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Peter J Hammond

(This paper also appears as CRETA Discussion paper 89)

October 2024 No: 1524

Warwick Economics Research Papers

ISSN 2059-4283 (online)
ISSN 0083-7350 (print)

Bayesian Rationality with Subjective Evaluations in Enlivened Decision Trees

Peter J. Hammond: `p.j.hammond@warwick.ac.uk`

CRETA (Centre for Research in Economic Theory and Applications)
University of Warwick, Coventry CV4 7AL, UK.

2024 October 27th, typeset from `ratEnlivenCRETA.tex`

Abstract: A decision-making agent is usually assumed to be Bayesian rational, or to maximize subjective expected utility, in the context of a completely and correctly specified decision model. Following the discussion in Hammond (2007) of Schumpeter's (1911, 1934) concept of entrepreneurship, and of Shackle's (1953) concept of potential surprise, this paper considers enlivened decision trees whose growth over time cannot be accurately modelled in full detail. An enlivened decision tree involves more severe limitations than model mis-specification, unforeseen contingencies, or unawareness, all of which are typically modelled with reference to a universal state space large enough to encompass any decision model that an agent may consider. We consider three motivating examples based on: (i) Homer's classic tale of Odysseus and the Sirens; (ii) a two-period linear-quadratic model of portfolio choice; (iii) the game of Chess. Though our novel framework transcends standard notions of risk or uncertainty, a form of Bayesian rationality is still possible. Instead of subjective probabilities of different models of a classical finite decision tree, we show that Bayesian rationality and continuity imply subjective expected utility maximization when some terminal nodes have attached real-valued subjective evaluations instead of consequences. Moreover, subjective evaluations lie behind, for example, the kind of Monte Carlo tree search algorithm that has been used by some powerful chess-playing software packages. `[215 words]`

Keywords: Prerationality, consequentialist decision theory, entrepreneurship, potential surprise, enlivened decision trees, subjective evaluation of continuation subtrees, Monte Carlo tree search.

JEL Classification: D81, D91, D11, D63.

Prologue

Grau, teurer Freund, ist alle Theorie;
Grün des Lebens gold'ner Baum.[1]
— Mephistopheles in Goethe's *Faust*, Part I.[2]

. . . he said that to finish [the] poem he could not get along without the house because down in the cellar there was an Aleph. He explained that an Aleph is one of the points in space that contains all other points.

The Aleph's diameter was probably little more than an inch, but all space was there, actual and undiminished. Each thing (a mirror's face, let us say) was infinite things, since I distinctly saw it from every angle of the universe. I saw the Aleph from every point and angle, and in the Aleph I saw the earth and in the earth the Aleph and in the Aleph the earth; I saw my own face and my own bowels; I saw your face; and I felt dizzy and wept, for my eyes had seen that secret and conjectured object whose name is common to all men but which no man has looked upon — the unimaginable universe. I felt infinite wonder, infinite pity. . . .
for Cantor's Mengenlehre,[3] [Aleph, or $\aleph$] is the symbol of transfinite numbers, of which any part is as great as the whole.

Out on the street, going down the stairways inside Constitution Station, riding the subway, every one of the faces seemed familiar to me. I was afraid that not a single thing on earth would ever again surprise me; I was afraid I would never again be free of all I had seen. Happily, after a few sleepless nights, I was visited once more by oblivion.

— Excerpts from Jorge Luis Borges *El Aleph* (1945), translated by Norman Thomas Di Giovanni in collaboration with the author.

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[1] "Grey, dear friend, is all theory; green the golden tree of life."

[2] The subject of this paper provided the content for my last seminar at Stanford before retiring in early 2007. A day or two beforehand, Kenneth Arrow left me a phone message asking if I had been inspired by this quotation from Goethe. While my answer had to be negative, I was left feeling that this should have been the source of my inspiration.

[3] "Mengenlehre" is "set theory" in German.

1 Background and Outline

1.1 Justifying Bayesian Rationality

In decision theory, Bayesian rationality is the hypothesis that a decision making agent's choices are those whose consequences, which are generally lotteries with both risky and uncertain outcomes, maximize the expected value of a Bernoulli utility function $Y \ni y \mapsto u(y) \in \mathbb{R}$ defined on the non-empty consequence domain Y. For risky consequences which emerge from what Anscombe and Aumann (1963) describe as a "roulette lottery", there is by definition an "objective" or hypothetical probability $\pi(\omega) \in [0, 1]$ of each lottery outcome ω in a non-empty finite sample space Ω. For uncertain consequences which emerge from what Anscombe and Aumann (1963) describe as a "horse lottery", Bayesian rationality requires there to be a "subjective" or personal probability $p(s) \in [0, 1]$ of each lottery outcome or state s in a non-empty finite state space S. A general "Anscombe–Aumann" lottery specifies, for each possible outcome $s \in S$ of a horse lottery with subjective probabilities, a suitable roulette lottery with objective probabilities $\lambda_s(y)$ over consequences y in the non-empty consequence domain Y. Then the appropriate expected utility maximand is the double sum $\sum_{s \in S} p(s) \sum_{y \in Y} \lambda_s(y) u(y)$ involving products of both objective and subjective probabilities.

Past work has offered normative justifications for Bayesian rational behaviour in decision trees based upon:

1. either the "consequentialist" hypothesis set out in Hammond (1988a, b; 1998a, b; 1999) requiring that the range of possible Anscombe–Aumann consequence lotteries which result from prescribed behaviour in any finite decision tree, including any continuation decision tree, should be explicable as the value of a suitably defined choice function defined on the relevant domain of non-empty finite feasible sets of consequence lotteries;

2. or, more recently, the associated concept of prerationality (Hammond, 2022) applied to weak base preference relations $\succsim$ defined on lottery consequence domains.

When either consequentialism or prerationality is assumed, justifying Bayesian rationality does require one additional well known continuity axiom. This axiom applies to preferences over each "Marschak (1950) triangle" which, given a triple $\{\lambda, \mu, \nu\}$ of roulette lotteries of which no two are indifferent, is defined as the set $\Delta(\{\lambda, \mu, \nu\})$ of all probability mixtures $q_\lambda \lambda + q_\mu \mu + q_\nu \mu$ of the three lotteries, where the three probability weights q_λ, q_μ, q_ν are all non-negative with $q_\lambda + q_\mu + q_\nu = 1$. After suitable relabelling, one can assume that the strict preference relation $\succ$ satisfies $\lambda \succ \mu$, $\lambda \succ \nu$, and $\mu \succ \nu$. Then, given the corresponding weak preference relation $\succsim$, the continuity axiom requires that the two sets

$$\{\alpha \in [0,1] \mid \alpha\lambda + (1-\alpha)\nu \succsim \mu\} \quad \text{and} \quad \{\alpha \in [0,1] \mid \mu \succsim \alpha\lambda + (1-\alpha)\nu\}$$

should both be closed subsets of the unit interval $[0,1] \subset \mathbb{R}$. Dropping continuity would allow some kind of lexicographic preference relation over lotteries which is not Bayesian rational.

Some of these earlier papers justifiying Bayesian rationality also invoked the assumption of dynamic consistency. This assumption requires intended or planned behaviour at the later decision nodes of a tree T to match actual behaviour. Yet in reality actual behaviour is determined only at a decision node of T that happens to be the initial node n_0 of T. At any decision node n of tree T that is not the initial node n_0 of T, behaviour is specified by treating n as the initial node of the *continuation subtree* $T_{\geq n}$ that results from eliminating all the nodes of tree T which do not weakly succeed n. Then, at decision nodes n which come strictly later than the initial node n_0 of T, by specifying actual behaviour without reference to previous intentions or plans, dynamic inconsistency between actions and plans or intentions is entirely ruled out. In this way, dynamic consistency is satisfied by construction.

1.2 Bounded Rationality? Or Bounded Modelling?

> "All models are wrong, but some are useful."
> — George Box (1919–2013)

Human ingenuity has led at least some of us to create puzzles and other decision problems in order to amuse or instruct each other. Many children, and some adults, derive satisfaction from solving jigsaw puzzles, or from learning how not to lose at noughts and crosses, otherwise known as tic-tac-toe. Other people try crossword puzzles, or sudoku, or Rubik's cube. Generations of students take courses in mathematics during which they are expected to learn by solving, or understanding the solutions to, progressively more demanding exercises. In each of these examples the challenge is to find a perfect solution to a well defined decision problem.

Typical decision problems, however, are not like puzzles or mathematical exercises. Indeed, they are very often far too challenging for full Bayesian rationality to be possible. This recognition, of course, was a key motivation for Simon (1955, 1957) to introduce his concepts of bounded rationality and satisficing. Yet satisficing seems hard to motivate except as the result of some compromise which emerges when the benefits of a more intensive search for a Bayesian optimal decision have been traded off against the additional cost of that search. Thus, satisficing seems to apply to the choice of what decision model to analyse rather than to the choice of what decision to make within a given model that is being analysed. For this reason, it seems that a more satisfactory fundamental concept may be that of a bounded model.

Yet a decision-making agent who uses a bounded model may well fail to recognize the all too likely need to revise that model whenever it is no longer possible to ignore the fact that some previously unmodelled pertinent event has occurred. Amongst other possibilities, this event could simply arise from developing an improved model of the original decision problem.

1.3 Enlivened Decision Trees

So, motivated by several examples set out in later sections, including the game of Chess, this paper argues that past work on Bayesian rationality in decision trees is seriously limited in its relevance. This is because of the failure to recognize any possibility that a decision maker's decision tree may be subject to "enlivenment" in the sense of enriching revisions that are needed in order to recognize events which, though they cannot be excluded as possibilities before the end of the original tree is reached, have had to be excluded from the original decision tree. In complicated decision problems such as those involved in playing Chess, typically such exclusions are in practice unavoidable because, even if the enlivening events are not entirely unforeseeable, they have had to be neglected because of computational or other practical modelling limitations.

In order to allow the decision tree to change, even unpredictably, a framework with "enlivened" decision trees is proposed. An entirely myopic agent who follows the old adage "Don't cross your bridges before you come to them" — which Savage (1963, p. 16) in particular has discussed — will act as though this enlivening is totally irrelevant. This leads to the agent lurching from one model to the next, displaying hubris throughout.

Of course many future decisions and their uncertain consequences cannot be modelled in any detail. Nevertheless, an agent with even a little sophistication should recognize that what matters for any one decision is the current expectation of what, when viewed in retrospect, its ultimate *ex post* value will be. Following ideas that Koopmans (1964) and Kreps (1990, 1992) developed in order to discuss the preference for flexibility, an agent should seek to determine these expected valuations as reliably as possible, using whatever limited evidence is deemed to be relevant, as well as what can be handled within whatever bounded resources the agent can afford to allocate to decision analysis. See also Dekel *et al.* (2001, 2005) and many successors for related ideas in the context of decision making with unforeseen contingencies whose possibility is, nevertheless, foreseen by an apparently omniscient and hubristic decision analyst. See also the work on unawareness in decisions and games by, inter alia, Schipper (2014a, b), Halpern and Rêgo (2014), Grant *et al.* (2015a, b), and especially the related work on growing awareness and reverse Bayesianism by Vierø (2009, 2021) and by Karni and Vierø (2013, 2015, 2017).[4]

We emphasize that the present paper differs from this earlier work on unforeseen contingencies or unawareness by not relying on the existence of any "augmented conceivable state space" of the kind defined in Karni and Vierø (2017, p. 304). Instead, initially we allow the relevant state space to grow entirely unpredictably as a result of the dynamic process that we call "enlivenment". Specifically, though a decision-making agent may be aware of the possibility of their own unawareness, they are unable even to formulate a practical model which is based, as usual, on a comprehensive space of all conceivably possible states. This enrichment of the previous concepts of unforeseen contingencies or unawareness, which was introduced informally in Hammond (2007), is inspired in part on Schumpeter's (1911, 1934) concept of entrepreneurship, as well as Shackle's (1953) concept of potential surprise. The concept of an enlivened decision tree was motivated in part by the classical example of Odysseus and the Sirens discussed in Section 3.

[4]Other relevant work on unawareness includes the papers published in the special issue of *Mathematical Social Sciences* edited by Schipper (2014a), as well as those cited in Vierø (2021).

That said, a completely specified enlivened decision tree, which is never subject to any further enlivenment, could be regarded as falling within a universal augmented conceivable state space. As discussed above, the existence of such a universal state space raises conceptual problems. To avoid these, we consider recursively enlivened decision trees which are fully enlivened because their growth can never be fully described in a single universal model. Nevertheless, even a fully enlivened tree can still be reduced to a simply enlivened tree with random outcomes that, instead of consequence lotteries, are subjective evaluations attached to the terminal nodes of a truncated decision tree. As discussed in Section 8.3, this approach to valuing a continuation subtree which can never be completely modelled was the basis of the successful Deep Fritz and then Stockfish open source engines for computer chess. Eventually, however, Stockfish has been supplanted by AlphaZero which results from a special kind of artificial intelligence.

1.4 Outline

Section 2 briefly reviews some distinctions between unbounded and bounded rationality, including as prominent examples of the latter Simon's concept of procedural rationality, as well as Manzini and Mariotti's (2007) "rational shortlist" method.

Next, Section 3 revisits the well known Homeric example of Odysseus and the Sirens. Previous work such as Strotz (1956), Pollak (1968), Hammond (1976) and Elster (1979) has typically regarded this as a prominent example of changing tastes, illustrating the distinction between naïve choice and sophisticated choice, as well as the value of commitment devices. Here, by contrast, the example is viewed as a mythical decision tree which the sorceress Kirke (or Circe) enlivened as the sage advice that she was offering Odysseus progressed through several stages.[5]

The next two Sections 4 and 5 focus on two particular examples. The first is a consumer who, as an investor, chooses a portfolio of financial assets in order to maximize a two-period quadratic utility function subject to a linear budget constraint. Enlivening this consumer's decision problem could merely affect parameter values, but it could also allow the possibility that new commodities, which may even not yet have been invented, could become relevant.

[5]In Homeric Greek, the spelling of her name is *Κίρκη*.

The second example in Section 5 involves the game of Chess, whether played by computers or by humans. It offers a cursory explanation of how Monte Carlo tree simulation can allow a computer algorithm to evaluate positions that arise after possible future moves have been analysed in detail as far as possible. It also presents a brief case study of a particularly unfortunate human move, described at the time as "the blunder of the century". This is seen as one particularly prominent player's failure to revise his bounded model of how the game was likely to proceed.

The main idea of the paper is set out and developed in Sections 6–8. First, Section 6 provides a summary of the key concepts we need to describe Bayesian rationality in classical "unenlivened" finite decision trees. A key tool used in later analysis is the evaluation $v(T)$ of any decision tree T. This is defined as the normalized expected utility generated by any consequence lottery that can result from deciding optimally at each decision node of T. This expected utility or evaluation can be calculated by backward recursion, starting at each terminal node of T, which has a specified consequence.

The focus of Section 7 is on a special kind of "simple" enlivenment that transforms an original decision tree T with finite graph (N, E) into an "enlivened" tree T^+ with an extended finite graph (N^+, E^+), where $N \subset N^+$. A simple enlivenment involves introducing into each directed edge $n \to n'$ in a non-empty finite subset of E an extra "enlivenment" edge $e_m^- \to e_m^+$ consisting of two extra nodes. The first of these is a "pre-enlivenment" event node e_m^- lying between the two nodes n and n' on a directed edge $n \to n'$ at which an uncertain deviation may or may not occur, according to the result of a horse lottery. There, if a deviation does not occur, the node e_m^- is immediately succeeded by node n' of the original unenlivened tree T. Alternatively, if a deviation does occur, it is to the extra "post-enlivenment" node e^+. This is the initial node of an arbitrary finite continuation subtree $T^+_{\geq e^+}$ that gets appended to T when T is enlivened to the new tree T^+.

An agent whose decisions in an enlivened decision tree T^+ are fully Bayesian rational is effectively acting as an unboundedly rational agent would if the enlivened decision tree really were the true and complete model of their decision problem. This complete model plays the role of the "augmented conceivable state space" considered by Karni and Vierø (2017, p. 304), amongst others, whose use was criticised in Section 1.3. Such a space seems close in spirit to the fictional device that Borges calls the "Aleph".

Instead, Section 8 weakens Bayesian rationality when facing an enlivened decision tree to a much less demanding requirement. This allows an enlivened decision tree, which is usually impossible to model in full detail, to be truncated at one or more "terminal evaluation nodes" where the relevant continuation decision tree has been reduced to a single terminal node. Moreover, this terminal node is given, instead of a consequence lottery, a subjective evaluation in the form of a real number that equals the agent's subjectively expected evaluation that would result from the unmodelled consequences of entering the truncated continuation decision tree. In this case, even when the agent cannot avoid remaining entirely unaware of the myriad details of the fully enlivened tree, with all its possible enlivened consequences, the evaluation of each possible terminal node $\bar{n}$ in a truncated decision tree, which is the evaluation of the continuation tree $T_{\geq \bar{n}}$ that was removed by truncation, can be treated as a special kind of uncertain state of the world.

Section 8.6 then states the main result of the paper that characterizes Bayesian rationality in this new setting. The subsequent Section 8.7 compares the arbitrariness of utilities and subjective probabilities in our model of Bayesian rationality with enlivened evaluations to the arbitrariness of those concepts in the Anscombe and Aumann (1963) model of subjective probability.

The concluding Section 9 starts, in Section 9.2, by analysing briefly the concept of "reverse Bayesianism" due to Vierø (2009, 2021) and Karni and Vierø (2013, 2015, 2017), as was mentioned in Section 1.3. Next, Section 9.3 offers a brief discussion of recent work by Ullmann-Margalit (2006), Paul (2014, 2015a, b, c) and other philosophers who have introduced the concept of a "transformative experience". Finally, Section 9.4 mentions some other possible extensions and conclusions.

2 Beyond Unbounded Rationality

2.1 Unbounded versus Procedural Rationality

Simon's (1955, 1957) famous concept of "bounded rationality" may perhaps best be defined by its negation. Decision agents who are *unboundedly rational* make perfect decisions based on perfect models of all the possible acts they could choose, along with all their potential consequences. The result could be the rather disturbing kind of complete model so artfully described in Jorges Luis Borges' short story "El Aleph", from which extracts are quoted in the prologue.

The definition of unbounded rationality in perfect models remains the same no matter whether the consequences are certain (determinate), or else, using the terminology due to Anscombe and Aumann (1963): (i) risky, with hypothetical "objective" probabilities as in a roulette lottery; (ii) uncertain, with personal or "subjective" probabilities as in a horse lottery. Such unbounded rationality would threaten to make games as complicated and enthralling as chess or Go no more interesting than the children's game of noughts and crosses, also known as "tic-tac-toe".[6] And there would be no such thing as the "law of unintended consequences"; every possible consequence should be calculated, making it in some sense intentional, even as the perhaps unfortunate outcome of a risky decision.

In addition to bounded rationality, Simon advanced the important related idea of "procedural rationality". This recognizes that decision *procedures* could be rational, even if they lead to decisions that are irrational in the sense of violating unbounded rationality. He emphasized concepts like *aspiration level*, along with *satisficing*. The latter appears to mean finding a decision that reaches the aspiration level, and making a decision that seems good enough rather than optimal. But optimal (or even just flexible) search suggests that if the aspiration level is reached quickly and easily, it is too undemanding and so should be raised.

2.2 Rational Shortlists

The normative framework we propose, by contrast, suggests that satisficing behaviour should occur, not within a given decision model, but in choosing how much detail to include within the model. Then ultimately behaviour should be optimal relative to whatever bounded model has been selected for analysis.

In the case of decision problems whose acts have only determinate consequences, the idea of a bounded model is neatly captured by the "rational shortlist" method introduced by Manzini and Mariotti (2007) to discuss the concept of "sequential rationalizability". Their idea is that, given a large feasible set of options, at an first initial stage the agent could shortlist a rel-

[6] Note that in Chess, the "Lomonosov tablebases" that are distributed online at `http://tb7.chessok.com/` currently specify perfect play starting from any legally possible position *provided* that there is a total of no more than seven pieces of either colour left on the board, including both kings. The usual game of Chess starts, of course, with each of the two players having 16 pieces on the board.

atively small subset for later serious consideration. Moreover, this shortlist should be small enough to make finding a fully optimal decision amongst those that are shortlisted a manageable decision problem. Thus, any shortlist can be thought of as a bounded model of the feasible set. Also, when it is recognized that observation and/or computation can be costly, work on "rational inattention" inspired by Sims (2003, 2011) and by Hansen and Sargent (2007) considers what bounded decision model may be optimal.

The choice of shortlist can be supposed to emerge rather arbitrarily, even randomly, from some kind of boundedly rational search procedure. Of course, some options may be much more likely to be shortlisted than others. Also, if the composition of the shortlist is regarded as random, the different random variables indicating whether each option belongs to the shortlist may well be correlated.

Once the shortlist has been determined at the first stage, however, it is entirely reasonable to assume that, at a subsequent second stage, the agent indeed selects an optimal element among those that have been shortlisted. That is, choice from within the shortlist satisfies what Simon (1955, 1957) would call "substantitive rationality".

2.3 Other Bounded Decision Models

Shortlisting can be viewed as a particular form of procedural rationality, involving a two-stage procedure. The main point to be made here, however, is that whatever the shortlist may be, it represents a *bounded* model of the full decision problem. Indeed, limitations like the inability of computers to play chess perfectly apply to all difficult decision problems, including most of those that arise in life rather than in the oversimplified models that are typically analysed and applied by economists and other decision scientists. For this reason, any model we use to inform our decision-making should be flexible enough to allow graceful adaptation to potential changes that any practical model must otherwise ignore.

Suppose an effort really is made to take Simon's "procedural rationality" idea as seriously as possible. Specifically, it is presumably interesting to explore the implications of assuming that:

1. agents' time, attention, and computational resources are far too limited for all but simplified models;

2. and in fact they confine themselves to bounded models which are sufficiently simple that they really can find the decision that is optimal within the confines of their bounded model.

Once one recognizes, however, that the model which an agent uses for making decisions is bounded, then one must also recognize that events may eventually force consideration of an expanded or "enlivened" model that includes unmodelled changes.

3 Odysseus and the Sirens Revisited

3.1 A Naïve Sailor's Model

As our first "classical" example of an enlivened decision tree, we reconsider the Homeric myth of Odysseus and the Sirens. According to this epic myth, naïve sailors whose shortest sea route passed near the Sirens' island had perhaps in the past used a simple decision model like the one illustrated in Figure 1. Specifically, they acted as though they thought that their choice was between:

- either going near the Sirens' island and reaching their destination early by a direct route;
- or avoiding the Sirens' island and arriving late after a detour.

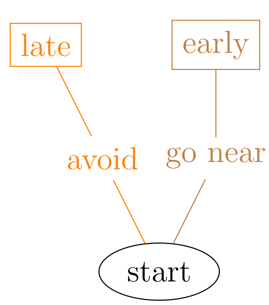

Figure 1: Naïve Sailor

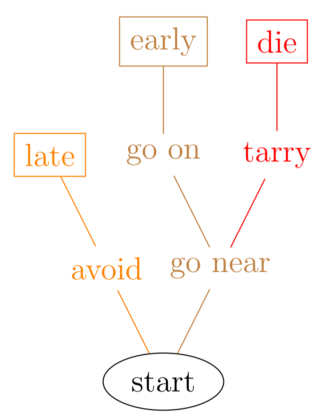

Figure 2: Sophisticated Sailor

3.2 A Sophisticated Sailor's Model

According to Homer, however, Odysseus has the sorceress Kirke as a supernaturally well-informed adviser. She warned Odysseus that the Sirens' singing had the power to lure unwary sailors to their deaths, and that the meadows on the Sirens' island were littered with sailors' bones. So if any naïve sailor came within earshot of the Sirens by choosing go near in the decision tree shown in Figure 1, they would find themselves facing instead the decision node marked go near of the enlivened and so expanded decision tree shown in Figure 2. At this node in the enlivened tree, their apparent choice would be:

- either go on home after hearing the Sirens,
- or tarry, enchanted by their singing, and die on their island, before ever reaching their intended destination.

Of course, the added feature was that, after hearing the Sirens, no sailor had ever exercised enough will-power to escape the island. This is the essential characteristic of what, in Hammond (1976), was called "potential addict" example of changing tastes. Faced with the decision tree of Figure 2, a sophisticated sailor who understands the persuasive power of the Sirens' singing would avoid their island and stay out of earshot, even at the cost of only reaching their intended destination after a significant delay.

3.3 Kirke's First Enlivened Model for Odysseus

Kirke's advice was not confined to a warning, however. Rather routine and unheroic stories about avoiding the Sirens' island and getting back to Ithaca somewhat late by a roundabout route do not constitute memorable epics. Instead Kirke drew attention to the possibility of sailing safely past the Sirens' island, provided the precaution was taken of stopping all the sailors' ears with wax. Thus, after deciding to approach the Sirens' island, but before getting within earshot, the choice at the node go near in Figure 3 would be:

- either wax all the crew's ears (including those of Odysseus himself), so none of them hears the Sirens;
- or use no wax, like earlier naïve sailors whose bones now litter the Sirens' meadow.

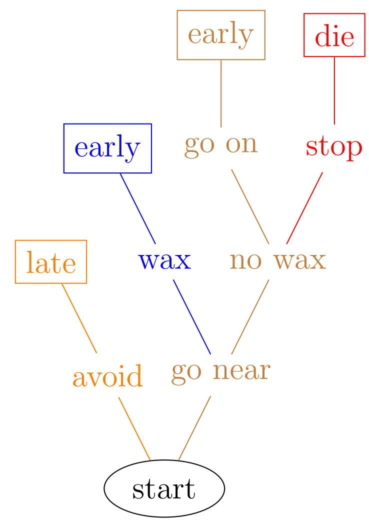

Figure 3: Kirke's First Model

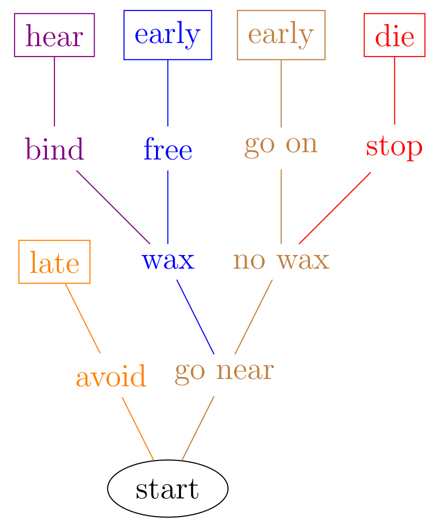

Figure 4: Kirke's Final Model

3.4 Kirke's Final Enlivened Model for Odysseus

A much more interesting epic, however, is the one that Homer has given us. Homer had Kirke advise Odysseus on an even better course of action which allowed Odysseus, at least, to hear the Sirens and yet escape with his life. Indeed, Odysseus was advised that, in addition to arranging for the ears of all his crew to be waxed, he should have himself bound tightly to the mast. Also, his crew should be given strict instructions that, in response to any pleas for release that they see Odysseus making, not only should these pleas be ignored, but also the tightness of his bounds should be increased even more. Thus, Kirke's final model for Odysseus includes an extra choice node marked wax in Figure 4, where the choice is:

- either bind Odysseus to the mast, with ears unwaxed so he can hear the Sirens,
- or leave Odysseus free but with ears waxed like the rest of the crew.

3.5 Toward Enlivened Decision Trees

The earlier naïve sailors whose bones littered the Sirens' meadow had a model like that in Figure 1. Once they had heard the Sirens' singing and so learned of their existence, they may have realized too late that a more appropriate model would have been like that in Figure 2. Odysseus (and his crew) were fortunate enough to be provided with a much more useful model, going even beyond Figure 2 to Figure 3 in the first instance, then ultimately to Figure 4.

Each decision tree in Figures 1–4 is lifeless when considered in isolation. The four trees together, however, tell an epic tale of learning. But it is *not* the usual statistical model of learning more and more about the state of the world within a fixed sample space. Rather, the set of possibilities is expanding, as more and more possibilities are included in the enriched model. By introducing the term "enlivened tree", I have not resisted the temptation to draw an analogy with a live growing tree. Nor of suggesting a strong analogy to the works of Schumpeter (1911, 1934) on innovation, and of Shackle (1953) on "potential surprise" — see Hammond (2007) for further discussion.

4 A Linear–Quadratic Portfolio Problem

4.1 A Two-Period Portfolio Problem

Our first example concerns a consumer with a two-period Bernoulli utility function that takes the quadratic form

$$u(\mathbf{x}_1, \mathbf{x}_2) = -\tfrac{1}{2}(\mathbf{x}_1 - \mathbf{a}_1)^\top \mathbf{Q}_1(\mathbf{x}_1 - \mathbf{a}_1) - \tfrac{1}{2}(\mathbf{x}_2 - \mathbf{a}_2)^\top \mathbf{Q}_2(\mathbf{x}_2 - \mathbf{a}_2) \quad (1)$$

Here $\mathbf{x}_1$ and $\mathbf{x}_2$ denote finite-dimensional consumption vectors in the two periods, which may possibly have different dimensions, whereas $\mathbf{a}_1$ and $\mathbf{a}_2$ are corresponding parameter vectors. Furthermore, assume that $\mathbf{Q}_1$, $\mathbf{Q}_2$ are symmetric and positive definite square matrices of appropriate dimension.

Suppose that the consumer faces two budget constraints, one each period, which can be written as

$$\mathbf{p}_1^\top \mathbf{x}_1 + \mathbf{q}^\top \mathbf{b} = m_1 \quad \text{and} \quad \mathbf{p}_2^\top \mathbf{x}_2 = m_2 + \mathbf{r}^\top \mathbf{b} \quad (2)$$

where $\mathbf{b}$ denotes a finite-dimensional portfolio vector of net asset holdings at the end of period 1, with $\mathbf{q}$ as the asset price vector in period 1, then $\mathbf{r}$ as the gross return vector. Of course $\mathbf{p}_1$ and $\mathbf{p}_2$ denote commodity price vectors each period, both assumed to be strictly positive, whereas $m_1, m_2 \in \mathbb{R}$ are outside wealth transfers. We allow $\mathbf{a}_2$, $\mathbf{r}$ and m_2 all to be uncertain, but treat $\mathbf{p}_2$ as certain, just as Hicks (1946) did when he used point expectations of future prices in his theory of temporary equilibrium.[7] For simplicity we also assume that the symmetric matrix $\mathbf{Q}_2$ is known in period 1. Finally, we assume that the random gross return vector $\mathbf{r}$ is stochastically independent of both random variables $\mathbf{a}_2$ and m_2.

4.2 The Second-Period Optimum

By the start of period 2, we assume that the parameter vector $\mathbf{a}_2$, the gross return vector $\mathbf{r}$, and unearned income m_2 have all become known, along with the portfolio vector $\mathbf{b}$ which is pre-determined by the consumer's own choice in period 1, Accordingly, the consumer's second-period optimization problem, which is independent of whatever $\mathbf{x}_1$ is chosen in period 1, reduces to

$$\max_{\mathbf{x}_2} \left\{-\tfrac{1}{2}(\mathbf{x}_2 - \mathbf{a}_2)^\top \mathbf{Q}_2 (\mathbf{x}_2 - \mathbf{a}_2)\right\} \quad \text{subject to} \quad \mathbf{p}_2^\top \mathbf{x}_2 = m_2 + \mathbf{r}^\top \mathbf{b} \tag{3}$$

To solve this constrained maximization problem, introduce the Lagrangian

$$\mathcal{L}_{\lambda_2}(\mathbf{x}_2) = -\tfrac{1}{2}(\mathbf{x}_2 - \mathbf{a}_2)^\top \mathbf{Q}_2 (\mathbf{x}_2 - \mathbf{a}_2) - \lambda_2(\mathbf{p}_2^\top \mathbf{x}_2 - m_2 - \mathbf{r}^\top \mathbf{b}) \tag{4}$$

Then $\mathcal{L}_{\lambda_2}(\mathbf{x}_2)$ is concave as a function of $\mathbf{x}_2$. So it is maximized at any point $\mathbf{x}_2$ that satisfies the first-order condition

$$\mathbf{0} = \mathcal{L}'_{\lambda_2}(\mathbf{x}_2) = -(\mathbf{x}_2 - \mathbf{a}_2)^\top \mathbf{Q}_2 - \lambda_2 \mathbf{p}_2^\top \tag{5}$$

Because $\mathbf{Q}_2$ is assumed to be positive definite and so invertible, this first-order condition is evidently equivalent to

$$(\mathbf{x}_2 - \mathbf{a}_2)^\top = -\lambda_2 \mathbf{p}_2^\top \mathbf{Q}_2^{-1} \tag{6}$$

or, after transposing and rearranging, to

$$\mathbf{x}_2 = \mathbf{a}_2 - \lambda_2 \mathbf{Q}_2^{-1} \mathbf{p}_2 \tag{7}$$

Substituting this into the budget equation in (3) gives

$$\mathbf{p}_2^\top \mathbf{x}_2 = \mathbf{p}_2^\top (\mathbf{a}_2 - \lambda_2 \mathbf{Q}_2^{-1} \mathbf{p}_2) = m_2 + \mathbf{r}^\top \mathbf{b} \tag{8}$$

[7] For a somewhat similar idea, see Myerson (1983).

implying that

$$\lambda_2 = \frac{\mathbf{p}_2^\top \mathbf{a}_2 - m_2 - \mathbf{r}^\top \mathbf{b}}{\mathbf{p}_2^\top \mathbf{Q}_2^{-1} \mathbf{p}_2} \tag{9}$$

Note that the solution λ_2 exists because $\mathbf{p}_2 \neq \mathbf{0}$ and $\mathbf{Q}_2$ is positive definite. Finally, we can combine (9) with (7) to determine the optimal demand vector, which is

$$\mathbf{x}_2^* = \mathbf{a}_2 - \frac{\mathbf{p}_2^\top \mathbf{a}_2 - m_2 - \mathbf{r}^\top \mathbf{b}}{\mathbf{p}_2^\top \mathbf{Q}_2^{-1} \mathbf{p}_2} \mathbf{Q}_2^{-1} \mathbf{p}_2 \tag{10}$$

Of course, for this solution to be economically sensible, we should require that $\lambda_2 \geq 0$, or equivalently, that $\mathbf{p}_2^\top \mathbf{a}_2 \geq m_2 + \mathbf{r}^\top \mathbf{b}$. Because this inequality involves the asset vector $\mathbf{b}$ chosen in the first period, we will return to this issue later after deriving the consumer's optimal decisions in the first period.

Note that this solution implies that *ex post*, after $\mathbf{a}_2$, $\mathbf{r}$ and m_2 have all become known and $\mathbf{x}_2^*$ has been chosen optimally, equations (7) and (9) imply that the consumer's maximized second period utility is

$$\begin{aligned} -\tfrac{1}{2}(\mathbf{x}_2^* - \mathbf{a}_2)^\top \mathbf{Q}_2 (\mathbf{x}_2^* - \mathbf{a}_2) &= -\tfrac{1}{2}\lambda_2^2 \mathbf{p}_2^\top \mathbf{Q}_2^{-1} \mathbf{Q}_2 \mathbf{Q}_2^{-1} \mathbf{p}_2 \\ &= -\frac{(\mathbf{p}_2^\top \mathbf{a}_2 - m_2 - \mathbf{r}^\top \mathbf{b})^2}{2\mathbf{p}_2^\top \mathbf{Q}_2^{-1} \mathbf{p}_2} \end{aligned} \tag{11}$$

4.3 First-Period Expected Utility

Coming back to the first period, we have assumed that $\mathbf{p}_2$ and $\mathbf{Q}_2$ are both known in advance. So after using (11), the *ex ante* expected value of the intertemporal Bernoulli utility function (1) can be expressed as the function

$$v(\mathbf{x}_1, \mathbf{b}) = -\tfrac{1}{2}(\mathbf{x}_1 - \mathbf{a}_1)^\top \mathbf{Q}_1 (\mathbf{x}_1 - \mathbf{a}_1) - \frac{\mathbb{E}(\mathbf{p}_2^\top \mathbf{a}_2 - m_2 - \mathbf{r}^\top \mathbf{b})^2}{2\mathbf{p}_2^\top \mathbf{Q}_2^{-1} \mathbf{p}_2} \tag{12}$$

of the first-period choice variables $\mathbf{x}_1$ and $\mathbf{b}$. The numerator of the fraction in the second term of the right-hand side of (12) can be expanded as

$$\begin{aligned} &\mathbb{E}(\mathbf{p}_2^\top \mathbf{a}_2 - m_2 - \mathbf{r}^\top \mathbf{b})^2 \\ &\qquad = \mathbb{E}(\mathbf{p}_2^\top \mathbf{a}_2 - m_2)^2 - 2\mathbb{E}[(\mathbf{p}_2^\top \mathbf{a}_2 - m_2)(\mathbf{r}^\top \mathbf{b})] + \mathbb{E}(\mathbf{r}^\top \mathbf{b})^2 \end{aligned} \tag{13}$$

Let $\bar{\mathbf{a}}_2 := \mathbb{E}\mathbf{a}_2$, $\bar{m}_2 := \mathbb{E}m_2$ and $\bar{\mathbf{r}} := \mathbb{E}\mathbf{r}$ denote the respective means, all of which are assumed to exist. Our assumption that $\mathbf{r}$ is stochastically independent of $\mathbf{a}_2$ and m_2 implies that the middle term on the right-hand side of

(13) reduces to

$$\mathbb{E}[(\mathbf{p}_2^\top \mathbf{a}_2 - m_2)(\mathbf{r}^\top \mathbf{b})] = (\mathbf{p}_2^\top \bar{\mathbf{a}}_2 - \bar{m}_2)(\bar{\mathbf{r}}^\top \mathbf{b}) \tag{14}$$

As for the last term on the right-hand side of (13), note that

$$(\mathbf{r}^\top \mathbf{b})^2 = (\mathbf{b}^\top \mathbf{r})\,(\mathbf{r}^\top \mathbf{b}) = \mathbf{b}^\top (\mathbf{r}\,\mathbf{r}^\top)\mathbf{b} \quad \text{and so} \quad \mathbb{E}(\mathbf{r}^\top \mathbf{b})^2 = \mathbf{b}^\top \mathbf{R}\mathbf{b} \tag{15}$$

where $\mathbf{R}$ denotes the symmetric square matrix $\mathbb{E}[\mathbf{r}\mathbf{r}^\top]$ of second moments of returns, which we also assume exists. The moment matrix $\mathbf{R}$ is positive definite under the assumption that the second moment $\mathbb{E}(\mathbf{r}^\top \mathbf{b})^2$ of the return to any portfolio $\mathbf{b} \neq \mathbf{0}$ is always positive.

Substituting from (14) and (15) in (13) gives

$$\begin{aligned} \mathbb{E}(\mathbf{p}_2^\top \mathbf{a}_2 - m_2 - \mathbf{r}^\top \mathbf{b})^2 &= \mathbb{E}(\mathbf{p}_2^\top \mathbf{a}_2 - m_2)^2 - 2(\mathbf{p}_2^\top \bar{\mathbf{a}}_2 - \bar{m}_2)\bar{\mathbf{r}}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{R}\mathbf{b} \\ &= c + (\mathbf{b}^* - \mathbf{b})^\top \mathbf{R}(\mathbf{b}^* - \mathbf{b}) \end{aligned} \tag{16}$$

where $\mathbf{b}^{*\top}\mathbf{R} = (\mathbf{p}_2^\top \bar{\mathbf{a}}_2 - \bar{m}_2)\bar{\mathbf{r}}^\top$, implying that $\mathbf{b}^* = \mathbf{R}^{-1}\bar{\mathbf{r}}(\mathbf{p}_2^\top \bar{\mathbf{a}}_2 - \bar{m}_2)$, and also

$$c = \mathbb{E}(\mathbf{p}_2^\top \mathbf{a}_2 - m_2)^2 - \mathbf{b}^{*\top}\mathbf{R}\mathbf{b}^* = \mathbb{E}(\mathbf{p}_2^\top \mathbf{a}_2 - m_2)^2 - (\mathbf{p}_2^\top \bar{\mathbf{a}}_2 - \bar{m}_2)^2 \bar{\mathbf{r}}^\top \mathbf{R}^{-1}\bar{\mathbf{r}} \tag{17}$$

Finally, therefore, after ignoring an irrelevant additive constant, the consumer's first-period maximand can be written as the quadratic form

$$v(\mathbf{x}_1, \mathbf{b}) = -\tfrac{1}{2}(\mathbf{x}_1 - \mathbf{a}_1)^\top \mathbf{Q}_1(\mathbf{x}_1 - \mathbf{a}_1) - \tfrac{1}{2}(\mathbf{b}^* - \mathbf{b})^\top \mathbf{S}(\mathbf{b}^* - \mathbf{b}) \tag{18}$$

where $\mathbf{S} := \mathbf{R}/\mathbf{p}_2^\top \mathbf{Q}_2^{-1}\mathbf{p}_2$.

4.4 The First-Period Optimization Problem

The consumer's first-period optimization is therefore to maximise the function (18) w.r.t. $\mathbf{x}_1$ and $\mathbf{b}$, subject to the budget constraint $\mathbf{p}_1^\top \mathbf{x}_1 + \mathbf{q}^\top \mathbf{b} = m_1$. We solve this constrained maximization problem by introducing the Lagrangian

$$\begin{aligned} \mathcal{L}_{\lambda_1}(\mathbf{x}_1, \mathbf{b}) = -\tfrac{1}{2}(\mathbf{x}_1 - \mathbf{a}_1)^\top \mathbf{Q}_1(\mathbf{x}_1 - \mathbf{a}_1) - \tfrac{1}{2}(\mathbf{b}^* - \mathbf{b})^\top \mathbf{S}(\mathbf{b}^* - \mathbf{b}) \\ - \lambda_1(\mathbf{p}_1^\top \mathbf{x}_1 + \mathbf{q}^\top \mathbf{b} - m_1) \end{aligned} \tag{19}$$

which is concave as a function of $(\mathbf{x}_1, \mathbf{b})$, so is maximized w.r.t. $(\mathbf{x}_1, \mathbf{b})$ when the first-order conditions

$$\mathbf{0} = \mathcal{L}'_{\lambda_1,\mathbf{x}_1} = -(\mathbf{x}_1 - \mathbf{a}_1)^\top \mathbf{Q}_1 - \lambda_1 \mathbf{p}_1^\top \quad \text{and} \quad \mathbf{0} = \mathcal{L}'_{\lambda_1,\mathbf{b}} = (\mathbf{b}^* - \mathbf{b})^\top \mathbf{S} - \lambda_1 \mathbf{q}^\top \tag{20}$$

are both satisfied. Because both $\mathbf{Q}_1$ and $\mathbf{S}$ are positive definite and so invertible, these first-order conditions are equivalent to

$$(\mathbf{x}_1 - \mathbf{a}_1)^\top = -\lambda_1 \mathbf{p}_1^\top \mathbf{Q}_1^{-1} \quad \text{and} \quad (\mathbf{b}^* - \mathbf{b})^\top = \lambda_1 \mathbf{q}^\top \mathbf{S}^{-1} \tag{21}$$

or, after transposing and rearranging, to

$$\mathbf{x}_1 = \mathbf{a}_1 - \lambda_1 \mathbf{Q}_1^{-1} \mathbf{p}_1 \quad \text{and} \quad \mathbf{b} = \mathbf{b}^* - \lambda_1 \mathbf{S}^{-1} \mathbf{q} \tag{22}$$

Substituting these into the budget equation gives

$$\mathbf{p}_1^\top \mathbf{x}_1 + \mathbf{q}^\top \mathbf{b} = \mathbf{p}_1^\top (\mathbf{a}_1 - \lambda_1 \mathbf{Q}_1^{-1} \mathbf{p}_1) + \mathbf{q}^\top (\mathbf{b}^* - \lambda_1 \mathbf{S}^{-1} \mathbf{q}) = m_1 \tag{23}$$

implying that

$$\lambda_1 = \frac{\mathbf{p}_1^\top \mathbf{a}_1 + \mathbf{q}^\top \mathbf{b}^* - m_1}{\mathbf{p}_1^\top \mathbf{Q}_1^{-1} \mathbf{p}_1 + \mathbf{q}^\top \mathbf{S}^{-1} \mathbf{q}} \tag{24}$$

Note that this is well defined because $\mathbf{p}_1 \neq \mathbf{0}$ and $\mathbf{q} \neq \mathbf{0}$, whereas both symmetric matrices $\mathbf{Q}_1$ and $\mathbf{S}$ are positive definite, and so invertible. Finally, we can use (22) and (24) in order to determine the optimal commodity and asset demand vectors, which are

$$\mathbf{x}_1^* = \mathbf{a}_1 - \frac{\mathbf{p}_1^\top \mathbf{a}_1 + \mathbf{q}^\top \mathbf{b}^* - m_1}{\mathbf{p}_1^\top \mathbf{Q}_1^{-1} \mathbf{p}_1 + \mathbf{q}^\top \mathbf{S}^{-1} \mathbf{q}} \mathbf{Q}_1^{-1} \mathbf{p}_1 \tag{25}$$

$$\text{and} \quad \mathbf{b} = \mathbf{b}^* - \frac{\mathbf{p}_1^\top \mathbf{a}_1 + \mathbf{q}^\top \mathbf{b}^* - m_1}{\mathbf{p}_1^\top \mathbf{Q}_1^{-1} \mathbf{p}_1 + \mathbf{q}^\top \mathbf{S}^{-1} \mathbf{q}} \mathbf{S}^{-1} \mathbf{q} \tag{26}$$

Of course, for this solution to be economically sensible, we should require that $\lambda_1 \geq 0$, or equivalently, that

$$\mathbf{p}_1^\top \mathbf{a}_1 + \mathbf{q}^\top \mathbf{b}^* = \mathbf{p}_1^\top \mathbf{a}_1 + \mathbf{q}^\top \mathbf{R}^{-1} \bar{\mathbf{r}} \, (\mathbf{p}_2^\top \bar{\mathbf{a}}_2 - \bar{m}_2) \geq m_1 \tag{27}$$

Furthermore, for the second-period solution we found previously to be economically sensible, we should require also that $\mathbf{p}_2^\top \mathbf{a}_2 \geq m_2 + \mathbf{r}^\top \mathbf{b}$. Because the two random variables $\mathbf{p}_2^\top \mathbf{a}_2 - m_2$ and $\mathbf{r}^\top \mathbf{b}$ are independent, this requirement implies that there must be a real number α for which, given the optimal choice of $\mathbf{b}$, one has

$$\mathbf{p}_2^\top \mathbf{a}_2 - m_2 \geq \alpha \geq \mathbf{r}^\top \mathbf{b} \tag{28}$$

for almost all possible values of the random pair $(\mathbf{p}_2^\top \mathbf{a}_2 - m_2, \mathbf{r}^\top \mathbf{b}) \in \mathbb{R}^2$.

4.5 An Enlivened Decision Problem

To enliven this linear–quadratic decision model, we consider the possibility that unforeseeable changes occur after the pair $(\mathbf{x}_1, \mathbf{b})$ has already been chosen in period 1. In general, there could be a new second period objective

$$-\tfrac{1}{2}(\mathbf{x}_2^+ - \mathbf{a}_2^+)^\top \mathbf{Q}_2^+ (\mathbf{x}_2^+ - \mathbf{a}_2^+) \tag{29}$$

in which the dimension of the vectors $\mathbf{x}_2^+$, $\mathbf{a}_2^+$ and the corresponding dimension of the positive definite square matrix $\mathbf{Q}_2^+$ may have increased, perhaps because of new commodities. Of course, the second-period budget constraint must also change; we write it as

$$\mathbf{p}_2^{+\top} \mathbf{x}_2^+ \leq \mathbf{r}^\top \mathbf{b} + m_2 \tag{30}$$

with the same asset vector $\mathbf{b}$ as before, since that is already determined by the consumer's decisions in period 1. The joint distribution of $(\mathbf{a}_2^+, m_2, \mathbf{r})$ may also change, as indeed it must if the dimension of $\mathbf{a}_2^+$ exceeds that of $\mathbf{a}_2$.

If these changes could be known in advance, then in period 1 the consumer would face the problem of maximizing, instead of the quadratic evaluation function $v(\mathbf{x}_1, \mathbf{b})$ defined by (18), a revised quadratic objective function

$$v^+(\mathbf{x}_1, \mathbf{b}) = -\tfrac{1}{2}(\mathbf{x}_1 - \mathbf{a}_1)^\top \mathbf{Q}_1 (\mathbf{x}_1 - \mathbf{a}_1) - \tfrac{1}{2}(\mathbf{b}^{+*} - \mathbf{b})^\top \mathbf{S}^+ (\mathbf{b}^{+*} - \mathbf{b}) \tag{31}$$

of the same choice variables $\mathbf{x}_1$ and $\mathbf{b}$, subject to the same first-period budget constraint $\mathbf{p}_1^\top \mathbf{x}_1 + \mathbf{q}^\top \mathbf{b} = m_1$ as in (2). What has changed, however, are the vector parameter $\mathbf{b}^{+*}$ and matrix parameter $\mathbf{S}^+$ which appear in the last term of (31), whose changes are now entirely unpredictable. Enlivenment requires recognizing that these parameters must be treated as themselves uncertain. A Bayesian rational consumer who remains convinced that some quadratic model is still appropriate will, by definition, hold some subjective probability beliefs concerning the unpredictable pair $(\tilde{\mathbf{b}}^*, \tilde{\mathbf{S}})$ of parameters that characterize each member of the parametric family of quadratic evaluation functions

$$\tilde{v}(\mathbf{x}_1, \mathbf{b}; \tilde{\mathbf{b}}^*, \tilde{\mathbf{S}}) \equiv -\tfrac{1}{2}(\mathbf{x}_1 - \mathbf{a}_1)^\top \mathbf{Q}_1 (\mathbf{x}_1 - \mathbf{a}_1) - \tfrac{1}{2}(\tilde{\mathbf{b}}^* - \mathbf{b})^\top \tilde{\mathbf{S}} (\tilde{\mathbf{b}}^* - \mathbf{b}) \tag{32}$$

Rationality, in the sense of subjective expected utility maximization, requires optimal policy in period 1 to maximize the expected value $\hat{\mathbb{E}}[\tilde{v}(\mathbf{x}_1, \mathbf{b}; \tilde{\mathbf{b}}^*, \tilde{\mathbf{S}})]$ of the function (32) w.r.t. probabilistic beliefs concerning the parameter pair

$(\tilde{\mathbf{b}}^*, \tilde{\mathbf{S}})$. Such an expectation, however, after ignoring an irrelevant additive constant, can be expressed in the convenient form

$$\hat{\mathbb{E}}[\tilde{v}(\mathbf{x}_1, \mathbf{b}; \tilde{\mathbf{b}}^*, \tilde{\mathbf{S}})] \equiv -\tfrac{1}{2}(\mathbf{x}_1 - \mathbf{a}_1)^\top \mathbf{Q}_1 (\mathbf{x}_1 - \mathbf{a}_1) - \tfrac{1}{2}(\hat{\mathbf{b}}^* - \mathbf{b})^\top \hat{\mathbf{S}} (\hat{\mathbf{b}}^* - \mathbf{b}) \quad (33)$$

This involves the appropriate subjective expected value $\hat{\mathbf{S}} := \hat{\mathbb{E}}[\tilde{\mathbf{S}}]$ of the random matrix $\tilde{\mathbf{S}}$. Note that the matrix $\hat{\mathbf{S}}$ is positive definite, and so invertible, as the expected value of the random positive definite matrix $\tilde{\mathbf{S}}$. This allows the vector $\hat{\mathbf{b}}^*$ to be chosen uniquely so that it satisfies the first-order condition $\hat{\mathbf{S}}\,\hat{\mathbf{b}}^* = \hat{\mathbb{E}}[\tilde{\mathbf{S}}\,\tilde{\mathbf{b}}^*]$, implying that

$$\hat{\mathbf{b}}^* = \hat{\mathbf{S}}^{-1}\,\hat{\mathbb{E}}[\tilde{\mathbf{S}}\,\tilde{\mathbf{b}}^*] = (\hat{\mathbb{E}}[\tilde{\mathbf{S}}])^{-1}\,\hat{\mathbb{E}}[\tilde{\mathbf{S}}\,\tilde{\mathbf{b}}^*] \quad (34)$$

5 Computer Chess

5.1 Simplified Chess

Consider the decision problem faced by a chess player who has to choose a move when confronted by a known position denoted by n_0. To specify this position requires saying whose turn it is to move, and what piece, if any, occupies each of the 64 squares on the board.[8] Then let $N_1 := N_{+1}(n_0)$ denote the set consisting of all those positions that can be reached by a move which is legal in position n_0.

Recall that, in the game of Chess, a player's King is in check just in case it is attacked by an opponent's piece, in the sense that, in the absence of an intervening move, that piece could capture the King. A player's move is legal only if it does not leave that player's King in check. If the player whose turn it is to move has no legal move, then: (i) that player has been checkmated and loses the game if that player's King in check; (ii) there is a stalemate and the game is a draw if that player's King is not in check.

Following the famous result of Zermelo (1913), as well as von Neumann's (1928) pioneering analysis of maximin or minimax strategies in two-person

[8] Actually, even in a simplified version of chess — without either clocks that are used to enforce limits on each player's total thinking time, or drawing rules that go beyond stalemate, threefold repetition, or perpetual check — the rules of chess specify that: (i) castling is disallowed if either the king or relevant rook has ever been moved previously; (ii) a pawn can capture an opposition pawn *en passant*, but only immediately after the pawn that is about to be captured has advanced two squares from its initial position. So there are many chess positions whose full description requires significantly more information.

"zero-sum" games of perfect information, given best play by both the White and Black players, there is an objective *result function*

$$N_1 \ni n_1 \mapsto r^+(n_1) \in \{W, D, L\} \tag{35}$$

This function maps each possible position $n_1 \in N_1$ to a determinate *result* $r^+(n_1) \in \{W, D, L\}$ of the game that, for the player who is about to move, is either a win (W), or a draw (D), or a loss (L). This result can be converted into a payoff using a scoring rule such as 1 for a win for White, or -1 for a win for Black, but 0 for a draw. Then, given that a subgame of Chess starts from the position n, the result of best play by both players in the subgame will be given by an objective *evaluation function*

$$N_1 \ni n_1 \mapsto v(n_1) \in \{1, 0, -1\} \tag{36}$$

For the player whose turn it is to move at n_0, a move from n_0 to n_1 is optimal if and only if:

1. n_1 maximizes the evaluation function $v(n_1)$ in case it is White's turn to move at n_0;

2. n_1 minimizes the evaluation function $v(n_1)$ in case it is Black's turn to move at n_0.

The objective normalized valuation function in (36) can only be computed, however, for a few relatively simple positions where:

- either it can be proved that, in a small number of moves, one side can force a win due to checkmate, or else, should they wish, a draw due to either (i) stalemate; (ii) a threefold repetition of the position; (iii) perpetual check;

- or alternatively, there are no more than 7 pieces on the board, including both Kings, in which case the Lomonosov "endgame tablebase" software cited in footnote 5 of Section 2.1 will specify what is the result of the game if both players follow maximin strategies.

Thus, in choosing what move to make at n_0, and so what should be the next position $n_1 \in N_1$ on the board, a player is typically forced to come up with subjective beliefs regarding the payoff function. These beliefs can be guided by looking ahead a few moves. But unless one can calculate with certainty a way to force a simple position whose evaluation is definitely known, ultimately one has to assign such evaluations to many such positions a few moves ahead. In this way, one constructs a subjective *evaluation function* mapping chess positions into subjectively expected payoffs. Computer chess programs for doing this involve algorithms that are good, even superhuman, but are still necessarily imperfect. Currently some of the most effective software uses an algorithm based on *Monte Carlo tree search* (MCTS), which is further discussed in Section 8.3 — see Browne *et al.* (2012) for a general survey that has been widely cited in the computer science literature. Applied to Chess, in order to evaluate a given position n, MCTS considers many simulated continuation subgames that all start in position n, but then introduces a little carefully controlled randomness into the routine for choosing each ensuing move. Then the final evaluation of any position n is the average score over all the simulated games that start in position n.

5.2 Real Chess

Real chess is considerably more complicated. For one thing, a player about to move can claim a draw by demonstrating that the next move can be chosen either to repeat the same position a third time, or so that both players will have made at least 50 moves without either a piece being captured or a pawn being moved. Also, the game usually ends with either: (i) one player who is losing choosing to resign; or (ii) with both players agreeing to a draw when both judge that they have an insufficient chance of winning.

Finally, there are time limits monitored by a chess clock, or actually a coupled pair of clocks, one for each player, which displays how much remaining total time that player has available before the next time control. Whenever either player has just made a move, they can press a lever that simultaneously stops their own clock and starts the opponent's. These additional considerations make the description of any chess position n rather more complicated, since it must include, for instance, how much more time each player can use before they would lose on time.

5.3 Human Failure in a Bounded Model

Human chess experts exercise their skill by focusing attention on only a small number of plausible moves in each position. Given any legal chess position n_0, consider the set $N_1 := N_{+1}(n_0)$ of all possible positions n_1 that can result after a legal move to n_1 is made from the position n_0. Chess experts discern that many members n_1 of N_1, though allowable, are too inferior to deserve much, if any, consideration. Of course, human chess experts are also very good at judging the value of any position n_1 that they might think of moving to. In this sense, they have good bounded models.

But, being merely human, even the very best players' models and evaluations of different positions may sometimes be grossly deficient. Witness how in 2006 Vladimir Kramnik, then the world champion, committed the "blunder of the century" by overlooking a checkmate in one move, which led to an immediate loss. This blunder was during the second game of a match of six games played against the computer program Deep Fritz.[9]

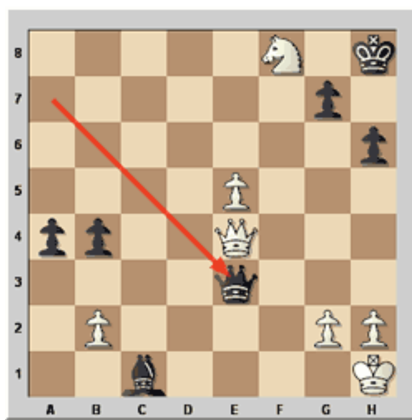

Figure 5: Deep Fritz v. Kramnik, Game 2

In this game, Deep Fritz was playing with the White pieces. Its last move before the position shown in Figure 5 was its 34th. The move was 34. Ne6×f8. This notation signifies that White's knight, which had been on square e6, was used to take the Black piece, actually a rook, which had been on square f8. In response, Kramnik (as Black) blundered horribly by playing the queen move 34 . . . Qa7–e3, as indicated by the arrow in Figure 5, thereby reaching the position shown in that Figure. Whereupon the computer program Deep Fritz promptly indicated that its next move, the queen move 35. Qe4–h7, would win at once by giving checkmate for White.

There must be thousands if not millions of chess players not nearly as strong as Kramnik who, if they were to be shown the position immediately

[9]See `https://en.chessbase.com/post/how-could-kramnik-overlook-the-mate-`

before the move 34 . . . Qa7–e3, would certainly notice that White was threatening to make the move 35. Qe4–h7 checkmate. So how did Kramnik overlook it?

A good clue likes in the following observation that Kramnik himself offered during the press conference that was held minutes after the game ended:

> "It was actually not only about the last move. . . . I calculated the line many, many times, rechecking myself. I already calculated this line when I played 29 . . . Qa7, and after each move I was recalculating, again, and again, and finally I blundered mate in one. Actually it was the first time that it happened to me, and I cannot really find any explanation. I was not feeling tired, I think I was calculating well during the whole game . . . It's just very strange, I cannot explain it."

One way to interpret this is that Kramnik as Black, already when choosing his 29th move, had been planning to make what turned out subsequently to be the disastrous move 34 . . . Qa7–e3 which is shown in Figure 5. In the end it was as if Kramnik had become so fixated on this plan that he restricted himself to a bounded model which made no provision for the possibility of being checkmated immediately after making that fatal move.

6 Bayesian Rationality in Decision Trees

6.1 Roulette Lotteries

Following the terminology of Anscombe and Aumann (1963), given any nonempty set Z, let $\Delta(Z)$ denote the set of all *roulette lotteries* or *simple probability measures*. These take the form of functions $Z \ni z \mapsto \lambda(z) \in [0, 1]$ with a *finite support* $\operatorname{supp} \lambda \subseteq Z$ such that

$$\lambda(z) > 0 \Longleftrightarrow z \in \operatorname{supp} \lambda \quad \text{and} \quad \sum_{z \in Z} \lambda(z) = \sum_{z \in \operatorname{supp} \lambda} \lambda(z) = 1 \tag{37}$$

Then, given any $z \in Z$, let $Z \ni z' \mapsto \delta_z(z') \in \Delta(Z)$ denote the unique *degenerate* lottery that satisfies $\delta_z(z) = 1$. Also, whenever Z is a finite set, let $\Delta^0(Z)$ denote the set of *fully supported* lotteries $\lambda \in \Delta(Z)$ that satisfy $\operatorname{supp} \lambda = Z$, or equivalently, $\lambda(z) > 0$ for all $z \in Z$. Finally, given any

$\lambda, \mu \in \Delta(Z)$ and any scalar $\alpha \in [0,1]$, let $\nu := \alpha\,\lambda + (1-\alpha)\,\mu \in \Delta(Z)$ denote the *lottery mixture* $Z \ni z \mapsto \nu(z) \in [0,1]$ which, for all $z \in Z$, satisfies

$$\nu(z) = [\alpha\,\lambda + (1-\alpha)\,\mu](z) = \alpha\,\lambda(z) + (1-\alpha)\,\mu(z) \tag{38}$$

6.2 Anscombe–Aumann Consequence Lotteries

The hypothesis of Bayesian rationality, or subjective expected utility maximization, applies when there is a non-empty *state space* S of possible states of the world s on which the subjective probability mapping $S \ni s \mapsto p(s) \in [0.1]$ is defined, where $\sum_{s \in S} p(s) = 1$. Following Anscombe and Aumann (1963) once again, we assume that S is finite. Also, following their terminology which was described in Section 1.1, the random process of determining an uncertain state of the world will be described as a "horse lottery".

Bayesian rationality concerns preferences over Anscombe–Aumann consequence lotteries. By definition, these may involve both risk, due to roulette lotteries, and uncertainty, due to horse lotteries. Let Y denote a non-empty consequence domain, and $\Delta(Y)$ the domain of roulette lotteries over Y. Given the finite set S of states s and the consequence domain Y, for each state $s \in S$, let Y_s be a copy of Y.[10] Then let

$$L^S(Y) := \prod_{s \in S} \Delta(Y_s) = \{\langle \lambda_s \rangle_{s \in S} \mid \forall s \in S : \lambda_s \in \Delta(Y_s)\} \tag{39}$$

denote the space of *Anscombe–Aumann lotteries*, or *AA lotteries*, in the form of lists $\langle \lambda_s \rangle_{s \in S}$ or mappings $S \ni s \mapsto \lambda_s \in \Delta(Y)$. Each such mapping specifies a combination of, first, a horse lottery that determines a state $s \in S$, followed second by a state-dependent roulette lottery λ_s that determines a consequence $y \in Y$.

[10] This is the case of a *state-independent consequence domain*, which we assume in order to simplify notation. The more general case of a *state-dependent consequence domain* occurs when Y_s depends on s. In this case, let $Y^{\cup} := \cup_{s \in S} Y_s$ denote the *union domain* of all consequences y that are feasible in some state $s \in S$. Then there may be a *state-dependent utility function* $D \ni (s,y) \mapsto u(s,y) \to \mathbb{R}$ defined on the domain $D := \{(s,y) \in S \times Y^{\cup} \mid y \in Y_s\}$ of feasible state–consequence pairs. Such state-dependent utility functions have been studied in Drèze (1962), Karni (1985), and Drèze and Rustichini (2004). See Hammond (1998b, 1999, 2022) for a unified treatment which generalizes the case when the consequence domain is state-independent, and which derives a state-independent utility function even when the consequence domain is state-dependent.

6.3 Choice from Pair Sets and Base Preferences

Let $\mathcal{F}(L^S(Y))$ denote the family of non-empty finite subsets of the AA-lottery domain $L^S(Y)$. A *choice function* is a mapping

$$\mathcal{F}(L^S(Y)) \ni F \mapsto C(F) \in \mathcal{F}(L^S(Y)) \tag{40}$$

that, for each non-empty *feasible set* $F \in \mathcal{F}(L^S(Y))$, determines a non-empty *choice set* $C(F) \in \mathcal{F}(L^S(Y))$ satisfying $C(F) \subseteq F$.

Corresponding to any choice function $F \mapsto C(F)$, its values when F is a *pair set* with $\#F = 2$ determine a strict preference relation $\succ_C$, a strict dispreference relation $\prec_C$, and an indifference relation $\sim_C$. These three relations are defined so that for each pair $\lambda^S, \mu^S \in L^S(Y)$, one has

$$\lambda^S \left\{ \begin{array}{c} \succ_C \\ \sim_C \\ \prec_C \end{array} \right\} \mu^S \quad \text{according as} \quad C(\{\lambda^S, \mu^S\}) = \left\{ \begin{array}{c} \{\lambda^S\} \\ \{\lambda^S, \mu^S\} \\ \{\mu^S\} \end{array} \right\} \tag{41}$$

Underlying the choice function $F \mapsto C(F)$ specified by (40), there is a single corresponding binary weak preference relation $\succsim_C$ on $L^S(Y)$, called the *base relation*. For each pair $\lambda^S, \mu^S \in L^S(Y)$, this base relation satisfies

$$\lambda^S \succsim_C \mu^S \iff \lambda^S \in C(\{\lambda^S, \mu^S\}) \iff \lambda^S \succ_C \mu^S \text{ or } \lambda^S \sim_C \mu^S \tag{42}$$

Finally, we mention the corresponding weak dispreference relation $\precsim_C$ defined so that

$$\lambda^S \precsim_C \mu^S \iff \mu^S \in C(\{\lambda^S, \mu^S\}) \iff \lambda^S \prec_C \mu^S \text{ or } \lambda^S \sim_C \mu^S \tag{43}$$

Evidently both the weak preference relation $\succsim_C$ and the weak dispreference relation $\precsim_C$ are *complete* in the sense that, for each pair $\lambda^S, \mu^S \in L^S(Y)$, one has:

1. either $\lambda^S \succsim_C \mu^S$ or $\mu^S \succsim_C \lambda^S$ or both;
2. either $\lambda^S \precsim_C \mu^S$ or $\mu^S \precsim_C \lambda^S$ or both.

6.4 Bayesian Rationality and Expected Utility

Let $F \mapsto C(F)$ be any choice function satisfying (40) and $C(F) \subseteq F$ for all $F \in \mathcal{F}(L^S(Y))$ that corresponds to the base preference relation $\succsim_C$ defined on the space $L^S(Y)$ of AA lotteries λ^S by (41). Then the mapping

$L^S(Y) \ni \lambda^S \mapsto U^S(\lambda^S) \in \mathbb{R}$ is a *utility function* which *represents* the preference relation $\succsim_C$ on $L^S(Y)$ just in case, for all $\lambda^S, \mu^S \in L^S(Y)$, one has

$$\lambda^S \succsim_C \mu^S \Longleftrightarrow U^S(\lambda^S) \geq U^S(\mu^S) \tag{44}$$

Given the specified non-empty finite set S of uncertain states of the world, an *interior subjective probability mass function* is a mapping $S \ni s \mapsto \mathbb{P}(s) \in (0,1]$ that satisfies $\sum_{s \in S} \mathbb{P}(s) = 1$. As discussed in Section 5.4 of Hammond (1998b) and in Hammond (2022), the restriction to positive probabilities is to avoid the difficulties that arise in continuation subtrees of a decision tree when zero probabilities are allowed.

Then the choice function $F \mapsto C(F)$, together with the associated base preference relation $\succsim_C$ on the space $L^S(Y)$, are both *Bayesian rational* just in case there exist a probability mass function $S \ni s \mapsto \mathbb{P}(s) \in (0,1]$, as well as a *Bernoulli utility function* $Y \ni y \mapsto u(y) \to \mathbb{R}$, such that the preference relation $\succsim_C$ on $L^S(Y)$ is represented by the *von Neumann subjective expected utility function* defined for all AA lotteries $\lambda^S = \langle \lambda_s \rangle_{s \in S}$ by the double sum

$$U^S(\lambda^S) = \sum_{s \in S} \mathbb{P}(s) \sum_{y \in Y} \lambda_s(y) u(y) \tag{45}$$

in the sense that (44) is satisfied.

6.5 Normalizing Utility

Recall that, as explained in Section 6.1, for each $y \in Y$, we use δ_y to denote the unique *degenerate* probability measure in $\Delta(Y)$ that satisfies $\delta_y(\{y\}) = 1$. To avoid trivialities, we assume that there exist at least three consequences $\underline{y}, y^0, \overline{y}$ in the domain Y such that, for the three corresponding degenerate lotteries $\delta_{\overline{y}}, \delta_{y^0}, \delta_{\underline{y}}$, the base strict preference relation $\succ$ satisfies the strict preference property $\delta_{\overline{y}} \succ \delta_{y^0} \succ \delta_{\underline{y}}$.

Consider any Bernoulli utility function $Y \ni y \mapsto u(y) \to \mathbb{R}$ and the associated preference relation $\succsim$ on $\Delta(Y)$ that satisfies

$$\lambda \succsim \mu \Longleftrightarrow U(\lambda^S) \geq U(\mu^S) \tag{46}$$

for the *von Neumann objective expected utility function* on $\Delta(Y)$ defined by

$$U(\lambda) = \sum_{y \in Y} \lambda(y) u(y) \tag{47}$$

As explained in Hammond (1998a, Section 2.3), assuming Bayesian rationality, the ratio $\dfrac{u(y^0) - u(\underline{y})}{u(\overline{y}) - u(\underline{y})}$ of utility differences equals, in economists' terminology, the constant marginal rate of substitution along an indifference curve between shifts in probability: (i) from consequence $\underline{y}$ to y^0; (ii) from consequence $\underline{y}$ to $\overline{y}$. This leads us to say that two Bernoulli utility functions $y \mapsto u(y)$ and $y \mapsto \tilde{u}(y)$ are *equivalent* just in case, for every triple $\underline{y}, y^0, \overline{y}$ of consequences in Y satisfying $\delta_{\overline{y}} \succ \delta_{y^0} \succ \delta_{\underline{y}}$ and so $u(\overline{y}) > u(y^0) > u(\underline{y})$, the corresponding ratios of utility differences satisfy

$$\frac{u(y^0) - u(\underline{y})}{u(\overline{y}) - u(\underline{y})} = \frac{\tilde{u}(y^0) - \tilde{u}(\underline{y})}{\tilde{u}(\overline{y}) - \tilde{u}(\underline{y})} \tag{48}$$

But (48) holds for every triple $\underline{y}, y^0, \overline{y}$ satisfying $u(\overline{y}) > u(y^0) > u(\underline{y})$ if and only if there exist an additive constant $\alpha \in \mathbb{R}$ and a positive multiplicative constant $\rho \in \mathbb{R}$ such that, for all $y \in Y$, one has

$$\tilde{u}(y) = \alpha + \rho u(y) \tag{49}$$

Now, given any pair $\underline{u}, \bar{u}$ of real numbers with $\bar{u} > \underline{u}$ and any Bernoulli utility function $Y \ni y \mapsto u(y) \to \mathbb{R}$, there exist two unique constants α and $\rho > 0$ such that the transformed utility function defined by (49) is a *normalized* utility function that satisfies $\tilde{u}(\underline{y}) = \underline{u}$ and $\tilde{u}(\overline{y}) = \overline{u}$. Indeed the two constants we need are given by

$$\rho = \frac{\overline{u} - \underline{u}}{u(\overline{y}) - u(\underline{y})} \quad \text{and then} \quad \alpha = \overline{u} - \rho u(\overline{y}) = \underline{u} - \rho u(\underline{y}) \tag{50}$$

From now on let u denote the unique Bernoulli utility function $Y \ni y \mapsto u(y) \to \mathbb{R}$ whose expected values defined by (47) satisfy (46), and which has been normalized to satisfy

$$u(\underline{y}) = \underline{u} \quad \text{and} \quad u(\overline{y}) = \overline{u} \tag{51}$$

6.6 Finite Decision Trees

In mathematics, a *graph* (N, E) is a non-empty set N of vertices or *nodes* n whose pairs may or may not be connected by edges $e = (n, n')$ with $n \neq n'$ in a specified set $E \subseteq N \times N$. The graph (N, E) is *finite* just in case the set

N of nodes is finite, which implies that the set E of edges is also finite . The graph (N, E) is *directed* just in case there is an antisymmetric and complete binary relation $>_{+1}$ on E — i.e., for each edge $(n, n') \in E$, either $n >_{+1} n'$, or $n' >_{+1} n$, but not both. The sequence $(n_1, n_2, \ldots, n_\ell)$ of ℓ nodes in N is a *path* of length $\ell \in \mathbb{N}$ in the directed graph (N, E) just in case $n_{k+1} >_{+1} n_k$ for $k = 1, 2, \ldots, \ell - 1$.

The graph (N, E) is a *directed tree* just in case there is a unique *initial node* $n_0 \in N$ (or root, or seed, or entry point) such that, for every other node $n \in N \setminus \{n_0\}$ of the graph, there is a unique path $(n_0, n_1, n_2, \ldots, n)$ which starts at node n_0 and ends at node n. In the case of a directed tree, we write each edge (n, n') of E in the form $n \to n'$, where $n' >_{+1} n$.

Let $T = (N, E)$ denote any finite directed tree. Given any node $n \in N$, say that:

1. any node $n' \in N$ (ultimately) *succeeds* n just in case either $n' = n$, or else there is a path $(n_1, n_2, \ldots, n_\ell)$ of nodes in N such that $n_1 = n$ and $n_\ell = n'$;

2. the *continuation subtree* $T_{\geq n} = (N_{\geq n}, E_{\geq n})$ is the unique tree in which:

   - $N_{\geq n}$ is the set of all nodes in the set N that succeed n;
   - $E_{\geq n}$ is the restriction $E \cap (N_{\geq n} \times N_{\geq n})$ of edges in E to directed pairs of nodes that both succeed n;

3. any other node $n' \in N$ *immediately succeeds* n just in case the ordered pair (n, n') is a directed edge of T, and then let

$$N^{+1}_{\geq n} := \{n' \in N \mid (n, n') \in E\} = \{n' \in N \mid n' >_{+1} n\} \tag{52}$$

   denote the set of all the immediate successors of n.

A *finite decision tree* is a finite directed tree $T = (N, E)$ in which, following the terminology and Anscombe and Aumann (1963), the set N of nodes is partitioned into four pairwise disjoint subsets:

1. the set N^d of *decision nodes* n at each of which the decision-making agent must choose an edge $n \to n'$ emanating from n — i.e., where $n' \in N^{+1}_{\geq n}$, as defined in (52);

2. the set N^c of *chance nodes* n where an edge emanating from n is determined randomly by a *roulette lottery* $N^{+1}_{\geq n} \ni n' \mapsto \pi(n'|n) \in (0,1]$ with the property that all specified probabilities are positive;[11]

3. the set N^e of *event nodes* where an edge emanating from n is determined by a *horse lottery* in which, for some non-empty set S of *states of the world*, the non-empty event $S_{\geq n} \subseteq S$ is partitioned into the collection $\{S_{\geq n'} \mid n' \in N^{+1}_{\geq n}\}$ of non-empty pairwise disjoint sub-events;

4. the non-empty set N^t of *terminal nodes* n at which $N^{+1}_{\geq n} = \emptyset$, so no edge emanates, and which are each mapped to an *Anscombe–Aumann consequence lottery* $\gamma(n) = \langle \gamma_s \rangle_{s \in S_{\geq n}} \in L^{S_{\geq n}}(Y)$ whose outcomes y belong to the specified consequence domain Y.

Given the consequence domain Y and state space S, let $\mathcal{T}^S(Y)$ denote the collection of all decision trees T whose terminal nodes $n \in N^t$ are mapped to AA consequence lotteries $\lambda^E \in L^E(Y)$ which, for some non-empty *event* $S_{\geq n} \subseteq S$, belong to $L^{S_{\geq n}}(Y)$. Say that a decision tree $T \in \mathcal{T}^S(Y)$ is:

- *deterministic* just in case $N^c = N^e = \emptyset$;
- *risky* just in case $N^c \neq \emptyset$ but $N^e = \emptyset$;[12]
- a *Savage tree* just in case $N^e \neq \emptyset$ but $N^c = \emptyset$;
- an *Anscombe–Aumann tree* just in case $N^c \neq \emptyset$ and $N^e \neq \emptyset$.

6.7 Truncated Decision Trees and Influence Diagrams

Let T be any decision tree in the domain $\mathcal{T}(Y)$, whose graph (N, E) consists of the set N of nodes together with the set E of directed edges $n \to n'$ that satisfy $n, n' \in N$ with $n \neq n'$. A *truncation* $\hat{T}$ of T is a decision tree with graph $(\hat{N}, \hat{E})$ where $\hat{N}$ is a non-empty subset of N, and

$$\hat{E} = \{n \to n' \in E \mid n, n' \in \hat{N}\} \tag{53}$$

[11] See Hammond (1988b) for an explanation of why, if there is a chance node $n \in N^c$ and a node $n' \in N^{+1}_{\geq n}$ at which $\pi(n'|n) = 0$, then all consequence lotteries must be indifferent.

[12] Raiffa (1968) focused on risky decision trees with *pecuniary consequences* in the form of payoffs measured in dollars.

That is, the set of edges $\hat{E}$ in $\hat{T}$ consists of those directed edges $n \to n'$ in E which join vertices satisfying $n, n' \in \hat{N}$.

We have assumed that any decision tree $T \in \mathcal{T}^S(Y)$ is finite, as is the continuation subtree $T_{\geq n}$ for any $n \in N$. Nevertheless, each continuation subtree can be arbitrarily large, as can the set M_n of moves $n \to n'$ in E that are possible at each node n, whether node n is a decision, chance, or event node. So generally it is impossible to represent all the nodes in a continuation subtree $T_{\geq n}$ in a single diagram of that subtree.

Instead of trying to represent graphically any full continuation subtree $T_{\geq n}$, we follow the standard method described by Howard and Matheson (2005), which uses influence diagrams to compress what would otherwise be the often extraordinarily convoluted full representation of a decision tree. In the simple cases economists usually consider, they often describe influence diagrams as showing "time lines", each of which represents the typical member of what may be a large set of several paths through the tree. Instead of ending in one specific consequence, such influence diagrams will usually end in a variable consequence.

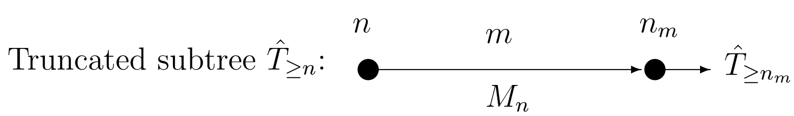

Figure 6: Influence Diagram for the Truncated Subtree $\hat{T}_{\geq n_m}$

Figure 6 shows an influence diagram for the truncation $\hat{T}_{\geq n}$ of any *continuation subtree* $T_{\geq n}$ that follows any particular node $n \in N$. Attached to each terminal node $n_m \in N^{+1}_{\geq n}$ with $m \in M_n$ of this truncated subtree, there is a corresponding continuation subtree $\hat{T}_{\geq n_m}$.

6.8 Evaluations of Unenlivened Continuation Subtrees

Consider the orthodox "unenlivened" decision model which is represented by any finite decision tree T in the domain $\mathcal{T}^S(Y)$ of trees with state space S and lottery consequences in the domain $L^{S'}(Y)$ for some non-empty $S' \subseteq S$. Working backwards, as usual in dynamic programming, we can use a recursive procedure to calculate the *evaluation* $v(T_{\geq n})$ or subjectively expected continuation value of reaching any node $n \in N$, which is the initial node of the continuation subtree $T_{\geq n}$.

In the finite decision tree T, the backward recursion starts at any terminal node $n \in N^t$. As discussed in Section 6.6, the specified consequence of reaching any terminal node $n \in N^t$ of tree T is the AA consequence lottery $\gamma(n) = \langle \gamma_s \rangle_{s \in S_{\geq n}} \in L^{S_{\geq n}}(Y)$. Then the evaluation $v(T_{\geq n})$ of the subtree $T_{\geq n}$, whose only node is the terminal node n, is specified by (45) as the double sum

$$v(T_{\geq n}) = U^{S_{\geq n}}(\gamma(n)) = \sum_{s \in S_{\geq n}} \mathbb{P}(s) \sum_{y \in Y} \gamma_s(y) u(y) \tag{54}$$

At any previous node $n \notin N^t$, the value $v(T_{\geq n})$ of reaching node n and continuation subtree $T_{\geq n}$ depends upon the set $\{v(T_{\geq n'}) \mid n' \in N^{+1}_{\geq n}\}$ of values at all the nodes $n' \in N^{+1}_{\geq n}$ which immediately succeed n. There are three cases to consider, depending upon whether n is a chance, event, or decision node. In the first case when n is a chance node whose immediate successors $n' \in N^{+1}_{\geq n}$ occur with respective specified probabilities $\pi(n'|n)$, the relevant recursion takes the form

$$v(T_{\geq n}) = \sum_{n' \in N^{+1}_{\geq n}} \pi(n'|n) v(T_{\geq n'}) \tag{55}$$

The second case occurs when n is an event node, each of whose immediate successors $n' \in N^{+1}_{\geq n}$ determines which is the relevant cell of the partition $\{S_{\geq n'} \mid n' \in N^{+1}_{\geq n}\}$ of the event $S_{\geq n}$ into pairwise disjoint sets. In this case the objectively specified probabilities $\pi(n'|n)$ that appear in (55) need to be replaced by subjective conditional probabilities $p(n'|n)$ derived from the relevant subjective probabilities $\mathbb{P}(s)$ for different states $s \in S_{\geq n}$. Because of our requirement that $\mathbb{P}(s) > 0$ for all $s \in S$, these conditional probabilities $p(n'|n)$ are all well defined, and can be calculated as

$$p(n'|n) = \sum_{s \in S_{\geq n'}} \mathbb{P}(s) \Big/ \sum_{s \in S_{\geq n}} \mathbb{P}(s) \tag{56}$$

So, when n is an event node with subjective probabilities $p(n'|n)$ given by (56) rather than a chance node with hypothetical or objective probabilities $\pi(n'|n)$, the previous formula (55) is changed to

$$v(T_{\geq n}) = \sum_{n' \in N^{+1}_{\geq n}} p(n'|n) v(T_{\geq n'}) \tag{57}$$

In the third and final case when n is a decision node, we apply the *optimality principle* of stochastic dynamic programming. This requires any current *optimal decision* $n^* \in N^{+1}_{\geq n}$ to be the first step toward achieving the highest possible expected value resulting from an appropriate plan for all subsequent decisions. Consider the induction hypothesis that, for each node $n' \in N^{+1}_{\geq n}$, the value $v(T_{\geq n'})$ is the maximum possible evaluation the agent can achieve by choosing an optimal decision at each decision node of $T_{\geq n'}$. This is trivally true when n' is a terminal node, so there is no decision to make at node n'. If this hypothesis is true at each node $n' \in N^{+1}_{\geq n}$, then any optimal decision at node n must be to move to an immediately succeeding node n^* which maximizes the evaluation $v(T_{\geq n'})$ with respect to n' subject to $n' \in N^{+1}_{\geq n}$. In other words, one must satisfy

$$n^* \in \arg\max_{n' \in N^{+1}_{\geq n}} v(T_{\geq n'}) \tag{58}$$

So the appropriate recursion when n is a decision node is

$$v(T_{\geq n}) = v(T_{\geq n^*}) = \max_{n' \in N^{+1}_{\geq n}} v(T_{\geq n'}) \tag{59}$$

Together, therefore, the four equations (54), (55), (57), and (59) do indeed determine $v(T_{\geq n})$ by backward recurrence in the four different cases.

7 Trees with Simply Enlivened Consequences

7.1 Simple Enlivenment as Enrichment

It is time to return to the main task in the rest of the paper. This is to explain how a relevant version of the Bayesian rationality hypothesis can be applied even to enlivened decision trees. Given any decision tree $T \in \mathcal{T}^S(Y)$, we begin by defining a *simple enlivenment* as a suitable modification of any particular "unenlivened" continuation subtree $T_{\geq n}$ whose initial node is n, as defined in Section 1.1. This simple enlivenment will then result from adding an appropriate set of extra nodes to $T_{\geq n}$. This, after all, is what happened when we described the enlivened decision trees in the three examples of Sections 3, 4, and 5.

Indeed, in the model of Odysseus and the Sirens set out in Section 3, there were three stages of enlivenment which come from adding new nodes one step at a time as one progresses through the succession of four trees illustrated in Figures 1–4. Second, in Section 4, the enlivened two-period linear-quadratic portfolio problem which is described in Section 4.5 is the result of adding extra dimensions to the original problem set out in Section 4.1. Third, the Chess "blunder of the century" described in Section 5.3 seems to have resulted from Kramnik as Black calculating the likely consequences of his planned moves using a bounded model which excludes the move 35. Qe4–h7 that Deep Fritz, playing White, used to deliver checkmate. In this sense, Deep Fritz was using an enlivened version of the bounded model which Kramnik had been using, with 35. Qe4–h7 as an extra possible move which is included in the enriched model, but not in Kramnik's bounded model.

Suppose now that the agent's original unenlivened continuation subtree $T_{\geq n}$ is simply enlivened to a new enlivened subtree $T^+_{\geq n}$ which extends $T_{\geq n}$ by recognizing new possibilities that had previously been excluded. Consider any new information which, when the agent is at n, arrives in time to change the modelled feasible set $M_n := N^{+1}_{\geq n}$ of possible immediate successors of n. These are the nodes n_m which can be reached by moves $n \to n_m$ for $m \in M_n$ that are possible at the initial node n of $T_{\geq n}$. As a result, the set M_n is replaced by a new expanded feasible set M^+_n which is incorporated in a new tree T^+. We argue that this expansion is really a trivial enlivenment of T because, before making a decision at node n, the agent has time to recognize that the feasible set of moves $n \to n_m$ at node n is M^+_n rather than only the subset M_n. And this, of course, is exactly what the agent should do.

So, to conclude, a necessary condition for the agent to be able to decide to move to a node n_m with $m \in M^+_n$ in the expanded continuation subtree $T^+_{\geq n}$ is that node n_m must be included in the agent's original continuation subtree $T_{\geq n}$. This is also sufficient; if node n_m is in the tree $T^+_{\geq n}$, then the agent, when at node n, could in principle choose to move there. Accordingly, we assume that M_n remains unaffected by any enlivenment of the continuation subtree $T_{\geq n}$. Only after a chosen decision node $n_m \in M_n$ has already been reached can any enlivening of the initial continuation subtree $T_{\geq n}$ occur.

Similarly, even if the initial node n of the original continuation subtree $T_{\geq n}$ is not a decision node, we still insist that simple enlivenment can never occur *at* node n. Instead, it must occur at a new node which belongs to $T^+_{\geq n}$ but not to $T_{\geq n}$.

7.2 Enlivenment Edges

The focus now is on a special kind of simple enlivenment that enriches an original continuation decision subtree $T_{\geq n}$ with initial decision node n and finite graph $(N_{\geq n}, E_{\geq n})$. As in Section 7.1, let M_n denote the set of all possible moves m along a directed edge $n \to n'_m$ in $E_{\geq n}$. Let M_n^+ denote a non-empty subset of M_n having the property that for each $m \in M_n^+$, an extra *enlivenment edge* $e_m^- \to e_m^+$ consisting of two extra nodes e_m^- and e_m^+ is introduced into the corresponding directed edge $n \to n_m$ of the original subtree $T_{\geq n}$. For each $m \in M_n^+$, this extra directed edge joins an earlier *pre-enlivenment node* e_m^- to a later *post-enlivenment node* e_m^+. Each pre-enlivenment node e_m^- appears in the middle of the unique corresponding edge $n \to n_m$, and is an event node where an uncertain deviation may or may not occur. Then, for each $m \in M_n^+$, the pre-enlivenment node e_m^- has two immediate successors that belong to $N_{+1}(e_m^-)$ in the enlivened continuation subtree $T_{\geq n}^+$. Specifically, at the new pre-enlivenment node event node e_m^-:

1. First, in case the horse lottery at e_m^- results in there being no deviation, the relevant immediate successor of e_m^- is the node $n_{e_m^-}$. Subsequently the continuation subtree $T^+_{\geq n_{e_m^-}}$ would be a copy of the subtree $T_{\geq n_m}$ that emanates from n_m in the original unenlivened tree T, unless there is a later enlivenment node in that subtree.

2. Second, in case the horse lottery at e_m^- results in a deviation, the relevant immediate successor of e_m^- is the *post-enlivenment node* e_m^+. This is the initial node of a finite continuation subtree $T^+_{\geq e_m^+}$ that gets appended to $T_{\geq n}$ at e_m^+ when $T_{\geq n}$ is enlivened to the new tree $T_{\geq n}^+$.

7.3 Enlivened Consequences and States

Before we can explore the implications of Bayesian rationality in the enlivened continuation subtree $T_{\geq n}^+$, we need to extend the domains of both the Bernoulli utility function $Y \ni y \mapsto u(y) \to \mathbb{R}$ which has been normalized to satisfy (51), and the finite state space S. For each move $m \in M_n^+$ at which the possibility of enlivenment is modelled, and so for each corresponding continuation subtree $T^+_{\geq e_m^+}$, there is a finite enriched domain Y_m^+ of consequences y and an enriched domain S_m^+ of possible states s that could feasibly result from following an appropriate path from e_m^+ to a terminal node. Then the *enlivened consequence domain* and *enlivened state space* are defined as the unions

$$Y^+ = \cup_{m \in M_n^+} Y_m^+ \quad \text{and} \quad S^+ = \cup_{m \in M_n^+} S_m^+ \tag{60}$$

over the finite collection of, respectively, all possible enriched consequence domains Y_m^+ and all possible enriched state spaces S_m^+. Let $\mathcal{T}^{S^+}(Y^+)$ denote the relevant domain of possible enlivened decision trees with the enlived consequence domain and state space. Let $L^{S^+}(Y^+)$ denote the relevant space of possible enlivened AA consequence lotteries.

Suppose that we now repeat the construction set out in Section 6.5 of the normalized Bernoulli utility function $Y \ni y \mapsto u(y) \to \mathbb{R}$ that satisfies (51), but for the extended domain Y^+ rather than the original domain Y. The result will be the enlivened Bernoulli utility function

$$Y^+ \ni y \mapsto u^+(y) \to \mathbb{R} \tag{61}$$

whose expectation represents the extended base preference relation $\succsim^+$ on the lottery domain $\Delta(Y^+)$. Moreover, because the two particular consequences $\underline{y}, \overline{y}$ used in the earlier normalization (51) are in Y^+ as well as in Y, we can the impose the obvious counterpart

$$u^+(\underline{y}) = \underline{u} \quad \text{and} \quad u^+(\overline{y}) = \overline{u} \tag{62}$$

of that earlier normalization. The resulting function (61) will then extend $Y \ni y \mapsto u(y) \to \mathbb{R}$ in the sense that the latter is the restriction to Y of the former.

Similarly, the construction of the subjective probabilities $\mathbb{P}(s)$ of each state $s \in S$ can be extended to a construction of the enlivened subjective probabilities $\mathbb{P}^+(s)$ of each state $s \in S^+$.

7.4 Bayesian Rationality in Simply Enlivened Trees

Let T denote any decision tree in the domain $\mathcal{T}^S(Y)$, Consider once again any enlivenment $T^+_{\geq n}$ of its continuation decision subtree $T_{\geq n}$ with, for any $m \in M_n = N^{+1}_{\geq n}$, its typical continuation subtrees $T^+_{\geq e^-_m}$ and $T^+_{\geq e^+_m}$. Note that both these subtrees belong to the enlivened domain $\mathcal{T}^{S^+}(Y^+)$ of finite decision trees. Our assumption that $T^+_{\geq n}$ is finite evidently implies that the set M_n is finite.

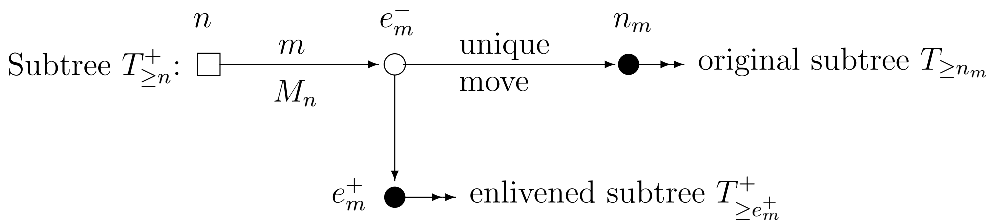

Figure 7: Influence Diagram for the Enlivened Subtree $T^+_{\geq n}$

Figure 7 shows the result of modifying the influence diagram of Figure 6 in order to accommodate the typical enlivenment edge $e_m^- \to e_m^+$ that appears in the enlivened continuation subtree $T^+_{\geq n}$, as described in Section 7.2, where:

- $M_n = N^{+1}_{\geq n}$ is the set of moves m that are feasible at the node n;
- $T^+_{\geq n}$ is truncated by detaching, for each $m \in M_n^+$, all of the nodes in both of the two alternative continuation subtrees $T_{\geq n_m}$ and $T^+_{\geq e_m^+}$.

Note that, in the case of an influence diagram derived from a truncated decision tree, such as that shown in Figure 7, there will be paths through the tree that, when truncated at their penultimate node, which is either n_m or e_m^+, end at the initial node of a variable continuation subtree which is equal to either the original subtree $T_{\geq n_m}$ or the enlivened subtree $T^+_{\geq e_m^+}$.

Consider also the evaluations $v(T_{\geq n})$ that were constructed in Section 6.8 by backward recurrence, using whichever of the four equations (54), (55), (57), and (59) applies to each successive node n. Bayesian rationality implies that exactly the same construction can be applied in order to reach enlivened evaluations $v^+(T^+_{\geq n'})$ at each node n' within the enlivened continuation subtree $T^+_{\geq e_m^+}$, for each $m \in M_n$.

Indeed, suppose we construct, for each $m \in M_n$ and corresponding event node e_m^- of $T^+_{\geq n}$, the evaluation $v^+(T^+_{\geq e_m^-})$ of the enlivened continuation decision tree $T^+_{\geq e_m^-}$. In order to do so, we apply the refined form of Bayesian rationality specified in Hammond (2022). It implies that, for each $m \in M_n$, there exists a unique subjective probability $\eta_m \in (0,1)$ that enlivenment occurs at the event node e_m^-, and so a probability $1 - \eta_m \in (0,1)$ that enlivenment does not occur at e_m^-. But the original continuation subtree $T_{\geq n_m}$ remains unenlivened, so its enlivened evaluation $v^+(T_{\geq n_m})$ equals its unenlivened evaluation $v(T_{\geq n_m})$. So, applying the obvious counterpart of rule (57) for the particular event node e_m^- of $T^+_{\geq n}$ tells us that

$$v^+(T^+_{\geq e_m^-}) = (1 - \eta_m)\, v(T_{\geq n_m}) + \eta_m\, v^+(T^+_{\geq e_m^+}) \tag{63}$$

Finally, in order to find the enlivened evaluation $v^+(T^+_{\geq n})$ of the entire enlivened continuation subtree $T^+_{\geq n}$, we apply the obvious modification for this enlivened tree of whichever of the three rules (55), (57), and (59) is relevant at node n, according as it is a chance, event, or decision node.

8 Trees with Terminal Evaluation Nodes

8.1 Difficulties in Evaluating Continuation Subtrees

The conditionally expected enlivened evaluation associated with entering any continuation subtree $T^+_{\geq n}$ of a simple enlivenment T^+ of the original decision tree $T \in \mathcal{T}^S(Y)$ is $v^+\big(T^+_{\geq n}\big)$. In principle, this can be calculated by following the procedure of backward recursion that was set out in Section 7.4.

Consider any node $m \in M'_n$ that, by definition, immediately succeeds n and has a potential enlivenment edge $e_m^- \to e_m^+$ inserted within the edge $n \to m$ of the original tree T. Then the backward recursion is trivial in the very special case when the enlivened subtree $T^+_{\geq e_m^+}$ shown in Figure 7 has only one terminal node n_m^t, whose specified enlivened consequence $\gamma^+(n_m^t)$ is a lottery over the enlivened consequence domain Y^+. This, of course, implies that $v^+\big(T^+_{\geq n}\big)$ is the enlivened expected utility $\mathbb{E}_{\gamma^+(n_m^t)}[u^+(y^+)]$ of this enlivened lottery consequence.

Carrying out the required backward recursion computation obviously becomes much more challenging, if not practically impossible, as at least some of the enlivened subtrees $T^+_{\geq e_m^+}$ shown in Figure 7 become more complicated. The set of moves $m \in M'_n$, each with its own subtree $T^+_{\geq e_m^+}$, may also become unmanageably large.

8.2 Recursive Enlivenments and Recursive Valuation

Even worse, suppose we go beyond the simply enlivened trees defined which were the focus of Section 7. Suppose instead we allow recursively enlivened trees that, by definition, themselves may be further enlivened. Then obviously it soon becomes practically impossible to consider all relevant possibilities.

In the first simple case, the continuation subtree $T^{+}_{\geq e^{+}_{m}}$ starting after the enlivening event at e^{-}_{m}, for any $m \in M'_{n}$, is always a decision tree in which no further enlivenment could occur. Yet in the account of the Homeric example in Section 3, the decision tree that Kirke described to Odysseus was enlivened in several successive stages. Indeed, there were subsequent enlivenings of decision trees that had emerged only after previous enlivenings. This illustrates the possibility of *recursive enlivenment.* And of course anybody who has ever played Chess at any level beyond the most basic also knows that recursive enlivenment affects a player's evolving understanding of the game being played, and so of how to evaluate any position in that game.

Even more troubling, as the game of Chess illustrates, the backward recursion procedure may already have become computationally infeasible when trying to find the evaluation function $v(T_{\geq n})$ for subtrees $T_{\geq n}$ of the original unenlivened tree T. Indeed, in Chess and Go, the number of relevant trees that ideally should be evaluated is finite, even if astronomical. Though in the case of Chess, as discussed in footnote 5 of Section 2.1, the relevant recursive calculation has been done completely for all positions that start with no more than 7 out of 32 pieces on the board, including of course both kings. Nevertheless, in real life, the collection of potentially relevant trees may be too large to constitute a set, let alone a measurable space on which one can define a probability measure. In general, however, as was already discussed in Sections 1.2 and 1.3, an agent's need to truncate the original unmanageable decision model so it becomes manageable could motivate that agent to consider enlivenments of that original model as its limitations become apparent later on.

8.3 Monte Carlo Tree Search

> Long before computer games became popular recreations, mathematicians viewed games as models of decision making. The general understanding of decisions, however, has been impeded

by the ambiguity of some of the basic components of game-tree search. In particular, the static evaluation function, or determination of a node's merit based on directly detectable features, has never been adequately defined. The expected-outcome model proposes that the appropriate value to assign a node is the expected value of a game's outcome given random play from that node on.
— from the abstract to Bruce Abramson's (1987) Ph.D. dissertation, eventually published as Abramson (1991).

Exploiting Abramson's key idea described in the quotation above helped to inspire a generation of computer programmes that:

1. in the case of Chess, led to the Stockfish software engine that would easily beat any human player over any sufficiently long run of games;

2. but in the case of Go, was unable to defeat the best human players.

Eventually, this generation of algorithms became superseded by AlphaZero, based on the kind of genetic algorithm that has become a key part of what has come to be known as "artificial intelligence". Such algorithms have proved far better at Chess than programs of the Stockfish generation, while finally becoming able to beat the best humans at Go. The key paper by Silver et al. (2018), however, reports that idea of Monte Carlo tree search (MCTS) remains part of AlphaZero.

8.4 Decision Trees with Terminal Evaluation Nodes

Algorithms based on MCTS suggest a key idea for simplifying the typical simply enlivened decision tree T^+, or its typical continuation subtree $T^+_{\geq n}$, that is represented by the influence diagram illustrated in Figure 7. This idea is to truncate the representative continuation subtree $T^+_{\geq e^+_m}$ of that diagram so it becomes simply one terminal node with, instead of a lottery consequence, an estimate $\tilde{v}^+ \left(T^+_{\geq e^+_m}\right)$ of its normatively appropriate evaluation $v^+ \left(T^+_{\geq e^+_m}\right)$. The result is the influence diagram shown in Figure 8, which is based on a truncation of T^+. Indeed, in essence this procedure reduces the enlivened decision tree T^+ to the trivial case mentioned in Section 8.1, with the difference that the single consequence lottery $\gamma^t(n^t_m)$ at each terminal node n^t_m gets replaced by a subjective evaluation $\tilde{v}^+_m \in \mathbb{R}$. So these results allow the construction of a final reduced enlivened tree in which each enlivened node e_m^+ is a terminal node, to which the subjective evaluation $\tilde{v}_m^+$ is attached as a real-valued generalized consequence.

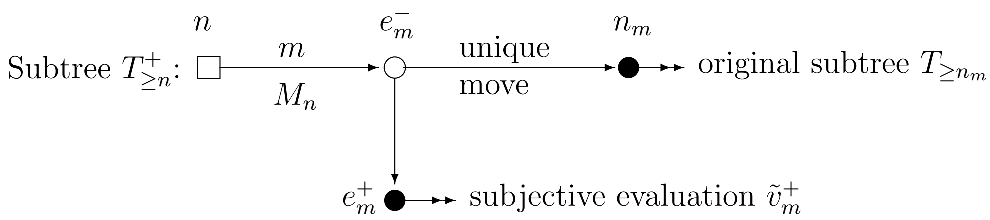

Figure 8: Influence Diagram for the Fully Reduced Subtree $T^+_{\geq n}$

Given any enlivened continuation subtree $T^+_{\geq n}$ that results from such recursive enlivenment, provided it is still finite, in principle one could still employ standard backward recursion in order to calculate the appropriate valuation attached to each continuation subtree $T^+_{\geq e_m^+}$ of $T^+_{\geq n}$. Yet after the reduction of tree-valued to real-valued states, the relevant real values should be those that emerge from any such calculation, including allowing for any possible recursion. In this way, using the subjective evaluations $\tilde{v}_m^+$ obviates entirely any need for any recursive calculation, except to the extent that such calculations may help to produce normatively superior beliefs about what subjective evaluations $\tilde{v}_m^+$ should be attached to at least some of the relevant continuation subtrees $T^+_{\geq n_m}$ of T^+.

8.5 Reformulation and Generalization

The difficulties noted in Section 8.2 make it practically imperative to truncate the agent's decision tree, at least in some cases. This creates the need to attach subjective evaluations to at least some of the terminal nodes of a truncated decision tree. Indeed, in the case of games as complicated as Chess or Go, there will be cases when the agent is forced to attach to every terminal node a subjective evaluation rather than a consequence in the form of a definite result. This helps explain why some form of Monte Carlo tree search algorithm has played such a key role in improving algorithms for playing Chess or Go.

So the time has come to reformulate and generalize our formulation of a finite decision tree in order to accommodate terminal nodes that have subjective evaluations rather than consequences attached. This can be accomplished by modifying the definition of decision tree in Section 6.6 to recognize, in addition to terminal nodes $n \in N^t$ that are mapped to Anscombe–Aumann consequence lotteries $\gamma(n) \in L^{S_{\geq n}}(Y)$, a new category N^e of *terminal evaluation nodes* n which, like other terminal nodes, also satisfy $N^{+1}_{\geq n} = \emptyset$, but which are mapped directly to subjective evaluations $v(n) \in \mathbb{R}$.

Let $\mathcal{T}^{*S}(Y^S)$ denote the domain of finite decision trees that results when the original domain $\mathcal{T}^S(Y^S)$ is expanded to include trees that admit this kind of terminal evaluation node. It is evidently equivalent to the domain $\mathcal{T}^S(Y^{*S})$ of finite decision trees that results when the original consequence domain Y is expanded to become $Y^* = Y \cup \mathbb{R}$ by allowing real-valued subjective evaluations to count as consequences.

Having extended the consequence domain in this way, the next step is to extend the construction in Section 6.8 of the evaluation $v(T_{\geq n})$ of each continuation subtree $T_{\geq n}$ of each decision tree $T \in \mathcal{T}^S(Y^S)$. Let $v^*(T_{\geq n})$ denote the result of the extended construction, which is an evaluation defined for each continuation subtree $T_{\geq n}$ of each decision tree $T \in \mathcal{T}^S(Y^{*S})$. These evaluations should still satisfy the four equations (54), (55), (57), and (59), though with each $v(T_{\geq n})$ and each $v(T_{\geq n'})$ replaced by $v^*(T_{\geq n})$ and $v^*(T_{\geq n'})$ respectively. The only new feature is that, at any terminal evaluation node n with subjective evaluation $v(n)$, the equation (54) should be replaced by the obvious

$$v^*(T_{\geq n}) = v(n) \tag{64}$$

8.6 Bayesian Rationality with Subjective Evaluations

The main conclusion of the paper can be stated in the following proposition, which establishes Bayesian rationality with subjective evaluations:

Proposition 1. *Suppose that base preferences defined on the enlivened domain $L^{S^+}(Y^+)$ of all possible Anscombe/Aumann consequence lotteries are both prerational and continuous w.r.t. objective probabilities for the domain $\mathcal{T}^S(Y^{*S})$ of all possible enlivened decision trees T that include terminal evaluation nodes. Then behaviour at any decision node of any tree $T \in \mathcal{T}^S(Y^{*S})$, together with the subjective evaluation $v^*(T_{\geq n})$ attached to any continuation subtree $T_{\geq n}$ of any tree $T \in \mathcal{T}^S(Y^{*S})$, together satisfy* (54) *at any terminal evaluation node n with subjective evaluation $v(n)$, as well as the four equations* (54), (55), (57), *and* (59), *though with each $v(T_{\geq n})$ and each $v(T_{\geq n'})$ replaced by $v^*(T_{\geq n})$ and $v^*(T_{\geq n'})$.*

8.7 Limitations of Bayesian Rationality

A more refined concept of rationality than mere Bayesian rationality would presumably require a rational agent to use a normatively justified Bernoulli utility function defined on the consequence domain, together with normatively justified subjective probabilities over uncertain states of the world. Thus, this richer concept of rationality would go beyond mere prerationality.

Prerationality in enlivened trees is no less limited. Indeed, rationality should require the agent's subjective evaluations at terminal evaluation nodes to be normatively justified, in addition to the agent's normalized Bernoulli utility function and subjective probabilities.

Of course, the agent's estimates of the relevant subjective probabilities and subjective evaluations may well be improved by procedures such as Monte Carlo tree search, as considered in Section 8.3, and/or techniques in management science such as scenario planning.

9 Extensions and Conclusions

9.1 Consequence Nodes and Menu Consequences

In Hammond and Troccoli Moretti (2024) we consider finite decision trees which, in addition to decision, chance, event, and terminal nodes, also have consequence nodes. Then the consequence of reaching a terminal node n is, instead of a single consequence lottery, a lottery over the stream of consequences and consequence lotteries that accumulate along the unique path through the decision tree that ends at node n. The implications of allowing consequence nodes are then routine unless the consequence of reaching any consequence node depends on the continuation decision tree emanating from that node. Even then, however, the result that continuity and prerationality together imply Bayesian rationality remains valid; all that changes is that the domain of relevant consequence streams becomes much richer.

For trees with consequence nodes, there is an obvious extension of the rules set out in the four equations (54), (55), (57), and (59) of Section 6.8. This extension treats the case when node n is a consequence node. With this extension, the characterization in Section 8.6 of Bayesian rationality with subjective evaluations still applies.

9.2 Reverse Bayesianism

"Reverse Bayesianism" was described in the series of joint papers by Karni and Vierø, as well as those by Vierø on her own, that were cited in Section 1.3. For the general finite decision trees considered here, reverse Bayesianism is the result saying that, if you condition the probabilities of different consequences in any enlivened decision tree T^+ on the event that enlivening does not occur, the result should be the corresponding probabilities in the original unenlivened decision tree T.

9.3 Transformative Experiences

The concept of transformative experiences arose in philosophy thanks to Paul (2014, 2015a, b, c), though similar ideas were discussed earlier in Ullmann-Margalit's (2006) paper on "big decisions" whose first characteristic (p. 158) is that they must be "transformative, or 'core affecting'". As Paul (2015b, p. 761) writes:

> a transformative experience is . . . both radically new to the agent and changes her in a deep and fundamental way; there are experiences such as becoming a parent, discovering a new faith, emigrating to a new country, or fighting in a war. . . .
>
> An *epistemically* transformative experience is an experience that teaches you something you could not have learned without having that kind of experience. Having that experience gives you new abilities to imagine, recognize, and cognitively model possible future experiences of that kind. A *personally* transformative experience changes you in some deep and personally fundamental way, for example, by changing your core personal preferences or by changing the way you understand your desires and the kind of person you take yourself to be. A *transformative experience*, then, is an experience that is both epistemically and personally transformative.
>
> The main problem with transformative decisions is that our standard decision models break down when we lack epistemic access to the subjective values for our possible outcomes.

As two philosophically prominent examples of transformative experiences, she considers the decisions of whether to become a vampire (!) or to have a child.[13] See also the discussion by Pettigrew (2015), Barnes (2015), and Campbell (2015), as well as Bykvist and Stefánsson (2017). The claim we make here is that any transformative experience involves a decision tree that becomes enlivened in case it includes the agent's decision of whether to undergo the experience.

9.4 A Possible Conclusion

The widely quoted aphorism due to the statistician George Box that was quoted at the head of Section 1.2 should remind us of the inevitable limitations that will occur in any formal model of all but the most trivial decision problems. Indeed, examples that extend in time from Homer's *Odyssey* to modern algorithms for playing chess demonstrate that, for an agent who has one or more decisions to make, the usefulness of any model is all too likely to be temporary. This paper has begun an investigation of what decision is rational for an agent who recognizes this fundamental difficulty. Specifically, it is argued that the best which such an agent can do, in effect, is to construct a probabilistic model of what the ultimate ex post evaluation of each possible decision could be, and then to maximize the expectation of this evaluation.

Acknowledgements: The research reported here was supported from 2007 to 2010 by a Marie Curie Excellence Chair funded by the European Commission under contract number MEXC-CT-2006-041121. Many thanks also to Kenneth Arrow, Ken Binmore, and Burkhard Schipper for their helpful suggestions, as well as to Marcus Miller and Joanne Yoong for enlivened discussions, while absolving them of all responsibility for my errors or omissions. Many thanks also for their patient attention and encouragement to audiences at: the GSB/Economics Department theory seminar at Stanford University (March 2007); the London School of Economics conference on preference change (May 2009); the University of Warwick (November 2010); CORE at the Université Catholique de Louvain (September 2011); the Hausdorff Center for Mathematics in Bonn (August 2013); the Università Cattolica del Sacro Cuore in Milan (October 2013); the conference on unawareness in game theory at the University of Queensland (February 2014); the conference on game theory and applications at the University of Bristol (May 2016); the

[13]Perhaps it would make a bit more sense to ask if it would be better to start practising vampirism, or occasionally adopting the feeding habits of a vampire bat.

Microeconomics Work in Progress online seminar at the University of Warwick (March 2021); the workshop on "Issues in Dynamic Decision Theory" at the University of Konstanz (July 2023); and the Centre d'Économie de la Sorbonne in Paris (November 2023). Finally, the prolonged process of producing the current version of this paper was greatly helped by discussions with Agustín Troccoli Moretti, my coauthor on a related project, as well as with Pablo Beker and Kirk Surgener, my colleagues at Warwick.

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