$ and $\sum_k \text{pr}(y_k) V_k = V_0^u$ implies $\sum_k \text{pr}(y_k) F(\pi(y_k), V_k) \leq F(\pi^0, V_0^u)$. Therefore we cannot have an ex-ante improvement by Proposition 3. Next we need to show that if there exist an ex-ante Pareto improvement, then cav $F(\pi^0, V_0^u)$ is greater than $F(\pi^0, V_0^u)$. If F coincide with cav F at the point (π^0, V_0^u), then when $\sum_k \text{pr}(y_k)\pi(y_k) = \pi^0$ and $\sum_k \text{pr}(y_k) V_k = V_0^u$ we have $\sum_k \text{pr}(y_k) \text{cav } F(\pi^k, V_k) \leq \text{cav } F(\pi^0, V_0^u) = F(\pi^0, V_0^u)$, as the concavification is concave. But by definition cav $F(\pi^k, V_k) \geq F(\pi^k, V_k)$ for all k, therefore $\sum_k \text{pr}(y_k) F(\pi^k, V_k) \leq F(\pi^0, V_0^u)$. Finally, assume the concavification of F is greater than F at (π^0, V_0^u). Carathéodory's theorem states that if a point m of $\mathbb{R}^3$ lies in the convex hull of the graph of F, then m lies in a 2-simplex with vertices in the graph of F. As the concavification of F is the boundary of the closure of the convex hull of F, cav $F(\pi^0, V_0^u)$ belongs to the boundary of a 2-simplex with vertices in the graph of F. Therefore $(\pi^0, V_0^u, \text{cav}F(\pi^0, V_0^u)) = \sum_{k=1}^3 \alpha_k m_k$, with $\sum_k \alpha_k = 1$, $\alpha_k \geq 0$ for all k, and where $m_k = (\pi_k, V_k, F(\pi_k, V_k))$ is a point in the graph of F for $k = 1, 2, 3$. As cav $F(\pi^0, V_0^u) > F(\pi^0, V_0^u)$, setting $\text{pr}(y_k) = \alpha_k$ we obtain that $\sum_k \text{pr}(y_k) F(\pi^k, V_k) > F(\pi^0, V_0^u)$. □

[22] Otherwise the optimization problem of agent u has no solution.

Appendix E Proof of Proposition 5

We start with the first part of the proposition. Let's work with the model in terms of contingent consumption. Assume the equilibrium under π is full Pareto efficient. Full Pareto optimality in our economy is characterized by full consumption smoothing, i.e. $x_0^h = x_s^h$ for $s = 1, 2$ for all h. From agents' optimality conditions it is easy to see that this allocation implies that equilibrium prices are $(p_0, p_1, p_2) = (1, \pi, 1 - \pi)$, and $x_1^c = \frac{1}{2}(w_0^c + \pi w_1^c + (1 - \pi)w_2^c)$. In equilibrium, as $R(\pi) = 1$, we must have $x_1^c \geq w_1^c + \theta_1 + \theta_3$, hence:

$$\frac{1}{2}(w_0^c + \pi w_1^c + (1 - \pi)w_2^c) \geq w_1^c + \theta_1 + \theta_3. \tag{11}$$

Solving the inequality in (11) for π we obtain:

$$\pi \geq \frac{2\theta_3 + 2(\theta_1 + w_1^c) - w_0^c - w_2^c}{w_1^c - w_2^c} = \underline{\pi}_1. \tag{12}$$

Assume now that $\pi \geq \underline{\pi}_1$. From inequality (12) we can go back to (11) and therefore the allocation $x_s^h = \frac{1}{2}\left(w_0^h + \pi w_1^h + (1 - \pi)w_2^h\right)$ for $s = 0, 1, 2$ and for all h satisfy both lower bounds on c's period 1 consumption, and jointly with prices $(p_0, p_1, p_2) = (1, \pi, 1 - \pi)$ is the only equilibrium when the belief is π. As there is full smoothing across time and states of the world, the equilibrium is fully Pareto efficient.

Now we prove the inverse of the second part of the proposition. First, notice that in any equilibrium where c's lower bound on state one consumption is not binding, then the lower bound on state 2 consumption cannot be binding. As the lower bound on x_2^c is the smallest of the two lower bound, if it were binding, then c could increase her utility by marginally reducing x_1^c and increasing x_2^c to bring both consumptions closer together. Assume now that in equilibrium the lower bounds on c's state one consumption is not binding (and therefore the lower bound on x_2^c is also not binding), then from the optimality conditions of both agents we have:

$$\frac{u'(x_s^h)}{u'(x_0^h)} = \frac{p_s}{\pi_s} \text{ for } s = 1, 2, \ \forall h. \tag{13}$$

Equation (13) implies that the equilibrium allocation satisfies $x_0^h = x_1^h = x_2^h$ for all h. We obtain full Pareto efficiency of the equilibrium. □

Appendix F Proof of Proposition 6

Assume, as a way of contradiction, that there exist $\widehat{x}$ and π^1 such that $V(\widehat{x}^h, \pi^1) \geq V(x^h, \pi^1)$ for all h and with strict inequality for some h. By convexity of CFS(R), if there exist such Pareto improvement, then there exist a marginal improvement. Let's compute the directional derivative of $V(x^u, \pi^1)$:

$$D_\alpha V_0^u = v'(x_0^u)\alpha_0 + \pi^1 v'(x_1^u)\alpha_1 + (1 - \pi^1)v'(x_2^u)\alpha_2, \tag{14}$$

where x_s^h is h's equilibrium consumption in state s in the uninformative equilibrium, with period 0 denoted by $s = 0$. For u to be indifferent with respect to the uninformative equilibrium we need $D_\alpha V_0^u = 0$. When u is indifferent we can solve equation (14) for α_0:

$$\alpha_0 = \frac{-\left(\pi^1 v'(x_1^u)\alpha_1 + (1 - \pi^1)v'(x_2^u)\alpha_2\right)}{v'(x_0^u)}. \tag{15}$$

Computing the directional derivative for c, forcing changes in consumption to be feasible, we obtain:

$$D_\alpha V_0^c = -v'(x_0^c)\alpha_0 - \pi^1 v'(x_1^c)\alpha_1 - (1-\pi^1)v'(x_2^c)\alpha_2. \tag{16}$$

Feasible changes that leave u indifferent have to satisfy (15). Replacing (15) into (16):

$$D_\alpha V_0^c = -\alpha_1\pi^1 v'(x_1^u)\left(\frac{v'(x_1^c)}{v'(x_1^u)} - \frac{v'(x_0^c)}{v'(x_0^u)}\right) - \alpha_2(1-\pi^1)v'(x_2^u)\left(\frac{v'(x_2^c)}{v'(x_2^u)} - \frac{v'(x_0^c)}{v'(x_0^u)}\right).$$

If in the uninformative equilibrium all constrains are binding, then $x_0^h \neq x_1^h \neq x_2^h$. In particular we have $x_1^c > x_2^c > x_0^c$ and $x_0^u > x_2^u > x_1^u$. As v' is decreasing:

$$\frac{v'(x_s^c)}{v'(x_s^u)} - \frac{v'(x_0^c)}{v'(x_0^u)} < 0 \text{ for } s = 1, 2. \tag{17}$$

Therefore, for c to be better off we need at least one $\alpha_s > 0$. This means increasing u's consumption in one of the states in period 1, or reducing c's consumption. But as both lower bounds are assumed to be binding, those changes are not constrained feasible. There is no constrained feasible allocations that attains a Pareto improvement.

If only the constraint on x_1^c is binding in the uninformative equilibrium, then $x_0^h = x_2^h$ for all h, and $x_1^c > x_0^c$. In this case equation (17) still holds for $s = 1$. For $s = 2$ the expression is equal to zero. This implies that c is made better off if and only if $\alpha_1 > 0$, i.e. if and only if u consumption in $s = 1$ is increased. This change is not constrained feasible and we reach the same conclusion: there is no constrained feasible allocations that attains a Pareto improvement. □

Appendix G Proof of Theorem 7

Assume without loss of generality that $R(\pi^1) \geq R(\pi^0)$. Then $\text{CFS}(R(\pi^1)) \subseteq \text{CFS}(R(\pi^0))$. By Proposition 6 there is no feasible allocation in $\text{CFS}(R(\pi^0))$, and so in $\text{CFS}(R(\pi^1))$, that Pareto dominates the uninformative equilibrium when utility is computed using π^1.

Assume that $R(\pi^k) \leq R(\pi^0)$ for $k = 1, 2, 3$. Let x be the uninformative equilibrium allocation. As $x_1^h \neq x_0^h$ for all h, x is not in the Pareto set under any belief. This means that for π^k there exist a different allocation, call it $\widehat{x}$, in the Pareto set, i.e. such that $V(\widehat{x}^h, \pi^k) \geq V(x^h, \pi^k)$ for all h with strict inequality for some h. If $\widehat{x} \in \text{CFS}(R(\pi^k))$ for all k then the uninformative equilibrium is not ex-post constrained Pareto efficient. If $\widehat{x} \notin \text{CFS}(R(\pi^k))$ for some k, consider the allocation $\widetilde{x} = \lambda x + (1-\lambda)\widehat{x}$ with $\lambda \in [0, 1]$. For λ sufficiently close to one, we can make $\widetilde{x} \in \text{CFS}(R(\pi^k))$ and $\widetilde{x} \neq x$. Strict convexity of preferences tells us that $V(\widetilde{x}^h, \pi^k) \geq V(x^h, \pi^k)$ for all h with strict inequality for some h. Thus the uninformative equilibrium is not ex-post constrained Pareto efficient. □

Appendix H Properties of the equilibrium gross interest rate

Lemma 3. *If (A2) is satisfied with equality, i.e.* $w_1^c + \theta_1 = w_2^c + \theta_2$*, then* R *is increasing in* $(0, \underline{\pi})$*.*

Proof. Assume $w_1^c + \theta_1 = w_2^c + \theta_2$ and $\pi \in (0, \underline{\pi})$.[23] Then when both constraints are binding there is full consumption smoothing in period 1. This implies $q_1/q_2 = \pi/(1-\pi)$, or $R = \pi/q_1$. From u's FOC $Rv_1' - v_0' = 0$, where $v_s' \equiv v'(x_s^u)$. As both constraints are binding $x_s^u = w_s^u - \theta_s - \theta_3 R$ for $s = 1, 2$; and

[23] it is easy to check that when $w_1^c + \theta_1 = w_2^c + \theta_2$, then $\underline{\pi}_2 = \underline{\pi}_1$.

$x^u_0 = w^u_0 + \theta_3 + q_1\theta_1 + q_2\theta_2 = w^u_0 + \theta_3 + (\pi\theta_1 + (1-\pi)\theta_2)R^{-1}$. Let $\phi(R,\pi) \equiv Rv'_1 - v'_0$, then using the implicit function theorem:

$$R' = -\frac{\phi_\pi}{\phi_R} = \frac{v''_0(\theta_1 - \theta_2)R^{-1}}{v'_1 + Rv''_1(-\theta_3) - v''_0\left(\frac{-(\pi\theta_1+(1-\pi)\theta_2)}{R^2}\right)} > 0, \tag{18}$$

where the inequality follows from both the numerator and denominator in (18) being positive. Assumption $w^c_1 + \theta_1 = w^c_2 + \theta_2$ implies $\theta_1 - \theta_2 = w^c_2 - w^c_1 < 0$. As $R > 0$ and $v'' < 0$, the numerator is positive. As $\theta_3 > 0$ and $\theta_s \leq 0$ for $s = 1, 2$; the denominator is also positive. □

It is straightforward to show that for all $\pi \geq \underline{\pi}$: $R(\pi) = 1$. The following lemma shows that for all other beliefs the gross interest rate is strictly less than one.

Lemma 4. *If $\pi < \underline{\pi}$, then $q_1 > \pi$, $q_2 \geq 1 - \pi$, and $R(\pi) < 1$.*

Proof. Equilibrium prices follow from u's first order conditions, which are sufficient conditions for utility maximization under our assumptions. These can be written as:

$$\begin{aligned} q_1 &= \pi\frac{v'(x^u_1)}{v'(x^u_0)}, \\ q_2 &= (1-\pi)\frac{v'(x^u_2)}{v'(x^u_0)}. \end{aligned} \tag{19}$$

When $\pi < \underline{\pi}$, the first best allocation is not constrained feasible and so we need to have: $x^c_1 > x^c_0$ and $x^c_2 \geq x^c_0$. This is equivalent to: $x^u_1 < x^u_0$ and $x^u_2 \leq x^u_0$. As $v'' < 0$, from (19) we see that $q_1 > \pi$ and $q_2 \geq 1 - \pi$, and so $q_1 + q_2 > 1$. This implies $R = (q_1 + q_2)^{-1} < 1$. □

When (A2) is satisfied with equality Lemma 3 gives us continuity of R. If (A2) is satisfied with strict inequality the following lemma gives us continuity of R in an interval around $\underline{\pi}$.

Lemma 5. *There exist a neighborhood around $\underline{\pi}$ such that the equilibrium gross interest rate is a continuous function of π.*

Proof. If (A2) holds with equality the result follows from Lemma 3. Assume (A2) holds with strict inequality and let's work with the model in terms of contingent consumption.

When $\pi = \underline{\pi}$ equilibrium prices are the solution to the system:

$$\begin{aligned} \pi v'(x^u_1) - p_1 v'(x^u_0) &= 0, \\ (1-\pi)v'(x^u_2) - p_2 v'(x^u_0) &= 0. \end{aligned}$$

When (A2) holds with strict inequality, the lower bound on x^c_1 is strictly above the lower bound on x^c_2, thus: $x^u_1 = w^u_1 - \theta_1 - \theta_3/(p_1+p_2)$, $x^u_2 = x^u_0 = (w^u_0 + p_2 w^u_2 + p_1(\theta_1 + \theta_3/(p_1+p_2)))/(1+p_2) > w^u_2 - \theta_2 - \theta_3/(p_1+p_2)$, where the last inequality follows from the fact that at $\pi = \underline{\pi}$, the equilibrium features full consumption smoothing.

The prices p_1 and p_2 are the endogenous variables, and π is a parameter. This system has a solution at $\pi = \underline{\pi}$, given by $p_1 = \pi$ and $p_2 = 1 - \pi$. If we can show that the determinant of the Jacobian of endogenous variables is not zero at $\pi = \underline{\pi}$, then the implicit function theorem will give us continuity of equilibrium prices, and therefore continuity of the equilibrium gross interest rate, in a neighborhood of $\underline{\pi}$. The Jacobian of endogenous variables is given by:

$$\begin{pmatrix} \pi v''_1\frac{\partial x^u_1}{\partial p_1} - v'_o - p_1 v''_0\frac{\partial x^u_0}{\partial p_1} & \pi v''_1\frac{\partial x^u_1}{\partial p_2} - p_1 v''_0\frac{\partial x^u_0}{\partial p_2} \\ (1-\pi)v''_2\frac{\partial x^u_2}{\partial p_1} - p_2 v''_0\frac{\partial x^u_0}{\partial p_1} & (1-\pi)v''_2\frac{\partial x^u_2}{\partial p_2} - v'_o - p_2 v''_0\frac{\partial x^u_0}{\partial p_2} \end{pmatrix}, \tag{20}$$

where $v'_s \equiv \frac{\partial u(x^u_s)}{\partial x^u_s}$. Evaluating (20) at $\pi = \underline{\pi}$, where $x^u_0 = x^u_1 = x^u_2$, $q_1 = \pi$ and $q_2 = (1-\pi)$, it simplifies to:

$$\begin{pmatrix} \underline{\pi} v''_1 \left(\frac{\partial x^u_1}{\partial p_1} - \frac{\partial x^u_0}{\partial p_1} \right) - v'_o & \underline{\pi} v''_1 \left(\frac{\partial x^u_1}{\partial p_2} - \frac{\partial x^u_0}{\partial p_2} \right) \\ 0 & -v'_o \end{pmatrix}, \tag{21}$$

The determinant of (21) is given by:

$$-\underline{\pi} v''_1 v'_0 \left(\frac{\partial x^u_1}{\partial p_1} - \frac{\partial x^u_0}{\partial p_1} \right) + \left(v'_0\right)^2 > 0, \tag{22}$$

where the sign of the expression in (22) follows from strict concavity of v, and $\frac{\partial x^u_1}{\partial p_1} - \frac{\partial x^u_0}{\partial p_1} = \frac{\theta_3 - \theta_1}{2-\underline{\pi}} > 0$, as $\theta_3 > 0$ and $\theta_1 \leq 0$.[24] □

Finally we show that (A4) implies that R is not differentiable at $\underline{\pi}$.

Lemma 6. *If $w^c_0 - w^c_2 \neq 2(\theta_1 + \theta_3)$, then R is not differentiable at $\underline{\pi}$.*

Proof. The right derivative of R at $\underline{\pi}$ is equal to zero as $R(\pi) = 1$ for all $\pi \geq \underline{\pi}$. If (A2) is satisfied with equality, the result follows from Lemma 3. Otherwise, in a interval around $\underline{\pi}$ the equilibrium gross interest rate can be define as:

$$R(\pi) = \begin{cases} R^{FB}(\pi), & \text{if } \pi > \underline{\pi}, \\ R^B(\pi), & \text{if } \pi \leq \underline{\pi}. \end{cases}$$

where R^{FB} is the interest rate under first best equilibrium prices, i.e. $R^{FB}(\pi) = 1$ for all π. On the other hand, $R^B(\pi) = (p_1 + 1 - \pi)^{-1}$, where p_1 given by the solution of u's first order conditions when $p_2 = 1 - \pi$ and $x^u_0 = x^u_2$.

This definition help us see that $R'_-(\underline{\pi}) = R'^B(\underline{\pi})$. Using the fact that $R^B(\underline{\pi}) = 1$, $R'^B(\underline{\pi}) = p'_1(\underline{\pi}) - 1$. Therefore we need to show that the first derivative of p_1 with respect to π is different from one. At $\pi = \underline{\pi}$ equilibrium prices follow from the system shown in the proof of Lemma 5. These prices feature $p_2 = 1 - \underline{\pi}$. Using the implicit function theorem:

$$\frac{\partial p_1}{\partial \pi} = \frac{-\left(v'_1 + \pi v''_1 \frac{\partial x^u_1}{\partial \pi} - p_1 v''_0 \frac{\partial x^u_0}{\partial \pi} \right)}{\pi v''_1 \frac{\partial x^u_1}{\partial p_1} - v'_0 - p_1 v''_0 \frac{\partial x^u_0}{\partial p_1}}.$$

In the limit $v'_0 = v'_1$, and it is easy to check that $\frac{\partial x^u_1}{\partial \pi} = -\frac{\partial x^u_1}{\partial p_1}$. Therefore $\partial p_1 / \partial \pi = 1$ if and only if $\frac{\partial x^u_0}{\partial \pi} = -\frac{\partial x^u_0}{\partial p_1}$. In equilibria where $p_2 = 1 - \pi$ and the constraint on x^c_1 is binding we have:

$$x^u_0 = \frac{1}{2-\pi} \left(w^u_0 + (1-\pi) w^u_2 + p_1 \left(\theta_1 + \frac{\theta_3}{p_1 + 1 - \pi} \right) \right).$$

Taking derivatives with respect to p_1 and π:

$$\frac{\partial x^u_0}{\partial p_1} = \frac{1}{2-\pi} \left(\theta_1 + \frac{\theta_3}{p_1 + 1 - \pi} \right) - \frac{1}{2-\pi} \left(\frac{p_1 \theta_3}{(p_1 + 1 - \pi)^2} \right),$$
$$\frac{\partial x^u_0}{\partial \pi} = \frac{x^u_0}{2-\pi} + \frac{1}{2-\pi} \left(-w^u_2 + \frac{p_1 \theta_3}{(p_1 + 1 - \pi)^2} \right).$$

So, $\frac{\partial x^u_0}{\partial \pi} = -\frac{\partial x^u_0}{\partial p_1}$ if and only if:

$$x^u_0 = w^u_2 - \theta_1 - \theta_3,$$
$$w^u_0 + (1-\underline{\pi}) w^u_2 + \underline{\pi}\theta_1 + \underline{\pi}\theta_3 = (2-\underline{\pi})(w^u_2 - \theta_1 - \theta_3),$$
$$w^u_0 - w^u_2 = -2(\theta_1 + \theta_3).$$

□

[24] $\frac{\partial x^u_1}{\partial p_1} = \frac{\theta_3}{(p_1+p_2)^2}$; $\frac{\partial x^u_0}{\partial p_1} = \frac{\theta_3 p_2 + \theta_1 (p_1+p_2)^2}{(1+p_2)(p_1+p_2)^2}$. Evaluating at $\underline{\pi}$, where $p_1 + p_2 = 1$: $\frac{\partial x^u_1}{\partial p_1} = \theta_3$; $\frac{\partial x^u_0}{\partial p_1} = \frac{\theta_1 + (1-\underline{\pi})}{2-\underline{\pi}}$.

Appendix I Proof of Proposition 9

Here we provide a proof for the case when (A2) is satisfied with strict inequality. First we will argue that f and its argmax are continuous functions at $\underline{\pi}$.

Lemma 7. *Fix π_0. Let π be in the interval around $\underline{\pi}$ where R is continuous and assume f is defined at $f(\pi, \pi^0)$. The function $f(\pi, \pi^0)$ is continuous in π. Let x^* be the argmax of $f(\pi, \pi^0)$, then x^* is a continuous function of π.*

Proof. The function v is strictly increasing, this implies that the last constraint in the definition of f in equation (5) will always bind at a solution. When that equation holds with equality we can solve it for x_0^c obtaining x_0^c as a function of π, π^0 and x_s^c for $s = 1, 2$:

$$x_0^c = g(x_1^c, x_2^c, \pi, \pi^0) \equiv w - v^{-1}\left(V^u(\pi^0) - \sum_s \pi_s v(w - x_s^c)\right).$$

The function f coincides with the function f^1 defined as:

$$f^1(\pi, \pi^0) = \Big\{ \underset{\{x_1^c, x_2^c\}}{\text{Max}} \quad v(g(x_1^c, x_2^c, \pi, \pi^0)) + \sum \pi_s v(x_s^c)$$
$$\text{s. t.} \quad x_s^c \geq w_s^c + \theta_s + \theta_3 R(\pi) \text{ for } s = 1, 2 \Big\}.$$

We will argue that f^1 is continuous. Consider the correspondence:

$$\Gamma(\pi) = \left\{(x_1^c, x_2^c) \in [0, w]^2 \,|x_s^c - w_1^c - \theta_1 - \theta_3 R(\pi) \geq 0 \ \forall s = 1, 2\right\},$$

Theorem 2.2. in chapter 7 in De la Fuente [8] tells us that if the functions defining $\Gamma(\pi)$ are continuous, concave in (x_1^c, x_2^c) for a given π, if $\Gamma(\pi)$ is compact and if there exist $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \in \Gamma(\pi)$ such that $\widehat{x}_s - w_1^c - \theta_1 - \theta_3 R(\pi) > 0 \ \forall s = 1, 2$, then $\Gamma(\pi)$ is continuous at π. Clearly the functions defining Γ are continuous and concave in consumption, and given a belief the set $\Gamma(\pi)$ is compact. Assumption (A3) implies $w_1^c + \theta_1 + \theta_3 R(\pi) < w$, therefore the point $\widehat{x} = (w, w)$ satisfy both conditions with strict inequality. Hence $\Gamma(\pi)$ is continuous in π.

As the function v is continuous and strictly increasing, its inverse is continuous. This gives us continuity of the objective function defining f^1. By Berge's theorem of the maximum the function $f^1(\pi, \pi^0)$ is continuous in π and its argmax is nonempty and upper hemicontinuous. As f^1 and f (and their argmax) are equivalent, we obtain continuity of f and upper hemicontinuity of x^*.

Now we will argue that x^* is single valued for a given π, therefore obtaining continuity of x^*. For this we work directly with f. The maximization problem in f is characterized by a constraint set that is convex and a objective function that is strictly concave in consumption for a given π, therefore x^* is the unique optimal solution. □

Proof of Proposition 9:

The derivative of f with respect to π (where it exists) is given by:

$$\frac{\partial f(\pi, \pi^0)}{\partial \pi} = v(x_1) - v(x_2) - \theta_3 R'(\pi)(\lambda_1 + \lambda_2) + \frac{v'(x_0)}{v'(w - x_0)}\left(v(w - x_1) - v(w - x_2)\right),$$

where λ_1 and λ_2 are the Lagrange multipliers associated with the constraints on x_1^c and x_2^c respectively. For all beliefs strictly above $\underline{\pi}$, $R'(\pi) = 0$. At $\underline{\pi}$ the right derivative of R is equal to zero and the left derivative is non-negative. Solving the maximization problem defining f we obtain:

$$\lambda_1 = \frac{v'(x_0^c)}{v'(w - x_0^c)}\pi v'(w - x_1^c) - \pi v'(x_1^c),$$

a continuous function of π. If both lower bounds on consumption are not binding, then $\lambda_1 = \lambda_2 = 0$, and it is straightforward to show that the Planner assigns an allocation featuring full smoothing across time and states. At $\underline{\pi}$ the CFS is a proper subset of CFS($R(\pi^0)$) ($R(\pi^0) < 1$). Therefore at $\underline{\pi}$ we must have $\lambda_1 > 0$, otherwise the first best allocation is constrained feasible in the uninformative equilibrium, violating constrained Pareto optimality of the uninformative equilibrium. By continuity, for π slightly below $\underline{\pi}$ it is still true that $\lambda_1 > 0$. If $R'_-(\underline{\pi}) \neq 0$, where R'_- denotes the left derivative of R, then we have:

$$v'_-(\underline{\pi}, \pi^0) - v'_+(\underline{\pi}, \pi^0) = -\theta_3 R'_-(\underline{\pi})(\lambda_1 + \lambda_2) < 0.$$

$R'_-(\underline{\pi})$ is different from zero by Lemma 6. We want to show that there exist $\epsilon > 0$ such that:

$$f(\underline{\pi}, \pi^0) < 0.5 f(\underline{\pi} + \epsilon, \pi^0) + 0.5 f(\underline{\pi} - \epsilon, \pi^0) \tag{23}$$

Taking a first order Taylor approximation around $\underline{\pi}$ we can write $f(\underline{\pi} + \epsilon)$ and $f(\underline{\pi} - \epsilon)$ as:

$$f(\underline{\pi} + \epsilon, \pi^0) = f(\underline{\pi}, \pi^0) + f'_+(\underline{\pi}, \pi^0)\epsilon + o(\epsilon)$$
$$f(\underline{\pi} + \epsilon, \pi^0) = f(\underline{\pi}, \pi^0) - f'_-(\underline{\pi}, \pi^0)\epsilon + o(\epsilon)$$

where $o(\epsilon)$ represents the remainder. If (23) is not true then:

$$\begin{aligned} 0 &\geq \epsilon(f'_+(\underline{\pi}, \pi^0) - f'_-(\underline{\pi}, \pi^0)) + o(\epsilon) \\ 0 &\geq (f'_+(\underline{\pi}, \pi^0) - f'_-(\underline{\pi}, \pi^0)) + \frac{o(\epsilon)}{\epsilon} \end{aligned} \tag{24}$$

When $R'_- \neq 0$, $f'_+(\underline{\pi}, \pi^0) - f'_-(\underline{\pi}, \pi^0) > 0$. If the approximation error in the Taylor approximation is non-negative, then (24) is a contradiction. If the approximation error is negative, then as $\frac{o(\epsilon)}{\epsilon}$ tends to zero as ϵ goes to zero, for ϵ small enough we can make the modulus of $\frac{o(\epsilon)}{\epsilon}$ smaller than $f'_+(\underline{\pi}, \pi^0) - f'_-(\underline{\pi}, \pi^0)$ and again we obtain a contradiction. □

The next lemma provides a sufficient condition for f to be defined at $(\underline{\pi}, \pi^0)$.

Lemma 8. *There exist $\delta > 0$ such that if $\pi^0 \in (\underline{\pi} - \delta, \underline{\pi})$, then v is defined at $(\underline{\pi}, \pi^0)$*

Proof. Let $x_s^u(\pi)$ be u's equilibrium allocation when the belief is π for $s = 0, 1, 2$. When the belief is equal to $\underline{\pi}$ we have: $x_s^u(\underline{\pi}) = w_1^u - \theta_1 - \theta_3$ for $s = 0, 1, 2$, by definition of $\underline{\pi}$. This allocation is constrained feasible by Proposition 1. Define $\underline{\alpha}^u$ as:

$$\underline{\alpha}^u(\pi) = v^{-1}\left(V(x^u(\pi), \pi) - v(x_1^u(\underline{\pi}))\right),$$

i.e. $\underline{\alpha}^u(\pi)$ is the period 0 consumption that leaves u indifferent between the equilibrium under belief π, and consuming $(\underline{\alpha}^u(\pi), x_1^u(\underline{\pi}), x_1^u(\underline{\pi}))$. Clearly $\underline{\alpha}^u(\underline{\pi}) = x_0^u(\underline{\pi})$, hence $\underline{\alpha}^u(\underline{\pi}) \in (0, w)$. If the equilibrium interest rate is continuous in π, so it is each component of the vector of equilibrium allocations. Therefore $\underline{\alpha}^u(\pi)$ is continuous in π. This implies that there exist a $\delta > 0$ such that for all $\pi \in (\underline{\pi} - \delta, \underline{\pi})$: $\underline{\alpha}^u(\pi) \in (0, w)$. Hence the set defined by the constraints in the definition of f is not empty for all $\pi^0 \in (\underline{\pi} - \delta, \underline{\pi})$. Continuity of R in an interval around $\underline{\pi}$ is proved in Lemma 5 in appendix H. □

Appendix J Proof of Theorem 10

First we argue that if π^0 is close to $\underline{\pi}$ then $\underline{\alpha}(\pi^0)$ and $\overline{\alpha}(\pi^0)$ are well defined.

Lemma 9. *There exist a $\delta > 0$ such that if $\pi^0 \in (\underline{\pi} - \delta, \underline{\pi})$, then $\underline{\alpha}(\pi^0)$ and $\overline{\alpha}(\pi^0)$ are in $[0, w]$.*

Proof. $\underline{\alpha}(\pi^0) \in [0, w]$ follows from Lemma 8, as $\underline{\alpha}(\pi^0) = w - \underline{\alpha}^u(\pi^0)$. Define $\overline{\alpha}^u$ as:

$$\overline{\alpha}^u(\pi) = v^{-1}\left(\frac{V(x^u(\pi), \pi)}{2}\right).$$

clearly $\overline{\alpha}^u(\underline{\pi}) = x^u_s(\underline{\pi})$ for $s = 0, 1, 2$, hence $\overline{\alpha}^u(\underline{\pi}) \in (0, w)$. If the equilibrium interest rate is continuous in π, so it is each component of the vector of equilibrium allocations. Therefore $\overline{\alpha}^u(\pi)$ is continuous in π. This implies that there exist a $\delta > 0$ such that for all $\pi \in (\underline{\pi} - \delta, \underline{\pi})$: $\overline{\alpha}^u(\pi) \in (0, w)$. But then $\overline{\alpha}(\pi^0) = w - \overline{\alpha}^u(\pi^0) \in [0, w]$. □

Now we will show that $\overline{\pi}(\epsilon)$ is continuous and well defined around $\epsilon = 0$. For simplicity let's assume that $\pi^0(\epsilon) = \underline{\pi} - \epsilon$ with $\epsilon = [-\gamma, \underline{\pi}]$, where γ a small positive number such that $\underline{\pi} + \gamma < 1$. We do this to simplify notation and make π^0 differentiable at $\epsilon = 0$.

Lemma 10. *Assume that (A2) holds with equality, or that (A2) holds with strict inequality and (A4) holds ($R'_-(\underline{\pi}) \neq 0$), then there exist a $\widehat{\epsilon} > 0$ such that $\overline{\pi}(\epsilon)$ is continuous for all $\epsilon \in [0, \widehat{\epsilon})$.*

Proof. We start with the case when (A2) holds with strict inequality. Let v^{-1} be the inverse of v. By the inverse function theorem v^{-1} is continuously differentiable. The belief $\overline{\pi}(\epsilon)$ is the π that solves the following equation:

$$w - v^{-1}\left(\frac{V^u(\pi^0(\epsilon))}{2}\right) = w^c_1 + \theta_1 + \theta_3 R(\pi). \tag{25}$$

At $\epsilon = 0$ the solution to (25) is $\pi = \underline{\pi} \in (0, 1)$. In appendix H we argued that for all $\pi \leq \pi^0$ the equilibrium gross interest rate is given by R^B. We also showed that R^B is differentiable at $\underline{\pi}$, and its derivative is, by definition, equal to $R'_-(\underline{\pi})$. Therefore we can rewrite (25) as:

$$w - v^{-1}\left(\frac{V^u(\pi^0(\epsilon))}{2}\right) = w^c_1 + \theta_1 + \theta_3 R^B(\pi). \tag{26}$$

As all the functions in (26) are continuously differentiable and $R^{B'}$ is different from zero at $\epsilon = 0$ by (A4), the implicit function theorem gives us continuity of $\overline{\pi}(\epsilon)$ in an interval around $\epsilon = 0$. Continuity tells us that $\overline{\pi}(\epsilon) \in (0, 1)$ for all ϵ in an interval around $\epsilon = 0$. When (A2) is satisfied with equality, monotonicity of R^B allows us to use the implicit function theorem and draw the same conclusions. □

Properties of g:

Notice that $g(a, 0) = 0$ for all a. When $\epsilon = 0$, then $\pi^0 = \overline{\pi} = \underline{\pi}$, therefore $\overline{V}(0) = V_0(0) = \underline{V}(0)$. Next, we will show that the right derivative of g with respect to ϵ at $(a, 0)$ is zero for all a.

Claim 1. $\lim_{\epsilon \to 0^+} g_\epsilon(a, \epsilon) = 0$ *for all a.*

Proof. Taking the first derivative of g with respect to ϵ:

$$\frac{\partial g(a, \epsilon)}{\partial \epsilon} = a\overline{V}'(\epsilon) + (1 - a)\underline{V}'(\epsilon) - V_0'(\epsilon).$$

Remember that the functions $\overline{V}(\epsilon)$, $\underline{V}(\epsilon)$ and $V_0(\epsilon)$ are given by:

$$\begin{aligned}\overline{V}(\epsilon) &= 2v(\overline{x}(\epsilon)),\\ \underline{V}(\epsilon) &= v(\underline{x}(\epsilon)) + v(w_1^c + \theta_1 + \theta_3),\\ V_0(\epsilon) &= v(x_0(\epsilon)) + \sum_s \pi_s^0 v(x_s(\epsilon)).\end{aligned} \tag{27}$$

where $x_0(\epsilon)$ is c's uninformative equilibrium allocation in period 0 when prior is $\pi^0 = \underline{\pi} - \epsilon$, similarly $x_s(\epsilon)$ is c's uninformative equilibrium allocation in state s. Consumption levels $\overline{x}(\epsilon)$ and $\underline{x}(\epsilon)$ are defined by:

$$\begin{aligned}2v(w - \overline{x}(\epsilon)) &= v(w - x_0(\epsilon)) + \sum_s \pi_s^0(\epsilon) v(w - x_s(\epsilon)),\\ v(w - \underline{x}) + v(w_1^u - \theta_1 - \theta_3) &= v(w - x_0(\epsilon)) + \sum_s \pi_s^0(\epsilon).v(w - x_s(\epsilon))\end{aligned} \tag{28}$$

Differentiating the functions defined in (27) with respect to ϵ:

$$\begin{aligned}\overline{V}'(\epsilon) &= 2v'(\overline{x}(\epsilon))\overline{x}'(\epsilon),\\ \underline{V}'(\epsilon) &= v'(\underline{x}(\epsilon))\underline{x}'(\epsilon),\\ V_0'(\epsilon) &= v'(x_0(\epsilon))x_0'(\epsilon) + \sum_s \pi_s^0 v'(x_s(\epsilon))x_s'(\epsilon) - v(x_1(\epsilon)) + v(x_2(\epsilon)).\end{aligned} \tag{29}$$

Using (28) we can get an expressions for $\overline{x}'(\epsilon)$ and $\underline{x}'(\epsilon)$:

$$\begin{aligned}-2v'(w - \overline{x}(\epsilon))\overline{x}'(\epsilon) =& - v'(w - x_0(\epsilon))x_0'(\epsilon) - \sum_s \pi_s^0 v'(w - x_s(\epsilon))x_s'(\epsilon) +\\ & v(w - x_1(\epsilon)) - v(w - x_2(\epsilon)),\\ -v'(w - \underline{x}(\epsilon))\underline{x}'(\epsilon) =& - v'(w - x_0(\epsilon))x_0'(\epsilon) - \sum_s \pi_s^0 v'(w - x_s(\epsilon))x_s'(\epsilon) +\\ & v(w - x_1(\epsilon)) - v(w - x_2(\epsilon)).\end{aligned} \tag{30}$$

When ϵ goes to 0^+: $x_0 = x_1 = x_2 = \overline{x} = \underline{x}$. Let $\phi(0^+) \equiv \lim_{\epsilon\to 0^+} \phi(\epsilon)$. In the limit we can rewrite (30) as:

$$\begin{aligned}2\overline{x}'(0^+) &= x_0'(0^+) + \sum_s \underline{\pi}_s x_s'(0^+),\\ \underline{x}'(0^+) &= x_0'(0^+) + \sum_s \underline{\pi}_s x_s'(0^+).\end{aligned} \tag{31}$$

Evaluating (29) at $x_0 = x_1 = x_2 = \overline{x} = \underline{x}$ and using (31) we have:

$$\overline{V}'(0^+) = V_0'(0^+) = \underline{V}'(0^+).$$

□

Next, we show that $g(1, 0)$ is a strict local minimum.

Claim 2. *If* $\lim_{\epsilon\to 0^+} (x_0'(\epsilon) - x_1'(\epsilon)) \neq 0$ *holds, then* $\lim_{\epsilon\to 0^+} g_{\epsilon\epsilon}(1, \epsilon) > 0$.

Proof. Notice that $g_{\epsilon\epsilon}(1, \epsilon) = \overline{V}''(\epsilon) - V_0''(\epsilon)$. Let us now compute the second derivatives of V_0, and $\overline{V}$:

$$\begin{aligned}V_0''(\epsilon) =& v''(x_0(\epsilon))x_0'(\epsilon)^2 + v'(x_0(\epsilon))x_0''(\epsilon) + \sum_s \pi_s^0 \left(v''(x_s(\epsilon))x_s'(\epsilon)^2 + v'(x_s(\epsilon))x_s''(\epsilon)\right) +\\ & 2\sum_s (-1)^s v'(x_s(\epsilon))x_s'(\epsilon),\\ \overline{V}''(\epsilon) =& 2v''(\overline{x}(\epsilon))\overline{x}'(\epsilon)^2 + 2v'(\overline{x}(\epsilon))\overline{x}''(\epsilon).\end{aligned}$$

where we have used the fact that the second derivative of π^0 with respect to ϵ is zero. The second derivative of $\overline{x}(\epsilon)$ has to satisfy:

$$
\begin{aligned}
&2v''(w-\overline{x}(\epsilon))\overline{x}'(\epsilon)^2 - 2v'(w-\overline{x}(\epsilon))\overline{x}''(\epsilon) = v''(w-x_0(\epsilon))x_0'(\epsilon)^2 - \\
&v'(w-x_0(\epsilon))x_0''(\epsilon) + \sum_s \pi_s^0\left(v''(w-x_s(\epsilon))x_s'(\epsilon)^2 - v'(w-x_s(\epsilon))x_s''(\epsilon)\right) + \\
&2\sum_s (-1)^s v'(w-x_s(\epsilon))x_s'(\epsilon).
\end{aligned} \tag{32}
$$

Now we can compute $\overline{V}''(0^+) - V_0''(0^+)$:

$$
\begin{aligned}
\overline{V}''(0^+) - V_0''(0^+) =& v''(x_0(0^+))\left(2\overline{x}'(0^+)^2 - x_0'(0^+)^2 - \sum_s \pi_s^0 x_s'(0^+)^2\right) + \\
& v'(x_0(0^+))\left(2\overline{x}''(0^+) - x_0''(0^+) - \sum_s \pi_s^0 x_s''(0^+)\right) - \\
& 2v'(x_0(0^+))\sum_s(-1)^s x_s'(0^+).
\end{aligned} \tag{33}
$$

Using (32):

$$
\begin{aligned}
2\overline{x}''(0^+) - x_0''(0^+) - \sum_s \pi_s^0 x_s''(0^+) =& \frac{v''(w-x_0(0^+))}{v'(w-x_0(0^+))}\Bigg(2\overline{x}'(0^+)^2 - x_0'(0^+)^2 - \\
& \sum_s \pi_s^0 x_s'(0^+)^2\Bigg) + 2\sum_s(-1)^s x_s'(0^+).
\end{aligned} \tag{34}
$$

Equations (33) and (34) imply:

$$
\overline{V}'' - V_0'' = \left(v''(x_0(0^+) + v'(x_0(0^+))\frac{v''(w-x_0(0^+))}{v'(w-x_0(0^+))}\right)\left(2\overline{x}'(0^+)^2 - x_0'(0^+)^2 - \sum_s \pi_s^0 x_s'(0^+)^2\right).
$$

As $v'' < 0$, $g_{\epsilon\epsilon}(1, 0^+) > 0$ if and only if $2\overline{x}'(0^+)^2 - x_0'(0^+)^2 - \sum_s \pi_s^0 x_s'(0^+)^2 < 0$. Assume (A2) is satisfied with strict inequality, then around $\underline{\pi}$ we have $x_0 = x_2$, and $x_0' = x_2'$ (See appendix H). Using (31) to replace $\overline{x}'(0^+)$:

$$
\begin{aligned}
x_0'(0^+)^2 + \sum_s \pi_s^0 x_s'(0^+)^2 - \frac{1}{2}\left(x_0'(0^+) + \sum_s \pi_s^0 x_s'(0^+)\right)^2 &> 0, \\
(2-\pi)x_0'(0^+)^2 + \pi x_1'(0^+)^2 - \frac{1}{2}\left((2-\pi)x_0'(0^+) + \pi x_1'(0^+)\right)^2 &> 0, \\
(2-\pi)\pi x_0'(0^+)^2 + (2-\pi)\pi x_1'(0^+)^2 - 2(2-\pi)\pi x_0'(0^+)x_1'(0^+) &> 0, \\
x_0'(0^+)^2 + x_1'(0^+)^2 - 2x_0'(0^+)x_1'(0^+) &> 0, \\
\left(x_0'(0^+) - x_1'(0^+)\right)^2 &> 0.
\end{aligned} \tag{35}
$$

If (A2) is satisfied with equality instead, then we have $x_1 = x_2$ and $x_1' = x_2'$, and the same conclusion follows. □

Now we characterize when is it that condition $\lim_{\epsilon\to 0^+}(x_0'(\epsilon) - x_1'(\epsilon)) \neq 0$ holds.

Claim 3. $\lim_{\epsilon\to 0^+}(x_0'(\epsilon) - x_1'(\epsilon)) = 0$ *if and only if* $R_-'(\underline{\pi}) = 0$.

Proof. In an interval around $\underline{\pi}$ equilibrium prices are such that $p_2 = 1-\pi$ and R follows from u's F.O.C:

$$
Rv_1' - v_0' = 0,
$$

where $v'_s = \partial u(x^u_s)/\partial x^u_s$. Differentiating this F.O.C. with respect to ϵ and taking the limit as ϵ approaches 0^+:

$$-R'_-(\underline{\pi})v'_1 + v''_1 x'_1(0^+) - v''_0 x'_0(0^+) = 0,$$

$$-v'_0 R'_-(\underline{\pi}) + v''_0(x'_1(0^+) - x'_0(0^+)) = 0.$$

Therefore $x'_1(0^+) = x'_0(0^+)$ if and only if $R'_-(\underline{\pi}) = 0$. □

From Claim 3 we see that (A2) being satisfied with equality, or assumption (A4) are sufficient conditions for g to attain a strict local minimum at $(1, 0)$. This gives us the following lemma:

Lemma 11. *If (A2) is satisfied with equality, or if (A4) holds, then* $\lim_{\epsilon\to 0} \tau = (1 - \tau_3)\overline{a} < 1 - \tau_3$

Proof. The second derivative of g with respect to ϵ: $g_{\epsilon\epsilon} = a\overline{V}''(\epsilon) + (1 - a)\underline{V}''(\epsilon) - V''_0(\epsilon)$ is continuous in a. As $g_{\epsilon\epsilon}(1, 0) > 0$, continuity in a implies that there exist $\overline{a} < 1$ such that $g_{\epsilon\epsilon}(\overline{a}, 0) > 0$. Using claims 1 and 2, $g(\overline{a}, \epsilon)$, as a function of ϵ, attains a strict local minimum at $(\overline{a}, 0)$ with $g(\overline{a}, 0) = 0$. Therefore on a neighborhood of $\epsilon = 0$, $g(\overline{a}, \epsilon) > 0$ for all ϵ. Thus $\overline{a}\overline{V}(\epsilon) + (1 - \overline{a})\underline{V}(\epsilon) - V_0(\epsilon) > 0$ for all ϵ in the interval. Solving for $\overline{a}$:

$$1 > \overline{a} > \frac{V_0(\epsilon) - \underline{V}(\epsilon)}{\overline{V}(\epsilon) - \underline{V}(\epsilon)} \ \forall \epsilon \in [0, \overline{\epsilon}). \tag{36}$$

But (36) implies that the limit of $\widehat{\tau}/(1 - \tau_3)$ as ϵ tends to 0 is strictly lower than one, giving us the result in the lemma. □

<u>Proof of Theorem 10</u>:

Set $\mu < (1 - \tau_3)\frac{1-\overline{a}}{2}$, where $\overline{a} = \lim_{\epsilon\to 0} \widehat{\tau}/(1 - \tau_3) < 1$. For this μ there exist a $\widehat{\delta}$ such that for all $\epsilon \in [0, \widehat{\delta})$, $|\widehat{\tau} - \overline{a}(1 - \tau_3)| < \mu$. Similarly, for this μ there exist a $\widetilde{\delta}$ such that for all $\epsilon \in [0, \widetilde{\delta})$, $|\widetilde{\tau} - (1 - \tau_3)| < \mu$. Set $\delta = \min\{\widehat{\delta}, \widetilde{\delta}\}$, then for all $\epsilon \in [0, \delta)$ we have:

$$\begin{aligned} \overline{a}(1 - \tau_3) - \mu < \widehat{\tau} < \overline{a}(1 - \tau_3) + \mu, \\ (1 - \tau_3) - \mu < \widetilde{\tau} < (1 - \tau_3) + \mu. \end{aligned} \tag{37}$$

Equation (37) imply that for all such ϵ, $\widehat{\tau} < \widetilde{\tau}$ as $a(1 - \tau_3) + \mu < (1 - \tau_3) - \mu$. □

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