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When do co-located firms selling identical products thrive?

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Abstract

When consumers only see prices once they visit stores, and some consumers have time to comparison shop, co-location commits stores to compete and lower prices, which draws consumers away from isolated stores. Profits of co-located firms are a single-peaked function of the number of shoppers—co-located firms thrive when there are some shoppers, but not too many. When consumers know in advance whether they have time to shop, effects are enhanced: co-located stores may draw enough shoppers to drive the expected price paid by a non-shopper below that paid when consumers do not know if they will have time to shop.

1 Introduction

Lincoln Square Mall (LSM), built in Urbana in the mid 1960s, was one of the first downtown fully-enclosed malls in the United States. Since then, a series of three "anchor" department stores and two groceries failed, and until the recent entry of a Common Ground Co-op, no major store had located there (http://deadmalls.com/malls/lincoln_square_mall.html). Standard spatial theories of firm competition suggest that the stores *should* have thrived. Indeed, LSM is located at a prime population center, access is easy, rents were low, and there were no nearby competing stores.

In sharp contrast, on the outskirts of neighboring Champaign, far from population centers, Walmart and Meijer profitably co-exist on opposing sides of the same street, as do Lowes and Home Depot. Standard theory suggests that competition between closely-located firms selling almost identical products *should* drive their profits down. In the canonical spatial model (Tirole 2001, p. 281), an increase in market share from locating closer to a rival is more than offset by the heightened price competition—firms do best to separate maximally.

Why then despite violating this prescription, have Walmart and Meijer thrived? The phenomena illustrated by these anecdotes is widespread. A large literature highlights the hollowing out of town centers (e.g., Iaria 2014), and co-location of competing firms at strip malls (e.g., Page and Tessier 2007), with policy responses seeking to offset this (e.g., the 1996 Town Centres First policy in England).

Our paper asks: when do co-located firms selling identical products earn higher profits than isolated firms? We model a spatial city structure, with one firm located at one end of a line, and two firms located at the other. Consumers, located inbetween, must decide where to make their purchases. Were consumers to see prices at the outset, the co-located firms would compete prices down to marginal cost, attracting consumers from far away,[1] but earning zero profits. Thus, with observable prices, co-location always harms stores.

However, in practice, almost all consumers only see prices set when they enter a store, and some consumers only have time to go to one store. But, then while the isolated firm will monopoly price its captured customers, the co-located firms will still compete on price to attract shoppers who have time to comparison shop. This makes consumers willing to travel further to shop at co-located stores, but the price competition is less severe than when prices

[1]Nystrom (1915) is perhaps the first to observe that "Stores that sell exactly the same kinds of goods and ... are clearly competitive do not merely divide the business that was formerly done by one store. When there is known to be competition this in itself attracts trade, and people come from farther away" (p.144).

are ex-ante observable. This opens up the possibility that the co-located stores may profit from their implicit commitment to lower prices. Our paper characterizes how the price competition between co-located firms—which hinges on the attribute composition of consumers who go to the cluster—interacts with the price-elasticity of the consumers' decisions of where to shop to determine the profits of isolated and co-located firms. Greater shares of comparison shoppers, which cause the co-located stores to offer stochastically lower prices (Rosenthal 1980; Varian 1980), and lower travel costs both induce more-distantly located consumers go to the co-located stores, at the expense of the isolated store.

Determining whether the co-located stores do better than the isolated store is more subtle, hinging on how many consumers comparison shop. To see why, observe that if no one has time to compare prices, then the co-located stores monopoly price, each earning half the profit of the isolated store. We prove that from this base, as the share of consumers who are "shoppers" rise, so do co-located store profits even though they compete more aggressively.

The economic logic underlying this result is keen. With few shoppers, distance is the key driver of where individuals shop. As a result, the measure of consumers drawn to the cluster by increased price competition is very elastic—marginal consumers have similar travel costs regardless of where they go. Thus, the price competition due to more shoppers draws enough customers to more than offset the reduced profit per customer, causing co-located firm profits to rise. Conversely, as the proportion of shoppers goes to one, co-located stores compete all profits away, earning less than the isolated store. When individual consumer demand is linear, profits of co-located stores are a concave, single-peaked function of the share of consumers who are shoppers. We show that for any given proportion of shoppers between 0 and 1, co-located stores make more profit than the isolated store when travel costs are below a threshold. With linear demand, this threshold is single-peaked in the proportion of shoppers, so that, fixing travel costs, the profits of co-located stores are highest when there are some comparison shoppers, but not too many. Our model is consistent with the empirical evidence that travel costs are a salient factor in shopping decisions (Marshall and Pires 2018), and that they have fallen in recent decades.[2]

Our base setting considers consumers who only learn whether they can comparison shop *after* they arrive at a location, for example, because they don't know how much time they have until then. By way of illustration, a family shopping with young children may not

[2]For example, according to AAA, the average composite cost per mile for driving 15,000 miles/year has dropped from \$0.47 in 1999 to \$0.40 (1999 dollar, \$0.62 nominal) in 2019, i.e., an almost 15% drop on top of the additional amenities of newer cars that make driving more enjoyable. Data source: https://exchange.aaa.com/automotive/driving-costs/#.XxiAwp5Kg2w

know how much time they will have to shop—there is a chance that kids may scream about going to a second store—and this, together with the uncertain length of time that it takes to find items at the first store visited, may determine whether they have time to shop at multiple stores. We contrast this setting with one where consumers know whether they have time to comparison shop *before* making travel choices, in order to glean how this consumer information affects outcomes.

When consumers know at the outset whether they have time to shop, the share of shoppers at the cluster always rises because shoppers gain more from price competition. In turn, relative to the base setting, this enhances price competition between co-located firms, which draws more consumers overall, always harming the isolated firm. In contrast, the impact of consumers knowing their shopping type in advance on co-located firm profit hinges on the number of shoppers. We prove that co-located firms earn higher profits when consumers know their shopping types in advance *if* there are sufficiently few shoppers, but they are harmed by the enhanced competition, when there are too many. With few shoppers, the expected price paid at a cluster store by a consumer who *knows* he lacks time to comparison shop falls below that paid by a consumer in the base setting who could be a shopper—so, a consumer who *knows* he has no time to shop is more willing to travel to cluster stores than a consumer in the base setting who does not know. The opposite happens when most consumers are shoppers, so an uninformed consumer expects to be a shopper, and hence expects to get the lowest price offered by the co-located stores. The consequences for co-located firm profits of having consumers know their shopping types in advance follow directly.

Finally, we show that our qualitative findings extend when we fully endogenize the spatial location of the three stores. To avoid edge effects associated with a line, we use a circle spatial structure. We show that equilibrium features a cluster of two stores and a third maximally-separated store when there are some shoppers, but not too many, and travel costs are intermediate. This reflects that when travel costs are extremely low, clustering incentives are so strong that all firms co-locate; while when travel costs are very high, the price competition due to clustering does not draw enough additional customers. We then characterize when the co-located stores earn more than the isolated store.

Literature. The vast spatial industrial organization literature in which greater spatial separation enhances firm profit is well-known. Our model builds on the search-cost literature that gives rise to price dispersion (Shilony 1977; Rosenthal 1980; Varian 1980; Burdett and Judd, 1983; Stahl, 1989, 1996; Ellison and Ellison, 2009; Ellison and Wolitzky, 2012; Ronayne 2018). Armstrong (2016) surveys recent advances in directed search models with heterogeneous sellers. Our paper integrates spatial features to costly search models and investigates the consequences of information structures for consumers (what they know about prices and their abilities to search when making shopping decisions). Constantinou and Bernhardt (2018) show how, when consumers do not see prices before making shopping decisions, firms may have incentives to offer price-matching guarantees that result in a prisoner's dilemma in which guarantees hurt all firms.

Fischer and Harrington (1996) spawned an agglomeration literature in which firms selling heterogeneous products can either co-locate or locate far from each other, when consumers do not know their product valuations before visiting a store. They argue that firms selling more differentiated products have greater incentives to co-locate.

The two closest papers to ours are Parakhonyak and Titova (2018) and Non (2010).[3] The search technologies in their models have costless recall and no notion of distance, features that fit online markets better. Concretely, in Parakhonyak and Titova, ex-ante homogeneous consumers incur identical *fixed* costs of visiting a cluster; and in Non, shoppers costlessly search all firms at all locations, while non-shoppers incur identical *fixed* travel and store-entering costs. In contrast, in our model of spatially-separated brick-and-mortar stores, consumers differ in their travel costs and can only shop at one location.

Parakhonyak and Titova build on the differentiated good framework of Fischer and Harrington (1996), analyzing a directed search, matching model with multiple clusters of different exogenous sizes. Larger clusters offer more variety and lower prices, so all consumers search them first (and are more likely to purchase there immediately or later return to them, leaving smaller clusters unsearched), ensuring that their firms earn higher profits. It follows that were Parakhonyak and Titova to endogenize cluster size, all stores would locate at the same cluster.

Like us, Non considers firms that sell identical products. She endogenizes firm choices of whether to locate in a cluster or to isolate. With no other source of consumer heterogeneity, Non predicts that when clustered and isolated firms co-exist in equilibrium, they must charge the same expected price (to attract some non-shoppers), which does not describe the real-world pricing of brick-and-mortar stores, where clustering leads to lower prices. In contrast, we model the spatial travel structure that is central to town-center considerations, analyzing how consumers' information and location characteristics interact to determine the profits of isolated and co-located firms selling identical products. We then extend our base

[3] See also Dudey (1990), who builds a model of Cournot competition at a location, where more firms at a location is assumed to draw more consumers, and Page and Tassier (2007) who build an 'ecological model' of location dynamics in which firms co-locate because they 'fit together' for some reason.

model to identify when clusters and isolated stores arise endogenously in equilibrium.

2 Model

A continuum of consumers, indexed by their locations x, is uniformly distributed on the unit interval. Three profit-maximizing stores sell a homogeneous good. Stores L_1 and L_2 are co-located at 0, and store R is located at 1. Marginal costs of production are normalized to zero. Consumers only have time to go to one location. Traveling distance d costs $\alpha t(d)$, with $\alpha > 0$, $t(0) = 0$, and $t'(\cdot) > 0$, $t''(\cdot)$, $t'''(\cdot) \geq 0$. Consumers do not see prices set by stores until they visit. Fraction $\mu \in [0, 1]$ of consumers are shoppers who can costlessly visit both co-located stores and purchase from the store offering the lower price; while fraction $1 - \mu$ are non-shoppers who only have time to visit one co-located store. Each consumer has a continuously differentiable, strictly decreasing demand $D(p)$. We assume that revenues $R(p) = pD(p)$ are strictly single-peaked with a global maximum at the monopoly price p^m. We assume that consumer surplus at price p^m is high enough that all consumers want to visit a store; and that consumer surplus given a price of 0 is finite.

Stores L_1 and L_2 set prices p_1 and p_2 and store R sets price p_R. The price set by a store is only observed when a consumer enters the store. A shopper who travels distance x to location 0 obtains payoff $\max\{\int_{p_1}^{\infty} D(p)dp, \int_{p_2}^{\infty} D(p)dp\} - \alpha t(x)$, while a non-shopper who visits L_i gets $\int_{p_i}^{\infty} D(p)dp - \alpha t(x)$, and a customer who visits R receives $\int_{p_R}^{\infty} D(p)dp - \alpha t(1 - x)$.

We consider two different informational settings for consumers.

Unknown types. In the unknown type setting, consumers do not know whether they will have time to comparison shop until after they select shopping locations. Thus, consumers only learn whether they can comparison shop at location 0 after they have arrived. For example, consumers may only learn whether they have time to go to a second store after visiting the first store.

Shoppers search both L_1 and L_2, see their prices, and then make their purchases at the store with the lower price. Non-shoppers see the price set by the sole store that they visit and then make purchases. Consumers who go to the isolated store R see its price and make purchases. With unknown types, a consumer's equilibrium choice of whether to visit the cluster of stores at location 0 or the isolated store at location 1 only depends on her location.

Known types. In the known type setting, prior to making a travel decision, a consumer learns whether she will have time to comparison shop. Now, a consumer's equilibrium choice of where to go hinges on both her location and on whether or not she will have time to comparison shop. In particular, a shopper is more willing than a non-shopper to travel to the cluster of stores, because she knows she will get the best deal. This complicates analysis, forcing us to impose more structure. With known types, we assume:

Assumption 1 *Travel costs are linear or quadratic, $t(x) = \alpha x$ or $t(x) = \alpha x^2$, where $\alpha > 0$.*

Assumption 2 ***(Stahl 1989)*** *$pR'(p)/R^2(p)$ is decreasing.*

Endogenous Co-Location. Our base formulation takes as exogenous a configuration in which two stores cluster at one location and the other store is maximally separated from those co-located stores. In the endogenous co-location setting we characterize when this configuration emerges as an equilibrium outcome. We focus on the setting where consumers do not know their types before visiting a location.

We first characterize equilibrium outcomes when the only feasible store locations are at the endpoints of the unit line. We then consider a richer setting in which stores are free to locate anywhere on a unit circle populated by a uniformly-distributed population of consumers. It costs a consumer $c(t) = \alpha t^2$ to travel distance t, where $\alpha > 0$. The three stores simultaneously choose where to locate on the unit circle before making their pricing choices, and consumers see the store locations, but not their prices or their shopping types before choosing where to travel. With no notion of left and right, we now label the stores s_i with associated location choices d_i, $i = 1, 2, 3$. The other assumptions are as in our base model, so pricing by stores mirrors that in the base model.

To ease presentation, we assume that α is sufficiently small that all consumers purchase in the equilibrium in which two stores co-locate. A grossly sufficient condition for this is

Assumption 3 *$\alpha \leq 16 \int_{p^m}^{\infty} D(p)dp$.*

This assumption ensures that even when store locations are $(d_1, d_2, d_3) = (0, 1/2, d_3 > 1/2)$, a consumer located at $1/4$ wants to shop, i.e. $\alpha(1/4)^2 \leq \int_{p^m}^{\infty} D(p)dp$.

3 Analysis with Unknown Types

In our unknown type setting, a strategy for a consumer is a function mapping her location x into a choice of whether to go to location 0 or location 1 to shop, and a probability of going to firm L_1 if she learns that she is a non-shopper at location 0, where a consumer who observes lowest price p purchases $D(p)$. The (possibly mixed) strategy of store $j \in \{L_1, L_2, R\}$ is given by a cdf $F_j(p)$ over the price set $p \in [0, \infty)$.

Because some consumers know the prices at both cluster stores, there is a unique Perfect Bayesian equilibrium (Baye et al. 1992; Johnen and Ronayne 2019). In equilibrium, co-located stores employ the same pricing strategy $F(p)$, and non-shoppers are equally likely to visit each co-located store. Store R sets the monopoly price p^m and earns $R^m \equiv R(p^m)$ from each of its customers. Lemma 1 characterizes pricing by co-located stores.[4] This result mirrors those in Varian (1980) and Rosenthal (1980).

Lemma 1 ***(Rosenthal 1980, Varian 1980)*** *In equilibrium, if $\mu \in (0,1)$, then stores at location 0 use a mixed pricing strategy over $p \in [b(\mu), p^m]$ with cumulative distribution function:*

$$F(p;\mu) = 1 - \left[\Big(\frac{1-\mu}{2\mu}\Big)\Big(\frac{R^m}{R(p)} - 1\Big)\right],$$

where $b(\mu)$ is the unique solution to $R(b(\mu)) = \left[\frac{1-\mu}{1+\mu}\right] R(p^m)$. If $\mu = 0$, then both stores set $p = p^m$; and if $\mu = 1$, then both stores set $p = 0$.[5]

The surplus that a consumer expects from shopping at location 0 is[6]

$$(1-\mu)\int_{b(\mu)}^{\infty} F(p;\mu)D(p)dp + \mu \int_{b(\mu)}^{\infty} \left[1 - (1 - F(p;\mu))^2\right] D(p)dp, \tag{1}$$

reflecting that $min\{p_1, p_2\} \sim 1 - (1 - F(p;\mu))^2$. Re-arranging this expected surplus yields

$$\int_{b(\mu)}^{p^m} F(p;\mu)D(p)dp + \int_{p^m}^{\infty} D(p)dp + \mu \int_{b(\mu)}^{p^m} F(p;\mu)(1 - F(p;\mu))D(p)dp. \tag{2}$$

The expected consumer surplus at location 1 is $\int_{p^m}^{\infty} D(p)dp$. Thus, the additional consumer surplus gain from visiting the co-located stores rather than R is

$$A(\mu) \equiv \int_{b(\mu)}^{p^m} F(p;\mu)D(p)dp + \mu \int_{b(\mu)}^{p^m} F(p;\mu)(1 - F(p;\mu))D(p)dp. \tag{3}$$

[4]If consumers see prices before making travel choices, then co-location always harms stores, as they marginal cost price, earning zero profits. The isolated firm solves a standard optimization problem that accounts for the impact of its price on decisions of consumers of where to travel, earning positive profits.

[5]Stahl (1989) essentially endogenizes the reservation price for non-shoppers, which we assume to be large enough that they only visit one cluster store. We use Stahl's presentation of the equilibrium strategy.

[6]Equation (1) uses integration by parts of expected consumer surplus: $\int_{b(\mu)}^{\infty} \left(\int_{p}^{\infty} D(x)dx\right) dF(p;\mu) = \int_{b(\mu)}^{\infty} D(p)F(p;\mu)dp$.

The first term is the consumer surplus that even non-shoppers gain because cluster stores charge less than the monopoly price. The second term is the added surplus that shoppers get. A consumer located at x goes to location 0 if and only if

$$A(\mu) \geq \alpha\, T(x), \quad \text{where } T(x) \equiv t(x) - t(1-x).$$

As long as $\alpha > \overline{\alpha}(\mu) \equiv A(\mu)/t(1) > 0$ travel costs are high enough that some consumers visit R. When this is so, the location of the marginal consumer going to location 0 is

$$x^*(\mu, \alpha) = T^{-1}(A(\mu)/\alpha). \tag{4}$$

Because $T(x)$ and $A(\mu)$ are increasing, $x^*(\mu, \alpha)$ is increasing in μ and decreasing in α. With no shoppers, $x^*(0, \alpha) = T^{-1}(A(0)/\alpha) = T^{-1}(0) = 1/2$.

We now compare the profits of the monopolist and cluster stores. The monopolist gets

$$\pi_R = (1 - x^*)R^m.$$

A cluster store's expected profit is the same at all prices prescribed by its mixed strategy, including the monopoly price p^m. But if a cluster store charges p^m, only non-shoppers buy from it. A fraction $x^*(\mu, \alpha)$ of all consumers go to location 0, a fraction $(1 - \mu)$ of them are non-shoppers, and half will visit each store. Thus, the expected profit of a co-located store is

$$\pi_L(p^m) = x^*(\mu, \alpha)\, \frac{1-\mu}{2}\, R^m.$$

The profit of a cluster store exceeds that of the monopolist ($\pi_L(p^m) > \pi_R$) if and only if

$$x^*(\alpha, \mu) > \frac{2}{3-\mu} \in (2/3, 1). \tag{5}$$

The inequality looks simple because the complexities of search and profit maximization by firms are embedded in x^*. But, x^* depends on $A(\mu)$, which depends on $F(p; \mu)$ in Lemma 1.

Proposition 1 *(1) Increases in the share of shoppers (μ) raise a cluster store's profit $\pi_L(\mu)$ when there are few shoppers ($\mu \approx 0$), but reduce its profit when there are many shoppers ($\mu \approx 1$). When demand is linear, $\pi_L(\mu)$ is a concave, single-peaked function of μ.*

(2) For any $\mu \in (0, 1)$, there exists an $\underline{\alpha}(\mu) \in (0, \overline{\alpha}(\mu))$, such that the profit of a co-located store is strictly higher than that of the isolated store if and only if $\alpha < \underline{\alpha}(\mu)$. When demand is linear, $\underline{\alpha}(\mu)$ is a single-peaked function of μ.

Increasing the share μ of shoppers has two effects on cluster store profits. The direct effect is to reduce their profits by increasing their competition with each other. The indirect effect is to draw more customers away from R. The proposition shows that the indirect strategic effect dominates when there are few shoppers—increasing μ from 0 raises co-located store profits by increasing their incentives to undercut each other, which then draws more consumers. If $\mu = 0$, the co-located firms monopoly price, each earning half the profit of firm R. From this base, increasing μ *always* raises profits of cluster firms precisely *because* they compete more aggressively on price. With few shoppers, distance is the key determinant of where to go, and the marginal consumers have similar travel costs regardless of where they shop. As a result, the measure of consumers drawn to the cluster by increased price competition is initially very elastic, even though pricing remains close to monopoly. This price competition due to more shoppers draws enough customers to more than offset the second-order reduced profit per customer, causing profits of co-located stores to rise. When, instead, almost all consumers are shoppers, pricing approaches marginal costs at the cluster, and their profits go to zero.

4 Extensions: Known Types and Endogenous Location

4.1 Known Types

We now show how the nature of consumer information affects equilibrium outcomes. Specifically, we derive how the qualitative properties of equilibrium outcomes are affected when consumers know whether they have time to comparison shop *before* deciding whether to go to isolated store R or the co-located store.

If the co-located stores believe that a fraction μ' of their customers are shoppers, then their pricing is given by Lemma 1 with distribution $F(p;\mu')$ and boundary $b(\mu')$, which depend on μ', not μ. The equilibrium value of μ' is determined by μ and travel cost parameter α.

A shopper located at x goes to the co-located stores at 0 if and only if

$$A_s(\mu') = \int_{b(\mu')}^{p^m} \left[1 - (1 - F(p;\mu'))^2\right] D(p)dp \geq \alpha T(x) = \alpha(2x - 1), \tag{6}$$

and a non-shopper located at x goes to location 0 if and only if

$$A_n(\mu') = \int_{b(\mu')}^{p^m} F(p;\mu')D(p)dp \geq \alpha T(x). \tag{7}$$

All consumers collect the same consumer surplus at store R. However, shoppers gain more surplus than non-shoppers from going to location 0, i.e., $A_s(\mu') > A_n(\mu')$, as shoppers pay the lowest price. That is, $1-(1-F(p;\mu'))^2 > F(p;\mu')$. From (6) and (7), one can solve for the marginal shopper $s(\mu',\alpha)$ and non-shopper $n(\mu',\alpha)$:

$$s(\mu',\alpha) = min\{\frac{\alpha + A_s(\mu')}{2\alpha}, 1\} > \ n(\mu',\alpha) = \frac{\alpha + A_n(\mu')}{2\alpha} > \frac{1}{2} \quad \text{(indifference conditions).} \tag{8}$$

Thus, a shopper located at x goes to location 0 if and only if $x \leq s(\mu',\alpha)$, and a non-shopper goes if and only if $x \leq n(\mu',\alpha)$. In equilibrium, beliefs are consistent with strategies. Thus, the fraction of consumers at location 0 who are shoppers is

$$\mu'(\mu,\alpha) = \frac{\mu}{\mu + (1-\mu)\ n(\mu',\alpha)/s(\mu',\alpha)} > \mu \qquad \text{(belief consistency),} \tag{9}$$

where the inequality follows from $n(\mu',\alpha)/s(\mu',\alpha) < 1$. It follows from $\mu' > \mu$ that price competition among co-located stores is more severe when consumers know their shopping types in advance. In our base setting, $x^*(\mu,\alpha) = \frac{\alpha+A(\mu)}{2\alpha} < s(\mu',\alpha)$. Thus, a greater share of shoppers go to location 0 than non-shoppers or consumers in our base setting who do not know their types when deciding where to go. It also follows directly that relative to our base setting, the isolated store is harmed when consumers know in advance whether they have time to shop, as it reduces their customer base:

$$x^*(\mu,\alpha) = \frac{\alpha + A(\mu)}{2\alpha} < \mu s^*(\mu',\alpha) + (1-\mu)n^*(\mu',\alpha) = \frac{\alpha + \mu A_s(\mu') + (1-\mu)A_n(\mu')}{2\alpha}.$$

Definition 1 *An equilibrium is given by a belief–price distribution pair,* $\{\mu' \equiv \mu^*(\mu,\alpha), F(p;\mu')\}$, *that satisfy equations* (8) *and* (9). *At* $\mu^*(\mu,\alpha)$, *denote* $s^*(\mu,\alpha) \equiv s(\mu^*(\mu,\alpha),\alpha)$ *and* $n^*(\mu,\alpha) \equiv n(\mu^*(\mu,\alpha),\mu)$.

A lower ratio of the indifferent marginal non-shopper to the indifferent marginal shopper $(n(\mu',\alpha)/s(\mu',\alpha))$ raises the ratio of shoppers to non-shoppers (μ') at the co-located stores, reducing prices—in the sense of the first order stochastic dominance. Lower prices, in turn, increase the incentives of all consumers to visit the co-located stores. Conversely, if shoppers respond more sharply than non-shoppers to reductions in prices at the co-located stores, the ratio of shoppers to non-shoppers rises at the co-located stores (n/s falls, so that μ' rises). Thus, the actions of co-located stores and consumers feature strategic complementarities. The question then arises as to whether those strategic complementarities can result in multiple equilibria—e.g., an equilibrium in which relatively few shoppers go to location zero, resulting in relatively high prices, and confirming the optimality of relatively few shoppers going to location zero; and an equilibrium in which the opposite holds.

We next prove that shoppers are never so much more sensitive to price changes than non-shoppers that multiple equilibria can arise.

Proposition 2 *A unique equilibrium exists. The equilibrium values $\mu^*(\mu,\alpha)$, $s^*(\mu,\alpha)$ and $n^*(\mu,\alpha)$ weakly increase in μ and decrease in α, strictly so for $s^*(\mu,\alpha) < 1$. Further, as travel costs rise, the ratio of shoppers to non-shoppers visiting the co-located stores falls: $s^*(\mu,\alpha)/n^*(\mu,\alpha)$ decreases in α.*

The proof uses Assumption 2 to bound the relative gains from being a shopper. The gain associated with being a shopper—more choice—is greatest for low prices with the lowest probability (i.e., prices close to $b(\mu')$). This reflects that $A_s(\mu') - A_n(\mu') = \int_{b(\mu')}^{p^m}[F(p;\mu') - F(p;\mu')^2]D(p)dp$, whose argument takes the form $z - z^2$, which has a derivative $1 - 2z$ that decreases in z. Assumption 2 bounds the degree of convexity of demand, which captures the marginal benefit associated with getting a lower price, and hence the relative sensitivity of shoppers vs. non-shoppers to the improved prices associated with an increased μ'.

The proposition further establishes that as the population share μ of shoppers rises, so does the proportion $\mu^*(\mu,\alpha)$ of consumers at the cluster who are shoppers. In turn, prices at the cluster fall in a first order stochastic dominance sense, inducing both more shoppers and non-shoppers to visit the cluster. Conversely, increasing travel costs, α, reduces the shopping price elasticity, increasing the impact of consumer location on the choice of where to shop, reducing the share of shoppers at the cluster.

As before, the profit of a cluster store is

$$\pi_L(p^m) = n^*(\mu,\alpha)\frac{1-\mu}{2}R^m, \tag{10}$$

while the isolated store earns

$$\pi_R = (1 - \mu s^*(\mu,\alpha) - (1-\mu)n^*(\mu,\alpha))R^m. \tag{11}$$

Since $s^*(\mu,\alpha)$ and $n^*(\mu,\alpha)$ increase in μ, the isolated store R's profit falls in μ. Further, R is hurt if consumers know their types before deciding where to shop: it collects the same profit per consumer but its consumer base falls, i.e., $\mu s^*(\mu,\alpha) + (1-\mu)n^*(\mu,\alpha) > x(\mu,\alpha)$ for $\mu > 0$. This reflects that the co-located stores draw a higher mix of shoppers when consumers know their own types, leading to stochastically better prices, and hence more consumers.

The profits of co-located stores are higher when consumers know whether they are shoppers at the outset than when they do not if and only if $n^*(\mu,\alpha) > x(\mu,\alpha)$.

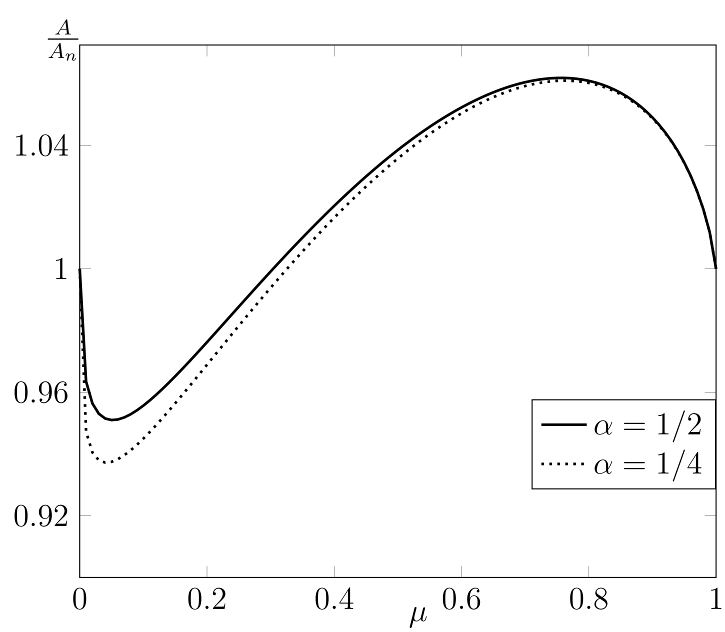
Proposition 3 *A cluster store's profit (π_L) increases in μ if there are few shoppers ($\mu \approx 0$), but decreases if there are many shoppers ($\mu \approx 1$).*

With few shoppers ($\mu \approx 0$), more non-shoppers than unknown types visit the cluster: $n^*(\mu, \alpha) > x(\mu, \alpha)$. Conversely, with many shoppers ($\mu \approx 1$), $n^*(\mu, \alpha) < x(\mu, \alpha)$. Thus, co-located stores earn higher profits when consumers know in advance whether they are shoppers if there are sufficiently few shoppers, but they are harmed if there are too many shoppers.

Proving that $n^*(\mu, \alpha) > x(\mu, \alpha)$ when μ is small is challenging—we must show that $A_n(\mu^*(\mu)) < A(\mu)$ in a neighborhood of $\mu = 0$, even though $A_n(\mu^*(\mu)) = A(\mu)$ at $\mu = 0$. In effect, we must sign the derivative $A'(\mu) - A'_n(\mu^*(\mu))\frac{d\mu^*(\mu)}{d\mu}$ at $\mu = 0$. This is tricky because (1) μ^* is an equilibrium object, with the properties that $\lim_{\mu\to 0} \frac{\mu^*(\mu)}{\mu} = \lim_{\mu\to 0} \frac{d\mu^*(\mu)}{d\mu} = 1$; and (2) $\lim_{\mu\to 0} A'(\mu) = \lim_{\mu\to 0} A'_n(\mu^*(\mu)) = \infty$. Thus, to sign the derivative, one must identify the rates of convergence. A key is to show that $0 < \lim_{z\to 0} \sqrt{z}A'_n(z) < \lim_{z\to 0} \sqrt{z}A'_s(z) < \infty$.

Figure 1 depicts $A(\mu)/A_n(\mu^*(\mu, \alpha))$ for different values of μ and α, for linear demand $D(p) = \beta - (\beta/2)p$. With linear demand, a single-crossing property holds, strengthening the results in Proposition 3: there exists a $\bar{\mu} \in (0, 1)$ such that $A(\mu) < A_n(\mu^*(\mu, \alpha))$ if and only if $\mu < \bar{\mu}$.

Figure 1: $A(\mu)/A_n(\mu^*(\mu))$ with linear demand $D(p) = 6 - 3p$.



For co-located stores, similar direct (increased price competition) and indirect effects (reduced external competition) exist as in the unknown type setting via the positive effect of μ on $\mu^*(\mu, \alpha)$. Once more, a cluster store's profit first rises in μ when there are few shoppers, reflecting the high sensitivity to increased price competition of a consumer's choice of where to shop when travel distances to stores are very similar. Proposition 3 reveals that the qualitative implications are reinforced if consumers know in advance whether they have time to comparison shop. With few shoppers, price competition is so enhanced by the higher endogenous share of shoppers at the cluster relative to μ that the expected price paid at a cluster store by a consumer who *knows* he does not have time to comparison shop falls below that paid by a consumer in the base setting who could turn out to be a shopper. This means that when μ is small, even a consumer who knows he does not have time to comparison shop is willing to travel farther than a consumer in the base setting, i.e., $n^*(\mu, \alpha) > x(\mu, \alpha)$. However, with enough shoppers, this inequality is reversed—the heightened value of likely securing the lowest price more than offsets the higher expected prices that obtain when consumers do not know whether they will have time to shop. It follows that the profit of co-located stores is higher when consumers know their types in advance when the fraction of shoppers μ is small, but not when it is large.

Figure 2: Profit at cluster by informational setting with $D(p) = 6 - 3p$ and $\alpha = 1/4$

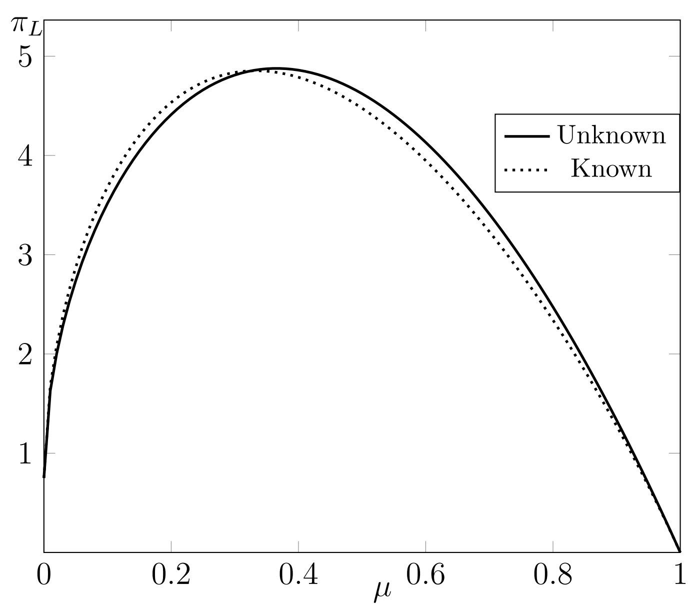


Figure 2 illustrates the profits of co-located stores in the two information settings for a linear demand. The single-crossing property in Figure 1 is inherited by the profit function, so that co-located stores are better off when consumers know their types if and only if $\mu < \bar{\mu}$.

4.2 Endogenous Co-Location

To this point, our analysis has taken the locations of stores as given. The questions that we next explore are: when will an equilibrium configuration emerge with co-location of two stores at one location, and one store that is maximally separated from the co-located stores; and how do the profits of the co-located and isolated stores compare? We first consider the simple case where the only feasible store locations are at 0 and 1 on the unit line. We then consider a richer setting in which stores can locate anywhere on the unit circle. The unit circle avoids the standard edge effects associated with 'captive audiences' on the unit line when there are more than two stores, and stores are free to locate anywhere.[7]

Equilibria with two possible locations. Clearly, in any (pure strategy) equilibrium with three stores and only two locations, at least two stores co-locate. Thus, without loss of generality, suppose that stores L_1 and L_2 co-locate at 0. The next proposition provides answers to the two core questions: when does R separate by locating at 1 in equilibrium? and when will R optimally separate from L_1 and L_2, yet earn lower profits?

Proposition 4 *There exist cutoffs α_l and α_h on the travel cost parameter α with $0 < \alpha_l < \alpha_h$ such that in equilibrium: (i) R co-locates with L_1 and L_2 at 0 if $\alpha < \alpha_l$, but separates to locate at 1 if $\alpha > \alpha_l$; and (ii) R earns strictly lower profits than stores L_1 and L_2 if $\alpha \in (\alpha_l, \alpha_h)$, but strictly higher profits if $\alpha > \alpha_h$.*

Proof: Store R's profit from locating at 0 would be $\frac{(1-\mu)R^m}{3}$, whereas at location 1, R would receive $(1 - x^*(\alpha, \mu))R^m$. Hence, R is strictly better off at location 1 when $x^*(\alpha, \mu) < \frac{2+\mu}{3}$, and strictly worse off if the inequality is reversed. Thus, equilibrium involves co-location of all three stores if $x^*(\alpha, \mu) > \frac{2+\mu}{3}$, but equilibrium involves R locating at 1 if the inequality is reversed (and both location combinations are equilibrium configurations if $x^*(\alpha, \mu) = \frac{2+\mu}{3}$). From (5), the co-located stores earn more profits than R at location 1 if $x^*(\alpha, \mu) > \frac{2}{3-\mu}$, where we note that $\frac{2}{3-\mu} < \frac{2+\mu}{3}$. The results then follow since $x^*(\alpha, \mu)$ decreases in α. $\square$

Intuitively, if travel costs are sufficiently low, all three stores co-locate because too few consumers would go to an isolated store. As the travel cost parameter α is increased, the

[7] The 'captive audiences' introduce asymmetries that result in mixing in equilibrium that has no natural analogue in our spatial store location setting.

profits from all three stores co-locating are unaffected, but the profits from isolating rise because consumer location would determine the shopping choice of more consumers. As α rises, eventually store R chooses to isolate itself rather than continue to incur the competition associated with having three stores at one location. At this point R earns lower profits that the co-located stores because as soon as it leaves location 0, competition is reduced there, increasing the profits of L_1 and L_2. Finally, as travel costs rise further, the market share and profits of co-located stores fall, and the market share and profits of R rise, and eventually these profits cross.

We next show that the same economic forces drive the endogenous location of stores when, rather than being restricted to two locations, they are free to locate anywhere on the unit circle. The one difference is that when travel costs are high, so the profits of an isolated store exceed those at a cluster, the stores all choose to isolate. We assume quadratic travel costs.

Equilibrium on the circle. In this setting the three stores simultaneously choose where to locate on the unit circle. Define $A(\mu)$ to be the consumer surplus gain from visiting a cluster of 2 stores rather than an isolated store, and let $\pi_j(d_1, d_2, d_3)$ denote store s_j's profit as a function of the store locations (d_1, d_2, d_3).

We first establish that if two stores co-locate, then the third store's best response is either also to co-locate, or to maximally separate from them, i.e., the circle analogue of locating at the opposite end of the unit line.

Lemma 2 *Let stores s_1 and s_2 cluster (without loss of generality) at 0. Then if store s_3's best response is to isolate, it maximally isolates at $d_3 = 1/2$.*

Proof: Suppose store s_3 locates at $d_3 \neq 0$. Denote the indifferent consumer between d_3 and 1 by x_+ (when it exists), and denote the indifferent consumer between 0 and d_3 by x_-. Then

$$x_-^2 - (d_3 - x_-)^2 = A(\mu)/\alpha \Rightarrow x_- = \frac{d_3}{2} + \frac{A(\mu)}{2\alpha d_3},$$

and

$$(1 - x_+)^2 - (x_+ - d_3)^2 = A(\mu)/\alpha \Rightarrow x_+ = \frac{1 + d_3}{2} - \frac{A(\mu)}{2\alpha(1 - d_1)}.$$

Thus, s_3's market share is $(x_+ - d_3) + (d_3 - x_-)$, so its profits are

$$\pi_3(0, 0, d_3) = \frac{1}{2}\left(1 - \frac{A(\mu)}{\alpha d_3(1 - d_3)}\right) R^m, \tag{12}$$

which are maximized by $d_3 = 1/2$. $\square$

We next establish the necessary and sufficient conditions for store s_2 to want to co-locate with store s_1 at 0 when store s_3 locates at 1/2 rather than isolate. By locating at 1/2, store s_3 minimizes the attraction to s_2 of isolating, as doing so would leave s_2 a market share of only $\frac{1}{4}$. Thus, this maximizes the attraction of co-locating with store s_1.

Lemma 3 *If s_1 locates at 0 and s_3 locates at 1/2, then s_2's best response is to locate at 0 if and only if $\alpha \leq 4(1-\mu)A(\mu)/\mu$.*

Proof: If s_2 co-locates at 0, it earns profit

$$\pi_2(0,0,1/2) = \left(\frac{1}{4} + \frac{A(\mu)}{\alpha}\right)(1-\mu)R^m.$$

If s_2 instead isolates, then a best response is to locate at 1/4, obtaining a market share of 1/4 and a profit of $\frac{R^m}{4}$. Thus, it is optimal to colocate if and only if

$$\left(\frac{1}{4} + \frac{A(\mu)}{\alpha}\right)(1-\mu) \geq \frac{1}{4}.$$

Solving yields

$$\alpha \leq \frac{4(1-\mu)A(\mu)}{\mu}.$$

□

Whenever, travel costs are so high that $\alpha > \frac{4(1-\mu)A(\mu)}{\mu}$, there is an equilibrium in which the three stores isolate. Finally, we identify sufficient conditions for store s_3 to want to separate away from the cluster of stores s_1 and s_2 at 0, and hence, by Lemma 2, to locate at $d_3 = 1/2$. The condition below in Lemma 4 is also necessary if a consumer located at 1/2 would travel to 0 to shop if all three stores co-locate there (rather than not shop).

Lemma 4 *If stores s_1 and s_2 co-locate at 0, then s_3's best response is to locate at $d_3 = 1/2$ if $\alpha \geq 12A(\mu)/(1+2\mu)$.*

Proof: If s_3 colocates with s_1 and s_2 at 0, then its profit of $\pi_3(0,0,0)$ is at most $\frac{(1-\mu)R^m}{3}$ (with equality if a consumer located at 1/2 would shop there). If s_3 instead isolates, then by Lemma 3 it locates to $d_3 = 1/2$. From equation (12), it earns profit

$$\pi_3(0,0,1/2) = \left(\frac{1}{2} - \frac{2A(\mu)}{\alpha}\right)R^m.$$

Thus, $\pi_3(0,0,1/2) \geq \pi_3(0,0,0)$ if $\alpha \geq \frac{12A(\mu)}{1+2\mu}$. □

When $\alpha < \frac{12A(\mu)}{1+2\mu}$, travel costs are so low that it breaks the equilibrium in which two firms co-locate at 0 and the other maximally separates to 1/2. When this is so, co-location at 0 draws enough consumers that the unique equilibrium is for all stores to co-locate there.

Putting Lemmas 3 and 4 together yields Proposition 5. Moreover, direct calculations yield a condition ensuring non-emptiness and a cutoff such that co-located stores earn higher profit than the isolated store.

Proposition 5 *It is an equilibrium for stores s_1 and s_2 to co-locate at 0 and s_3 to maximally separate at 1/2 if*

$$\underline{\alpha} = 12A(\mu)/(1+2\mu) \leq \alpha \leq 4(1-\mu)A(\mu)/\mu = \bar{\alpha}.$$

This condition is non-empty if and only if $\mu \leq \frac{1}{2}(\sqrt{3}-1)$.

When the co-location equilibrium exists, a co-located store earns a higher profit than the isolated store if $\alpha \in [\underline{\alpha}, \hat{\alpha})$ where $\hat{\alpha} = 4(3-\mu)A(\mu)/(1+\mu)$, and $\hat{\alpha} < \bar{\alpha}$ if and only if $\mu < 1/3$.

Thus, two stores co-locating and one store maximally separating is an equilibrium when (i) the travel cost parameter α is neither too small, nor too large, and (ii) the number of shoppers, μ, is not too large. The economics underlying these conditions is sharp. When α is smaller, so is the isolated store's market share. As a result, ceteris paribus, when $\mu > 0$, there is a lower threshold $\underline{\alpha}$ such that if the travel cost parameter is below this threshold, the third store also prefers to co-locate at 0. It follows that when the cluster equilibrium exists, the form that it takes is (generically) unique—it features full clustering of all three stores when travel costs are sufficiently low; and when travel costs exceed the $\underline{\alpha}$ threshold, it features clustering of two stores, and maximal separation by the third, isolated store.

Similarly, there is an upper threshold, $\bar{\alpha}$, such that if α exceeds this threshold, then co-location does not attract enough consumers to compensate for the profits dissipated by the price competition. When α exceeds $\bar{\alpha}$, stores always isolate in equilibrium. Finally, with enough shoppers, as α rises, a threshold is reached at which the equilibrium jumps from all stores co-locating to all stores isolating. This reflects that many shoppers so enhance price competition when two stores co-locate that it shrinks the market share of an isolated firm by enough that it also wants to co-locate. Thus, the outcome with two co-located stores and one maximally isolated store that is the focus of our earlier analysis emerges endogenously in equilibrium when travel costs are intermediate and there are some shoppers, but not too many.

To understand the characterization of the relative profits of the cluster and isolated store, first observe that at $\underline{\alpha}$, the isolated store is indifferent to maximal separation and all stores co-locating; but when it isolates, it reduces price competition at the cluster, raising the profits of the two remaining co-located stores above its isolation profits, analogously to the setting with only two possible locations. As the travel cost parameter is increased further, the profits of the isolated store rise, while those of the co-located stores fall, reflecting that increased travel costs reduce the market share of the co-located stores (without affecting pricing). The question becomes whether or not they cross before reaching $\bar{\alpha}$, as in the simple two location setting. It turns out that the crossing occurs at $\hat{\alpha}$, which is less than $\bar{\alpha}$ if the share of shopers is small enough: when $\mu < 1/3$, the competition at the cluster softens, making the cluster less appealing to consumers. When $\alpha \in (\hat{\alpha}, \bar{\alpha})$ consumers are sufficiently sensitive to the lower cluster prices to make co-locating preferable to isolating for stores s_1 and s_2. However, their shopping elasticity is not so high to make enough consumers switch from the isolated store to the cluster given the level of intra-cluster competition. Conversely when $\mu > 1/3$, there are enough shoppers to so intensify intra-cluster competition that even at $\bar{\alpha}$, enough consumers switch away from the isolated store to lower its profits below the co-located stores.

5 Conclusion

Standard spatial theory suggests that firms selling similar products maximize profits by separating maximally. Nonetheless, in recent years, stores like Lowes and Home Depot that sell very similar products have thrived despite co-locating (at fringes), while stores in city centers that face limited local competition have had troubles. We note that when most consumers only see prices once they visit a store and some consumers have time to comparison shop then co-location commits stores to compete and lower prices, which draws more consumers.

Our central finding is that co-located firms thrive when there are some shoppers, but not too many. With few shoppers, the measure of consumers drawn to the co-located stores is very price elastic because travel costs differ only modestly for the marginal consumer. Thus, the marginal value of commitment to slightly lower prices is high. These effects are enhanced if consumers know in advance whether they will have time to comparison shop. Indeed, price competition at co-located stores may rise by enough that the expected price paid by a non-shopper falls below that paid when consumers do not know if they will have time to shop. The flip side is that with too many shoppers, price competition grows so fierce that the high numbers of customers drawn fail to offset the reduced profit per customer.

6 References

Armstrong, Mark. 2017. "Ordered Consumer Search." *Journal of the European Economic Association* 15: 989-1024.

Baye, Michael, Dan Kovenock, and Casper De Vries. 1992. "It takes two to tango: equilibria in a model of sales." *Games and Economic Behavior* 4: 493-510.

Bernhardt, Dan, Evangelos Constantinou, and Mehdi Shadmehr. 2021. "When do co-located firms selling identical products thrive?" Working Paper.

Burdett, Ken and Ken Judd. 1983. "Equilibrium Price Dispersion." *Econometrica* 51: 955-69.

Constantinou, Evangelos and Dan Bernhardt. 2018. "The Price Matching Dilemma." *International Journal of Industrial Organization* 59: 97-113.

Dudey, Marc. 1990. "Competition by Choice: The Effect of Consumer Search on Firm Location Decisions." *American Economic Review* 80: 1092-1104.

Ellison, Glenn. 2005. "A Model of Add-on Pricing." *Quarterly Journal of Economics* 120: 585-637.

Ellison, Glenn and Sara Ellison. 2009. "Search, Obfuscation, and Price Elasticities on the Internet." *Econometrica* 77: 427-52.

Ellison, Glenn and Alexander Wolitzky. 2012. "A Search Cost Model of Obfuscation." *RAND Journal of Economics* 43: 417-441.

Fischer, Jeffrey and Joseph Harrington. 1996. "Product Variety and Firm Agglomeration." *RAND Journal of Economics* 27: 281-309.

Iaria, Alessandro. 2014. "Consumer Benefit of Big-Box Supermarkets: The Importance of Controlling for Endogenous Entry." Mimeo.

Johnen, Johannes, and David Ronayne. 2019. "The Only Dance in Town: Unique Equilibrium in a Generalized Model of Price Competition." Mimeo.

Marshall, Guillermo, and Tiago Pires. 2018. "Measuring the Impact of Travel Costs on Grocery Shopping." *Economic Journal* 128: 2538-57.

Non, Marielle. 2010. "Isolation or Joining a Mall? On the Location Choice of Competing Shops." mimeo.

Nystrom, Paul. 1915. *Economics of retailing.* Ronald Press Company.

Page, Scott and Troy Tassier. 2007. "An Ecological Model of Firm Entry and Exit: Why Chains Beget Chains." *Journal of Economic Dynamics and Control* 31: 3427-58.

Parakhonyak, Alexei and Maria Titova. 2018. "Shopping Malls, Platforms, and Consumer Search." *International Journal of Industrial Organization* 58: 183-213.

Ronayne, David. 2018. "Price Comparison Websites." Mimeo, Oxford University.

Rosenthal, Robert W. 1980. "A Model in Which an Increase in the Number of Sellers Leads to a Higher Price." *Econometrica* 48: 1575-1579.

Shilony, Yuval. 1977. "Mixed pricing in oligopoly." *Journal of Economic Theory* 14: 373-388.

Stahl, Dale. 1989. "Oligopolistic Pricing with Sequential Consumer Search." *American Economic Review* 79: 700-12.

Stahl, Dale. 1996. "Oligopolistic pricing with heterogeneous consumer search." *International Journal of Industrial Organization* 14: 243-68.

Tirole, Jean. 2001. *The Theory of Industrial Organization.* MIT Press.

Varian, Hall R. 1980. "A Model of Sales." *American Economic Review* 70: 651-659.

7 Appendix

Proof of Proposition 1: We begin with two preliminary lemmas.

Lemma 5 *(i) $A'(\mu) > 0$ with $\lim_{\mu\to 0} A'(\mu) = \infty$ and $\lim_{\mu\to 1} A'(\mu) = 0$; (ii) $A(\mu)$ is strictly concave when demand is linear, $D(p) = \beta_1 - \beta_2\, p$.*

Proof. See our working paper (Bernhardt et al. 2021). ■

Lemma 6 *For $x^* < 1$, $x^*(\mu)$ is a strictly increasing function of μ, with $\lim_{\mu\to 0} \frac{dx^*(\mu)}{d\mu} = \infty$ and $\lim_{\mu\to 1} \frac{dx^*(\mu)}{d\mu} = 0$. If demand is linear, then $x^*(\mu)$ is strictly concave.*

Proof: Recall that $\alpha\, T(x^*(\mu)) = A(\mu)$ when $x^* < 1$. Thus,

$$\frac{dx^*(\mu)}{d\mu} = \frac{1}{\alpha\, T'(x^*(\mu))}\, \frac{dA(\mu)}{d\mu} > 0, \tag{13}$$

and

$$\frac{d^2x^*(\mu)}{d\mu^2} = \frac{1}{T'(x^*(\mu))}\, \left(\frac{1}{\alpha}\frac{d^2A(\mu)}{d\mu^2} - T''(x^*(\mu))\, \left(\frac{dx^*(\mu)}{d\mu}\right)^2\right), \tag{14}$$

Moreover, $T''(x) = t''(x) - t''(1-x)$. Because $t'''(x) \geq 0$ and $x^* \geq 1/2$, we have $T''(x^*) \geq 0$. The results then follow from Lemma 5. □

We now use these lemmas to prove Proposition 1.

Part 1. Observe that

$$\frac{d\pi_L(\mu)}{d\mu} = \frac{R^m}{2}\, \left(\frac{dx^*(\mu)}{d\mu}\,(1-\mu) - x^*(\mu)\right). \tag{15}$$

Thus, using Lemma 6,

$$\begin{aligned}
\lim_{\mu\to 0} \frac{d\pi_L(\mu)}{d\mu} &= \frac{R^m}{2}\, \left(\lim_{\mu\to 0}\frac{dx^*(\mu)}{d\mu} - x^*(0)\right) = \infty \\
\lim_{\mu\to 1} \frac{d\pi_L(\mu)}{d\mu} &= \frac{R^m}{2}\, (0 - x^*(1)) < 0.
\end{aligned}$$

Moreover, differentiating (15),

$$\frac{d^2\pi_L(\mu)}{d\mu^2} = \frac{R^m}{2}\, \left(\frac{d^2x^*(\mu)}{d\mu^2}\,(1-\mu) - 2\,\frac{dx^*(\mu)}{d\mu}\right).$$

From Lemma 6, $x^*(\mu)$ increases in μ. Thus, for μ sufficiently small that $x^*(\mu) < 1$, from Lemma 6, with linear demand, $\frac{d^2x^*(\mu)}{d\mu^2} < 0$, and hence $\frac{d^2\pi_L(\mu)}{d\mu^2} < 0$. Otherwise, $x^*(\mu) = 1$, and thus $\frac{d^2\pi_L(\mu)}{d\mu^2} = 0$.

Part 2. At $\underline{\alpha}(\mu)$, $x(\mu, \alpha(\mu)) = 2/(3-\mu)$. By the implicit function theorem (evaluated at $x^*(\mu, \alpha)$)

$$\alpha'(\mu) = \frac{1}{\partial x^*/\partial\alpha}\left(\frac{2}{(3-\mu)^2} - \frac{\partial x^*}{\partial\mu}\right). \tag{16}$$

We claim that $\alpha'(\mu)$ satisfies a single crossing property: $\alpha'(\mu) > 0$ for $\mu < \tilde{\mu}$, and $\alpha'(\mu) < 0$ for $\mu > \tilde{\mu}$. To see this, first notice that in equilibrium $x^*(\mu, \alpha) = T^{-1}(A(\mu)/\alpha)$, and thus

$$\frac{\partial x^*(\mu, \alpha)}{\partial\alpha} = -\frac{A(\mu)}{\alpha^2 T'(A(\mu)/\alpha)} < 0,$$

since $T'(x) > 0$. Moreover, $dx^*(x, \alpha)/d\mu > 0$ (from (13)), and when $A(\mu)$ is concave and $T(x)$ is convex, we have $d^2x^*(x, \alpha)/d\mu^2 < 0$ (from (14)).

Thus, the denominator in (16) is negative such that $\alpha'(\mu) > 0$ if $2/(3-\mu)^2 - \partial x^*/\partial\mu < 0$; and $\alpha'(\mu) < 0$ otherwise. The first term $2/(3-\mu)^2$ increases in μ, while $\partial x^*/\partial\mu$ decreases in μ (since $d^2x^*/d\mu^2 < 0$). Thus, $2/(3-\mu)^2 - \partial x^*/\partial\mu$ increases in μ: if $2/(3-\tilde{\mu})^2 - \partial x^*/\partial\mu|_{\mu=\tilde{\mu}} = 0$ at $\tilde{\mu}$, then $2/(3-\mu)^2 - \partial x^*/\partial\mu < 0$ for all $\mu < \tilde{\mu}$, and $2/(3-\mu)^2 - \partial x^*/\partial\mu > 0$ for all $\mu > \tilde{\mu}$.

For $\mu \approx 0$, we have $\alpha'(\mu) > 0$, since $\partial x^*/\partial\mu \to \infty$ (from (13)) and $2/(3-\mu)^2 \to 2/9$ as $\mu \to 0$. If $\tilde{\mu} \in (0, 1)$, then $\alpha(\mu)$ increases for $\mu < \tilde{\mu}$, peaks at $\tilde{\mu}$, and decreases for $\mu > \tilde{\mu}$. If $\tilde{\mu} \notin (0, 1)$, then $\alpha'(\mu) > 0$ for all μ and the peak is reached at $\mu = 1$. □

Proof of Proposition 2: From equation (8),

$$\frac{s(\mu', \alpha)}{n(\mu', \alpha)} = \frac{\alpha + A_s(\mu')}{\alpha + A_n(\mu')}. \tag{17}$$

From equation (9),

$$\frac{s(\mu', \alpha)}{n(\mu', \alpha)} = \frac{\mu'(1-\mu)}{(1-\mu')\,\mu}. \tag{18}$$

Combining equations (17) and (18) yields that μ' is consistent with equilibrium if and only if

$$g(\mu'; \alpha, \mu) = \frac{\mu'}{1-\mu'}\frac{1-\mu}{\mu} - \frac{\alpha + A_s(\mu')}{\alpha + A_n(\mu')} = 0.$$

Observe that $g(0; \alpha, \mu) < 0 < g(1; \alpha, \mu)$. Thus, at least one solution exists. The solution is unique when $\partial g(\mu'; \alpha, \mu)/\partial\mu' > 0$. The solution is unique, and thus a unique equilibrium cutoff $\mu^*(\mu, \alpha)$ exists, when $\partial g(\mu'; \alpha, \mu)/\partial\mu' > 0$, which the next Lemma proves.

Lemma 7 $\frac{\partial g(\mu';\alpha,\mu)}{\partial \mu'} > 0.$

Proof. See our working paper. ■

Next, consider the comparative statics results. The comparative statics with respect to μ are:

$$\frac{\partial \mu^*(\mu,\alpha)}{\partial \mu} = -\frac{\partial g/\partial \mu}{\partial g/\partial \mu'}\bigg|_{\mu'=\mu^*(\mu,\alpha)} = \frac{\mu^*/((1-\mu^*)\mu^2)}{\partial g/\partial \mu'|_{\mu'=\mu^*(\mu,\alpha)}} > 0 \quad (\text{since } \partial g/\partial \mu' > 0). \tag{19}$$

Since $A_s(\mu')$ and $A_n(\mu')$ strictly increase in μ', $s(\mu^*(\mu),\alpha)$ and $n(\mu^*(\mu),\alpha)$ increase in μ.

The comparative statics with respect to α are:

$$\frac{\partial \mu^*(\mu,\alpha)}{\partial \alpha} = -\frac{\partial g/\partial \alpha}{\partial g/\partial \mu'}\bigg|_{\mu'=\mu^*(\mu,\alpha)} = -\frac{(A_s(\mu^*)-A_n(\mu^*))/(\alpha+A_n(\mu^*))^2}{\partial g/\partial \mu'|_{\mu'=\mu^*(\mu,\alpha)}} < 0,$$

because we have established that $\partial g/\partial \mu' > 0$ and $A_s(\mu)-A_n(\mu) > 0$. From (8), $s(\mu^*(\alpha),\alpha) = 1/2 + A_s(\mu^*(\alpha))/2\alpha$ and $n(\mu^*(\alpha),\alpha) = 1/2 + A_n(\mu^*(\alpha))/2\alpha$. Because $\mu^*(\alpha)$ decreases in α and $A_s(\mu)$ and $A_n(\mu)$ both increase in μ,

$$\frac{\partial s(\mu^*(\alpha),\alpha)}{\partial \alpha}, \ \frac{\partial n(\mu^*(\alpha),\alpha)}{\partial \alpha} < 0.$$

Finally, in equilibrium $\frac{s(\mu^*(\alpha),\alpha)}{n(\mu^*(\alpha),\alpha)} = \frac{\mu^*(\alpha)}{1-\mu^*(\alpha)}\frac{1-\mu}{\mu}$ from Bayes rule (9). Thus,

$$\frac{\partial s(\mu^*(\alpha),\alpha)/n(\mu^*(\alpha),\alpha)}{\partial \alpha} = \frac{(1-\mu)}{\mu(1-\mu^*)^2}\frac{\partial \mu^*(\alpha)}{\partial \alpha} < 0. \qquad \square$$

Proof of Proposition 3: We begin with a lemma.

Lemma 8 *(i)* $\lim_{\mu\to 0}\frac{\mu^*(\mu)}{\mu} = 1$ *and* $\lim_{\mu\to 1}\frac{\mu^*}{\mu} = 1$*. (ii)* $\lim_{\mu\to 0}\frac{d\mu^*(\mu)}{d\mu} = \lim_{\mu\to 1}\frac{d\mu^*(\mu)}{d\mu} = 1.$

Proof. See our working paper. ■

Part 1. From (10),

$$\frac{\partial \pi_L}{\partial \mu} = \frac{R^m}{2}\left((1-\mu)\,\frac{\partial n(\mu^*(\mu),\alpha)}{\partial \mu^*}\frac{d\mu^*(\mu)}{d\mu} - n(\mu^*,\alpha)\right). \tag{20}$$

We first show that $\lim_{\mu\to 0}\frac{\partial \pi_L}{\partial \mu} > 0$. From (8), $\partial n(\mu^*(\mu),\alpha)/\partial\mu' = A_n'(\mu^*)/2\alpha$. From part (i) of Lemma 8, $\lim_{\mu\to 0} n(\mu^*(\mu),\alpha) = n(0,\alpha) = 1/2$. From part (ii) of Lemma 8, $\lim_{\mu\to 0}\frac{d\mu^*(\mu)}{d\mu} = 1$. Substituting these into equation (20) yields

$$\lim_{\mu\to 0}\frac{\partial \pi_L}{\partial \mu} = \frac{R^m}{2}\ \left(\lim_{\mu\to 0}\frac{A_n'(\mu^*)}{2\alpha} - \frac{1}{2}\right).$$

Thus, it suffices to show that $\lim_{\mu\to 0}\frac{A_n'(\mu^*)}{2\alpha} = \infty$. From part (i) of Lemma 8,

$$\begin{aligned}\lim_{\mu\to 0} A_n'(\mu^*(\mu)) = \lim_{\mu\to 0} A_n'(\mu) &= \lim_{\mu\to 0}\int_{b(\mu)}^{p^m}\frac{(1-F(p;\mu))}{\mu(1-\mu)}D(p)dp\\ &= \lim_{\mu\to 0}\int_{b(\mu)}^{p^m}\frac{(1-F(p;\mu))}{\mu}D(p)dp)\\ &= \infty\ \ \text{(from Lemma 5)}.\end{aligned}$$

Next, consider $\mu\to 1$. From part (i) of Lemma 8, $\lim_{\mu\to 1}\mu^* = 1$. Thus, $\lim_{\mu\to 1} n(\mu^*,\alpha) = n(1,\alpha)\geq 1/2$. Moreover, from (8), $\partial n(\mu^*,\alpha)/\partial\mu^* = A_n'(\mu^*)/2\alpha$. Thus,

$$\lim_{\mu\to 1}\frac{\partial \pi_L}{\partial \mu} = \frac{R^m}{2}\ \left(\lim_{\mu\to 1}\,(1-\mu)\frac{A_n'(\mu)}{2\alpha} - n(1,\alpha)\right).$$

Using

$$\lim_{\mu\to 1}(1-\mu)A_n'(\mu) = \lim_{\mu\to 1}\int_{b(\mu)}^{p^m}\frac{(1-F(p;\mu))D(p)dp}{\mu} = 0\ \ \text{(by Lemma 1)}.$$

Thus,

$$\lim_{\mu\to 1}\frac{\partial \pi_L}{\partial \mu} = -\frac{R^m}{2}n(1,\alpha) < 0. \qquad \square$$

Part 2. First, we prove the case of $\mu\approx 0$. Then, we prove the case of $\mu\approx 1$.

Case of $\mu\approx 0$. We show that there exists $\epsilon > 0$ such that if $\mu\in(0,\epsilon)$, then $A(\mu) - A_n(\mu^*(\mu)) < 0$. Observe that $A(0) - A_n(\mu^*(0)) = 0$. Thus, it suffices to show that

$$\lim_{\mu\to 0}\left\{\ \Delta(\mu)\equiv A'(\mu) - A_n'(\mu^*)\ \frac{d\mu^*(\mu)}{d\mu}\ \right\} < 0. \tag{21}$$

From (3), (6), and (7), we have:

$$A(\mu) = \mu A_s(\mu) + (1-\mu)A_n(\mu), \tag{22}$$

which implies

$$A'(\mu) = A_s(\mu) - A_n(\mu) + \mu A_s'(\mu) + (1-\mu)A_n'(\mu). \tag{23}$$

Substituting from (23) into (21), we have:

$$\Delta(\mu) = A_s(\mu) - A_n(\mu) + \mu(A_s'(\mu) - A_n'(\mu)) + A_n'(\mu) - A_n'(\mu^*)\ \frac{d\mu^*(\mu)}{d\mu}. \tag{24}$$

Next, we substitute for $\frac{d\mu^*(\mu)}{d\mu}$ into (24) and simplify. From equation (19)

$$\frac{d\mu^*(\mu)}{d\mu} = -\frac{\partial g/\partial\mu}{\partial g/\partial\mu'}\bigg|_{\mu'=\mu^*(\mu,\alpha)} = \frac{\frac{\mu^*}{1-\mu^*}\frac{1}{\mu^2}}{\frac{1-\mu}{\mu(1-\mu^*)^2} - \frac{(\alpha+A_n(\mu^*))A_s'(\mu^*)-(\alpha+A_s(\mu^*))A_n'(\mu^*)}{(\alpha+A_n(\mu^*))^2}}. \tag{25}$$

To ease exposition, we define $\delta(\mu)$ to be the denominator of the above expression:

$$\delta(\mu) \equiv \frac{\partial g}{\partial\mu'}\bigg|_{\mu'=\mu^*(\mu,\alpha)} = \frac{(1-\mu)}{\mu(1-\mu^*)^2} + \frac{(\alpha+A_s(\mu^*))A_n'(\mu^*) - (\alpha+A_n(\mu^*))A_s'(\mu^*)}{(\alpha+A_n(\mu^*))^2}.$$

Using $\lim_{\mu\to 0}\mu A_n'(\mu) = 0$, $\lim_{\mu\to 0}\frac{d\mu^*(\mu)}{d\mu} = 1$ and $\lim_{\mu\to 0}\frac{\mu^*(\mu)}{\mu} = 1$ from Lemma 8, we have:

$$\lim_{\mu\to 0}\mu^*\delta(\mu) = 1. \tag{26}$$

Next, substitute equation (25) into (24), and use $\delta(\mu)$ to obtain:

$$\begin{aligned}
\delta(\mu)\ \Delta(\mu) \;&=\; \delta(\mu)\ \{\ A_s(\mu) - A_n(\mu) + \mu(A_s'(\mu) - A_n'(\mu))\ \} \\
&+\; \frac{1-\mu}{\mu(1-\mu^*)^2}\ A_n'(\mu) - \frac{\mu^*}{1-\mu^*}\frac{1}{\mu^2}\ A_n'(\mu^*) \\
&-\; A_n'(\mu)\ \frac{(\alpha+A_n(\mu^*))A_s'(\mu^*) - (\alpha+A_s(\mu^*))A_n'(\mu^*)}{(\alpha+A_n(\mu^*))^2}
\end{aligned} \tag{27}$$

Our goal is to show that $\lim_{\mu\to 0}\delta(\mu)\ \Delta(\mu) < 0$. From Lemma 8, $\lim_{\mu\to 0} A_n(\mu) = \lim_{\mu\to 0} A_s(\mu) = 0$. We have $\lim_{\mu\to 0}\mu A_n'(\mu) = \lim_{\mu\to 0}\mu A_s'(\mu) = 0$. Thus, (27) simplifies to:

$$\begin{aligned}
\lim_{\mu\to 0}\mu^*\delta(\mu)\ \Delta(\mu) \;&=\; \lim_{\mu\to 0}\left\{\ \mu^*\frac{1-\mu}{\mu(1-\mu^*)^2}\ A_n'(\mu) - \frac{\mu^{*2}}{1-\mu^*}\frac{1}{\mu^2}\ A_n'(\mu^*)\ \right\} \\
&-\; \lim_{\mu\to 0}\ \mu^*\ A_n'(\mu)\ \frac{(\alpha+A_n(\mu^*))A_s'(\mu^*) - (\alpha+A_s(\mu^*))A_n'(\mu^*)}{(\alpha+A_n(\mu^*))^2}.
\end{aligned}$$

Rearranging μ^* terms, write $\lim_{\mu\to 0}\mu^*\delta(\mu)\Delta(\mu)$ as:

$$\begin{aligned}
&\lim_{\mu\to 0}\left\{\ \frac{\mu^*(1-\mu)}{\mu(1-\mu^*)^2}\ A_n'(\mu) - \frac{1}{1-\mu^*}\left(\frac{\mu^*}{\mu}\right)^2 A_n'(\mu^*)\ \right\} \\
&-\lim_{\mu\to 0}\ \sqrt{\frac{\mu^*}{\mu}}\ \sqrt{\mu}\ A_n'(\mu)\ \frac{(\alpha+A_n(\mu^*))\sqrt{\mu^*}A_s'(\mu^*) - (\alpha+A_s(\mu^*))\sqrt{\mu^*}A_n'(\mu^*)}{(\alpha+A_n(\mu^*))^2}.
\end{aligned} \tag{28}$$

Lemma 9 *(i)* $\lim_{z\to 0}\sqrt{z}\, b'(z) = -2R^m\ K_0$, *where* $0 < K_0 < \infty$, *(ii)* $\lim_{z\to 0}\sqrt{z}\ A_n'(z) = K_n$ *and* $\lim_{z\to 0}\sqrt{z}\ A_s'(z) = K_s$, *with* $0 < K_n < K_s < \infty$ *and* $K_n = \frac{4}{3}\ R^m\ D(p^m)\ K_0$.

Proof. See our working paper. ■

Lemma 9 shows that $\lim_{z\to 0}\sqrt{z}\ A_n'(z) = K_n$ and $\lim_{z\to 0}\sqrt{z}\ A_s'(z) = K_s$, with $0 < K_n < K_s$. Moreover, recall from part (i) of Lemma 8 that $\lim_{\mu\to 0}\frac{\mu^*(\mu)}{\mu} = 1$. Thus, (28) simplifies to:

$$\begin{aligned}
\lim_{\mu\to 0}\mu^*\delta(\mu)\ \Delta(\mu) &= \lim_{\mu\to 0}\left\{ \frac{\mu^*(1-\mu)}{\mu(1-\mu^*)^2}\ A_n'(\mu) - \frac{1}{1-\mu^*}\left(\frac{\mu^*}{\mu}\right)^2\ A_n'(\mu^*) \right\} - \frac{K_n(K_s-K_n)}{\alpha} \\
&\le \lim_{\mu\to 0}\left\{ \frac{\mu^*}{\mu(1-\mu^*)}\Big(A_n'(\mu) - \left(\frac{\mu^*}{\mu}\right)\ A_n'(\mu^*)\Big)\right\} - \frac{K_n(K_s-K_n)}{\alpha} \\
&= \lim_{\mu\to 0}\left\{\frac{\mu^*}{\mu^2(1-\mu^*)}\Big(\mu\ A_n'(\mu) - \mu^*\ A_n'(\mu^*)\Big)\right\} - \frac{K_n(K_s-K_n)}{\alpha}.
\end{aligned}$$

Lemma 10 $\lim_{z\to 0}\frac{d(zA_n'(z))}{dz} > 0$.

Proof. See our working paper. ■

Lemma 10 shows that $zA'(z)$ is increasing when z is sufficiently small. This together with $\mu^* \ge \mu$ and $\lim_{\mu\to 0}\mu^*/\mu = 1$ implies that $\mu\ A_n'(\mu) - \mu^*\ A_n'(\mu^*) \le 0$. Thus,

$$\lim_{\mu\to 0}\mu^*\delta(\mu)\ \Delta(\mu) < -\frac{K_n(K_s-K_n)}{\alpha} < 0 \ \Rightarrow\ \lim_{\mu\to 0}\delta(\mu)\Delta(\mu) = -\infty.$$

Case of $\mu \approx 1$. We begin with a lemma.

Lemma 11 $0 < \lim_{z\to 1} A_s'(z) < \lim_{z\to 1} A_n'(z) = \infty$.

Proof. See our working paper. ■

Now, because $A(1) - A_n(\mu^*(1)) = 0$, it suffices to show that

$$\lim_{\mu\to 1}\left\{\ \Delta(\mu) \equiv A'(\mu) - A_n'(\mu^*)\ \frac{d\mu^*(\mu)}{d\mu}\ \right\} < 0.$$

From (23) and $\lim_{\mu\to 1}(1-\mu)A_n'(\mu) = 0$, we have $\lim_{\mu\to 1} A'(\mu) = \lim_{\mu\to 1} A_s'(\mu)$. From Lemma 8, $\lim_{\mu\to 1} d\mu^*/d\mu = 1$. From Lemma 11 above, $0 < \lim_{z\to 1} A_s'(z) < \lim_{z\to 1} A_n'(z) = \infty$. Combining these results, we have

$$\lim_{\mu\to 1}\Delta(\mu,\mu^*(\mu)) = \lim_{\mu\to 1}\left[A_s'(\mu) - A_n'(\mu^*)\frac{d\mu^*(\mu)}{d\mu}\right] = -\infty.$$

This concludes the proof of Proposition 3. □