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Auctioning control and cash-flow rights separately*

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Abstract

We consider a classical auction setting in which an asset/project is sold to buyers who privately receive signals about expected payoffs, and payoffs are more sensitive to the signal of the bidder who controls the asset. We show that a seller can increase revenues by sometimes allocating cash-flow rights and control to different bidders, e.g., with the highest bidder receiving cash flows and the second-highest receiving control. Separation reduces a bidder's information rent, which depends on the importance of his private information for the value of his awarded cash flows. As project payoffs are most sensitive to the information of the bidder who controls the project, allocating cash flow to another bidder lowers bidders' informational advantage. As a result, when signals are close, the seller can increase revenues by splitting rights between the top two bidders.

Keywords: Control and cash flow rights; separation of rights; mechanism design; interdependent valuations

JEL classification: D44; D82

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1 Introduction

The starting point for our paper is a classical auction setting in which a seller seeks to sell a single asset/project to risk-neutral bidders who privately receive signals about the asset's expected future cash flows. When a bidder wins control, the asset's payoffs depend on both his signal and those of rival bidders. In this setting, the literature has focused on mechanisms in which only the bidder who controls the project receives cash flows, possibly splitting them with the seller, but no *other* bidder receives cash flows.

Our paper shows that a seller can do better by sometimes allocating control to one bidder and (some or all) cash flows to other bidders. We establish that as long as expected cash flows are more sensitive to the signal of the bidder who controls the project than to those of other bidders, expected seller revenues are strictly higher when the seller sometimes allocates control and cash-flow rights to *different* bidders, and that the bidder who should be assigned control may not be the one that would generate the highest project payoffs. Our paper is the first to propose a "separation" mechanism of this form, to establish its revenue advantages, and to identify the sources of those advantages.

To highlight how outcomes in our separation framework differ from those in the "no-separation" frameworks of existing studies, we focus on settings with ex-ante identical bidders. In the classical no-separation framework, under standard regularity conditions, the seller optimally awards both control and cash-flow rights to the highest bidder. This result reflects that (i) allocating control to the bidder with the highest signal maximizes social welfare, and (ii) allocating cash flows to the highest bidder reduces rents earned by bidders with lower signals, thereby minimizing bidders' total rents (reflecting the envelope theorem logic that rents earned by lower types cumulate to carry over to higher types). Our paper derives the surprising result that violating either (i) or (ii) can increase seller revenue. Our insight is that separating control and cash-flow rights between different bidders facilitates rent extraction. We show that one can always design a separation mechanism so that the benefit of separation strictly outweighs its costs.

To illustrate how separation mechanisms increase seller revenues, consider a simple example where two bidders receive independently-distributed signals t_1 and t_2. The asset generates cash flows of $v_i = \frac{1}{1+\rho}(t_i + \rho t_{-i})$ if $i \in \{1,2\}$ has control, where $\rho \in (0,1)$ means that cash flows are more sensitive to the controlling agent's signal than the rival bidder's signal. First consider standard English auctions where cash-flow and control rights are not separated. The equilibrium bidding strategy is $\beta_i(t_i) = t_i$, and the bidder with the higher signal $t_h > t_s$ wins. The seller's revenue is t_s and the winning bidder's payoff is $\frac{t_h - t_s}{1+\rho}$.

Now consider the following two-stage separation mechanism. The first stage is an "always-separating" English auction in which the highest bidder pays the exit price of the second-highest bidder and receives cash-flow rights, but, unlike in a standard no-separation auction, the second-highest bidder receives control. In the second stage, the seller offers the first-stage winner an option to override the first-stage outcome: the winner can pay the seller a small fixed extra payment of p_{extra} to acquire control, while still retaining all cash flows.

One can show that bidding $\beta_i(t_i) = t_i$ still constitutes an equilibrium to the first stage of our separation mechanism. Thus, considering the first stage outcome alone, seller revenue is the *same* as in the no-separation auction, but the winning bidder's payoff is reduced by a factor of ρ to $\rho\frac{t_h - t_s}{1+\rho}$: only bidders bear the efficiency loss from assigning control to the lower-valuation bidder. The second stage recovers some of this efficiency loss, leading to a Pareto improvement: as the asset generates more cash flows under the winning bidder's control, the winning bidder will pay to acquire control whenever the efficiency gain $(1-\rho)\frac{t_h - t_s}{1+\rho}$ exceeds the price p_{extra}. Both the seller and the winning bidder benefit, implying that expected seller revenue strictly exceeds that in the standard English auction.

These insights extend, holding for any number of bidders, general signal structures and valuation functions where cash flows strictly increase in the controller's signal and are more sensitive to the controller's signal than those of the other bidders. We allow p_{extra} to depend on the exit prices of losing bidders and derive the form that maximizes seller revenues. We also show that the bidding equilibrium is *ex-post* incentive compatible. That is, our separation mechanism has the virtue that it is an ex-post equilibrium (Bergemann and Morris (2008))—ex post, no bidder has any regrets—and it always generates (weakly and sometimes strictly) higher revenue than the no-separation mechanism, event by event.

The source for the benefits of separation is that the project's payoff is most sensitive to the information of the bidder who controls the project. As we know from standard auction theory, a bidder's information rent depends on the importance of his private information for project payoffs. Allocating both control and cash flow rights to the same bidder maximizes the bidder's informational advantage. Allocating cash flows, instead, to a bidder who does not control the project, reduces the sensitivity of project payoffs to this bidder's private information, lowering his informational advantage. When the two highest signals are sufficiently close, the inefficiency cost from allocating control to a lower signal bidder and the cost of increased bidder rents due to allocating cash flows to a lower signal bidder become arbitrarily small, leaving only the benefit from the reduced sensitivity of a bidder's payoff to his signal. Thus, when the difference in the two highest signals is small enough, separation dominates no-separation.

We then consider the possibility that the controller must receive a minimum share q of cash flows, for example to satisfy corporate regulations that mandate minimum equity stakes for control, or to assuage moral hazard concerns. We observe that one can split rights in two ways: instead of (1) always giving cash flow rights to the highest bidder and sometimes giving control to the second-highest bidder, one could (2) always give control to the highest bidder and sometimes give the second-highest bidder some cash flows. In each of these two separation mechanism designs, the highest bidder only receives all rights when his signal sufficiently exceeds the second highest. We show that for any $q < 1$, at least one of these two types of separation mechanisms can be designed to (i) be ex-post incentive compatible, and (ii) generate strictly higher expected revenues than no-separation English auctions.

Other researchers have examined settings where a single bidder splits cash flows with the seller. Ekmekci, Kos and Vohra (2016) consider the problem of selling a firm to a buyer who is privately informed about post-sale cash flows and the benefits of control. The seller can offer a menu of cash-equity mixtures, and the buyer must obtain a minimum equity claim to cash flows (with the seller retaining any residual cash flows) to gain control rights. They provide sufficient conditions for the optimal mechanism to take the form of a take-it-or-leave-it offer for either the minimum stake or for all shares. In contrast, we examine a multi-bidder setting in which the seller can allocate control and cash-flow rights to different bidders, showing that such separation *among bidders* increases seller revenues.

Mezzetti (2003; 2004) also studies two-stage mechanisms with interdependent valuations where first allocations are determined and then agents observe their outcome-decision payoffs, and transfers are determined depending on the information revealed in both stages. Mezzetti largely focuses on efficiency, showing that one can implement the ex-ante efficient allocation in an ex-post incentive compatible way using contingent transfers. He also shows that one can extract full rents if bidders' signals at the first-stage *perfectly determine* their realized outcome-decision payoffs at the second-stage. This determinability allows for a punishment mechanism that asks bidders to report their types at the first stage and their realized payoffs at the second stage; if a bidder misreports at the first stage, no one is subsequently fooled, so a designer can cross-check against bidders' reports to detect and punish lying at the first stage.

The bankruptcy resolution or private equity/venture capital settings that motivate our analysis do not feature deterministic relationships between signals and realized cash flows. Instead, our mechanism exploits the feature that when expected cash flows depend more strongly on the information of the bidder who controls the asset, any cash flows a bidder receives are less sensitive to his signal if he does not have control. The design exploits this lowered sensitivity to reduce a bidder's informational advantage by splitting control and cash flows, awarding control or cash flows to the second-highest bidder when signals are close, thereby raising expected seller revenues.

The literature has examined designs of no-separation auctions with common valuations in many settings. McAfee, McMillan, and Reny (1989) derive conditions under which, with common values, the optimal no-separation selling procedure is implemented by a simple mechanism in which a seller solicits reports from one bidder and offers the asset to another. Bergemann, Brooks, and Morris (2016) and Brooks and Du (2018) identify robust auctions in pure common value settings that yield maximum revenue guarantees. Lauermann and Speit (2023) study bidding in common-value auctions with an unknown number of bidders.

Other researchers have examined the consequences of separating ownership and control in the market for corporate control in the context of agency issues, free riding problems and information aggregation (see, e.g., Bagnoli and Lipman 1988, Ekmekci and Kos 2016, Voss and Kulms 2022). Our paper contributes to this literature by identifying an advantage of the separation of ownership and control from the perspective of optimal auction design.

Existing mechanism design approaches take the underlying property rights regime as fixed. One can interpret our analysis as treating the property rights regime as a design choice that is selected by the mechanism (in essence, an endogenously-determined property right). For instance, agents may have payoff-relevant private information, and their reported information affects the ownership and control structure. More narrowly, our separation mechanism extends existing frameworks by incorporating the assignments of cash flows and control to the mechanism design for a single asset.[1] In practice, a designer needs extensive leverage to be able to enforce the assignment of rights. One setting where this is so is that of a VC who seeks to exit an investment in a start-up. The VC has considerable leverage vis à vis the initial founders in determining whether or not the founders retain control.[2] The VC has two viable exit alternatives—(1) selling the firm back to its initial founders, in which case the founders receive both cash flows and control, or (2) selling the firm to an outside company, in which case the founders receive significant cash flows but not control (either continuing to work as employees, or leaving the firm). In essence, the VC holds an auction between the founders and outside firms, and separation sometimes occurs.[3] Corporate bankruptcy, where a judge has extensive leverage in determining bankruptcy allocations, is potentially another such setting.

2 The Model

There are $n > 1$ ex-ante identical bidders who bid for an asset/project. The project can be controlled (run) by only one bidder who generates a stream of future cash flows. The bidders and the seller are risk-neutral. Bidders do not discount future cash flows, whereas the seller values only current cash payments from the auction, discounting future cash flows to zero.

[1] For analyses of the ownership and control structure for multiple assets, see, e.g., the classical work of Grossman and Hart (1986) or Hart and Moore (1990).

[2] We thank Dima Leshchinskii for bringing this example to our attention.

[3] There are also settings where separation sometimes occurs, but the seller does not dictate terms. For example, an entrepreneur may "sell" a project idea to a syndicated VC group, where the lead VC is directly involved in the project management, while the other VCs only contribute funding in return for claims to future cash flows. Private equity clubs (e.g., club deals) and limited partner frameworks feature a similar separation, where the limited partners provide capital and other inputs, while the general partner runs the business.

Each bidder i receives a private signal t_i that is informative about the project's future cash flows. We also refer to t_i as bidder i's type. We use $\mathbf{t} \equiv (t_1, t_2, ..., t_n)$ to denote the vector of all bidder types and $\mathbf{t}_{-i} \equiv (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$ to denote the vector of bidder types other than i. We assume that signals are weakly affiliated, nesting independently-distributed signals as a special case. We use $f(\mathbf{t})$ to denote the joint density of $\mathbf{t}$. We assume that $f(\mathbf{t})$ is symmetric in its arguments, and uniformly continuous and strictly positive over its support $[\underline{t}, \bar{t}]^n$.

Valuations are interdependent: expected future cash flows from the project under bidder i's control, $v_i(t_1, ..., t_n)$, depend on the signals of all bidders. We assume v_i is nondecreasing in its arguments, twice continuously differentiable, and strictly increasing in t_i. Valuations are also symmetric:

$$v_i(t_1, ..., t_n) = u(t_i; \mathbf{t}_{-i}), \text{ for all } i, \tag{1}$$

where the function u is the same for each bidder and symmetric in its last $n-1$ components. Valuations satisfy a single-crossing condition: given any signal vector $\mathbf{t}$,

$$\frac{\partial v_i}{\partial t_i}(\mathbf{t}) \geq \frac{\partial v_j}{\partial t_i}(\mathbf{t}), \text{ for all } i \text{ and all } j \neq i. \tag{2}$$

The single-crossing condition implies that a bidder's signal has a greater influence on cash flows if he runs the project than if another bidder runs the project. Given the symmetry condition in (1), the single-crossing condition reduces to requiring that

$$u_1(t_1; t_2, ..., t_n) \geq u_2(t_2; t_1, ..., t_n), \tag{3}$$

where u_i is the derivative of u with respect to the i^{th} argument.

These assumptions on the signal distribution and valuations are standard in studies of auctions with interdependent values (see, e.g., Krishna (2010) or Vohra (2011)). We add a mild assumption that there exist signals $t_2 \geq t_3 \geq ... \geq t_n$ with $\bar{t} > t_2$ and $t_n > \underline{t}$, such that (3) holds as a strict inequality at $t_1 = t_2$:

$$u_1(t_2; t_2, t_3, ..., t_n) > u_2(t_2; t_2, t_3, ..., t_n). \tag{4}$$

We sometimes specialize to bidder valuations that are linear in the signals, as in Bergemann and Morris (2007), Bergemann, Shi and Valimaki (2009), or Gorbenko and Malenko (2022):

$$u(t_i, \mathbf{t_{-i}}) = A_n\big(t_i + \rho \sum_{j \neq i} t_j\big), \tag{5}$$

where $A_n \equiv \frac{1}{1+(n-1)\rho}$ is a normalizing parameter that sets $u(t, t, ..., t) = t$, and $\rho < 1$ implies that expected project payoffs are more sensitive to the controller's signal than to the signals of other bidders. We can rewrite this as $u = A_n \left(\rho \sum_j t_j + (1-\rho) t_i \right)$, i.e., a bidder's valuation is the sum of common value and private value components, where ρ measures the degree of common valuations: the higher is ρ, the less the assignment of control matters for cash flows.

Our key departure from the literature is to consider settings in which a seller can allocate control and cash-flow rights to different bidders. That is, a bidder who does not control the project may nonetheless receive some or all of the cash flows generated. Formally, we consider direct-revelation mechanisms that allow for such separation. Let $R_j(\mathbf{t}) \in [0, 1]$ be the probability bidder j is assigned control when bidders report $\mathbf{t}$. Let $Q_{ji}(\mathbf{t}) \in [0, 1]$ be the share of total cash flows that i gets when bidders report $\mathbf{t}$ and control is assigned to j.[4] Let $M_i(\mathbf{t})$ be i's expected cash payment to the seller when bidders report $\mathbf{t}$.

We require that

$$\sum_j R_j(\mathbf{t}) \leq 1, \quad \text{for all } \mathbf{t}, \tag{6}$$

and

$$\sum_i Q_{ji}(\mathbf{t}) = 1, \quad \text{for all } j \text{ and all } \mathbf{t}.^5 \tag{7}$$

We term our mechanism a "separation mechanism" to contrast with the standard "no-separation" mechanisms, which corresponds to a special case of our separation mechanism with $Q_{jj}(\mathbf{t}) = 1$ for all j and $Q_{ji}(\mathbf{t}) = 0$ for all $i \neq j$, for all $\mathbf{t}$.

We also impose a minimum control stake requirement for the bidder who is given control:

$$Q_{jj}(\mathbf{t}) \geq q \text{ for all } j \text{ and } \mathbf{t}, \tag{8}$$

where $q \in [0, 1)$. The advantages of separation hold regardless of whether $q = 0$ or $q > 0$. We allow for $q > 0$ to capture settings in which the bidder who receives control may need to retain a claim to cash flows, e.g., due to regulatory requirements, or to address moral hazard concerns (Ekmekci, Kos, and Vohra (2016)).

[4] If given report $\mathbf{t}$, j is never assigned control, then the value of $Q_{ji}(\mathbf{t})$ is irrelevant.

[5] The scenario $\sum_j R_j(\mathbf{t}) < 1$ corresponds to the seller retaining the project with some probability, which is suboptimal if the expected cash flows when all bidders receive the lowest signal $\underline{t}$ are sufficiently high. One can also weaken (7): $\sum_i Q_{ji}(\mathbf{t}) < 1$ would be outcome equivalent to $\sum_i Q_{ji}(\mathbf{t}) = 1$ but reducing $R_j(\mathbf{t})$.

Let $U_i(t_i, t_i'; \mathbf{t}_{-i})$ be bidder i's expected profit when he is type t_i and reports t_i', and all other bidders truthfully report $\mathbf{t}_{-i}$:

$$U_i(t_i, t_i'; \mathbf{t}_{-i}) \equiv \sum_j R_j(t_i'; \mathbf{t}_{-i}) Q_{ji}(t_i'; \mathbf{t}_{-i}) v_j(\mathbf{t}) - M_i(t_i'; \mathbf{t}_{-i}). \tag{9}$$

Here, $\sum_j R_j Q_{ji} v_j$ is the expected value of the cash flows awarded to bidder i, where the summation over j reflects that bidders other than i may run the project when i receives cash flows. The second term is the expected value of the payments that i makes to the seller.

Integrating (9) over $\mathbf{t}_{-i}$ yields bidder i's expected profit when he has type t_i but reports t_i' and all other bidders report truthfully:

$$\begin{aligned} \bar{U}_i(t_i, t_i') &= \int_{\Omega_{n-1}} \sum_j R_j(t_i'; \mathbf{t}_{-i}) Q_{ji}(t_i'; \mathbf{t}_{-i}) v_j(t_i; \mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i} \\ &\quad - \int_{\Omega_{n-1}} M_i(t_i'; \mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i}, \end{aligned} \tag{10}$$

where $f_{-i}(\mathbf{t}_{-i}|t_i)$ is the conditional marginal density of $\mathbf{t}_{-i}$ given t_i. Bidder i's expected profit is the expected value of cash flows received when he reports t_i' (the first line) net of his expected cash payment when he reports t_i' (the second line).

The equilibrium expected profit for bidder i of type t_i is $\bar{U}_i(t_i, t_i)$. Equilibrium requires that both the (interim) incentive compatibility condition,

$$\bar{U}_i(t_i, t_i) = \max_{t_i'} \bar{U}_i(t_i, t_i'), \tag{11}$$

and the (interim) individual rationality condition,

$$\bar{U}_i(t_i, t_i) \geq 0, \tag{12}$$

hold for all i and t_i. We later characterize equilibria that satisfy the stronger requirements of ex-post incentive compatibility and ex-post individual rationality.

The seller's expected revenue is the sum of the expected payments of all bidders:

$$\pi_s = \sum_{i=1}^{n} \int M_i(\mathbf{t}) f(\mathbf{t}) d\mathbf{t}. \tag{13}$$

We show that separation mechanisms that satisfy the feasibility conditions (6) and (7), the incentive compatibility and individual rationality conditions and the minimum control stake requirement can always be designed so as to generate higher expected seller revenue than their no-separation counterparts. Our characterizations hold regardless of whether bidder signals are correlated or independently distributed. With correlated signals, we know from Cremer and McLean (1988) that a seller can design a mechanism that exploits the correlation to achieve full extraction. However, such mechanisms require large side bets that may lead to large regrets, rendering an assumption of risk-neutral bidders problematic. This leads us to focus on separation mechanisms that either take an English-auction format or are direct-revelation mechanisms that are ex-post incentive compatible. We show they can always be designed to generate strictly higher expected revenues than English no-separation auctions.[6]

2.1 Discussion

The intuition for the advantages of separation can be understood by applying the envelope theorem to (10) and (11), which yields

$$\begin{aligned}\frac{d\bar{U}_i(t_i,t_i)}{dt_i} &= \int_{\Omega_{n-1}} \sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})\frac{\partial v_j(t_i;\mathbf{t}_{-i})}{\partial t_i} f_{-i}(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} \\ &\quad + \int_{\Omega_{n-1}} \Big[\sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})v_j(t_i;\mathbf{t}_{-i}) - M_i(\mathbf{t})\Big]\frac{df_{-i}(\mathbf{t}_{-i}|t_i)}{dt_i}d\mathbf{t}_{-i}, \qquad (14)\end{aligned}$$

where $\Omega_{n-1} \equiv [\underline{t},\bar{t}]^{n-1}$ is the space of integration over the signals of bidders other than i. The first term is the contribution to a bidder's rents from his private information regarding the value of the cash flows, while the second term is the contribution to bidder rents from correlation in bidder signals. The first term is the key for understanding the advantage of separation: as in the standard no-separation setting, allocating cash flows to a bidder i with signal t_i enables him to earn differential rents relative to when i has a lower signal, as reflected by the term $R_j(\mathbf{t})Q_{ji}(\mathbf{t})$; but, unlike in the no-separation setting, the differential rents are scaled by $\frac{\partial v_j(t_i;\mathbf{t}_{-i})}{\partial t_i}$. That is, bidder i's differential rents are weighted by the sensitivity of the value of his awarded cash flows to his signal when the project is run by bidder j. Because a bidder's signal has a greater influence on cash flows when he runs the project than if another

[6]Lopomo (2000) and Chung and Ely (2007) give general conditions under which the English no-separation auction yields the highest seller revenue among all ex-post incentive compatible no-separation mechanisms.

bidder does, awarding bidder i cash flows when the project is run by a different bidder reduces i's overall rents vis à vis awarding bidder i cash flows when i also runs the project himself.

Our model considers an impatient seller who does not value retention of cash flows. This assumption is standard in the security design literature (see, e.g., Biais and Mariotti 2005) where a seller owns an asset that generates future cash flows, but has a higher discount rate than buyers, creating gains to trade. Our insights on the advantages of separation extend if a seller is as patient as bidders, but has to raise cash to cover an upfront investment or other liquidity need. Existing studies have examined such settings when cash-flow rights and control are split between the seller and *one* bidder, with the seller giving both control and a share of cash-flows to the highest bidder in exchange for the cash needed for investment. Our insights apply here, too: a seller can do better by sometimes splitting a share of cash flows among multiple bidders—while retaining the remaining cash flows for herself, or by awarding control to a bidder who is not the highest bidder.

Our framework assumes that cash flows are contractible and are split in the form of equities. This assumption captures settings such as bankruptcy resolution, takeover auctions, private equity or venture capital where equity is commonly used. Our mechanism can be implemented by having bidders take equity shares or dual class shares where only one class of shares has control rights, as occurs in practice. In the market for corporate control, the use of equity is especially natural as regulations mandate a minimum equity share to gain control.

2.2 Two-stage separation mechanism when $q = 0$

In a standard **no-separation English auction**, the auctioneer continuously increases price and the auction stops when second highest bidder exits, with the winner pays that exit price. As is well known (see equation 6.5 in Krishna 2010), bidding strategies in the symmetric equilibrium take the following form: if bidders $k+1, k+2, ..., N$ have dropped out, with their exit prices revealing their signals t_{k+1},t_{k+2}, ..., t_N (strategies are monotone) to the remaining k active bidders, then the strategy of a remaining bidder i with signal t_i is to drop out at the price

$$\beta^k \left(t_i, t_{k+1}, ..., t_N\right) = u\left(t_i; t_i, ..., t_i, t_{k+1}, ..., t_N\right), \tag{15}$$

which is the expected value of the cash flows generated by bidder i when all k active bidders have signal t_i, while those bidders who exited have signals as revealed by their exit prices.

Next we describe our two-stage separation auction:

Definition 1 *(two-stage auction) The first stage is a standard English auction, i.e., the auctioneer continuously increases price and the auction stops when the next-to-last bidder exits.*

In the second stage, the seller offers the first-stage winner a choice of whether to receive cash-flow rights but give control rights to the highest losing bidder or to pay an additional fee to obtain both rights. If the winner only chooses cash flows then he pays the seller the exit price of the highest losing bidder and control is assigned to the highest losing bidder. If, instead, the winner chooses to obtain both rights then he pays the exit price of the highest losing bidder plus an extra payment of $p_{extra}(\cdot) \geq 0$*, where* $p_{extra}(\cdot)$ *can be any symmetric function of the exit prices of the losing bidders, but does not depend on the winner's bid.*

The auction rules, including $p_{extra}(\cdot)$, are public information before the first stage. The first stage is an always-separating English mechanism in which the second-highest bidder (who, in equilibrium, has the second-highest valuation) receives control and the highest bidder receives cash-flow rights. In the second stage, the seller offers the winner the option to override the first stage outcome by paying the seller an additional $p_{extra}(\cdot)$ to acquire control.

Define $\Delta(t_1, t_2; t_3, ..., t_n)$ to be the difference in expected cash flows from giving control to bidder 1 rather than bidder 2, when the other signals are $t_3, ..., t_n$:

$$\Delta\left(t_1, t_2; t_3, ..., t_n\right) \equiv u\left(t_1; t_2, t_3, ..., t_n\right) - u\left(t_2; t_1, t_3, ..., t_n\right).$$

$\Delta(t_1, t_2; t_3, ..., t_n)$ is the efficiency gain from allocating control to the higher bidder 1 rather than bidder 2 when $t_1 > t_2$. This gain weakly increases in t_1 and is nonnegative if $t_1 \geq t_2$.

Proposition 1 *Part A: In the symmetric equilibrium of the two-stage auction:*

(i) In the first-stage, bidding strategies are given by (15), as in a no-separation English auction, regardless of the functional form of $p_{extra}(\cdot)$*.*

(ii) In the second stage, without loss of generality let t_1 *be the first-stage winner's type and let* t_2 *be the highest losing bidder's type as inferred from the exit prices. The first-stage*

winner acquires control if and only if $\Delta(t_1, t_2; t_3, ..., t_n) \geq p_{extra}$.

Part B: Given any $p_{extra}(\cdot)$*, this equilibrium is ex-post incentive compatible.*

Proof: Consider an arbitrary bidder i with signal t_i when all other bidders follow their posited equilibrium strategies. Denote the highest of the other $n-1$ signals by t_h and let $\mathbf{t}_{-i-h}$ denote the vector of the other $n-2$ bidder types. We show bidder i is weakly better off following his equilibrium strategy for any realization of his rivals' signals.

In stage 2 only the winner's strategy is relevant, so assume without loss of generality that bidder i won the first stage (but he need not have followed his equilibrium strategy in the first stage). The difference in bidder i's profit from receiving both rights versus just receiving cash flow rights is $\Delta(t_i, t_h; \mathbf{t}_{-i-h}) - p_{extra}$. This difference in profits is positive if and only if $\Delta(t_i, t_h; \mathbf{t}_{-i-h}) \geq p_{extra}$. This establishes the optimality of the bidding strategy in stage 2.

In stage 1, we decompose analysis according to whether or not bidder i is the highest type.

Case 1: $t_i \geq t_h$. Then bidder i will win the first stage if he follows his equilibrium strategy. If i deviates and still wins the first stage, the deviation does not affect his profits because neither the winning price nor p_{extra} depend on his bid. If he deviates and loses the first stage, then his profit is zero, so deviation again is not profitable (since his equilibrium profit is nonnegative).

Case 2: $t_i < t_h$. Bidder i will lose the first stage if he follows his equilibrium strategy. If he deviates and still loses the first stage, then his profit is unaffected. If he deviates and wins the first stage, then his profit is negative if he does not pay p_{extra} in stage 2 (so bidder h retains control and the value of the cash flows under bidder h's control is less than bidder i's payment), and his profit is even more negative if he pays p_{extra} to obtain control (since bidder i generates even lower cash flows than bidder h and $p_{extra} \geq 0$).

Since a bidder's profit is zero if he exits at the lowest price of $u(\underline{t}; \underline{t}, ..., \underline{t})$,[7] the ex-post incentive compatibility of our mechanism also implies it is ex-post individually rational. ■

Proposition 1 implies that seller revenue is always weakly higher than in the standard English no-separation auction. In the first stage, the bidding strategy, and hence seller revenues, is the same as in the English auction where the outcome is efficient with the best bidder type

[7] In the measure zero event that all bidders exit at the lowest price and a bidder is (randomly) selected as the winner, his profit is nonnegative if he follows his equilibrium strategy in the second stage.

receiving both control and cash flow rights. The efficiency loss in our mechanism from assigning control to the second-most productive bidder is borne *entirely* by the winning bidder.

In the second stage, the seller offers the winner an opportunity to Pareto-improve on the first-stage outcomes, which benefits both the seller and the winner. The seller extracts additional rents from a winner whose type exceeds the second-highest type by enough that the winner would make an extra payment to obtain an efficient assignment of control. Thus, seller revenue is the same as in the no-separation auction if the winning bidder chooses not to pay p_{extra} to obtain control; and revenue is higher by p_{extra} if the winner pays to obtain control.

Via a simple choice of p_{extra}, our mechanism can always generate strictly greater expected seller revenues than the standard English auction in a robust, detail-free, way:

Result 1: Let p_{extra} be a constant. There exists a $p^* > 0$ such that for all $p_{extra} \in (0, p^*)$, the two-stage separation auction design generates strictly higher expected revenues than the no-separation English auction.

Proof: See the appendix. □

The intuition for Result 1 is simple: when the price offer is sufficiently small, it will be accepted with a strictly positive probability, leading to strictly higher expected revenue. We now derive the $p_{extra}(\cdot)$ that maximizes expected revenue in the two-stage mechanism. We solve for the optimal $p_{extra}(t_2, ..., t_n)$, where the types $t_2, ..., t_n$ of the losing bidders are inferred from their exit prices by inverting (15) and t_2 is the type of the highest loser.

Result 2: The seller's optimal choice of $p_{extra}(t_2, ..., t_n)$ is the monopoly price conditional on the highest signal t_1 being at least t_2:

$$p_{extra}^{optimal}(t_2, ..., t_n) = \Delta\left(t^{opt}, t_2; t_3, ..., t_n\right),$$

where

$$t^{opt} \equiv \arg\max_t \Delta\left(t, t_2; t_3, ..., t_n\right) \int_t^{\bar{t}} f_1\left(x|\mathbf{t}_{-1}\right) dx$$

and $f_1\left(\cdot|\mathbf{t}_{-1}\right)$ is the conditional marginal density of t_1 given the losing signals $\mathbf{t}_{-1}$,

$$f_1(x|\mathbf{t}_{-1}) \equiv \frac{f(x; \mathbf{t}_{-1})}{\int_{t_2}^{\bar{t}} f(s; \mathbf{t}_{-1}) ds}. \tag{16}$$

For example, consider two bidders with *i.i.d.* uniform signals on $[1,2]$. Let the project value if run by bidder i with signal t_i when the rival has signal t_{-i} be $v_i(t_i, t_{-i}) = \frac{2}{3}t_i + \frac{1}{3}t_{-i}$. In the first stage, bidder i truthfully exits at a price of t_i. Denote the loser's exit price (and type) by t_s. Conditional on t_s, the winner's type is uniformly distributed over $[t_s, 2]$. The monopoly price, and hence the optimal price for control rights, leaves the winner with type $\mathrm{E}[t|t \geq t_s] = \frac{t_s+2}{2}$ indifferent between accepting and declining. Hence,

$$p_{extra}^{optimal} = \Delta\left(\frac{t_s+2}{2}, t_s\right) = \frac{2-t_s}{6}.$$

The winner accepts this price offer with probability 0.5. Taking the unconditional expected value of this price offer yields that expected revenue in our two-stage auction exceeds that in no-separation English auction by $\frac{1}{18}$.[8] This expected revenue gain is realized despite a social welfare loss of $\frac{1}{36}$ due to the inefficient control assignment. Thus, bidder rents are reduced by $\frac{1}{18} + \frac{1}{36} = \frac{1}{12}$, or 38% of the no-separation mechanism rents of $\frac{2}{3}E\left[t_h - t_s\right] = \frac{2}{9}$.[9]

3 Separation mechanisms with minimum stake $q > 0$

We next analyze mechanisms in settings where the agent controlling the project must at least receive share $q \in (0,1)$ of cash flows.[10] This requirement captures the realistic features of Ekmekci, Kos, and Vohra (2016) that a bidder may need to retain a claim to cash flows in order to gain control, for example, due to regulatory requirements or to address moral hazard concerns. We assume that as long as the minimum stake is met, giving the controller a higher stake does not lead to higher cash flows. This is consistent with the empirical findings of Edmans, Gosling, and Jenter (2023) from their surveys on CEO compensation that the primary drivers of CEO effort are "intrinsic motivation" and "personal reputation."

[8]One can show that our two-stage auction with the optimal price offer maximizes expected revenue among all (interim) incentive compatible separation mechanisms in this example.

[9]To see the expected revenue increase, note that the offer is accepted with probability 0.5. As $t_s = \min(t_1, t_2)$, the expected revenue difference between our two-stage auction and the no-separation English auction is $0.5\frac{(2-E[\min(t_1,t_2)])}{6} = \frac{1}{18}$. Given t_s, control is assigned inefficiently if $t_h \in [t_s, \frac{2+t_s}{2}]$, with an expected social welfare loss of $\left(\frac{2}{3} - \frac{1}{3}\right)\frac{2-t_s}{4} = \frac{2-t_s}{12}$; since $t_h \in [t_s, \frac{2+t_s}{2}]$ with probability 0.5, the unconditional expected social welfare loss is $\mathrm{E}\left[\frac{2-t_s}{24}\right] = \frac{1}{36}$.

[10]While this section focuses on $q > 0$, our characterizations of separation mechanisms also apply to $q = 0$.

Technically, $q > 0$ reduces tractability of analyses of the extended two-stage auctions that generalize Definition 1 by bundling control with share q of cash flows. There are two complications. First, because the losing bidder receives allocations and his payment depends on his bid, truthful bidding is no longer an equilibrium. Second, whether the losing bidder keeps his allocation depends on whether the winning bidder accepts the price offer, which, in turn, depends on the winner's belief about the loser's type, and those beliefs will depend on the loser's bid. This introduces signaling considerations to the bidding.

These complications lead us to consider direct-revelation separation mechanisms in which the payments are adjusted so that truthful bidding is an equilibrium. We extend the direct-revelation mechanism of the standard no-separation English auction to accommodate the separation of control and cash allocations, while preserving ex-post incentive compatibility. This leads to two classes of mechanisms, A and B, which reflect two ways to divide control and cash flow rights. In Mechanism A, when the two highest reported types are sufficiently close, the second-highest bidder receives control and share q of cash flows, and the highest bidder receives share $1-q$ of cash flows. Thus, assignment of control is **inefficient**. Mechanism A generalizes the two-stage mechanism detailed in Proposition 1, reducing to it in the limit as q goes to zero. Mechanism B has the opposite design: when the reported types are sufficiently close, the highest bidder receives control and a share q of cash flows, and the second-highest bidder receiving share $1-q$ of cash flows. Thus, assignment of control is **efficient**.

3.1 Separation functions and their inverses

In both Mechanisms A and B, a "separation function" S determines how close the two highest types must be for separation to occur. A bidder i receives control and all cash flows if and only if his reported type satisfies $t'_i \geq S(\mathbf{t}'_{-i})$, where $S(\mathbf{t}'_{-i})$ weakly exceeds the highest component in $\mathbf{t}'_{-i}$; that is, $S(\mathbf{t}'_{-i})$ weakly exceeds the highest reported type of bidders other than i. It is useful to work with the inverse of the separation function, S^{-1}: a bidder i receives neither control nor cash flows if and only if his reported type satisfies $t'_i \leq S^{-1}(\mathbf{t}'_{-i})$. If, instead, t'_i is in an intermediate range so that $t'_i \in \left[S^{-1}\left(\mathbf{t}'_{-i}\right), S\left(\mathbf{t}'_{-i}\right)\right]$, bidder i receives partial allocations: either control plus a share of cash flows, or a share of cash flows but no control.

Formally we define separation functions and their inverses below, where we use s_i rather than t_i to denote a generic signal that is in $[\underline{t}, \bar{t}]$ but is not necessarily the signal of bidder i:

Definition 2 *(Separation function) For any $n-1$ signals $s_1, ..., s_{n-1}$, denote the highest signal by s_h and the second-highest by s_s. A "separation function" $S(s_1, ..., s_{n-1})$ is a symmetric function of $s_1, ..., s_{n-1}$ with $S(s_1, ..., s_{n-1}) \in [s_h, \bar{t}]$ that weakly increases in s_h.*

(Inverse of separation function) For $n = 2$, the "inverse" function $S^{-1}(s_1)$ is given by the smallest $s \in [\underline{t}, s_1]$ such that $S(s) \geq s_1$. For $n > 2$, $S^{-1}(s_1, ..., s_{n-1})$ is given by the smallest $s \in [s_s, t_h]$ such that $S(s_1, .., s_{h-1}, s, s_{h+1}, .., s_{n-1}) \geq s_h$, where $(s_1, .., s_{h-1}, s, s_{h+1}, .., s_{n-1})$ is the vector of $n-1$ signals formed by replacing s_h with s in $(s_1, ..., s_{n-1})$.

For example, for $n > 2$, a linear separation function is $S(s_1, ..., s_{n-1}) = s_h + w(\bar{t} - s_h)$ for $w \in [0, 1)$. Its inverse is $S^{-1}(s_1, ..., s_{n-1}) = \max\left\{s_h - \frac{w}{1-w}(\bar{t} - s_h), s_s\right\}$. The max operator in S^{-1} reflects that if bidder i's signal is not the highest then to receive an allocation i's signal must (i) not be too far below the highest, and (ii) exceed those of the other $n-2$ bidders.

In the analysis that follows we first use the separation functions to construct mechanisms A and B. We then derive the conditions under which each class of mechanism is ex post incentive compatible. Finally, we show that given the conditions for ex-post incentive compatibility, Mechanisms A and B can be designed to generate strictly higher expected revenues than no-separation English auctions.

3.2 Separation Mechanism with Inefficient Splitting

To begin, we define Mechanism A and determine when it is *ex-post* incentive compatible.

Definition 3 *(Mechanism A: inefficient splitting of rights) Let the reported types be $t'_1 \geq t'_2 \geq t'_3 \geq ... \geq t'_n$.*

If $t'_1 > S(\mathbf{t}'_{-1})$, bidder 1 receives control and all cash flows and pays

$$\begin{aligned} M_1 &= u(S(\mathbf{t}'_{-1}); t'_2, ..., t'_n) - (1-q)u(t'_2; S(\mathbf{t}'_{-1}), ..., t'_n) + (1-2q)u(t'_2; t'_2, ..., t'_n) \\ &\quad + qu(S^{-1}(\mathbf{t}'_{-1}); t'_2, ..., t'_n). \end{aligned} \tag{17}$$

If $t'_1 \leq S(\mathbf{t}'_{-1})$, bidder 2 receives control and a fraction q of cash flows, and bidder 1 *receives fraction* $1-q$ *of cash flows.*

$$\text{Bidder 1 pays:} \quad M_1 = (1-2q)\, u\left(t_2'; t_2', ..., t_n'\right) + qu\left(S^{-1}(\mathbf{t}'_{-1}); t_2', ..., t_n'\right) \tag{18}$$

$$\text{Bidder 2 pays:} \quad M_2 = qu\left(S^{-1}(\mathbf{t}'_{-2}); t_1', t_3', ..., t_n'\right). \tag{19}$$

All other bidders receive nothing and pay nothing.

When $q = 0$, Mechanism A reduces to the direct-revelation mechanism of our two-stage auction in Definition (1), which is ex-post incentive compatible as established earlier. To determine the conditions for ex-post incentive compatibility for general q, define

$$\rho_{\min} \equiv \min_{\mathbf{t}} \frac{\partial v_2(\mathbf{t})}{\partial t_1} / \frac{\partial v_1(\mathbf{t})}{\partial t_1}.$$

$\frac{\partial v_2(\mathbf{t})}{\partial t_1} / \frac{\partial v_1(\mathbf{t})}{\partial t_1}$ is the ratio of the influence of a bidder's signal on cash flows if another bidder runs the project relative to when he runs the project. With linear valuations, i.e., with $v_i\left(t_1, ..., t_n\right) = A_n\left(t_i + \rho \sum_{j \neq i} t_j\right)$, we have $\rho_{min} = \rho$. For general valuations, $\rho_{\min}$ is a measure of the minimum sensitivity of cash flows to the signals of bidders who do not run the project.

Proposition 2 *Suppose* ρ_{min} *is large enough and* q *is small enough that* $\rho_{\min} \geq \frac{q}{1-q}$*. Then for Mechanism A truthful reporting is an ex post equilibrium given any separation function* S*.*

Proof: See the appendix. □

The logic for ex-post incentive compatibility extends that for English no-separation auctions to settings where control and cash flow rights can be assigned to different bidders and multiple bidders may receive allocations. In the direct-revelation mechanism of the English no-separation auction, given other bidders' reports, a bidder's allocation and payment vary *discretely* (not continuously): they only depend on which of *two* "report-regions" a bidder's reported type is in, and a bidder receives allocations only when his report is the highest.

Mechanism A retains the feature that given other bidders' reports, bidder i's allocations and payments vary discretely—they do not change within a report-region—but it has *four* report-regions, reflecting the additional ways to separate control and cash flow allocations. In region 4, bidder i's report exceeds the second-highest report by enough that $t_i' > S(\mathbf{t}'_{-i})$, so bidder i receives control and all cash flows; in region 3, i's report is highest, but now

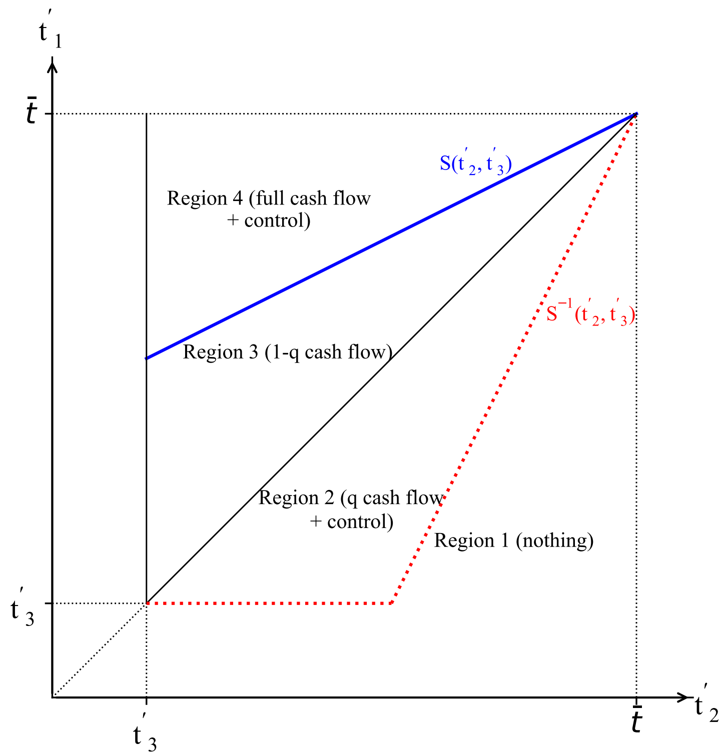

Figure 1: Illustration of the four payoff regions for Mechanism A, showing bidder 1's allocation given its report t_1' and those of the other bidders 2 and 3 when $t_2' \geq t_3'$. The blue solid line plots $S\left(t_2', t_3'\right)$ as a function of t_2' (given t_3'), and the red dotted line plots $S^{-1}\left(t_2', t_3'\right)$ as a function of t_2'. Bidder 1 receives allocations in regions 2, 3 and 4. The no-separation English mechanism corresponds to the special case where $S = s_h$, so that S and S^{-1} coincide on the 45 degree line, and regions 2 and 3 collapse to have zero-measure.

$t_i' \in \left(t_h', S(\mathbf{t}_{-i}')\right)$, where t_h' is the highest of the other reports, so bidder i receives share $1-q$ of cash flows but not control; in region 2, i's report is the second-highest but close enough to the highest, $t_i' \in \left(S^{-1}(\mathbf{t}_{-i}'), t_h'\right)$, so bidder i receives control and share q of cash flows; in region 1, t_i' is lower yet with $t_i' < S^{-1}(\mathbf{t}_{-i}')$, i.e., t_i' is either below the second-highest or sufficiently lower than the highest, so bidder i receives nothing. Figure 1 illustrates the four report-regions and allocations for three bidders and a linear separation function.

When a bidder's type is at the boundary of any two adjacent regions, the payments leave him indifferent to reporting any type in those regions. Thus, there is no incentive for local deviations, and when bidders bid truthfully, the differential rents earned at lower types add up and carry over to higher types. If a bidder's cash flow allocation changes across a boundary, his payment adjusts—if he would receive a share x of cash flows at a lower unit price by deviating to report a lower type, then when he bids his true type, he pays for the share x at that lower unit price; he only pays the higher unit price for the remaining cash flows that he would not receive if he deviated. Similarly, if control changes when a bidder's reported type switches from one region to a neighboring region thereby changing the value of cash flows, then the change in the value of the awarded cash flows is incorporated into the payments.

These payment features make deviations unprofitable if they are within a given report-region, or if a bidder's type is at the boundary of two regions and he deviates locally. However, global IC also requires deviations be unprofitable when a bidder's type falls in the interior of one region and he deviates to the interior of another region. Lemma 1 lets us circumvent having to exhaustively rule out the many possible such deviations:

Lemma 1 *Suppose $U(t,t')$ is a function of $t, t' \in [\underline{t}, \bar{t}]$ with the following properties:*

(a) U is differentiable with respect to t for all t';

(b) U is differentiable at $(t=t', t')$ except possibly for a countable set S^* of t'.

(c) $U(t,t)$ is continuous in t and $\frac{\partial}{\partial t'}U(t,t')|_{t=t'} = 0$ for all $t' \notin S^*$.

Sufficient condition for local IC to imply global IC: *If, for all t, $\frac{\partial}{\partial t}U(t,t')$ weakly increases in t', then $t'=t$ maximizes $U(t,t')$ over $t' \in [\underline{t}, \bar{t}]$ for all t.*

Necessary condition for global IC. *If $t'=t$ maximizes $U(t,t')$ for all t, then $\frac{\partial}{\partial t'}\frac{\partial}{\partial t}U(t,t')|_{t'=t} \geq 0$ at any t where U is twice differentiable at $(t, t'=t)$.*

Proof: See the appendix. □

Here we interpret $U(t,t')$ as the payoff of a type t bidder who reports t', conditional on the other bidders' (truthfully-reported) types.[11] The standard way to think about incentive compatibility is to examine what happens if we fix t and vary t'. This yields a necessary condition for incentive compatibility that the second-order condition be negative, $\frac{\partial^2}{\partial^2 t'}U(t,t') = \frac{\partial}{\partial t'}\left(\frac{\partial}{\partial t'}U(t,t')\right) \leq 0$. We instead fix t' and vary t. This yields a necessary condition that the cross-partial be positive, $\frac{\partial}{\partial t'}\left(\frac{\partial}{\partial t}U(t,t')\right) \geq 0$ when this derivative exists. Similarly, a standard

[11]One can also interpret $U(t,t')$ as a bidder's interim expected payoff and use Lemma 1 to derive conditions for interim IC. This approach is useful in settings where payoff functions are not affine so that one cannot appeal to the standard argument that the supremum of a family of affine functions is convex.

sufficient condition for local IC to imply global IC is that $\frac{\partial}{\partial t'}U(t,t')$ weakly *decrease* in t' over $t' \in [\underline{t}, \bar{t}]$; Lemma 1 instead requires that $\frac{\partial}{\partial t}U(t,t')$ weakly *increase* in t' over $t' \in [\underline{t}, \bar{t}]$.

The approach in Lemma 1 of fixing t' and varying t is useful for two reasons. First, fixing t' and varying t often eases analysis as $\frac{\partial}{\partial t}U(t,t')$ typically takes a simpler form than $\frac{\partial}{\partial t'}U(t,t')$. This reflects that reported types (t') affect auction outcomes (winning and allocations) and hence bidder payoffs, but true types (t) typically do not. Second, payoff functions often involve discontinuities with respect to t' (given the types of other bidders) when $t' \neq t$. Such discontinuities arise even in no-separation mechanisms; and in separation mechanisms there are four report-regions, leading to three boundary points where discontinuities may arise.

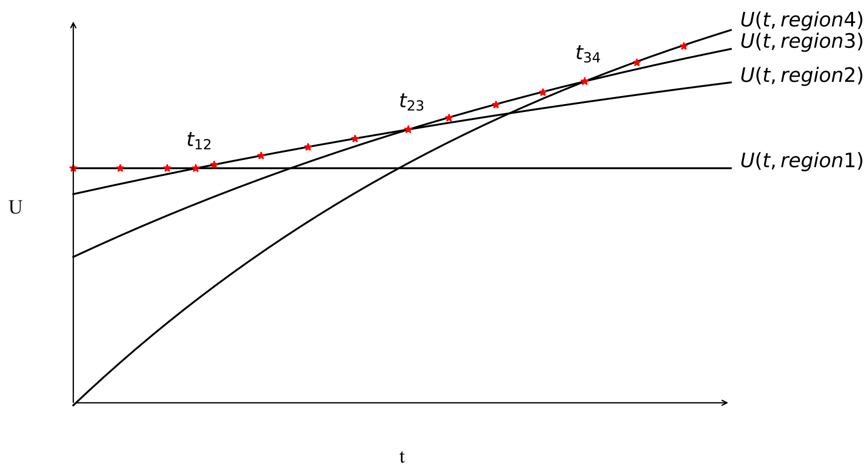

Figure 2: Illustration of how the single-crossing condition ensures global IC. The red stars indicate the upper contour of report payoffs (the payoff from truthful reporting) as a function of the bidder's type t.

The sufficient condition that $\frac{\partial}{\partial t}U(t,t')$ weakly increases in t' is a single-crossing condition. Figure 2 illustrates how the condition ensures global ex-post IC for Mechanism A. The figure plots four curves (in solid black), labeled $U(t, \text{region } 1)$ through $U(t, \text{region } 4)$, of a bidder's payoff as a function of the bidder's actual signal t when his reported type t' is in one of the four report-regions (given the other $n-1$ reports). Reflecting the single-crossing condition, the four curves exhibit increasing steepness. The upper envelope of these curves (in red stars) is: $U(t, \text{region } 1)$ for $t < t_{12}$, $U(t, \text{region } 2)$ for $t \in (t_{12}, t_{23})$, $U(t, \text{region } 3)$ for $t \in (t_{23}, t_{34})$, and $U(t, \text{region } 4)$ for $t > t_{34}$. At t_{34}, the bidder with type $t = t_{34}$ is indifferent between reporting region 3 or 4. Bidders are similarly indifferent at t_{23} and t_{12}. Clearly, when t is in region 4 so

$t > t_{34}$, the payoff of $U(t, \text{region } 4)$ from truthful reporting exceeds those of the other curves. Similarly, when t is in any other region, truthful reporting also leads to the highest payoff.

We now verify that Mechanism A satisfies the sufficient single-crossing condition of Lemma 1 when $\rho_{\min} \geq \frac{q}{1-q}$. Because a bidder's payoff does not vary with his report within a report-region, the single-crossing condition holds for reported types in the interior of a region, so we only need to check the three boundaries. Without loss of generality, consider bidder 1 and let bidder 2 have the largest signal of the other $n-1$ bidders. From (9), given $\mathbf{t}_{-1}$, we have

$$\frac{\partial}{\partial t_1} U_1(t_1, t_1'; \mathbf{t}_{-1}) = \sum_j R_j(t_1'; \mathbf{t}_{-1}) Q_{j1}(t_1'; \mathbf{t}_{-1}) \frac{\partial v_j(\mathbf{t})}{\partial t_1}. \tag{20}$$

Equation (20) shows that the single-crossing condition amounts to requiring a bidder's expected allocations weighted by the sensitivity of cash flows (possibly generated under another bidder's control) to the bidder's signal be non-decreasing in his reported type. When crossing the boundary from report-region 1 to 2, bidder 1's awarded cash flow share rises from zero to q, trivially satisfying single crossing. Next, when crossing from report-region 2 to 3, bidder 1's awarded cash flow share changes from q to $1-q$, but bidder 1 loses control to bidder 2, reducing the cash flow sensitivity to bidder 1's signal. Thus, single crossing requires

$$(1-q)\frac{\partial v_2(\mathbf{t})}{\partial t_1} \geq q \frac{\partial v_1(\mathbf{t})}{\partial t_1}$$

at all t_1, which is satisfied when $\rho_{\min} \geq \frac{q}{1-q}$. Third, when crossing from report-region 3 to 4, bidder 1's cash flow share increases from $1-q$ to 1, and bidder 1 also gains control from bidder 2, increasing cash flow sensitivity, trivially satisfying single crossing. Lastly, note that a bidder can ensure a nonnegative profit by reporting the lowest possible type $\underline{t}$. Thus, the ex-post incentive compatibility of Mechanism A also implies ex-post individual rationality (the same holds for Mechanism B).

For perspective, in standard no-separation mechanisms, a sufficient condition for local incentive compatibility to imply global incentive compatibility is that allocations be non-decreasing. The analogous condition in equation (20) for a separation mechanism is that allocations *weighted* by the sensitivity of cash flows to a bidder's signal be non-decreasing. As a result, in separation mechanisms, a bidder's allocation can *decrease* in his type if it is compensated by a gain in control (as in Mechanism B); conversely, increasing cash flow allocations may fail to ensure global IC if accompanied by a loss of control.

Separation mechanisms provide *two* ways to incentivize a high type bidder not to deviate to reporting a lower type: (i) as in a no-separation mechanism, rewarding a high report by assigning more allocations; (ii) rewarding a high report by assigning a higher probability of control (conditional on given allocations). The increased control rewards a high type because high types benefit more from running the project rather than having it run by a bidder with a lower signal. This latter channel is closed in no-separation mechanisms where the probability of control conditional on receiving cash flows is always one. With separation mechanisms, the conditional probability of control can be less than one and vary with a bidder's reported type. This gives a seller more leeway in the mechanism design, facilitating rent extraction.

3.3 Separation Mechanism with Efficient Splitting

We now define Mechanism B and characterize when it is ex-post incentive compatible:

Definition 4 *(Mechanism B: efficient splitting of rights) Let the reported types be $t_1' \geq t_2' \geq t_3' ... \geq t_n'$. Then*

If $t_1' > S(\mathbf{t}'_{-1})$, bidder 1 receives control and all cash flows, and pays

$$M_1 = (1-q)u(S(\mathbf{t}'_{-1}); t_2', ..., t_n') + (2q-1)u(t_2'; t_2', ..., t_n') + (1-q)u(t_2'; S^{-1}(\mathbf{t}'_{-1}), ..., t_n'). \quad (21)$$

If $t_1' \leq S(\mathbf{t}'_{-1})$, bidder 1 receives control and a share q of cash flows, and pays

$$M_1 = (2q-1)\, u\left(t_2'; t_2', ..., t_n'\right) + (1-q)\left[u\left(t_2'; S^{-1}\left(\mathbf{t}'_{-1}\right), ..., t_n'\right)\right], \quad (22)$$

while bidder 2 receives fraction $1-q$ of cash flows and pays

$$M_2 = (1-q)\, u\left(t_1'; S^{-1}\left(\mathbf{t}'_{-2}\right), t_3', ..., t_n'\right). \quad (23)$$

All other bidders receive nothing and pay nothing.

With Mechanism B, when the highest reported type t_1' does not exceed the second highest reported type t_2' by enough, i.e., when $t_1' < S(\mathbf{t}'_{-1})$, the bidder 1 who reports t_1' receives control and share q of cash flows, and the second highest bidder 2 gets share $1-q$. When, instead, the t_1' is sufficiently higher so that $t_1' \geq S(\mathbf{t}'_{-1})$, bidder 1 receives control and all cash flows.

The assignment of control is the opposite of Mechanism A, but it shares key features. In particular, there are four report-regions; and a bidder's payment and allocations do not depend on where his reported type is in a given report-region. When a bidder's type is at the boundary of adjacent regions, the payments leave him indifferent to reporting any type in those regions. Thus, there is no incentive for local deviations. Once more, if a bidder would receive a share x of cash flows at a lower unit price by deviating to report a lower type, then when he bids his true type, he pays for the share x at that lower unit price; he only pays the higher unit price for the remaining cash flows that he would not have received if he deviated.[12]

These features deliver the local ex-post IC of Mechanism B. The extra requirements that ensure global ex-post IC are the opposite of those for Mechanism A, reflecting the opposite control assignment and cash flow division upon splitting. We now need a measure of the *maximum* sensitivity to the signals of bidders who do not run the project:

$$\rho_{\max} \equiv \max_{\mathbf{t}} \frac{\partial v_2(\mathbf{t})}{\partial t_1} / \frac{\partial v_1(\mathbf{t})}{\partial t_1}.$$

Proposition 3 *Suppose that* $\rho_{\max} \leq \frac{q}{1-q}$. *Then for Mechanism B, truthful reporting is an ex-post equilibrium given any separation function* S.

We use (20) to show that Mechanism B under $\rho_{\max} \leq \frac{q}{1-q}$ satisfies single-crossing. As with Mechanism A, we only need to check the three boundaries. Single crossing trivially holds when crossing from report-region 1 to 2, or from 3 to 4, as both cash flow and control weakly increase in these two cases. Thus, the key case left to verify is when crossing from report-region 2 to 3, where bidder 1's awarded cash flow share changes from $1-q$ to q, and bidder 1 gains control from bidder 2, increasing the cash flow sensitivity to bidder 1's signal. For $q \geq 0.5$, the cash flow allocation weakly increases across the boundary so single crossing is trivially satisfied; even if $q<0.5$ so that cash flow allocation decreases, obtaining control can compensate for this decrease to satisfy single crossing, as long as

$$q\frac{\partial v_1(\mathbf{t})}{\partial t_1} \geq (1-q)\,\frac{\partial v_2(\mathbf{t})}{\partial t_1}$$

[12]If a bidder would receive more cash flows by deviating to reporting a lower type (which could be incentive compatible in the separation mechanism), the "remaining" cash flows would be negative. In this case, the bidder effectively "sells" the difference in the two cash flows back to the seller.

at all t_1, which holds when $\rho_{\max} \leq \frac{q}{1-q}$. The proof otherwise mirrors that for Mechanism A.

Increasing q makes incentive compatibility easier to satisfy for Mechanism B, but harder for Mechanism A. Intuitively, a higher q in Mechanism B and a lower q in A ensure that the highest bidder receives "enough" cash flows ($1-q$ for Mechanism A and q for B), which incentivizes truth telling. Similarly, in Mechanism B, a smaller ρ expands the range of q that satisfies $\rho_{\max} \leq \frac{q}{1-q}$, whereas in Mechanism A, a *larger* ρ expands the range of q that satisfies $\rho_{\min} \geq \frac{q}{1-q}$. The contrast reflects that ρ is inversely related to the significance of the allocation of control. As a result, reducing ρ in Mechanism B increases the cost of deviation by the bidder with the highest signal of submitting the second-highest bid and hence losing control, whereas a larger ρ in Mechanism A reduces the deviation gain of a bidder with the highest signal from reducing his bid to become the second-highest and hence gain control.

3.4 Revenue dominance of separation mechanisms

We now show that under the sufficient conditions for Mechanisms A and B to be *ex-post* incentive compatible, they can be designed to generate strictly higher expected revenues than no-separation English auctions.

Proposition 4 *Suppose* $\rho_{\min} \geq \frac{q}{1-q}$. *Then separation functions* S *exist for which Mechanism A generates strictly higher expected seller revenues than no-separation English auctions.*

Conversely, suppose $\rho_{\max} \leq \frac{q}{1-q}$. *Then separation functions* S *exist for which Mechanism B generates strictly higher expected seller revenues than no-separation English auctions.*

Proof: See the appendix. □

The broad intuition mirrors that for Result 1: when the two highest types are close enough, the seller strictly gains from separating cash-flow rights and control. More specifically, when the two highest signals are close, the cost of separation (of inefficient assignment in Mechanism A, and of increased bidder rents due to assigning cash flows to a lower signal bidder in Mechanism B) approaches zero, but the advantage of separation (from reducing a bidder's information advantage by reducing the sensitivity of cash flows to his signal) remains strictly positive. This does not change if $q > 0$: now when the two highest signals are close, the cost of separation still approaches zero, while the advantage of separation is scaled by a factor of $1-q$, hence remaining strictly positive.

On a technical level, $q>0$ introduces subtleties to establishing revenue dominance because, unlike with $q=0$, revenue from the separation mechanism does not dominate the no-separation mechanism profile-by-profile. Nonetheless, *expected revenue dominance* still obtains. To illustrate how we establish expected revenue dominance, we prove it here for Mechanism A in the two-bidder setting where bidders receive uniformly distributed i.i.d. signals t_1 and t_2, and have linear valuation functions $u(t_1;t_2)=\frac{1}{1+\rho}(t_1+\rho t_2)$ with $\rho\in(0,1)$.

Proof: Pick any $s^*\in(\bar{t},\underline{t})$. For any $\delta\in(0,\bar{t}-s^*]$, define the δ-separation function

$$\begin{aligned} S_\delta(s) &= s^*+\delta \text{ if } s\in[s^*,s^*+\delta]\,; \\ S_\delta(s) &= s \text{ if } s\notin[s^*,s^*+\delta]\,. \end{aligned}$$

The inverse separation function is

$$\begin{aligned} S_\delta^{-1}(s) &= s^* \text{ if } s\in[s^*,s^*+\delta]; \\ S_\delta^{-1}(s) &= s \text{ if } s\notin[s^*,s^*+\delta]\,. \end{aligned}$$

Separation occurs if and only if both t_1 and t_2 are in $[s^*,s^*+\delta]$. We show that for δ sufficiently small, the Mechanism A that uses the δ-separation function generates strictly higher expected seller revenues. Let $D(t_1,t_2)$ be the difference in seller revenue between the δ-separation mechanism and the no-separation English auction. Let $t_h\equiv\max\{t_1,t_2\}$ and $t_s\equiv\min\{t_1,t_2\}$ denote the larger and smaller signals.

Since $D(t_1,t_2)=0$ if $t_s\notin[s^*,s^*+\delta]$, we only need to consider signal realizations with $t_s\in[s^*,s^*+\delta]$. There are two cases according to whether or not $t_h\geq s^*+\delta$.

Case 1: $t_h\geq s^*+\delta$. Then in the δ-separation mechanism, the bidder with the high signal retains both cash flows and control. Hence, seller revenue is his payment in (17):

$$u(s^*+\delta;t_s)-(1-q)\,u(t_s;s^*+\delta)+(1-2q)\,u(t_s;t_s)+qu(s^*,t_s)\,.$$

Subtracting no-separation English auction revenues, $u(t_s;t_s)$, from this yields the difference:

$$D(t_1,t_2)=u(s^*+\delta;t_s)-(1-q)\,u(t_s;s^*+\delta)-2qu(t_s;t_s)+qu(s^*,t_s)\,. \tag{24}$$

On the right-hand side of (24), rewriting $u(s^* + \delta; t_s)$ as $(1 - q)\, u(s^* + \delta; t_s) + qu(s^* + \delta; t_s)$ and combining terms with coefficients $(1 - q)$ and q separately, (24) decomposes to

$$D(t_1, t_2) = D_{1a}(t_1, t_2) + D_{1b}(t_1, t_2),$$

where

$$\begin{aligned} D_{1a} &\equiv (1-q)\left[u(s^*+\delta; t_s) - u(t_s; s^*+\delta)\right] = (1-q)(u_1 - u_2)(s^* + \delta - t_s) && (25) \\ D_{1b} &\equiv q\left[u(s^*+\delta; t_s) - 2u(t_s; t_s) + u(s^*, t_s)\right] = qu_1(2s^* + \delta - 2t_s), && (26) \end{aligned}$$

and $u_1 = \frac{1}{1+\rho}$ and $u_2 = \frac{\rho}{1+\rho}$ are the derivatives of u with respect to its first and second arguments.

Case 2: $t_h \in [t_s, s^* + \delta)$. Seller revenue in the δ-separation mechanism is the sum of the higher bidder's payment in (18) plus the lower bidder's payment in (19):

$$(1 - 2q)u(t_s; t_s) + qu(s^*; t_s) + qu(s^*; t_h).$$

Subtracting no-separation English auction revenues, $u(t_s; t_s)$, from this yields the difference:

$$D = -2qu(t_s, t_s) + qu(s^*, t_s) + qu(s^*, t_h) \equiv D_2.$$

Since $t_h \geq t_s$, $u_1 > 0$ and $t_s - s^* \leq \delta$, we have

$$\begin{aligned} D_2 &\geq -2qu(t_s, t_s) + qu(s^*, t_s) + qu(s^*, t_s) = -2qu_1(t_s - s^*) \\ &\geq -2qu_1\delta. && (27) \end{aligned}$$

Integrate $D(t_1, t_2)$ over t_1 and t_2 to obtain the expected revenue difference:

$$\begin{aligned} E[D] &= 2f^* \int_{\underline{t}}^{\bar{t}} \int_{t_2}^{\bar{t}} D(t_1, t_2)dt_1dt_2 && (28) \\ &= 2f^* \int_{s^*}^{s^*+\delta} \int_{t_2}^{\bar{t}} D(t_1, t_2)dt_1dt_2, && (29) \end{aligned}$$

where $f^* \equiv 1/(\bar{t} - \underline{t})^2$ is the uniform density, the factor 2 reflects that t_1 and t_2 are equally likely to be the lower signal, and the bounds of integration for t_2 in (29) reflect that when

t_2 is the lower signal, $D(t_1, t_2) = 0$ if $t_2 \notin [s^*, s^* + \delta]$.[13]

Decompose the integration over $t_1 \in (t_2, \bar{t})$ into the sum of integrations over $t_1 \in (s^* + \delta, \bar{t})$ (i.e., case 1) and $t_1 \in (t_2, s^* + \delta)$ (i.e., case 2) to obtain

$$E[D] = 2\left(ED_{1a} + ED_{1b} + ED_2\right), \quad \text{where}$$

$$ED_{1i} \equiv f^* \int_{s^*}^{s^*+\delta} \int_{s^*+\delta}^{\bar{t}} D_{1i}(t_1, t_2)\, dt_1 dt_2, \quad \text{for } i = a, b \tag{30}$$

$$ED_2 \equiv f^* \int_{s^*}^{s^*+\delta} \int_{t_2}^{s^*+\delta} D_2(t_1, t_2)\, dt_1 dt_2. \tag{31}$$

To complete the proof, we show that for δ sufficiently small, (i) ED_{1a} exceeds a term that is positive and quadratic in δ, (ii) $ED_{1b} = 0$, and (iii) ED_2 exceeds a negative term that goes to zero faster than δ^2. We first bound ED_{1a} from below. Substituting the definition of D_{1a} in (25) into (30) yields

$$\begin{aligned} ED_{1a} &= f^*(1-q)(u_1 - u_2) \int_{s^*}^{s^*+\delta} \int_{s^*+\delta}^{\bar{t}} (s^* + \delta - t_2)\, dt_1 dt_2 \\ &= \frac{1}{2} f^*(1-q)(u_1 - u_2)(\bar{t} - s^* - \delta)\,\delta^2, \end{aligned} \tag{32}$$

where the last line uses $\int_{s^*+\delta}^{\bar{t}} (s^* + \delta - t_2)\, dt_1 = (s^* + \delta - t_2)(\bar{t} - s^* - \delta)$ and $\int_{s^*}^{s^*+\delta} (s^* + \delta - t_2)\, dt_2 = \frac{1}{2}\delta^2$. Further, when $\delta < \frac{\bar{t} - s^*}{2}$, we have $(\bar{t} - s^* - \delta) > \frac{1}{2}(\bar{t} - s^*)$, and thus (32) yields

$$ED_{1a} > \frac{1}{4} f^*(1-q)(u_1 - u_2)(\bar{t} - s^*)\,\delta^2. \tag{33}$$

Next we show $ED_{1b} = 0$. Substituting the definition of D_{1b} in (26) into (30) yields

$$\begin{aligned} ED_{1b} &= f^* q u_1 \int_{s^*}^{s^*+\delta} \int_{s^*+\delta}^{\bar{t}} (2s^* + \delta - 2t_2)\, dt_1 dt_2 \\ &= f^* q u_1 \int_{s^*+\delta}^{\bar{t}} \left[\int_{s^*}^{s^*+\delta} (2s^* + \delta - 2t_2)\, dt_2 \right] dt_1 = 0, \end{aligned}$$

[13] An alternative way to derive (28) is as follows:

$$\begin{aligned} E[D] &= \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{\bar{t}} D(t_1, t_2) f(t_1, t_2)\, dt_1 dt_2 = f^* \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{\bar{t}} D(t_1, t_2)\left(\mathbf{1}_{t_1 \geq t_2} + \mathbf{1}_{t_1 < t_2}\right) dt_1 dt_2 \\ &= 2f^* \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{\bar{t}} D(t_1, t_2) \mathbf{1}_{t_1 \geq t_2} dt_1 dt_2 = 2f^* \int_{\underline{t}}^{\bar{t}} \int_{t_2}^{\bar{t}} D(t_1, t_2) dt_1 dt_2, \end{aligned}$$

where the indicator function $\mathbf{1}_{t_1 \geq t_2}$ ($\mathbf{1}_{t_1 < t_2}$) is 1 when t_2 is the lower (higher) signal, and the factor 2 reflects that each bidder is equally likely to have the higher (or lower) signal.

where the second line switches the order of integration, and last line uses $\int_{s^*}^{s^*+\delta}(2s^*+\delta-2t_2)\,dt_2 = 0$. Lastly, we bound ED_2 from below. Substitute the lower bound on D_2 in (27) into (31):

$$\begin{aligned} ED_2 &\geq -2qu_1\delta f^* \int_{s^*}^{s^*+\delta}\int_{t_2}^{s^*+\delta} dt_1 dt_2 \\ &= -2qu_1\delta f^* \int_{s^*}^{s^*+\delta}(s^*+\delta-t_2)\,dt_2 = -qu_1 f^*\delta^3, \end{aligned}$$

where the last line uses $\int_{s^*}^{s^*+\delta}(s^*+\delta-t_2)dt_2 = \frac{1}{2}\delta^2$.

Thus, when δ is small, ED_{1a} exceeds a positive term that approaches zero at a rate of δ^2 (see (33) and note that $u_1 > u_2$ for all $\rho < 1$), $ED_{1b} = 0$ and ED_2 exceeds a negative term that approaches zero at a rate of δ^3. Hence, for δ sufficiently small, $ED_{1a}+ED_{1b}+ED_2 > 0$, i.e., the δ-separation mechanism generates strictly higher expected revenues. $\square$

In the Appendix, we prove Proposition 4 for general settings with an arbitrary number $n \geq 2$ of bidders, general valuation functions and general signal distributions that allow for signal correlation. We only impose the mild single-crossing condition in (4) that there exists a signal vector, say $t_2^*, ..., t_n^*$, with $\bar{t} > t_2^* \geq t_3^* \geq ... \geq t_n^* > \underline{t}$ for which strict single-crossing holds, so we consider a δ-separation mechanism that separates locally when signals are in a small neighborhood of this vector, i.e., separation occurs if and only if the two highest signals are in $[t_2^*, t_2^*+\delta]$, and the other signals are in the neighborhood of t_3^* through t_n^*.

The general proof follows a logic similar to that for the example. When $t_s \in [t_2^*, t_2^*+\delta]$ and t_h sufficiently exceeds t_s so that $t_h \in [t_2^*+\delta, \bar{t}]$, separation does not occur. In this instance, expected revenue conditional on t_s being in $[t_2^*, t_2^*+\delta]$ and a given t_h, exceeds that from no separation by an amount that goes to zero linearly with δ.[14] As we reduce δ to zero, the probability that $t_s \in [t_2^*, t_2^*+\delta]$ goes to zero linearly in δ, and the probability that $t_h \in [t_2^*+\delta, \bar{t}]$ *approaches a constant.* Thus, the contribution of this expected revenue surplus to the total expected revenue ($2ED_{1a} + 2ED_{1b}$ in the example) goes to zero at rate δ^2.

When, instead, $t_s \in [t_2^*, t_2^*+\delta]$ and t_h is close to t_s so that $t_h \in [t_s, t_2^*+\delta]$, separation occurs. For a given t_s and t_h, seller revenue in the separation mechanism can be less than

[14]The analogue in the two-stage mechanism with $q = 0$ is where the winning bidder makes an extra payment to receive control and all cash flows when his signal sufficiently exceeds the second highest. For p_{extra} (see Proposition 1) sufficiently small, a Taylor series expansion yields that if $\Delta(t_1, t_2; t_3, ..., t_n) = p_{extra}$, then $t_1 - t_2$ is proportional to p_{extra}.

that in the no-separation mechanism. However, as δ goes to zero, the amount by which expected seller revenue is lower does not exceed a term that is linear in δ (as in (27)). Because the probabilities that $t_s \in [t_2^*, t_2^* + \delta]$ and $t_h \in [t_s, t_2^* + \delta]$ *both go to zero linearly in* δ, the contribution of this to expected revenue ($2ED_2$ in the example) exceeds a negative term that goes to zero at the faster rate of δ^3 (rather than δ^2).

Summing these two contributions to revenues yields that expected revenue difference between the separation and no-separation mechanisms is strictly positive for δ sufficiently small.

3.5 Discussion

A large q (e.g., $q \geq 0.5$) always satisfies the premise that $\rho_{\max} \leq \frac{q}{1-q}$ for Mechanism B to be ex-post incentive compatible. In addition, a sufficiently large q would satisfy any minimum stake requirement—if (8) holds for a given q, then it holds for a lower q. Thus, *Mechanism B can always be designed to satisfy any minimum stake requirement and generate strictly higher expected seller revenues than no-separation English auctions by choosing q sufficiently high.*

In a working paper, we characterize when either Mechanism A or Mechanism B is optimal among all incentive compatible separation mechanisms. We focus on settings where cash flows are linear functions of *i.i.d.* signals that satisfy the standard monotone hazard condition. Reflecting the feature that the cost of inefficient control decreases in ρ, we prove that when ρ is sufficiently large and q is small, it is optimal to split by assigning control to the second-highest bidder and cash flow share $1 - q$ to the highest bidder. When, instead, ρ is small enough or q is large enough (e.g., $q \geq 0.5$ so the controller gets most of the cash flows), it is optimal to reverse this split of rights.

Our current model assumes that it is costless to run the project. Our mechanisms and findings extend immediately when running the project has a publicly-known cost. The working paper shows that our qualitative findings extend to a setting where bidders receive multi-dimensional signals. In this setting the cash flows generated by a bidder's control are the sum of a bidder-specific component and a common component (in essence our current model assumes the two components are perfectly correlated), and bidders are privately informed about each component as well as their costs of running the project.[15] We identify a class of separation mechanisms in which the three-dimensional signals can be reduced to a single dimension, rendering analysis tractable. We show that this mechanism class can generate both higher revenues and greater social welfare than no-separation mechanisms. This has implications for bankruptcy resolution where the court's strong bargaining power gives it substantial leeway in structuring allocations. Hart (2023) points out that there are different approaches to bankruptcy resolution reflecting different conflicting possible objectives— welfare maximization versus revenue maximization—and our analysis indicates that separation mechanisms can lead both to higher revenues and to higher social efficiency.

4 Conclusions

Our paper revisits the classical auction setting in which a seller seeks to sell a single asset/project to bidders who privately receive signals about the asset's future cash flows. The asset's payoffs hinge on both the signal of the bidder who controls the asset and those of rival bidders. The mechanism design literature has focused on optimal mechanisms in which control and cash flow rights are bundled together so that bidders who do not receive control receive no cash flows. We extend this framework by incorporating the assignment of control rights as part of our "separation mechanism". We show that a seller can increase expected revenues by sometimes separating the allocation of control and cash flow rights, i.e., by allocating cash flows to a bidder who does not control the project. Separating control and cash-flow rights helps rent extraction because a project's payoff is most sensitive to the signal of the bidder who runs the project. Allocating cash flows to another bidder reduces the sensitivity of their value to this bidder's signal, which redues bidders' overall information advantages. As a result, the seller can increase revenues by splitting rights between the top two bidders when their signals are close.

[15] A negative cost can capture a private benefit of control as in Ekmekci, Kos and Vohra (2016).

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6 Appendix

Proof of Result 1: From the strict single-crossing condition (4) there exists a signal vector $t_2^*, ..., t_n^*$, with $\bar{t} > t_2^* \geq t_3^* \geq ... \geq t_n^* > \underline{t}$. For any $\epsilon > 0$, define the set

$$H_\epsilon \equiv \left\{ \begin{array}{c} (t_3, ..., t_n) : \exists \ (x_3^*, ..., x_n^*) \text{ that is a permutation of } (t_3^*, ..., t_n^*) \text{ such that} \\ t_i \in [x_i^* - \epsilon, x_i^*] \text{ for all } i = 3, ..., n \end{array} \right\}.$$

H_ϵ includes all points $(t_3, ..., t_n)$ in an ϵ-neighborhood of $(t_3^*, ..., t_n^*)$ and their permutations.

From the continuity of $u(t_1; ..., t_n)$ and its derivatives, there exists an $\omega > 0$ and an $\epsilon \in (0, \min\{\bar{t} - t_2^*, t_n^* - \underline{t}\})$ such that for all $t_2 \in [t_2^*, t_2^* + \epsilon/2]$, $t_1 \in [t_2, t_2^* + \epsilon]$, and $t_3, ..., t_n \in H_\epsilon$, inequality (4) holds with:

$$u_1(t_1; t_2, t_3, ..., t_n) - u_2(t_2; t_1, t_3, ..., t_n) \geq \omega. \tag{34}$$

We now show that a price offer of $p^* \equiv \frac{\epsilon}{2}\omega > 0$ will be accepted with strictly positive probability, which establishes the result. To proceed, consider n signals $t_1, ..., t_n$, where (wlog) t_1 and t_2 are the highest and second-highest signals. Suppose t_2 is in the interval $\left[t_2^*, t_2^* + \frac{\epsilon}{2}\right]$, and $t_3, t_4, ..., t_n$ are in H_ϵ. When $t_1 \in [t_2^* + \epsilon, \bar{t}]$, we have

$$\begin{aligned} u(t_1; t_2, ..., t_n) - u(t_2; t_1, ..., t_n) &= \int_{t_2}^{t_1} \left(u_1(t; t_2, ..., t_n) - u_2(t_2; t, ..., t_n)\right) dt \\ &\geq (t_2^* + \epsilon - t_2)\,\omega \geq \frac{\epsilon}{2}\omega, \end{aligned}$$

Thus, for all $t_2 \in \left[t_2^*, t_2^* + \frac{\epsilon}{2}\right]$ and $t_1 \in [t_2^* + \epsilon, \bar{t}]$, a price offer of $p^* \equiv \frac{\epsilon}{2}\omega > 0$ is accepted. $\square$

Proof of Lemma 1: Define

$$D(t, t') \equiv U(t, t) - U(t, t'). \tag{35}$$

By construction, $D(t, t) = 0$ at all t. (35) immediately yields:

Claim 1: $t' = t$ maximizes $U(t, t')$ over $t' \in [\underline{t}, \bar{t}]$ for all given t if and only if

$$D(t, t') \geq 0 \quad \text{for all } t \text{ and } t'.$$

To economize on language, in the remainder of the proof when we mention t and t', we assume that $t, t' \in [\underline{t}, \bar{t}]$. Consider the lemma's premise that for all $t' \notin S^*$, U is differentiable at $(t = t', t')$ with $\frac{\partial}{\partial t'} U(t, t')|_{t=t'} = 0$. This premise implies that for all $t \notin S^*$, U is differentiable at $(t, t' = t)$ and $\frac{\partial}{\partial t'} U(t,t')|_{t'=t} = 0$. Thus, for any $t \notin S^*$ and t', we have

$$\begin{aligned}\frac{\partial}{\partial t} D\left(t,t'\right) &= \frac{d}{dt} U\left(t,t\right) - \frac{\partial}{\partial t} U(t,t') = \frac{\partial}{\partial t} U(t,\hat{t})|_{\hat{t}=t} + \frac{\partial}{\partial \hat{t}} U(t,\hat{t})|_{\hat{t}=t} - \frac{\partial}{\partial t} U(t,t') \\ &= \frac{\partial}{\partial t} U(t,\hat{t})|_{\hat{t}=t} - \frac{\partial}{\partial t} U(t,t'), \end{aligned} \tag{36}$$

where the term $\frac{\partial}{\partial \hat{t}} U(t,\hat{t})|_{\hat{t}=t}$ in the first line vanishes by the lemma's premise that U is differentiable at $(t = t', t')$ with $\frac{\partial}{\partial t'} U(t,t')|_{t=t'} = 0$ for all $t' \notin S^*$.

To establish the sufficient condition for global IC, we assume the lemma's premise that for all t, $\frac{\partial}{\partial t} U(t,t')$ weakly increases in t' over $t' \in [\underline{t}, \bar{t}]$). By this premise, if $t \geq t'$, then $\frac{\partial}{\partial t} U(t,\hat{t})|_{\hat{t}=t} \geq \frac{\partial}{\partial t} U(t,t')$. Now suppose $t \notin S^*$. Then (36) holds, which yields $\frac{\partial}{\partial t} D\left(t,t'\right) \geq 0$ for $t \geq t'$ and $t \notin S^*$. Because $D\left(t,t'\right)$ is continuous in t and S^* contains only countably many points, when we fix t', we have $D\left(t,t'\right) \geq D\left(t',t'\right) = 0$ for all $t \geq t'$. By the same logic, if $t \leq t'$ and $t \notin S^*$, we have $\frac{\partial}{\partial t} U(t,\hat{t})|_{\hat{t}=t} \leq \frac{\partial}{\partial t} U\left(t,t'\right)$, and (36) yields $\frac{\partial}{\partial t} D\left(t,t'\right) \leq 0$, and from this we have $D\left(t,t'\right) \geq D\left(t',t'\right) = 0$ for all $t \leq t'$. Thus, fixing t' and varying t, $D\left(t,t'\right)$ is minimized at $t = t'$. Therefore, $D\left(t,t'\right) \geq D\left(t',t'\right) = 0$ for all t. This, by the "if" part of Claim 1, establishes sufficiency.

Now we prove necessity. Referring to the lemma's premise, consider t where $U(t,t')$ is twice differentiable at $(t, t' = t)$. By the lemma's other premise that $t' = t$ maximizes $U(t,t')$ for all t, the first-order condition with respect to t' evaluated at $t' = t$, yields $\frac{\partial}{\partial t'} U(t,t')|_{t'=t} = 0$. Hence (36) holds. Differentiating both sides of (36) with respect to t (and evaluating at $t' = t$) yields:

$$\frac{\partial^2}{\partial^2 t} U(t,t')|_{t'=t} = \frac{\partial^2}{\partial t^2} U(t,t')|_{t'=t} + \frac{\partial^2}{\partial t' \partial t} U(t,t')|_{t'=t} - \frac{\partial^2}{\partial^2 t} U(t,t')|_{t'=t}. \tag{37}$$

The first and third terms in (37) cancel, so (37) reduces to

$$\frac{\partial^2}{\partial^2 t} D\left(t,t'\right)|_{t'=t} = \frac{\partial^2}{\partial t' \partial t} U(t,t')|_{t'=t}. \tag{38}$$

Furthermore, the lemma's premise that "for all given t, $t' = t$ maximizes $U(t,t')$ over $t' \in [\underline{t}, \bar{t}]$" and the "only if" part of Claim 1 imply that for all given t', $t = t'$ minimizes $D(t,t')$. The second-order condition (with respect to t) for minimization gives $\frac{\partial^2}{\partial^2 t} D\left(t,t'\right)|_{t=t'} \geq 0$ for all t'. This is equivalent to $\frac{\partial^2}{\partial^2 t} D\left(t,t'\right)|_{t'=t} \geq 0$ for all t, which leads to $\frac{\partial^2}{\partial t' \partial t} U\left(t,t'\right)|_{t'=t} \geq 0$ via (38). This establishes necessity. $\square$

Proof of Proposition 2:

We use Lemma 1 to prove that Mechanism A is ex-post incentive compatible, interpreting $U(t,t')$ as $U_i(t_i,t_i';\mathbf{t}_{-i})$ in (9), and taking $\mathbf{t}_{-i}$ as given. Without loss of generality, consider bidder $i=1$ and assume $t_2 \geq t_3 ... \geq t_n$.

Observe that Mechanism A satisfies the premises "a" – "c" of Lemma 1. Premise (a) holds since $U(t_1,t_1';\mathbf{t}_{-1})$ is differentiable with respect to t_1. Premise (b) also holds: S^* consists of three points, i.e., $S^* = \{S^{-1}(t_2,t_3,...,t_n), t_2, S(t_2,...,t_n)\}$, and $U(t_1,t_1';\mathbf{t}_{-1})$ is differentiable at $(t_1=t_1',t_1')$ for all $t_1' \notin S^*$. Finally, premise (c) is satisfied. In particular, $\frac{\partial}{\partial t_1'}U(t_1,t_1')|_{t_1=t_1'} = 0$ for all $t_1' \notin S^*$ because our mechanism is locally incentive compatible (see the main text). Further, we show $U(t_1,t_1)$ is continuous in t_1. This trivially holds if $t_1 \notin S^*$ so consider $t_1 \in S^*$ at the boundary between a report-region k and $k+1$. Then $U(t_1,t_1) = U(t_1,t_1'$ in region k)=$U(t_1,t_1'$ in region $k+1$). This, combined with $U(t_1,t_1')$ being continuous and differentiable in t_1 establishes that $U(t_1,t_1)$ is continuous in t_1.

The main text established that Mechanism A satisfies the single-crossing condition. Hence, by Lemma 1, the proposition follows. $\square$

Proof of Proposition 4: From the strict single-crossing condition (4) there exists a signal vector $t_2^*,...,t_n^*$, with $\bar{t} > t_2^* \geq t_3^* \geq ... \geq t_n^* > \underline{t}$. For any $\epsilon > 0$, define the set

$$H_\epsilon \equiv \left\{ \begin{array}{c} (t_3,...,t_n) : \exists \ (x_3^*,...,x_n^*) \text{ that is a permutation of } (t_3^*,...,t_n^*) \text{ such that} \\ t_i \in [x_i^* - \epsilon, x_i^*] \text{ for all } i = 3,...,n \end{array} \right\}.$$

H_ϵ includes all points $(t_3,...,t_n)$ in an ϵ-neighborhood of $(t_3^*,...,t_n^*)$ and their permutations (inclusion of permutations preserves the symmetry of the set H_ϵ).

From the continuity of $u(t_1;...,t_n)$ and its derivatives, there exists an $\omega > 0$ and an $\epsilon \in (0, \min\{\bar{t} - t_2^*, t_n^* - \underline{t}\})$ such that for all $t_2 \in [t_2^*, t_2^* + \epsilon]$, $t_1 \in [t_2, t_2^* + \epsilon]$ and $t_3,...,t_n \in H_\epsilon$, inequality (4) holds with:

$$u_1(t_1;t_2,t_3,...,t_n) - u_2(t_2;t_1,t_3,...,t_n) > \omega. \tag{39}$$

Fix such an ϵ and ω. For any $\delta \in (0,\epsilon]$ and any $n-1$ generic signals $s_1,...,s_{n-1}$, define the "δ-separation function" $S_\delta(s_1,...,s_{n-1})$ by:

$$\begin{aligned} S_\delta(s_1,...,s_{n-1}) &= t_2^* + \delta \text{ if } s_h \in [t_2^*, t_2^* + \delta] \text{ and } \mathbf{t}_{-h} \in H_\epsilon; \\ S_\delta(s_1,...,s_{n-1}) &= s_h \text{ otherwise.} \end{aligned}$$

Here s_h denotes the highest signal among $s_1, ..., s_{n-1}$, and $\mathbf{t}_{-h}$ is the vector of the $n-2$ signals other than s_h. The associated inverse δ-separation function is

$$\begin{aligned} S_\delta^{-1}(s_1, ..., s_{n-1}) &= t_2^* \text{ if } s_h \in [t_2^*, t_2^* + \delta] \text{ and } \mathbf{t}_{-h} \in H_\epsilon; \\ S_\delta^{-1}(s_1, ..., s_{n-1}) &= s_h \text{ otherwise.} \end{aligned}$$

Given signals $(t_1, t_2, ...t_n)$, let t_h denote the highest signal and t_s the second-highest and let $\mathbf{t}_{-h-s}$ be the vector of the other $n-2$ signals. Separation occurs if and only if t_h and t_s are in $[t_2^*, t_2^* + \delta]$ and $\mathbf{t}_{-h-s}$ is in H_ϵ. We show that for all δ sufficiently small, the Mechanism A that uses the δ-separation function generates strictly higher expected seller revenues than no-separation English auctions. Let $D(t_1, t_2, ...t_n)$ be seller revenue in the δ-separation mechanism minus that in the no-separation English auction. In the analysis that follows we only consider $t_s \in [t_2^*, t_2^* + \delta]$ and $\mathbf{t}_{-h-s} \in H_\epsilon$, because $D(t_1, t_2, ...t_n) = 0$ in all other scenarios. There are two relevant cases according to whether $t_h \geq t_2^* + \delta$ or not.

Case 1: $t_h \geq t_2^* + \delta$. Seller revenue in the δ-separation mechanism is the highest bidder's payment (17):

$$u(t_2^* + \delta; t_s, \mathbf{t}_{-h-s}) - (1-q)\, u(t_s; t_2^* + \delta, \mathbf{t}_{-h-s}) + (1-2q)\, u(t_s; t_s, \mathbf{t}_{-h-s}) + qu(t_2^*; t_s, \mathbf{t}_{-h-s}).$$

Revenue in a no-separation English auction is $u(t_s; t_s, \mathbf{t}_{-h-s})$. Algebra yields the difference (upon rewriting $u(t_2^* + \delta; t_s, \mathbf{t}_{-h-s})$ as $(1-q)\, u(t_2^* + \delta; t_s, \mathbf{t}_{-h-s}) + qu(t_2^* + \delta; t_s, \mathbf{t}_{-h-s})$):

$$D(t_1, t_2, ..., t_n) = D_{1a}(t_1, t_2, ..., t_n) + D_{1b}(t_1, t_2, ..., t_n),$$

where

$$D_{1a} \equiv (1-q)\left[u(t_2^* + \delta; t_s, \mathbf{t}_{-h-s}) - u(t_s; t_2^* + \delta, \mathbf{t}_{-h-s})\right] \quad (40)$$

$$D_{1b} \equiv q\left[u(t_2^* + \delta; t_s, \mathbf{t}_{-h-s}) - 2u(t_s; t_s, \mathbf{t}_{-h-s}) + u(t_2^*; t_s, \mathbf{t}_{-h-s})\right]. \quad (41)$$

Case 2: $t_h \in [t_s, t_2^* + \delta)$. Seller revenue in the δ-separation mechanism is the sum of the highest bidder's payment in (18) and the second highest bidder's payment in (19):

$$(1-2q)\, u(t_s; t_s, \mathbf{t}_{-h-s}) + qu(t_2^*; t_s, \mathbf{t}_{-h-s}) + qu(t_2^*; t_h, \mathbf{t}_{-h-s}).$$

Seller revenue in the no-separation English auction is $u\left(t_s; t_s, \mathbf{t}_{-h-s}\right)$. The difference is

$$D(t_1, t_2, ..., t_n) = q[u(t_2^*; t_s, \mathbf{t}_{-h-s}) + u(t_2^*; t_h, \mathbf{t}_{-h-s}) - 2u(t_s; t_s, \mathbf{t}_{-h-s})] \equiv D_2(t_1, t_2, ..., t_n). \quad (42)$$

Integrating $D(t_1, t_2, ..., t_n)$ over t_1 through t_n yields the expected revenue difference, which we rewrite as:

$$E\left[D\right] = n\left(n-1\right) \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2}^{\bar{t}} D(t_1, t_2, t_3, ..., t_n) f(\mathbf{t}) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n, \quad (43)$$

where $\Omega_{n-2} \equiv [\underline{t}, \bar{t}]^{n-2}$ is the space of integration for $t_3, ..., t_n$, and $\mathbf{1}_{t_3,...,t_n \in H_\epsilon}$ is an indicator function that equals 1 if $(t_3, ..., t_n) \in H_\epsilon$, and zero if $t_3, ..., t_n \notin H_\epsilon$ (recall $D = 0$ if $(t_3, ..., t_n) \notin H_\epsilon$). The integration limits in (43) imply that t_1 and t_2 are the highest and second-highest of the n signals, and this underlies the factor $n\left(n-1\right)$: the factor n reflects that any of the n signals, not necessarily t_1, can be the highest signal, and the factor $n-1$ reflects that any of the remaining $n-1$ signals, not necessarily t_2, can be the second-highest.[16]

Decomposing the integration over t_1 in (43) from t_2 to $\bar{t}$ into the sum of integrations from t_2 to $t_2^* + \delta$ and from $t_2^* + \delta$ to $\bar{t}$, yields

$$E\left[D\right] = n\left(n-1\right)\left(ED_{1a} + ED_{1b} + ED_2\right), \quad (44)$$

where

$$ED_{1i} \equiv \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2^*+\delta}^{\bar{t}} D_{1i}(t_1, t_2, t_3, ..., t_n) f(\mathbf{t}) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n, \text{ for } i = a, b. \quad (45)$$

$$ED_2 \equiv \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2}^{t_2^*+\delta} D_2\left(t_1, t_2, t_3, ..., t_n\right) f(\mathbf{t}) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n. \quad (46)$$

In line with t_1 and t_2 denoting the highest and second-highest of the n signals in (43), and hence in (45) and (46), in the rest of the proof, we set $t_h = t_1$ and $t_s = t_2$ in (40)–(42). Next, we show that for δ sufficiently small, ED_{1a} exceeds a term that is positive and quadratic in δ, and ED_{1b} and ED_2 each exceed a term that goes to zero at a rate faster than δ^2.

Step 1: By (40), (45) yields

$$ED_{1a} = (1-q) \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2^*+\delta}^{\bar{t}} \left(u(t_2^* + \delta; t_2, ..., t_n) - u(t_2; t_2^* + \delta, ..., t_n)\right) f(\mathbf{t}) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n. \quad (47)$$

[16] The factor $n\left(n-1\right)$ reflects the same logic as the factor 2 in the two-bidder example (see footnote 13).

In the integrand above, first subtracting and then adding $u(t_2; t_2, ..., t_n)$ yields

$$\begin{aligned} & u\left(t_2^*+\delta; t_2, ..., t_n\right) - u\left(t_2; t_2^*+\delta, ..., t_n\right) \\ = & \left[u\left(t_2^*+\delta; t_2, ..., t_n\right) - u(t_2; t_2, ..., t_n)\right] - \left[u\left(t_2; t_2^*+\delta, ..., t_n\right) - u(t_2; t_2, ..., t_n)\right] \\ = & \int_{t_2}^{t_2^*+\delta} \left(u_1\left(t; t_2, ..., t_n\right) - u_2\left(t_2; t, ..., t_n\right)\right) dt > \omega\left(t_2^*+\delta-t_2\right), \end{aligned} \tag{48}$$

where the inequality follows because $t_2 \in [t_2^*, t_2^*+\delta])$ and $t_3, ..., t_n \in H_\epsilon$, and hence (39) holds. Plugging (48) into (47) yields

$$\begin{aligned} ED_{1a} &> (1-q)\,\omega \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2^*+\delta}^{\bar{t}} \left(t_2^*+\delta-t_2\right) f(\mathbf{t}) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n \\ &\geq (1-q)\,\omega f_{\min} \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2^*+\delta}^{\bar{t}} \left(t_2^*+\delta-t_2\right) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n \\ &= (1-q)\,\omega f_{\min}\left(\bar{t}-t_2^*-\delta\right) \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \left(t_2^*+\delta-t_2\right) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_2 dt_3 ... dt_n \\ &= \frac{1}{2}(1-q)\,\omega f_{\min}\left(\bar{t}-t_2^*-\delta\right)\delta^2 \int_{\Omega_{n-2}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_3 ... dt_n, \end{aligned}$$

where $f_{\min} \equiv \min_{t_1,...,t_n} f(\mathbf{t})$ and the last line follows from $\int_{t_2^*}^{t_2^*+\delta} \left(t_2^*+\delta-t_2\right) dt_2 = \frac{1}{2}\delta^2$. Note that $\left(\bar{t}-t_2^*-\delta\right) > \frac{1}{2}\left(\bar{t}-t_2^*\right)$ if $\delta < \frac{\bar{t}-t_2^*}{2}$. Thus, for $\delta < \frac{\bar{t}-t_2^*}{2}$,

$$ED_{1a} > \left[\frac{1}{4}(1-q)\,\omega f_{\min}\left(\bar{t}-t_2^*\right) \int_{\Omega_{n-2}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_3 ... dt_n\right] \delta^2. \tag{49}$$

Step 2: Next we derive a lower bound on ED_{1b} in (45). Define

$$k_{\max}(t_3, t_4, ...t_n; \delta) \equiv \max_{t_1, t_2 \in \left[t_2^*, t_2^*+\delta\right]} u_1(t_1; t_2, ...t_n) \quad \text{and} \quad k_{\min}(t_3, t_4, ...t_n; \delta) \equiv \min_{t_1, t_2 \in \left[t_2^*, t_2^*+\delta\right]} u_1(t_1; t_2, ...t_n) \tag{50}$$

where u_1 is the derivative of u with respect to its first argument. For the first and second terms in the expression for D_{1b} in (41), since $t_2 \in [t_2^*, t_2^*+\delta]$, Taylor series expansions yield

$$u\left(t_2^*+\delta, t_2, ..., t_n\right) \geq u\left(t_2^*, t_2, ..., t_n\right) + k_{\min}\delta \text{ and } u\left(t_2, t_2, ..., t_n\right) \leq u\left(t_2^*, t_2, ..., t_n\right) + k_{\max}\left(t_2 - t_2^*\right).$$

Combining these two inequalities yields

$$u\left(t_2^*+\delta, t_2, ..., t_n\right) - 2u\left(t_2, t_2, ..., t_n\right) + u\left(t_2^*, t_2, ..., t_n\right) \geq q\left(k_{\min}\delta - 2k_{\max}\left(t_2-t_2^*\right)\right). \tag{51}$$

Substituting this bound into D_{1b} in (41), then (45) yields a bound for ED_{1b}:

$$\begin{aligned} ED_{1b} &\geq q\int_{\Omega_{n-2}}\int_{t_2^*}^{t_2^*+\delta}\int_{t_2^*+\delta}^{\bar{t}} \left(k_{\min}\delta - 2k_{\max}\left(t_2 - t_2^*\right)\right) f(\mathbf{t})\mathbf{1}_{t_3,\ldots,t_n\in H_\epsilon} dt_1 dt_2 dt_3 \ldots dt_n \\ &= q\int_{\Omega_{n-2}}\int_{t_2^*+\delta}^{\bar{t}}\int_{t_2^*}^{t_2^*+\delta} (k_{\min}\delta - 2k_{\max}(t_2 - t_2^*)) f(\mathbf{t})\mathbf{1}_{t_3,\ldots,t_n\in H_\epsilon} dt_2 dt_1 dt_3 \ldots dt_n, \quad (52)\end{aligned}$$

where the second line switches the order of integration between t_1 and t_2. Define

$$\hat{f}_{\min}\left(t_1, t_3, t_4, \ldots t_n; \delta\right) \equiv \min_{t_2\in\left[t_2^*, t_2^*+\delta\right]} f\left(t_1, t_2, \ldots t_n\right) \quad \text{and} \quad \hat{f}_{\max}(t_1, t_3, t_4, \ldots t_n; \delta) \equiv \max_{t_2\in\left[t_2^*, t_2^*+\delta\right]} f(t_1, t_2, \ldots t_n).$$

Note that $\hat{f}_{\max} \geq \hat{f}_{\min} > 0$. Because $\hat{f}_{\min}, \hat{f}_{\max}, k_{\min}, k_{\max}$ do not depend on t_2, the inside integration over t_2 in (52) yields:

$$\begin{aligned} \int_{t_2^*}^{t_2^*+\delta} \left(k_{\min}\delta - 2k_{\max}\left(t_2 - t_2^*\right)\right) f(\mathbf{t}) dt_2 &\geq \int_{t_2^*}^{t_2^*+\delta} \left(k_{\min}\delta\hat{f}_{\min}\right) dt_2 - \int_{t_2^*}^{t_2^*+\delta} 2k_{\max}\left(t_2 - t_2^*\right)\hat{f}_{\max} dt_2 \\ &= \left(\hat{f}_{\min}k_{\min} - \hat{f}_{\max}k_{\max}\right)\delta^2, \quad (53)\end{aligned}$$

where we use $\int_{t_2^*}^{t_2^*+\delta}\left(t_2 - t_2^*\right) dt_2 = \frac{1}{2}\delta^2$. Plugging (53) into (52) yields

$$ED_{1b} \geq -q\delta^2\int_{\Omega_{n-2}}\int_{t_2^*+\delta}^{\bar{t}} \left(\hat{f}_{\max}k_{\max} - \hat{f}_{\min}k_{\min}\right)\mathbf{1}_{t_3,\ldots,t_n\in H_\epsilon} dt_1 dt_3 \ldots dt_n. \quad (54)$$

Next we show as δ goes to zero, $\hat{f}_{\min}k_{\min} - \hat{f}_{\max}k_{\max}$ (which depends on $\delta, t_1, t_3, t_4, \ldots t_n$ but not t_2) approaches zero in a uniform sense:

Claim 1: For any constant $\kappa > 0$, there exists a $\hat{\delta}(\kappa) > 0$ such that $\hat{f}_{\max}k_{\max} - \hat{f}_{\min}k_{\min} < \kappa$ for all $\delta < \hat{\delta}(\kappa)$ and all $t_1, t_3, \ldots, t_n$.

Proof of Claim 1: Define $c_1 \equiv \max_{t_1,\ldots,t_n} \left|\frac{d}{dt_1}u_1\left(\mathbf{t}\right)\right|$ and $c_2 \equiv \max_{t_1,\ldots,t_n} \left|\frac{d}{dt_2}u_1(\mathbf{t})\right|$. Since u is twice continuously differentiable, c_1 and c_2 are well defined and bounded. From the definitions of $k_{\max}$ and k_{min} in (50) and the Taylor series expansion, we have $k_{\max} - k_{\min} \leq$

$(c_1 + c_2)\,\delta$.[17] Thus,

$$\hat{f}_{\max}k_{\max} - \hat{f}_{\min}k_{\min} = \left(\hat{f}_{\max} - \hat{f}_{\min}\right)k_{\max} + \hat{f}_{\min}(k_{\max} - k_{\min})$$
$$\leq (\hat{f}_{\max} - \hat{f}_{\min})k_{\max} + \hat{f}_{\min}(c_1 + c_2)\delta. \tag{55}$$

Note that $\hat{f}_{\min}$ in the second term on the right-hand side of (55) is bounded because

$$\hat{f}_{\min} \leq f_{\max} \equiv \max_{t_1,\ldots,t_n} f(\mathbf{t}). \tag{56}$$

Next we examine the first term on the right-hand side of (55). $k_{\max} \leq \max_{t_1,\ldots,t_n} u_1(t_1; t_2, \ldots, t_n)$ so $k_{\max}$ is bounded. Further, the model's premise that f is uniformly continuous yields for any constant $\kappa > 0$, there exists a $\delta^*(\kappa) > 0$ such that $\hat{f}_{\max} - \hat{f}_{\min} < \kappa$ for all $\delta < \delta^*(\kappa)$ and all $t_1, t_3, \ldots, t_n$. By the above arguments and (55), Claim 1 follows.

By Claim 1 and (54), for any $\kappa > 0$, there exists a $\hat{\delta}(\kappa) > 0$ such that for all $\delta < \hat{\delta}(\kappa)$:

$$ED_{1b} \geq -q\kappa\delta^2 \int_{\Omega_{n-2}} \int_{t_2^*+\delta}^{\bar{t}} \mathbf{1}_{t_3,\ldots,t_n \in H_\epsilon} dt_1 dt_3 \ldots dt_n$$
$$= -q\kappa\,(\bar{t} - t_2^* - \delta)\,\delta^2 \int_{\Omega_{n-2}} \mathbf{1}_{t_3,\ldots,t_n \in H_\epsilon} dt_3 \ldots dt_n, \tag{57}$$

where the second line comes from integrating over t_1. Setting $\kappa = \frac{1-q}{8q}\omega f_{\min} \frac{\bar{t}-t_2^*}{\bar{t}-t_2^*-\delta}$, (57) yields

$$ED_{1b} > -\frac{1}{8}(1-q)\,\omega f_{\min}\,(\bar{t} - t_2^*)\,\delta^2 \int_{\Omega_{n-2}} \mathbf{1}_{t_3,\ldots,t_n \in H_\epsilon} dt_3 \ldots dt_n,$$

which, combining with the lower bound on ED_{1a} in (49), yields for all δ sufficiently small:

$$ED_{1a} + ED_{1b} > \left[\frac{1}{8}(1-q)\,\omega f_{\min}\,(\bar{t} - t_2^*) \int_{\Omega_{n-2}} \mathbf{1}_{t_3,\ldots,t_n \in H_\epsilon} dt_3 \ldots dt_n\right]\delta^2. \tag{58}$$

Step 3: The definition of D_2 in (42) yields

$$D_2 = q[u(t_2^*, t_2, t_3, \ldots, t_n) + u(t_2^*, t_1, t_3, \ldots, t_n) - 2u(t_2, t_2, t_3, \ldots, t_n)]$$
$$\geq 2q[u(t_2^*, t_2, t_3, \ldots, t_n) - u(t_2, t_2, t_3, \ldots, t_n)]$$
$$= -2q\,[u(t_2, t_2, t_3, \ldots, t_n) - u(t_2^*, t_2, t_3, \ldots, t_n)] \geq -2qk^*\delta, \tag{59}$$

[17] To see this, assume that $k^\delta_{\max}$ is obtained at $(t_1, t_2) = (t_1^{***}, t_2^{***})$ so that $k^\delta_{\max} \equiv u_1(t_1^{***}; t_2^{***}, t_3, \ldots, t_n)$, and that $k^\delta_{\min}$ is obtained at $(t_1, t_2) = (t_1^{**}, t_2^{**})$ so that $k^\delta_{\min} \equiv u_1(t_1^{**}; t_2^{**}, t_3, \ldots, t_n)$. Then

$$k^\delta_{\max} - k^\delta_{\min} = u_1(t_1^{***}; t_2^{***}, t_3, \ldots, t_n) - u_1(t_1^{**}; t_2^{**}, t_3, \ldots, t_n)$$
$$= (u_1(t_1^{***}; t_2^{***}, t_3, \ldots) - u_1(t_1^{**}; t_2^{***}, t_3, \ldots)) + (u_1(t_1^{**}; t_2^{***}, t_3, \ldots) - u_1(t_1^{**}; t_2^{**}, t_3, \ldots))$$
$$\leq c_1|t_1^{***} - t_1^{**}| + c_2|t_2^{***} - t_2^{**}|.$$

where $k^* \equiv \max_{t_1,t_3,...,t_n} u_1(t_1,t_1,t_3,...,t_n)$, the first inequality follows from $t_1 \geq t_2$, and the second inequality follows from $t_2 - t_2^* \leq \delta$. Plugging (59) into (46) yields

$$\begin{aligned} ED_2 &\geq -2qk^*\delta \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2}^{t_2^*+\delta} f(\mathbf{t})\, \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n \\ &\geq -\left(2qk^* f_{\max} \int_{\Omega_{n-2}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_3 ... dt_n\right) \delta^3, \end{aligned} \tag{60}$$

where $f_{\max}$ is given in (56) and we use $\int_{t_2^*}^{t_2^*+\delta} \int_{t_2}^{t_2^*+\delta} f(\mathbf{t}) dt_1 dt_2 \leq f_{\max}\delta^2$.

Step 4: (58) and (60) show that when δ is small, $ED_{1a} + ED_{1b}$ exceeds a positive term that approaches zero at the rate δ^2, and ED_2 exceeds a negative term that approaches zero at the rate δ^3. Thus, for δ sufficiently small, $ED_{1a} + ED_{1b} + ED_2 > 0$: Mechanism A generates strictly higher expected revenues. □

Proof for Mechanism B: The argument for Mechanism B mirrors that for A. As with Mechanism A, we only consider $t_s \in [t_2^*, t_2^* + \delta]$ and $\mathbf{t}_{-h-s} \in H_\epsilon$ (because $D = 0$ otherwise), and again there are two cases according to whether $t_h \geq t_2^* + \delta$ or not.

Case 1: $t_h \geq t_2^* + \delta$. Revenue in the δ-separation mechanism B is the highest bidder's payment (21):

$$(1-q)\, u(t_2^* + \delta; t_s, \mathbf{t}_{-h-s}) + (2q-1)\, u(t_s; t_s, \mathbf{t}_{-h-s}) + (1-q)\, u(t_s; t_2^*, \mathbf{t}_{-h-s}).$$

Revenue in a no-separation English auction is: $u(t_s; t_s, \mathbf{t}_{-h-s})$. The revenue difference is

$$D(t_1, t_2, ..., t_n) = (1-q)\left[u(t_2^* + \delta; t_s, \mathbf{t}_{-h-s}) - 2u(t_s; t_s, \mathbf{t}_{-h-s}) + u(t_s; t_2^*, \mathbf{t}_{-h-s})\right].$$

Subtracting and then adding a term $(1-q)\, u(t_2^*; t_s, \mathbf{t}_{-h-s})$, we rewrite the above as

$$D(t_1, t_2, ..., t_n) = D_{1a}^B(t_1, t_2, ..., t_n) + D_{1b}^B(t_1, t_2, ..., t_n),$$

where

$$D_{1a}^B \equiv (1-q)\left[u(t_s; t_2^*, \mathbf{t}_{-h-s}) - u(t_2^*; t_s, \mathbf{t}_{-h-s})\right] \tag{61}$$

$$D_{1b}^B \equiv (1-q)\left[u(t_2^* + \delta; t_s, \mathbf{t}_{-h-s}) - 2u(t_s; t_s, \mathbf{t}_{-h-s}) + u(t_2^*; t_s, \mathbf{t}_{-h-s})\right]. \tag{62}$$

Case 2: $t_h \in [t_s, t_2^* + \delta)$. Seller revenue in the δ-separation mechanism B is the sum of the highest bidder's payment in (22) and the second highest bidder's payment in (23):

$$(2q-1)\, u(t_s; t_s, \mathbf{t}_{-h-s}) + (1-q)\left[u(t_s; t_2^*, \mathbf{t}_{-h-s})\right] + (1-q)\, u(t_h; t_2^*, \mathbf{t}_{-h-s})$$

Revenue in the no-separation English auction is $u\left(t_s, t_s, \mathbf{t}_{-h-s}\right)$. The revenue difference is

$$\begin{aligned} D(t_1,...,t_n) &= (1-q)\left(u(t_s;t_2^*,\mathbf{t}_{-h-s})+u(t_h;t_2^*,\mathbf{t}_{-h-s})-2u(t_s;t_s,\mathbf{t}_{-h-s})\right)\equiv D_2^B(t_1,...,t_n) \quad (63)\\ &\geq 2\left(1-q\right)\left(u\left(t_s;t_2^*,\mathbf{t}_{-h-s}\right)-u\left(t_s;t_s,\mathbf{t}_{-h-s}\right)\right), \quad (64)\end{aligned}$$

where the inequality follows since $t_h \geq t_s$ and u increases in its arguments.

Analogously with Mechanism A, the equations for the expected revenue differences (45)–(46) hold, where on the left-hand sides we replace ED_{1a}, ED_{1b}, and ED_2 with their Mechanism B counterparts ED_{1a}^B, ED_{1b}^B, and ED_2^B, and on the right-hand side we replace D_{1a} with D_{1a}^B, D_{1b} with D_{1b}^B, and D_2 with D_2^B. Similarly, (43) and (44) hold, with ED_{1a}^B,ED_{1b}^B, ED_2^B replacing ED_{1a}, ED_{1b}, ED_2 on the right-hand side of (44). Again, since t_1 and t_2 denote the highest and second-highest of the n signals in (43), we interpret t_h and t_s in (61)–(64) as t_1 and t_2.

We now show that for δ sufficiently small, ED_{1a}^B exceeds a term that is positive and quadratic in δ, and ED_{1b}^B and ED_2^B each exceed terms that go to zero at a rate faster than δ^2.

Step 1: Substituting the definition of D_{1a}^B in (61) into (45) yields

$$ED_{1a}^B \equiv (1-q)\int_{\Omega_{n-2}}\int_{t_2^*}^{t_2^*+\delta}\int_{t_2^*+\delta}^{\bar{t}}\left[u\left(t_2;t_2^*,...,t_n\right)-u\left(t_2^*;t_2,...,t_n\right)\right]f(\mathbf{t})\mathbf{1}_{t_3,...,t_n\in H_\epsilon}dt_1dt_2dt_3...dt_n. \quad (65)$$

In the integrand above, since $t_2 \in [t_2^*, t_2^*+\delta])$ and $t_3, ..., t_n \in H_\epsilon$, we have:

$$u\left(t_2;t_2^*,...,t_n\right)-u\left(t_2^*;t_2,...,t_n\right)=\int_{t_2^*}^{t_2}\left(u_1\left(t;t_2,...,t_n\right)-u_2\left(t_2;t,...,t_n\right)\right)dt \;>\; \omega\left(t_2-t_2^*\right),$$

where the inequality follows from (39). Then (65) yields

$$\begin{aligned} ED_{1a}^B &> (1-q)\,\omega f_{\min}\int_{\Omega_{n-2}}\int_{t_2^*}^{t_2^*+\delta}\int_{t_2^*+\delta}^{\bar{t}}\left(t_2-t_2^*\right)\mathbf{1}_{t_3,...,t_n\in H_\epsilon}dt_1dt_2dt_3...dt_n\\ &= (1-q)\,\omega f_{\min}\left(\bar{t}-t_2^*-\delta\right)\int_{\Omega_{n-2}}\int_{t_2^*}^{t_2^*+\delta}\left(t_2-t_2^*\right)\mathbf{1}_{t_3,...,t_n\in H_\epsilon}dt_2dt_3...dt_n\\ &= \frac{1}{2}\left(1-q\right)\omega f_{\min}\left(\bar{t}-t_2^*-\delta\right)\delta^2\int_{\Omega_{n-2}}\mathbf{1}_{t_3,...,t_n\in H_\epsilon}dt_3...dt_n,\end{aligned}$$

where $f_{\min} \equiv \min_{t_1,...,t_n} f(\mathbf{t})$, and we use $\int_{t_2^*}^{t_2^*+\delta}\left(t_2-t_2^*\right)dt_2 = \frac{1}{2}\delta^2$. Since $\left(\bar{t}-t_2^*-\delta\right) >$

$\frac{1}{2}\left(\bar{t}-t_2^*\right)$ if $\delta < \frac{\bar{t}-t_2^*}{2}$, inequality (49) holds for $\delta < \frac{\bar{t}-t_2^*}{2}$:

$$ED_{1a}^B > \left[\frac{1}{4}\left(1-q\right)\omega f_{\min}\left(\bar{t}-t_2^*\right)\int_{\Omega_{n-2}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_3...dt_n\right]\delta^2.$$

Thus, we get the same lower bound on ED_{1a}^B for Mechanism B as for Mechanism A.

Step 2: Comparing (62) with (41) shows that D_{1b}^B for Mechanism B (which is (62)) equals D_{1b} for Mechanism A (which is (41)) multiplied by a factor $\frac{1-q}{q}$. Thus, by the same logic as in Step 2 in the proof for Mechanism A, as δ goes to zero, D_{1b}^B exceeds a negative term that approaches zero faster than δ^2. Thus, for sufficiently small δ, (58) holds for Mechanism B:

$$ED_{1a}^B + ED_{1b}^B > \left[\frac{1}{8}\left(1-q\right)\omega f_{\min}\left(\bar{t}-t_2^*\right)\int_{\Omega_{n-2}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_3...dt_n\right]\delta^2. \tag{66}$$

Step 3: From the lower bound on D_2^B in (64), we have:

$$D_2^B \geq -2\left(1-q\right)\left(u\left(t_2;t_2,t_3,...,t_n\right)-u\left(t_2;t_2^*,t_3,...,t_n\right)\right) \geq -2\left(1-q\right)k^{**}\delta, \tag{67}$$

where $k^{**} \equiv \max_{t_1,t_2,t_3,...,t_n} u_2\left(t_1,t_2,t_3,...,t_n\right)$, and the inequality follows from $t_2 - t_2^* \leq \delta$. Plugging (67) into (46) yields

$$\begin{aligned} ED_2^B &\geq -2\left(1-q\right)k^{**}\delta\int_{\Omega_{n-2}}\int_{t_2^*}^{t_2^*+\delta}\int_{t_2}^{t_2^*+\delta} f\left(\mathbf{t}\right)\mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3...dt_n \\ &\geq -2\left(1-q\right)k^{**}f_{\max}\left(\int_{\Omega_{n-2}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_3...dt_n\right)\delta^3, \end{aligned} \tag{68}$$

where $f_{\max}$ is given in (56) and we use $\int_{t_2^*}^{t_2^*+\delta}\int_{t_2}^{t_2^*+\delta} f(\mathbf{t})dt_1 dt_2 \leq f_{\max}\delta^2$.

Step 4: (66) and (68) show that when δ is small, $ED_{1a}^B + ED_{1b}^B$ exceeds a positive term that approaches zero at the rate δ^2, and ED_2^B exceeds a negative term that approaches zero at the rate δ^3. Thus, for δ sufficiently small, $ED_{1a}^B + ED_{1b}^B + ED_2^B > 0$. □