WARWICK
ECONOMICS

Identification of preferences, demand and equilibrium with finite data

F. Kubler R. Malhotra & H. Polemarchakis

(This paper also appears as CRETA Discussion Paper No: 60)

July 2020 No: 1290

Warwick Economics Research Papers

ISSN 2059-4283 (online)
ISSN 0083-7350 (print)

Identification of preferences, demand and equilibrium with finite data

F. Kübler [1]   R. Malhotra [2]   H. Polemarchakis [3]

July 15, 2020

[1]University of Zürich; fkubler@gmail.com
[2]University of Warwick; r.malhotra@warwick.ac.uk
[3]University of Warwick; h.polemarchakis@warwick.ac.uk

Abstract

We give conditions under which an individual's preferences can be identified with finite data. First, we derive conditions that guarantee that a finite number of observations of an individual's binary choices identify preferences over an arbitrarily large subset of the choice space and allow one to predict how the individual shall decide when faced with choices not previously encountered. Second, we extend the argument to observations of individual demand. Finally, we show that finitely many observations of Walrasian equilibrium prices and profiles of individual endowments suffice to identify individual preferences and, as a consequence, equilibrium comparative statics.

Key words: identification, finite data, preferences, choices, demand, Walrasian equilibrium.

JEL classification: D80; G10.

With only a finite number of observations of an individual's choices, can we predict how the individual shall decide when faced with choices not previously encountered?

Variations of this question have been extensively discussed for the case of Walrasian demand, but most of the existing literature either focuses on the case of infinitely many observations or asks only whether observations are consistent with utility maximization. The seemingly very important question of what one can conclude about an individual's preferences from finitely many observations has largely been overlooked. In this paper, we pose exactly this question: what does one need to know about preferences a priori in order to be able to make non-trivial inference about the underlying data generating preference with only finitely many data points?

Revealed preference analysis, the weak axiom, was introduced by Samuelson (1938) as a necessary condition for demand data, a collection of pairs of prices and bundles of commodities, to be generated by the maximization of a preference relation subject to the budget constraint. Houthakker (1950) introduced the strong axiom, sufficient for demand data to be generated by preference optimization. Later, Afriat (1967) established the generalized axiom of revealed preference as necessary and sufficient for a finite set of demand data to be derived from the maximization of a preference relation or ordinal utility function.Forges and Minelli (2006), Reny (2015) and Nishimura, Ok, and Quah (2017) extended the argument to arbitrary data sets and choices. Varian (1982) used the results in Afriat (1967) to discuss the possibility of making statements about preferences from a finite number of observations.

Hurwicz and Uzawa (1971) and Mas-Colell (1977) gave necessary and sufficient conditions for the integrability of a demand function, the derivation of a generating ordinal utility function, and for a demand function to identify preferences. This answers our motivating question affirmatively for an infinite number of observations. However, the results say nothing for the case of a finite number of observations. It does not even address the question of asymptotics.

Mas-Colell (1978) gave sufficient conditions to ensure that, for a nested, increasing sequence of demand data that, at the limit, cover a dense subset of consumption choices, any associated sequence of preference relations converges to the unique preference relation that generated demand. In a recent paper, Chambers, Echenique, and Lambert (2019) considered the case of pairwise choice and showed that convergence fails when data sets are collections of choices from pairwise comparisons of alternatives become dense, generating preferences may not convergence to the unique underlying preference even if the underlying preference relation is continuous. Convergence obtains if the data satisfies a condition implied by, but weaker than monotonicity.

For choices generated by a monotone preference relation, convergence follows from Forges and Minelli (2006).

In this paper, we focus on a related, yet different question. We characterize frameworks where, after a sufficiently large, but finite number of observations of an individual's choices, one can identify the preference relation of the individual over a subset of the choice set, $\mathbf{X}' \subset \mathbf{X}$. One can think of the observations of being drawn randomly from some appropriate distribution over choice-sets. We require identification to be *effective* in the sense that strict preferences can be identified as strictly revealed preferred in the data. We refer to this as *finite identification* over the set $\mathbf{X}'$. In particular, we are interested in finite identification over binary sets. Given any two elements of the choice set x, y with $x \succ y$, when are preferences over $\{x, y\}$ finitely identified? That is, when can one infer from finitely many observations on previous choices that x must be strictly preferred to y? This question combines revealed preference analysis with identification: an algorithm that determines from the finitely many choices that it is impossible that $y \succeq x$.

We show that this finite identification over arbitrary binary sets implies that for any compact choice set, for a sufficiency large number of observations, one can identify preferences over an arbitrarily large subset of the choice set. Finally, we derive additional conditions that ensure that one can predict what the individual will choose from a choice-set not previously encountered.

We consider the binary choice problem in Chambers, Echenique, and Lambert (2019) and the Walrasian demand model in Mas-Colell (1978). For binary choice, we show that continuity and monotonicity of preferences are sufficient for finite identification. In this setting, finite identification implies directly that one can forecast choices from most binary choice sets. For Walrasian demand, we show that the assumptions in Mas-Colell (1977) suffice for finite identification. We also give an argument to show that, for budget sets not previously encountered, demand can be predicted with arbitrary accuracy.

A question that arises directly from our analysis is whether any of these results extent to an equilibrium setting. The lack of available data and problems with econometric estimation procedures notwithstanding, it is an important theoretical question whether the necessary information concerning the unobservable characteristics of individuals can be identified from their observable, market behavior. The identification of preferences from observed behavior has strong positive as well as normative implications. The transfer paradox, introduced by Leontief (1936), makes it clear that knowledge of the utility functions is necessary in order to identify welfare effects of transfers. More generally, explanation and prediction, as well as normative analysis, require individual or aggregate behavior, that is observable, to identify the fundamentals of the economy, that are not.

Brown and Matzkin (1990) showed how to extend Afriat (1967) to a framework where one has a finite number of observations on profiles of equilibrium endowments and Walrasian equilibrium prices. That is, observations on the equilibrium manifold. Chiappori et al. (2004) showed that the equilibrium manifold locally identifies individual preferences. In this paper we build on these results and show that individual preferences can be identified from finitely many observations of prices and distributions of endowments on the equilibrium manifold.

The rest of this paper we organize as follows. in section 1, we describe an abstract choice setting and make precise what we mean by finite identification. In section 2, we consider the case of binary choice. In section 3, we consider the case of Walrasian demand. In section 4 we discuss the equilibrium correspondence.

1 The setting

We start our analysis in a setting of individual choice. Subsequently, we extend the analysis to a setting in which only aggregate choices are observable.

Objects of choice are alternatives $x \in \mathbf{X} \subset \mathbb{R}^L$. An individual has complete and transitive preferences over $\mathbf{X}$ that we describe by $\succeq \subset \mathbf{X} \times \mathbf{X}$. We write $x \succeq y$ whenever $(x, y) \in \succeq$. Associated with the preference relation, $\succeq$, there is the strict preference relation, $\succ$ and the indifference relation, $\sim$.

For a collection of choice sets, $\mathcal{A}$, a choice function, $f : \mathcal{A} \to \mathbf{X}$ associates with every choice set, $\mathbf{A} \subset \mathbf{X}$, $\mathbf{A} \in \mathcal{A}$, an element, $f(\mathbf{A}) \in \mathbf{A}$. Observations are a collection of choice sets and associated choices $(\mathbf{A} \in \mathcal{A}, f(\mathbf{A}))$ or, for simplicity, $(\mathcal{A}, f)$. For a subset of the set of alternatives, $\mathbf{X}' \subset \mathbf{X}$, the restriction of the choice function to $\mathbf{X}'$ is defined on the restricted collection of choice sets $\mathcal{A}' = \{\mathbf{A} \in \mathcal{A} : \mathbf{A} \subset \mathbf{X}'.\}$. Observations are finite if $\mathcal{A}$ has finitely many elements.

A preference relation, $\succeq$ *rationalizes* observations $(\mathcal{A}, f)$ if, for all $\mathbf{A} \in \mathcal{A}$,

$$f(\mathbf{A}) \succeq y \quad \text{for all } y \in \mathbf{A}.$$

Evidently, without any restrictions on preferences, this is not of interest because indifference between all $x \in \mathbf{X}$ rationalizes any observations.

Within a restricted class of preference relations, rationalization is not granted, while multiple preference relation may rationalize the observations.

Within a class, $\mathcal{P}$, preferences over a set $\mathbf{X}' \subset \mathbf{X}$ are *identified* by observations $(\mathcal{A}, f)$ if, for any $\succeq, \succeq' \in \mathcal{P}$ that rationalize the observations $(\mathcal{A}, f)$,

$$x \succ y \Leftrightarrow x \succ' y \quad \text{for all } x, y \in \mathbf{X}'.$$[1]

So far, our treatment of identification is not constructive, and it is not restricted to finite data. Our definition of finite identification below is restricted to finite data and constructive. That is, we require that, within a class of preference relations, $\mathcal{P}$,

1. a finite set of observations $(\mathcal{A}, f)$ identify preferences over the set $\mathbf{X}'$, and,
2. identification is *effective* in the sense that given any finite set observations, $(\mathcal{A}', f')$ with

$$\mathcal{A}' \subset \mathcal{A} \cup \left(\cup_{x,y \in \mathbf{X}} \{x, y\}\right), \quad \text{and} \quad f'(\mathbf{A}) = f(\mathbf{A}), \text{ for } \mathbf{A} \in \mathcal{A},$$

   there exists an algorithm to determine, in finitely many steps, whether or not there exists any $\succeq \in \mathcal{P}$ such that $\succeq$ rationalizes $(\mathcal{A}', f')$.

Importantly, effective identification implies that if a finite set of observations $(\mathcal{A}', f)$ identify preferences over a set $\mathbf{X}' \subset \mathbf{X}$ then, for any $x, y \in \mathbf{X}'$ with $x \succ y$, one can determine this relation in finitely many steps: the observation $y \in f(\{x, y\})$ cannot be rationalized jointly with $(\mathcal{A}', f)$. We say that, given the observations $(\mathcal{A}', f)$, x is *revealed strictly preferred* to y and write

$$x \succ^{R(\mathcal{A}', f)} y,$$

which makes clear the dependence on the observations.

We are now in a position to define our first central concept.

Definition 1. Given a nested sequence of finite collections of choice sets, $\mathcal{A}_n$, $n = 1, \ldots$, $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, preferences over $\mathbf{X}' \subset \mathbf{X}$ are finitely identified within a class $\mathcal{P}$ by observations $((\mathcal{A}_n)_{n=1}^{\infty}, f)$ if there exists an n such that preferences over $\mathbf{X}'$ are effectively identified by observations $(\mathcal{A}_n, f)$. Equivalently, for all $x, y \in \mathbf{X}'$,

$$x \succ y \Leftrightarrow x \succ^{R(\mathcal{A}_n, f)} y.$$

The definition ensures not only identification, but also that one can learn the preferences exactly from the observed choices. As mentioned in the introduction it combines identification with revealed preference analysis. We

[1] We define identification with reference only to strict preference for reasons that shall be clear.

require that the observations imply that x must be revealed preferred to y for all $x, y \in \mathbf{X}'$.

As explained above, identification can only be meaningful if one restricts preferences. The preference relation $\succeq$ is upper semi-continuous if, for every $x \in \mathbf{X}$, the upper contour set, $\mathbf{R}_+(x) = \{y : y \succeq x\}$, is closed. It is continuous if for every $x \in \mathbf{X}$ the upper contour set as well as the lower contour set, $\mathbf{R}_-(x) = \{y : x \succeq y\}$, are closed. It is monotonically increasing (or simply monotone) if $x > y$ implies $x \succ y$. It is convex if $x \succeq y$ implies that $\lambda x + (1-\lambda)y \succeq y$ for all $\lambda \in [0,1]$. It is strictly convex if $x \succeq y$, $x \neq y$, implies that $\lambda x + (1-\lambda)y \succ y$ for all $\lambda \in (0,1)$. It is strongly montone-convex if it is montone and, for any convex $\mathbf{A} \subset \mathbf{X}$ and for $\bar{x} \in \arg\max_{x \in \mathbf{A}} \succeq$, for any $\epsilon > 0$, there is a $\delta > 0$, such that

$$\bar{x} \succ x' + \delta \mathbf{1} \quad \text{for all } x' \in \mathbf{A} \text{ with } \|\bar{x} - x'\| \geq \epsilon.$$

Finally, following Mas-Colell (1977), Remark 4, we say that continuous, monotone and convex preferences $\succeq$ are Lipschitzian, if, for every $r > 0$, there are numbers, $H > 0$ and $\epsilon > 0$, such that, if

$$x, y, z \in \mathbf{X}_r = \{w \in \mathbf{X} : \frac{1}{1+r}\mathbf{1} \leq w \leq (1+r)\mathbf{1}\}$$

with $x \sim y$, and $\|x - z\| < \epsilon$, then

$$\delta(x, \mathbf{R}_+(z)) \leq H\delta(y, \mathbf{R}_+(z)),$$

where for a set $\mathbf{A} \subset \mathbf{X}$, $\delta(x, \mathbf{A}) = \inf_{y \in \mathbf{A}} \|x - y\|$.

Throughout the paper, for a preference relation, $\succeq$ and a set, $\mathbf{A} \subset \mathbf{X}$,

$$\arg\max_{x \in \mathbf{A}} \succeq = \{x \in \mathbf{A} : x \succeq x' \text{ for all } x' \in \mathbf{A}\}.$$

Evidently, if the preference relation $\succeq$ is strictly convex, for any convex $\mathbf{A} \subset \mathbf{X}$, $\arg\max_{x \in \mathbf{A}} \succeq$ is either empty or a singleton.

2 Pairwise choice

We first consider the simple binary choice problem. That is,

$$\mathbf{A}_k = \{x_k, y_k\} \quad \text{with} \quad x_k, y_k \in \mathbf{X},$$

and

$$\mathcal{A}_n = \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}.$$

We assume that, as $n \to \infty$, the set $\cup_{i=1}^n \mathbf{A}_i$ becomes dense in $\mathbf{X} \times \mathbf{X}$.

Proposition 1. *On any $\{x, y\} \subset int(\mathbf{X})$, preferences are finitely identified within the class of monotone and continuous preferences.*

Proof. Suppose $x \succ y$ for some $\succeq$ that rationalizes the observations $(\mathcal{A}_n, f)$ for all n. As $\cup_{i=1}^n \mathbf{A}_i$ becomes dense, by continuity, there must exist a k with $\mathbf{A}_k = \{\bar{x}, \bar{y}\}$, such that $\bar{x} < x$ and $\bar{y} > y$, and

$$f(\mathbf{A}_k) = \bar{x}.$$

By monotonicity, any $\succeq'$ that rationalizes $(\mathcal{A}_k, f)$ must satisfy $x \succ' y$. □

The result can also be inferred from Chambers et al. (2019). The proof here is obviously much simpler, but it does not show identification over all of $\mathbf{X}$ in the limit. Obviously identification is effective since we can infer from the observations that there cannot be any monotone and continuous preference relation that is consistent with the data and that satisfies $y \succeq x$. It is clear that without restricting the set of preferences finite identification is impossible.

To what extend does this result allow us to predict choices? There are two ways to understand this question. First, what are the chances that we can predict choices if we randomly (uniformly) pick $x, y \in \mathbf{X}$? Theorem 1 below shows that, for sufficiently large n, the probability can be arbitrarily close to 1. Second, can we predict choices from an arbitrary (convex) set $\mathbf{A} \subset \mathbf{X}$? Clearly, if $\mathbf{A}$ is infinite, we will generally be unable to predict the exact choice from finitely many observations. However, Theorem 2 below shows that, for sufficiently large n, we can predict the choice with arbitrary finite accuracy.

Theorem 1. *For any $\epsilon > 0$, there exists a set $\mathbf{X}'$, such that $(\mathbf{X} \setminus \mathbf{X}')$ has Lebesgues measure less than ϵ and such that preferences over $\mathbf{X}'$ are finitely identified within the class of monotone and continuous preferences.*

Proof. For $x \in \mathbf{X}$ let $\mathbf{O}^+(x) = \{x' \in \mathbf{X} : x' \gg x\}$ and let $\mathbf{O}^-(x) = \{x' \in \mathbf{X} : x' \ll x\}$. These are open sets and, by monotonicity, subsets of the strict, upper and lower contour sets of x, respectively. For a point $(x, y) \in \succ \subset \mathbf{X} \times \mathbf{X}$, let $\mathbf{O}(x, y) = \mathbf{O}^+(x) \times \mathbf{O}^-(y)$, an open set that is a subset of $\succ$.

Importantly,

$$\cup_{(x,y) \in \succ} \mathbf{O}(x, y) \ = \succ .$$

It is clear to see that the left hand is a subset of $\succ$. For equality, note that, by continuity, if $x \succ y$, for $\epsilon > 0$ sufficiently small,

$$x \succ x - \epsilon \mathbf{1} \succ y + \epsilon \mathbf{1} \succ y,$$

and, therefore, $(x, y) \in \mathbf{O}(x - \epsilon\mathbf{1}, y + \epsilon\mathbf{1})$.

Next, let $\mathbf{K}$ be an arbitrary compact subset of $\succ$. It follows that the family of open sets $\mathbf{O}(x, y) \cap \mathbf{K}$, $(x, y) \in \succ$, forms an open cover of $\mathbf{K}$ and, since the latter is a compact set, there must a finite sub-cover. By Proposition 1, a finite a finite number of observations suffice to know the corresponding (x, y) of this sub-cover.

To finish the proof, observe that

$$\{(x, y) \in \mathbf{X} \times \mathbf{X} : x \succ y \text{ or } y \succ x\}$$

has full Lebesgue measure in $\mathbf{X} \times \mathbf{X}$ and that, for any $\epsilon > 0$, one can find a compact subset that has Lebesgue measure $(1 - \epsilon)$. □

Finally, our knowledge of preferences on a large set of possible choices allows us to forecast choices from arbitrary convex choice sets if preferences are convex.

Theorem 2. *Given any convex and compact* $\mathbf{B} \subset int(\mathbf{X})$*, for every* $\epsilon > 0$*, there exists an* n*, such that, for any strongly monotone-convex and continuous* $\succeq$ *and* $\succeq'$ *that rationalize* $(f, \mathcal{A}_n)$*, any* $x \in \arg\max_{x \in \mathbf{B}} \succeq$ *and* $x' \in \arg\max_{x \in \mathbf{B}} \succeq'$ *satisfy*

$$\|x - x'\| < \epsilon.$$

Proof. Define the upper envelope of $\mathbf{B}$ as

$$\mathbf{B}^+ = \{x \in \mathbf{B} : \text{there is no } x' \in \mathbf{B}, \text{with } x' > x\}.$$

For a sufficiently large number of points, n, there must be a $\bar{x} \in \mathbf{B}^+$ and observations $\{\bar{x}_j, x_j\} \in \mathcal{A}_n$, $j = 1, \ldots, k$, with $\bar{x}_j \in \mathbf{O}^-(\bar{x})$ and $\bar{x}_j = f(\{\bar{x}_j, x_j\})$ for all $j = 1, \ldots k$, such that

$$\mathbf{M} = \cup_{j=1,\ldots,k} \left(\mathbf{O}^-(x_j) \cap \mathbf{B}^+\right)$$

surrounds $\bar{x}$: for any $\tilde{x} \in \mathbf{B}^+$ with $\|\tilde{x} - \bar{x}\| > \epsilon$, there is no path connecting $\bar{x}$ and $\tilde{x}$ on $\mathbf{B}^+$ that does not intersect $\mathbf{M}$. It follows from the definition of strong monotone-convexity that if the global maximum of $\succ$ on $\mathbf{B}$ is $\bar{x}$ that for any ϵ and any $x' \in \mathbf{B}$ with $\|\bar{x} - x'\| \geq \epsilon$ there exists a point strictly above $\mathbf{B}^+$ which is strictly inferior to $\bar{x}$ and its distance to $\mathbf{B}^+$ can be bounded below by some δ uniformly. The Lebesgue measure of the set $\mathbf{O}^-(x) \cap \mathbf{B}^+$ can then be bounded away from zero.

□

3 Individual demand

We consider the setting in Mas-Colell (1978) and take $\mathbf{X} = \mathbb{R}^L_{++}$. Budget sets are

$$\mathbf{A} = \{x \in \mathbf{X} : p \cdot x \leq 1\}, \quad p \gg 0,$$

and individual, Walrasian demand choices, $f(\mathbf{A})$ define the demand function, $f : \mathbb{R}^L_{++} \to \mathbb{R}^L_{++}$ that satisfies Afriat (1967) inequalities. For a given monotone, strictly convex and continuous preference relation $\succeq$ we write $f^{\succeq}(.)$ to denote the Walrasian demand function that is generated by $\succeq$. We take $\mathcal{A}_n = \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$, where $\mathbf{A}_k$ is the budget set associated with prices p_k, and we we assume that prices $(p_k : k = 1, \ldots, n)$ become dense in $\mathbb{R}^L_{++}$.

The key to finite identification in this setting is to reduce it to the previous setting. For this, the notion of revealed preference we introduced earlier is important. Given $x, y \in int(\mathbf{X})$ we say that x is strictly revealed preferred to y if there are some $p_i \in \mathbb{R}^L_{++}$, $i = 1, \ldots, N$ so that $x = f(p_1)$, $x_i = f(p_i)$, $i = 2, \ldots, N-1$ and $p_i \cdot x_{i+1} < p_i \cdot x_i$, with $x_N = y$. Note that revealed preferences are transitive. The key to finite identification is to show that, if $x \succ y$, then x must be strictly revealed preferred to y. The following lemma is a slight variation of a result in Mas-Colell (1977), Remark 12. We provide a proof for completeness that closely follows Mas-Colell's proof.

Lemma 1. .

If $\succeq$ is continuous, Lipschitzian, monotone and strictly convex, then $x \succ y$ is and only if x is revealed preferred to y.

Proof. Given any $y \in int(\mathbf{X})$, define

$$T_y = \{z \in \mathbf{X} : y \text{ is not revealed preferred to } z\},$$

and define a new preference relation $\succ'$ by

$$u \succeq' v \text{ if } \begin{cases} u \in \operatorname{conv}(T_y \cup \mathbf{R}_+(v)) & \text{if } v \notin T_y \\ u \in T_y \cap \mathbf{R}_+(v) & \text{if } v \in T_y \end{cases},$$

where $\mathbf{R}_+(\cdot)$ denotes the upper contour set.

It can be verified that $\succeq'$ is upper-semi continuous, monotone and convex. We can define a demand correspondence for $\succeq'$ by

$$h(p) = \arg \max_{x \in \{p \cdot x \leq 1\}} \succeq' .$$

Crucially, it must be the case that $f^{\succeq}(\cdot) = h(\cdot)$. To prove this, note first that $h(\cdot)$ is non-empty for all p. Suppose that for some $p \in \mathbb{R}^L_{++}$,

$u \neq v = f^{\succeq}(p)$ but $u \in h^{\succeq'}(p)$. By the definition of $\succeq'$ this can only be the case if $u \in T_y$ but $v \notin T_y$. By the continuity of $f^{\succeq}(.)$ there must be a p' sufficiently close to p so that $f^{\succeq}(p) \notin T_y$ but $p' \cdot f^{\succeq}(p') > p' \cdot u$. This is a contradiction to transitivity since we would have that y is revealed preferred to $f^{\succeq}(p')$ which is revealed preferred to u and u cannot be in T_y.

Therefore, $\succeq$ and $\succeq'$ generate the same demand functions. By Theorem 2' in Mas-Colell (1977) this implies that they are the same preferences, but this is only possible if T_y is equal to the upper contour set of $\succeq$ at y. This proves the result.

□

With this the following theorem follows directly.

Theorem 3. *Given any $x, y \in int(\mathbf{X})$, preferences over $\{x, y\}$ are finitely identified within the class of continuous, strictly convex, monotone and Lipschitzian preferences.*

Proof. Suppose $x \succ y$; by Lemma 1 x is revealed preferred to y. By continuity, of demand and the fact that observations become dense, we must eventually observe prices for which x is revealed preferred to y and preferences over $\{x, y\}$ are finitely identified.

□

Note that the Theorem also follow from the asymptotic results in Mas-Colell (1978) (the same way that Proposition 1 follows from Forges and Minelli (2006) and Chambers et al. (2019)). Although the argument is somewhat lengthy it is useful to make it explicit using the result in Mas-Colell (1978). Suppose $x \succ y$ and take any sequence of preferences $\succeq'_k$ that rationalizes $(A_i, f(A_i))_{i=1}^k$ and satisfies $y \succeq'_k x$. Since we assume that preferences are continuous, strictly convex, monotone and Lipschitzian they generate Lipschitzian demand that satisfies a boundary condition and it follows from Mas-Colell (1978), Theorem 4 that $\succeq'_k \rightarrow \succeq$ in the topology of closed convergence. As Hildenbrand (1994) points out, when restricted to compact subsets of $\mathbb{R}^L_{++}$ convergence in the topology of closed convergence is equivalent to convergence in the Hausdorff distance. Recall that for two sets $\mathbf{A}, \mathbf{B} \subset \mathbb{R}^L$ the Hausdorff distance can be written as

$$d_H(\mathbf{A}, \mathbf{B}) = \inf\{\varepsilon \geq 0;\ \mathbf{A} \subseteq \mathbf{B}_\varepsilon \text{ and } \mathbf{B} \subseteq \mathbf{A}_\varepsilon\},$$

where, for any set $\mathbf{C}$

$$\mathbf{C}_\varepsilon := \bigcup_{x \in \mathbf{C}} \{z \in \mathbb{R}^L\ \ d(z, (x, y)) \leq \varepsilon\}.$$

But, since $x \succ y$, clearly, $(y, x) \in \succeq'_k$ while $(y, x) \notin \succeq$. Furthermore there must be an open ϵ neighborhood around (x, y) in $\mathbf{X} \times \mathbf{X}$ so that $x' \succ y'$ in that neighborhood. Therefore $(y, x) \notin \succeq_\epsilon$, and for all k, and $\succeq'_k \not\subseteq \succeq_\epsilon$ – a contradiction to the definition of convergence.

It remains to be shown that identification is effective. For this, consider the observations $(\mathcal{A}_k, f_k)$. By the argument in Afriat (1967), there is a solution to the system of inequalities

$$\phi_j \leq \phi_i + \lambda_i p_i(x_j - x_i), \quad \lambda_i > 0. \qquad (A_k)$$

and the utility function

$$u_k(x) = \min\{\ldots \phi_i + \lambda_i p_i(x - x_i), \ldots\}$$

rationalizes the observations. Augment the system by the inequality

$$u_y \leq \phi_i + p_0(x_i - y),$$

and note that, in the last inequality, prices are a variable, they are not part of the data as in the inequalities (A_k), while the variable λ_0 has been normalized to 1. By our argument above, for k large, $x \succ y$ implies that there must be a $x_1 \ll x$ but sufficiently close to x such that there can be no solution to the inequalities with $u_y > \phi_1$. Importantly, the fact that there is no solution does not only imply that there is no piece-wise linear utility function, but by Afriat's theorem it implies that there is no continuous and quasi-concave utility function.

Note that the assumption that preferences are Lipschitzian cannot be dispensed with. This is surprising since with a finite number of observations one cannot test whether preferences are Lipschitzian. Nor can one test whether they are strictly convex. However, only the assumption of Lipschtitzian and strictly convex preferences guarantees that for sufficiently many observations the Afriat-inequalities no longer have a solution.

Theorems 1 and 2 above now directly apply to the individual demand setting. Preferences are finitely identified over arbitrarily large (strict) subsets of any compact subset of $\mathbb{R}^L_{++}$, and we can forecast demand at any given price within ϵ as the set of observations becomes sufficiently large. This latter result is similar to the result Beigman and Vohra (2006) – they consider a more sophisticated setting of PAC-learning.

4 Equilibrium

So far, our analysis focused on the classical choice setting. More generally, economic theory derives relationships between the fundamentals of the economy, some of which may not be observable, and observed individual or aggregate behavior and prices. It is then of interest to ask whether what is observed can be used to deduce individual preferences. We tackle this issue in the classical setting of Walrasian equilibrium and assume that individual behavior, individual demand in particular, is not observable. Observations consist of equilibrium prices and individual endowments.

As Brown and Matzkin (1990) and Chiappori et al. (2004) point out, this is equivalent to assuming that the aggregate demand function, as a function of prices and profiles of individual incomes, is observable. However, if only aggregate demand behavior is observable, the identification of the preference relations of individuals is more surprising: it might be possible to disaggregate the observed behaviors into different families of individual demand functions generated by different profiles of utilities. Indeed, Chiappori et al. (2004) give an example with quasi - linear preferences that shows that this may be the case. But surprisingly, they also derive sufficient condition on aggregate demand that ensure that identification is possible.

For a finite number of observations the result seems even more surprising. How can one possibly infer from observations on aggregate demand that individual h prefers two apples to one banana? The key lies in the construction in Brown and Matzkin (1990) of necessary and sufficient conditions for observed aggregate demand to be rationalized by individual utility maximization. The requirement is that there exist individual demands that satisfy the strong axiom and add up to aggregate demand. With a large number of observations, the possible individual demands turn out to be restricted to a "small set " of possible demands, and, from this, one can infer individual preferences.

Instead of taking choice sets and resulting choices as fundamentals, we assume that one observes the aggregate demand of individuals, $h \in \mathbf{H}$ that we denote by $z(p, w^{\mathbf{H}})$. The profile of incomes $w^{\mathbf{H}} = (w^h)_{h\in\mathbf{H}}$ varies independently of prices, and aggregate demand is the sum of individual demand functions. A sequence of observations consists of $(p_i, w_i^{\mathbf{H}}, z(p_i, w_i^{\mathbf{H}}))_{i=1,\ldots,n}$.

Brown and Matzkin (1990) show that the model is testable. In order to define finite identification in this framework, there is one important qualification we have to make. Since we want to show that finitely many observations on aggregate demand *locally* identify preferences we have to assume that the $w^{\mathbf{H}}$ lie in some a compact subset $\mathbf{W} \subset \mathbb{R}^H_{++}$ with non-empty interior – if one allows boundary values the argument trivially reduces to the previous case, since aggregate demand and individual demand are identical if there is only one individual as Balasko (1999) pointed out.

To modify Definition 1 for this setting, consider a sequence $(p_i, w_i^{\mathbf{H}}, z(p_i, w_i^{\mathbf{H}}))$, $i = 1, \ldots$, $(p_i, w_i)^{\mathbf{H}} \in \mathbb{R}^L_{++} \times \mathbf{W}$ for all i, with $\{(p_i, w_i^{\mathbf{W}}), i = 1, \ldots n\}$ becoming dense in $\mathbb{R}^L_{++} \times \mathbf{W}$ as $n \to \infty$. For an individual, $\bar{h} \in \mathbf{H}$ preferences over $\mathbf{X}^{\bar{h}'} \subset \mathbf{X}^{\bar{h}}$ are finitely identified within a class $\mathcal{P}$, if there is an n such that, for all $x, y \in \mathbf{X}^{\bar{h}}$ $x \succ y$ implies that, for all $x_i^{\bar{h}} \in \mathbf{X}$, $p_i \cdot x_i^{\bar{h}} = w_i^{\bar{h}}$ for all $i = 1, \ldots, n$ and all $h \in \mathbf{H}$ with

$$\sum_{h \in \mathbf{H}} x_i^h = z(p_i, w_i^{\mathbf{H}}) \quad \text{for all} \quad i = 1, \ldots n$$

and with $(x_i^h, p_i)_{i=1\ldots n}$ satisfying SARP for all $h \in \mathbf{H}$, the revealed preference relation

$$x \succ^{\bar{h}R((A_i, x_i^{\bar{h}})_{i=1\ldots n})} y \quad \text{with } A_i = \{x : p_i \cdot x \leq w_i^{\bar{h}}\}$$

holds.

In other words, whenever there are individual demands that are consistent with SARP and add up to aggregate demand, x must be strictly revealed preferred to y by individual $\bar{h}$.

Are there conditions under which it is possible that preferences of some individual are finitely identifiable? Chiappori et al. (2004) consider the asymptotic case and derive such conditions. Building on this, and on Mas-Colell (1978) we can derive such conditions.

In order to proof our main result we need the following assumptions from Chiappori et al. (2004). These assumptions are directly on demand and not on preferences.

Assumption 1. 1. For every individual, $h \in \mathbf{H}$, the income effect for every commodity, $\partial z_l / \partial w^h$ is a twice differentiable function of income and

$$\frac{\partial^2 z_l}{\partial (w^h)^2} \neq 0.$$

2. For every individual, h there exist commodities, $m \neq 1$ and $n \neq 1$, such that

$$\frac{\partial}{\partial w^h}(\ln \frac{\partial^2 z_m}{(\partial w^h)^2}) \neq \frac{\partial}{\partial w^h}(\ln \frac{\partial^2 z_n}{(\partial w^h)^2}).$$

Assumptions 1.1 and 1.2 are from Chiappori et al. (2004) and they ensure that income effects do not vanish for any commodity, while there are two commodities for which the partial elasticities of the income effects with respect to revenue do not vanish. Assumption 1.2 implies that there are at least three commodities: $L \geq 3$; a different argument is required for economies with two commodities, $L = 2$.

Theorem 4. *Let $z(p, w^{\mathbf{H}})$ be a continuous aggregate demand function satisfying Assumption 1 being generated by preferences $(\succeq^h)^{h \in \mathbf{H}}$. Preferences of individual $\bar{h}$ are finitely identified over $\{x, y\}$, $x, y \in \mathbf{X}^h$ within the class of continuous, monotone, strictly convex and Lipschitzian preferences.*

To prove the result we need two lemmas that establish that preferences are uniquely identified in the limit.

Given any n, define $(\succeq_n^h)_{h\in\mathbf{H}}$ to be any profile of preferences consistent with the observations $(p_1, w_1^{\mathbf{H}}, z_1), \ldots, (p_n, w_n^{\mathbf{H}}, z_n)$ in the sense that there are allocations $x_i^{\mathbf{H}}$ that add up to observed aggregate demand, $\sum_{h\in\mathbf{H}} x_i^h = z_i$ for all i and that each x_i^h is rationalized by $\succeq_n^h$.

The following lemma follows directly from Mas-Colell (1978).

Lemma 2. *Suppose $(\succeq_n^h)^{h\in\mathbf{H}} \to (\succeq^h)^{h\in\mathbf{H}}$ in the topology of closed convergence. Then $z(p, w^1, \ldots, w^H)$ is generated by $(\succeq^h)^{h\in\mathbf{H}}$.*

Proof. Suppose $\succeq$ does not generate aggregate demand at a point $(p, w^{\mathbf{H}})$. Then there must be individual demands $\tilde{x}^h$, $\sum_h \tilde{x}^h \neq z(p, w^{\mathbf{H}})$ with each

$$\tilde{x}^h \in \arg \max_{x\in\{x:p\cdot x\leq w^h\}} \succeq^h .$$

As in Mas-Colell's proof of Lemma 3, by denseness, there is a sequence of observations $(p_n, w_n^{\mathbf{H}}) \to (p, w^{\mathbf{H}})$. Clearly since aggregate demand is continuous $z(p_n, w_n^{\mathbf{H}}) \to z(p, w^{\mathbf{H}})$ and therefore there must be allocations associated with a sub-sequence that satisfy $x_n^h \to x^h$ for all h, with $x_n^h \in \arg\max_{x\in\{x:p_n\cdot x\leq w_n^h\}} \succeq_n^h$ and with $\sum_h x^h = h(p, w^{\mathbf{H}})$. It suffices to show that for at least one h $x^h \succ \tilde{x}^h$. Since this now reduces to an individual problem the result follows from Lemma 3 in Mas-Colell (1978).

□

The following lemma is the main result in Chiappori et al. (2004).

Lemma 3. *Suppose $(\succeq_h)_{h\in\mathbf{H}}$ and $(\tilde{\succeq}_h)_{h\in\mathbf{H}}$ rationalize an aggregate demand function that satisfies Assumption 1.1.-1.4. Then $(\succeq_h)_{h\in\mathbf{H}} = (\tilde{\succeq}_h)_{h\in\mathbf{H}}$.*

The proof of Theorem 4 is as follows.

Proof. We first show that preferences are uniquely identified in the limit. This part of the proof is almost identical to the proof of Theorem 4 in Mas-Colell (1978) – we repeat the main steps for completeness. Suppose $\succeq_n^{\mathbf{H}}$ are as the sequences constructed above. Since the closure of the considered preferences are closed convergence compact the sequence must have an accumulation point $\succeq^*$. By Lemma 2 this generates aggregate demand. By Lemma 3 it is the unique preference relation that generates aggregate demand. Therefore $\succeq_n$ cannot have a different accumulation point.

As in the proof of Theorem 3 , convergence in the topology of closed convergence is equivalent to convergence in the Hausdorff metric, and $\succeq_n \to \succeq^*$ implies that at some finite n the true preferences and the constructed preferences must agree on a given strict comparison.

□

As in the case of individual demand, Theorems 1 and 2 can be directly applied to this setting of aggregate demand. A sufficiently large, finite set of observations allows us to know agents' preferences over large (infinite) sets and it allows us to predict their choices (within some ϵ) at prices not previously encountered.

Famously, for excess demand as a function of the prices of commodities, Debreu (1974) and Mantel (1974) provided negative results, which confirmed initial results in Sonnenschein (1973): as long as the number of individuals aggregated is large relative to the number of commodities, aggregate excess demand need not satisfy any restrictions beyond homogeneity and Walras' law as prices vary in a compact set of strictly positive prices; The overall impression from the work following the conundruM posed by Sonnenschein (1973) was that individual rationality fails to generate observable implications for a general specification of endowments and preferences as is standard in the theory of general competitive equilibrium; which was interpreted to confirm that the theory does not also explanation and prediction.

Alternatively, in critical and insightful contributions, Brown and Matzkin (1990) and Brown and Matzkin (1996) pointed out that (1) demand is not easily — or, for that matter, in principal — observable out of equilibrium; (2) prices are not the only variables that determine the demand of individuals. Prices movements reflect fluctuations in the fundamentals of an economy. The relationship between these fundamentals and the resulting equilibrium prices is a natural focus for empirical observation. What theory should aim at are observable relationships between equilibrium prices and fundamentals. Following a revealed preference approach, Brown and Matzkin (1990) gave a set of testable restrictions that apply to finite data sets.

Here, following Brown and Matzkin (1990), Brown and Matzkin (1996) and Chiappori, Ekeland, Kubler, and Polemarchakis (2004), one observes aggregate demand as, not only the prices of commodities, but, also, the endowments or incomes of individuals vary. Alternatively, one observes the set of equilibrium prices, locally, as a function of the endowments or incomes of individuals. We show that general equilibrium theory allows for explanation and prediction even with a finite set of observations.

References

S. Afriat. The construction of a utility function from demand data. *International Economic Review*, 8:67–77, 1967.

Y. Balasko. Deriving individual demand functions from the equilibrium manifold. Unpublished manuscript, 1999.

E. Beigman and R. Vohra. Non parametric learnability of income-lipschitz demand functions. Unpublished manuscriopt, 2006.

D. Brown and R. Matzkin. Recoverability and estimation of the demand and utility functions of traders when demands are unobservable. Unpublished manuscript, 1990.

D. Brown and R. Matzkin. Testable restrictions on the equilibrium manifold. *Econometrica*, 64:1249–1262, 1996.

C. Chambers, F. Echenique, and N. Lambert. Recovering preferences from finite data. *Econometrica*, xx:xxx–xxx, 2019.

P.-A. Chiappori, I. Ekeland, F. Kubler, and H. Polemarchakis. Testable implications of general equilibrium theory: a differentiable approach. *Journal of Mathematical Economics*, 40:105–119, 2004. URL http://www.polemarchakis.org/a64-tge.pdf.

G. Debreu. Excess demand functions. *Journal of Mathematical Economics*, 1:000–000, 1974.

F. Forges and E. Minelli. An extension of afriat's theorem to general budget sets. Unpublished manuscript, 2006.

W. Hildenbrand. *Market Demand*. Princeton University Press, 1994.

H. S. Houthakker. Revealed preference and the utility function. *Economica*, 17:159–174, 1950.

L. Hurwicz and H. Uzawa. On the integrability of demand functions. In L. Hurwicz J. Chipman and H. Sonnenschein, editors, *Preferences, Utility and Demand*. Harcourt Brace, Jovanovich, 1971.

V. Leontief. A note on the pure theory of transfer. In *Exploration in Economics: Notes and Essays in Honor of F. W. Taussig*. McGraw Hill, 1936.

R. Mantel. On the characterization of aggregate excess demand. *Journal of Economic Theory*, 7:348–353, 1974.

A. Mas-Colell. On the recoverability of consumers' preferences from demand behavior. *Econometrica*, 45:1409–1430, 1977.

A. Mas-Colell. On revealed preference analysis. *The Review of Economic Studies*, 45:121–131, 1978.

H. Nishimura, E. A. Ok, and J. K.-H. Quah. A comprehensive approach to revealed prefeference theory. *American Econoimc Review*, 107.:1239–1263, 2017.

Ph. J. Reny. A characterization of rationalizable consumer behavior. *Econometrica*, 83:175–192, 2015.

P. A. Samuelson. A note on the pure theory of consumers' behavior. *Economica*, 5:61–71, 1938.

H. Sonnenschein. Do walras' identity and continuity characterize the class of community excess demand functions? *Journal of Economic Theory*, 6: 345–354., 1973.

H. R. Varian. The nonparametric approach to demand analysis. *Econometrica*, 50:945–973, 1982.