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Hugo Molina & Ao Wang

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Vertical Bargaining under Uncertain Retailer Responsiveness: A Structural Approach *

PRELIMINARY AND INCOMPLETE

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Hugo Molina[†] Ao Wang[‡]

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Abstract

We develop an empirical framework to analyze vertical relationships with manufacturer-retailer bargaining. Our key innovation is the introduction of a novel Nash-in-Nash bargaining model that incorporates uncertainty in retailers' pricing responses to wholesale prices. This model extends existing Nash-in-Nash frameworks by relaxing assumptions about the timing of wholesale and retail price setting. We show that our model can be microfounded by a two-stage noncooperative game with delegated negotiations. We propose a two-step strategy that separably identifies bargaining and responsiveness parameters and implies a Generalized Method of Moments estimation procedure.

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[†]Université Paris-Saclay, INRAE, UR ALISS, 94205, Ivry-sur-Seine, France; hugo.molina@inrae.fr; https://sites.google.com/site/hugomolinaresearch/

[‡]University of Warwick, UK (ao.wang@warwick.ac.uk).

1 Introduction

In many industries, firms deal with retailers (or intermediaries) to distribute their products to consumers. Prominent examples include the food retail sector, the pharmaceutical industry, the cable television market, the smartphone industry, the car market, or the health care sector. These vertical structures often exhibit multilateral contracting with externalities, where competing upstream firms sell their products through the same competing downstream retailers.[1] Furthermore, the terms of trade are typically negotiated rather than being dictated by either the upstream or downstream firms. Over the past decade, the "Nash equilibrium in Nash bargains" (Nash-in-Nash) pioneered by Horn and Wolinsky (1988) has become a workhorse approach to model such complex vertical relations in a tractable way (Collard-Wexler, Gowrisankaran and Lee, 2019). This approach has not only influenced the literature on vertical contracting, but has also played a key role in a number of recent high-profile merger cases (Wright and Yun, 2020; Carlton, 2020).[2] In this framework, upstream and downstream firms engage in bilateral negotiations over wholesale prices of products, which serve as inputs in the retailers' pricing decisions. Existing works typically model the timing of these stages as either sequential or simultaneous. The sequential timing assumes that bargaining occurs before retail pricing decisions, whereas the simultaneous timing considers that both stages take place at the same time.

Researchers often lack direct observation of the true timing and thus make an assumption based on institutional knowledge or computational convenience (Lee, Whinston and Yurukoglu, 2021). However, recent research emphasizes that neither timing assumption is entirely accurate and this modelling choice cannot be settled by theoretical arguments alone, making it ultimately an empirical question (Crawford et al., 2018; Rogerson, 2020).[3] More importantly, the choice between sequential and simultaneous timing can have profound implications for a number of antitrust and regulatory issues such as merger control (see, e.g., Moresi, 2020; Bonnet, Bouamra-Mechemache and Molina, forthcoming).[4] Hence, misspecifying

[1]Contracting externalities arise from competition on both sides of the market, as the surplus from an agreement between an upstream and a downstream firm depends on the agreements formed by other pairs of upstream and downstream firms.

[2]See, e.g., Comcast/NBC in 2011 (Rogerson, 2014), Anthem/Cigna in 2017 (Sheu and Taragin, 2021), and AT&T/Time Warner in 2018 (Shapiro, 2021; Carlton et al., 2022).

[3]Rogerson (2020) states that: "Whether and to what extent firms in the real world, which have limited information and face costs of gathering and analyzing information, account for such effects (impact of bargaining outcomes on downstream price setting) is an empirical issue."

[4]For instance, Moresi (2020) demonstrates that a vertical merger lowers downstream prices under simultaneous timing, whereas it increases downstream prices under sequential timing when downstream firms are close competitors (see also Panhans, 2024).

the timing assumption may result in erroneous predictions and misleading policy recommendations.

In this article, we develop a framework of vertical relations with Nash-in-Nash bargaining that relaxes the timing assumption between wholesale and retail price setting decisions. Our key innovation is the introduction of retailer responsiveness parameters. These parameters measure the likelihood that retailers adjust their retail pricing decisions in response to out-of-equilibrium events during negotiations with manufacturers. Notably, the sequential and simultaneous timing assumptions correspond respectively to the responsive and unresponsive configurations of the responsiveness parameters, which are to be estimated from the data.[5] The uncertainty in the unresponsiveness preventing a retailer from responding to unexpected bargaining outcomes can arise from multiple sources. For instance, adjusting retail prices may involve non-trivial physical and managerial costs (e.g., Levy et al., 1997; Zbaracki et al., 2004). Alternatively, coordination failures within a multidivisionnal retailer may arise due to internal communication errors (e.g., Marschak and Radner, 1972; Williamson, 1981; Güth, Müller and Spiegel, 2006).[6] Building on Rey and Vergé (2020), we offer a noncooperative microfoundation for our Nash-in-Nash bargaining model with uncertainty in retailers' responsiveness, which provides support for its use in the analysis of vertically related markets.

We develop a two-step strategy to identify our parameters of interest including bargaining, responsiveness, and marginal cost parameters. This strategy implies a Generalized Method of Moments (GMM) estimation procedure. In the first step, given bargaining and responsiveness parameters along with wholesale prices, we invert the first-order conditions (FOCs) of the Nash-in-Nash bargaining solution to recover upstream product margins and, in turn, product-level marginal costs of production.[7] We show that these FOCs are linear in the upstream margins, and that invertibility hinges on the non-zero determinant of this linear system. We argue that such a condition generically holds and formalize this argument for sufficiently smooth demand systems (e.g., mixed-logit, probit, GEV). In the second step, we use instrumental variables (orthogonal to the unobserved marginal cost

[5] That is, there is no uncertainty about retailers' ability (responsive) or inability (unresponsive) to adjust their retail prices in response to out-of-equilibrium events during negotiations with manufacturers.

[6] That is, the retail pricing division of a retailer may simply fail to observe the outcomes negotiated by its bargaining division (e.g., van Damme and Hurkens, 1997; Güth, Müller and Spiegel, 2006).

[7] Wholesale prices can be either directly observed (e.g., Noton and Elberg, 2018) or inferred from the data. In the latter case (more common in existing works), the standard approach is to infer wholesale prices from estimates of consumer demand and the set of equations characterizing necessary conditions for a Nash equilibrium in retail pricess (e.g., Villas-Boas, 2007).

factors) to construct moment conditions and identify the model parameters. A key challenge is separately identifying the bargaining, responsiveness, and marginal cost parameters, as they jointly determine equilibrium wholesale prices. We show that exogenous demand shifters and rotators, as well as cost shifters, can provide identification power. As a result, standard instruments used in the literature (e.g., BLP-type instruments) remain valid. We conduct Monte Carlo simulations to provide supportive evidence for the validity of our approach.

Related literature. The present article contributes to the recent empirical literature on buyer-seller bargaining in vertical markets (see Lee, Whinston and Yurukoglu, 2021, for a comprehensive survey). Since Draganska, Klapper and Villas-Boas (2010), many articles have adopted the simultaneous timing assumption, either to account for retail price stickiness or to simplify the estimation of the Nash-in-Nash bargaining model (e.g., Ho and Lee, 2017; Crawford et al., 2018; Noton and Elberg, 2018; Sheu and Taragin, 2021). In contrast, another strand of the literature has opted for the sequential timing assumption on the ground that retail prices respond to changes in wholesale prices (e.g., Crawford and Yurukoglu, 2012; Yang, 2020; Bonnet, Bouamra-Mechemache and Molina, forthcoming).[8] We show, however, that this timing assumption is not innocuous and may have important consequences for estimates, counterfactual simulations, and policy implications (see also Moresi, 2020; Panhans, 2024; Bonnet, Bouamra-Mechemache and Molina, forthcoming). Our paper proposes a microfounded Nash-in-Nash bargaining model with uncertainty about retailers' responsiveness in retail pricing decisions that generalizes the timing assumption between wholesale and retail price settings. To the best of our knowledge, we are the first to make progress in this direction.

We also contribute to the literature by proposing a unified econometric framework and formally establishing the econometric properties of Nash-in-Nash bargaining models. Our sources of identification for the bargaining and responsiveness parameters relate to the research on firm conduct in differentiated product markets (e.g., Berry and Haile, 2014). Specifically, we exploit the variation in market conditions that are excluded from marginal costs to discriminate between models

[8] Dubois and Særthe (2020) also consider an empirical model of vertical relations with sequential timing, where downstream firms adjust their product assortment (instead of retail prices) in response to wholesale prices determined through the Nash-in-Nash bargaining solution. More generally, beyond buyer-seller bargaining, sequential timing has been widely used in empirical studies on vertical contracting (e.g., Brenkers and Verboven, 2006; Villas-Boas, 2007; Ho, 2009; Bonnet and Dubois, 2010; Goldberg and Hellerstein, 2013; Fan and Yang, 2020; Hristakeva, 2022).

of vertical relations that differ in both the distribution of bargaining power in the supply chain and the timing of wholesale and retail price setting.[9] As our framework encompasses existing Nash-in-Nash bargaining models (sequential or simultaneous timing), one can apply our strategy to solve the identification of the bargaining parameters in these models, a well-known challenge in this literature.[10]

Finally, our paper relates to the recent works in empirical industrial organization that explore the implications of organizational frictions for market outcomes such as prices and welfare (e.g., Crawford et al., 2018; Hortaçsu et al., 2024). Our Nash-in-Nash bargaining model incorporates uncertainty in retailers' responsiveness that measure their abilities of adjusting the retail pricing decisions in response to out-of-equilibrium events in the bargaining. The uncertainty can arise from multiple sources, such as non-trivial price adjustment costs (e.g., Levy et al., 1997; Aguirregabiria, 1999; Zbaracki et al., 2004; Reis, 2006; Goldberg and Hellerstein, 2013; Arcidiacono et al., 2020) and possible coordination failures (e.g., Marschak and Radner, 1972; Williamson, 1981; Güth, Müller and Spiegel, 2006). We do not take a specific stance on such sources and do not seek to endogenize them (which is an interesting avenue for future research).

The remainder of this article is organized as follows. Section 2 introduces our model of vertical relations with Nash-in-Nash bargaining and intra-retailer frictions. Section 3 studies the identification of our model, including a discussion on the sources of exogenous variations that help identifying the structural parameters. Section 4 presents Monte Carlo simulations. Section 5 concludes.

2 Model

2.1 Overview

Consider a market, indexed by t, where M upstream manufacturers, indexed by $m = 1, ..., M$, interact with R downstream retailers, indexed by $r = 1, ..., R$, to sell their products to consumers. To simplify the exposition, we assume that each consumer on the market chooses among a set $\mathscr{J} \equiv \{0, 1, \dots, J\}$ of differentiated

[9] The logic of using exogenous variation in firm markups to empirically distinguish between models of competition was pioneered by Bresnahan (1982) and extended by Berry and Haile (2014) to differentiated product markets (see Gandhi and Nevo, 2021, for a literature review). Aside from the buyer-seller bargaining literature, this approach has also been applied in vertical markets to test different forms vertical conduct (e.g., Brenkers and Verboven, 2006; Villas-Boas, 2007; Bonnet and Dubois, 2010; Bonnet et al., 2013; Duarte et al., 2024).

[10] As noted by Gowrisankaran, Nevo and Town (2015): "An interesting extension to explore in future work is formal identification of the bargaining weights." See also Lee, Whinston and Yurukoglu (2021) for an informal discussion of this identification issue.

products. It is straightforward to extend the arguments to the situation with a market-varying set of products. We use $\mathscr{J}_m$ to denote the set of products owned by manufacturer m, and $\mathscr{J}_r$ the set of products distributed by retailer r.

We consider profit-maximizing manufacturers and retailers. Define respectively the profit function of manufacturer m and retailer r in market t as follows:

$$\pi_{mt}((w_{jt})_{j\in\mathscr{J}_m};\mathbf{p}_t) \equiv \sum_{j\in\mathscr{J}_m} (w_{jt}-\mu_{jt})\, q_{jt}(\mathbf{p}_t) \tag{1a}$$

$$\pi_{rt}((w_{jt}+c_{jt})_{j\in\mathscr{J}_r};\mathbf{p}_t) \equiv \sum_{j\in\mathscr{J}_r} (p_{jt}-w_{jt}-c_{jt})\, q_{jt}(\mathbf{p}_t) \tag{1b}$$

where w_{jt} is the wholesale price of product j in market t, p_{jt} is its retail price, and $\mathbf{p}_t \equiv (p_{1t},\ldots,p_{Jt})$denotes the vector of retail prices. The terms μ_{jt} and c_{jt} are respectively the constant marginal cost of production and distribution for product j in market t.[11] Without loss of generality, we absorb c_{jt} in w_{jt} and μ_{jt} and normalize $c_{jt}=0$ for all $j\in\mathscr{J}$. Finally, $q_{jt}(\mathbf{p}_t)>0$ is the demand for product j in market t at prices $\mathbf{p}_t$. We stack $q_{jt}(\mathbf{p}_t)$ in a vector-valued function $\mathbf{q}_t(\mathbf{p}_t)\equiv(q_{1t}(\mathbf{p}_t),\ldots,q_{Jt}(\mathbf{p}_t))$.

Manufacturers and retailers make their decisions according to the following timing. In the first stage, manufacturers and retailers engage in simultaneous and secret bilateral negotiations to determine wholesale prices of products. In the second stage, each retailer r learns its state of nature, either "responsive" or "unresponsive". With probability ρ_r, retailer r is in a "responsive" state., where it can optimally adjust its retail prices upon observing an out-of-equilibrium bargaining outcome. Conversely, with probability $1-\rho_r$, retailer r is in a "unresponsive" state, where it is unable to optimally adjust its retail prices in response to unexpected bargaining outcomes. In the third stage, given contract secrecy and realized retailer-specific states, retailers engage in a simultaneous retail price competition.[12]

We use a new Nash-in-Nash bargaining model that accounts for the uncertainty about retailers' responsiveness states to determine bargaining outcomes in the first stage. Our framework generalizes existing Nash-in-Nash solutions in vertical relations that assume that a specific state of nature arises with certainty.

[11] We consider the case of constant marginal cost of production commonly used in empirical studies. The scenario of (dis)economies of scale is left for future research.

[12] Contract secrecy here implies that each retailer sets the retail prices of its products without observing the trading terms negotiated by its rivals in the first stage. This information structure is commonly referred to as "interim unobservability" in the vertical contracting literature (see, e.g., Rey and Vergé, 2004; Gaudin, 2019; Rey and Vergé, 2020). In this information structure, whether or not a retailer observes other retailers' realized states of responsiveness is irrelevant because of the unobservability of others' trading terms in the first stage.

Several articles consider sequential timing—wholesale prices are set before retail prices—which is equivalent to assuming that the "responsive"' state arises with probability $\rho_r = 1$ for all r (e.g., Crawford and Yurukoglu, 2012; Bonnet, Bouamra-Mechemache and Molina, forthcoming). Conversely, other articles in the literature adopt simultaneous timing—wholesale and retail prices are determined at the same time— corresponding to the configuration in which the "unresponsive" state arises with probability $1 - \rho_r = 1$ for all r (e.g., Draganska, Klapper and Villas-Boas, 2010; Ho and Lee, 2017; Crawford et al., 2018). To offer support for the reasonableness of our surplus division in the vertical chain, we provide a noncooperative formulation of our Nash-in-Nash bargaining solution. In Appendix A, we demonstrate that our solution concept coincides with the sequential equilibrium of a variant of the noncooperative game developed by Rey and Vergé (2020).

In what follows, we proceed in reverse order of timing to formalize each stage.

2.2 Stage 3: Retail price competition

Let $\mathbf{w}_{rt}$ and $\mathbf{w}_{-rt}$ denote the vectors of wholesale prices for the products distributed by retailer r and its rivals, respectively. Similarly, let $\mathbf{p}_{rt}$ and $\mathbf{p}_{-rt}$ represent the vectors of retail prices for the products distributed by retailer r and its rivals, respectively. Due to contract secrecy, retailer r sets $\mathbf{p}_{rt}$ to maximize profit, holding the belief that its rivals pay the equilibrium wholesale prices to manufacturers.[13] This retail pricing decision is also made conditional on the realized state of nature.

Formally, consider first the case where retailer r is in the "responsive" state, implying that it can optimally respond to any unexpected outcomes that occurred during its negotiations with manufacturers in stage 1. Thus, after observing $\mathbf{w}_{rt}$ and holding the belief that $\mathbf{w}_{-rt} = \mathbf{w}^*_{-rt}$, retailer r sets $\mathbf{p}_{rt}$ such that:

$$\mathbf{p}_{rt}(\mathbf{w}_{rt}; \mathbf{p}^*_{-rt}) \equiv \underset{\{p_{jt}\}_{j\in\mathcal{J}_r}}{\operatorname{argmax}} \sum_{j\in\mathcal{J}_r} (p_{jt} - w_{jt} - c_{jt}) q_{jt}(\mathbf{p}_{rt}, \mathbf{p}^*_{-rt}) \tag{2}$$

where the * superscripts denote the equilibrium values.

Consider now the case where retailer r is in the "unresponsive" state. As it cannot adjust its retail prices in response to unexpected bargaining outcomes, it is as if retailer r were unable to observe $\mathbf{w}_{rt}$ (in addition to $\mathbf{w}_{-rt}$). Hence, holding

[13] As retailer r receives no information about its rivals' bargaining outcomes (even when it observes an out-of-equilibrium event during its negotiations with manufacturers), there is no reason to revise its beliefs regarding $\mathbf{w}_{-rt}$ compared to those held during the bargaining stage (consistent with the "no-signaling-what-you-don't-know" condition of Fudenberg and Tirole, 1991).

the belief that $\mathbf{w}_{rt} = \mathbf{w}^*_{rt}$ and $\mathbf{w}_{-rt} = \mathbf{w}^*_{-rt}$, retailer r sets $\mathbf{p}_{rt}$ such that:

$$\mathbf{p}_{rt}(\mathbf{w}^*_{rt}; \mathbf{p}^*_{-rt}) \equiv \underset{\{p_{jt}\}_{j\in\mathscr{J}_r}}{\operatorname{argmax}} \sum_{j\in\mathscr{J}_r} (p_{jt} - w^*_{jt} - c_{jt}) q_{jt}(\mathbf{p}_{rt}, \mathbf{p}^*_{-rt}) \tag{3}$$

Two remarks are in order. First, due to contract secrecy, retailer's pricing behavior does not depend on other retailers' realized responsiveness states (see also footnote 12). Second, the researcher could recover retailers' price-cost margins (and marginal costs) from (2) and (3). These identified margins will be used to identify and estimate the parameters of the Nash-in-Nash bargaining in the first stage.[14] The standard assumption in empirical work is that an equilibrium is played in the data (see Berry, Levinsohn and Pakes, 1995, and the ensuing literature). Consequently, (2) and (3), as well as their corresponding first-order conditions, become observably equivalent when wholesale prices are at equilibrium:

$$q_{jt}(\mathbf{p}^*_t) \ + \sum_{k\in\mathscr{J}_{r(j)}} (p^*_{kt} - w^*_{kt} - c_{kt}) \, \frac{\partial q_{kt}}{\partial p_{jt}} = 0 \quad \forall j \in \mathscr{J}. \tag{4}$$

In Appendix B, we describe how to recover retailers' margins and marginal costs from (4).

2.3 Stage 2: Realization of retailers' responsiveness states

Prior to the retail price competition, each retailer learns its responsiveness state. With probability ρ_r, retailer r is in a "responsive" state, where it can optimally adjust its retail prices in response to any out-of-equilibrium outcome that may have occured during its negotiations with manufacturers. Conversely, with probability $1 - \rho_r$, retailer r finds itself in an "unresponsive" state, where it is unable to adjust its retail prices following unexpected bargaining outcomes. We do not impose restrictions on the correlation between retailers' state realizations.

The retailer-specific parameter ρ_r is interpreted as the likelihood that retailer r's pricing decisions are responsive to deviations in the bargaining outcomes for its products. We take a fairly agnostic stance about the sources of uncertainty surrounding the retailer's responsiveness in pricing decisions. It may stem from non-trivial price adjustment costs. For instance, Levy et al. (1997) document that physical price adjustment costs (e.g., labor costs of changing shelf prices)

[14] In some situations, the researcher directly observes retailers' markups (e.g., the Dominick's dataset in Goldberg and Hellerstein (2013)) and does not need to back out them from the pricing game in the third stage. However, one may still need (2) and (3) to characterize retailers' pricing in the out-of-equilibrium events in the Nash-in-Nash bargaining of the first stage.

account for 0.7% of revenues at five multistore supermarket chains ($105,887 per store per year). Additionally, Zbaracki et al. (2004) provide evidence that managerial costs associated with gathering information, making pricing decisions, and communicating these decisions within the retailer can be up to six times higher than physical costs. They further highlight that costs of renegotiating prices can reach up to twenty times the physical costs. Such costs are relevant in business-to-business contexts where retail prices are determined through bargaining, as in insurer-employer relationships (Ho and Lee, 2017). Besides, intra-retailer information frictions may also induce the uncertainty about the responsiveness. In a multidivisional retailer, coordination failures could arise between the bargaining division, in charge of negotiating wholesale prices, and the pricing division, which sets retail prices, due internal communication errors.[15] In this context, parameter ρ_r can be seen as an "all-or-nothing" signal technology: before setting retail prices, the pricing division either receives the (correct) outcomes negotiated by the bargaining division with a certain probability or no information at all.[16] We refer to Rogerson (2020) for a discussion on other types of within-firm frictions.

2.4 Stage 1: Manufacturer-retailer bargaining

In stage 1, manufacturers and retailers engage in bilateral negotiations over wholesale prices of products that account for the uncertain realizations of retailers' responsiveness states in stage 2. Formally, taking other wholesale prices as given, the equilibrium wholesale price of product $j \in \mathcal{J}_m \cap \mathcal{J}_r$ solves the following Nash bargaining problem:

$$w_{jt}^* \equiv \underset{w_{jt}}{\operatorname{argmax}} \ \left(\pi_{mt}(\rho_r) - \pi_{mt}^{-j}(\rho_r)\right)^{\lambda_j} \left(\pi_{rt}(\rho_r) - \pi_{rt}^{-j}(\rho_r)\right)^{1-\lambda_j} \tag{5}$$

where parameter λ_j is the bargaining weight of manufacturer m vis-à-vis retailer r in the negotiation over w_{jt}. The terms $\pi_{mt}(\rho_r)$ and $\pi_{rt}(\rho_r)$ represent respectively the profit that manufacturer m and retailer r get if an agreement is reached:

$$\pi_{mt}(\rho_r) = \rho_r \pi_{mt}(w_{jt}, \mathbf{w}_{mt,-j}^*, \mathbf{p}_{rt}(w_{jt}, \mathbf{w}_{rt,-j}^*; \mathbf{p}_{-rt}^*), \mathbf{p}_{-rt}^*) + (1-\rho_r)\pi_{mt}(w_{jt}, \mathbf{w}_{mt,-j}^*, \mathbf{p}_t^*)$$
$$\pi_{rt}(\rho_r) = \rho_r \pi_{rt}(w_{jt}, \mathbf{w}_{rt,-j}^*, \mathbf{p}_{rt}(w_{jt}, \mathbf{w}_{rt,-j}^*; \mathbf{p}_{-rt}^*), \mathbf{p}_{-rt}^*) + (1-\rho_r)\pi_{rt}(w_{jt}, \mathbf{w}_{rt,-j}^*, \mathbf{p}_t^*)$$

[15] Hortaçsu et al. (2024) provide direct evidence of pricing frictions due to miscoordination between different divisions within an airline company.

[16] This type of signal technology is employed in papers such as Rubinstein (1989), Laffont and Tirole (1993), van Damme and Hurkens (1997), and Güth, Müller and Spiegel (2006).

where $\mathbf{w}^*_{mt,-j}$ is the vector of equilibrium wholesale prices for all products owned by manufacturer m except product $j \in \mathscr{J}_m$, $\mathbf{w}^*_{rt,-j}$ is the vector of equilibrium wholesale prices for all products distributed by retailer r except product $j \in \mathscr{J}_r$, and $\mathbf{p}^*_t$ denotes the vector of equilibrium retail prices. Finally, the terms $\pi^{-j}_{mt}(\rho_r)$ and $\pi^{-j}_{rt}(\rho_r)$ denotes respectively the status quo payoffs of manufacturer m and retailer r in the event of bilateral disagreement, which are given by:

$$\pi^{-j}_{mt}(\rho_r) = \rho_r \pi^{-j}_{mt}(\mathbf{w}^*_{mt,-j}, \mathbf{p}^{-j}_{rt}(\mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) + (1-\rho_r)\pi^{-j}_{mt}(\mathbf{w}^*_{mt,-j}, \mathbf{p}^*_{-jt})$$
$$\pi^{-j}_{rt}(\rho_r) = \rho_r \pi^{-j}_{rt}(\mathbf{w}^*_{rt,-j}, \mathbf{p}^{-j}_{rt}(\mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) + (1-\rho_r)\pi^{-j}_{rt}(\mathbf{w}^*_{rt,-j}, \mathbf{p}^*_{-jt})$$

where $\mathbf{p}^*_{-jt}$ is the vector of equilibrium retail prices for all products except j, and $\mathbf{p}^{-j}_{rt}(\mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt})$ corresponds to the vector of out-of-equilibrium retail prices set by retailer r when product j is no longer offered in market t and other retailers' prices are $\mathbf{p}^*_{-rt}$.

We derive the first-order conditions of (5) that characterize the surplus division in market t: for $j \in \mathscr{J}$ with $j \in \mathscr{J}_m \cap \mathscr{J}_r$,

$$\begin{aligned}\sum_{k\in\mathscr{J}_m} (w^*_{kt} - \mu_{kt}) \Bigg[\frac{1-\lambda_j}{\lambda_j} \Big(q^*_{kt} - \rho_r q^{-j}_{kt}(\mathbf{p}^{-j}_{rt}(\mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) - (1-\rho_r) q^{-j}_{kt}(\mathbf{p}^*_{-jt}) \Big) q^*_{jt} \\ - \rho_r \sum_{h\in\mathscr{J}_r} \frac{\partial q_{kt}}{\partial p_{ht}} \frac{\partial p_{ht}}{\partial w_{jt}} \Big(\pi_{rt}(\mathbf{w}^*_{rt}; \mathbf{p}^*_t) - \pi^{-j}_{rt}(\rho_r) \Big) \Bigg] = \Big(\pi_{rt}(\mathbf{w}^*_{rt}; \mathbf{p}^*_t) - \pi^{-j}_{rt}(\rho_r) \Big) q^*_{jt},\end{aligned} \tag{6}$$

where q^{-j}_{kt} denotes the out-of-equilibrium demand for product k when product j is not offered in market t. The formulas and computational details can be found in Appendix C.

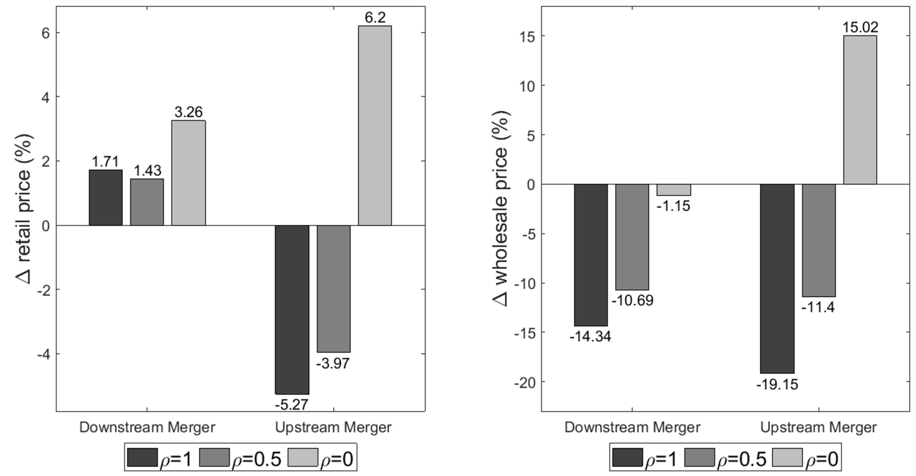
As previously outlined, ρ_r in (6) measure the likelihood that retailer r will be in the responsive state and adjust its retail pricing decisions in stage 3 in response to out-of-equilibrium events during negotiations with manufacturers. When $\rho_r = 1$ (responsiveness), both retailer r and its manufacturers anticipate that the outcomes of their wholesale negotiations will fully influence r's retail pricing decisions. In this case, (6) boils down to the first-order conditions of the Nash-in-Nash model with sequential timing (e.g., Crawford and Yurukoglu, 2012; Bonnet, Bouamra-Mechemache and Molina, forthcoming). When instead $\rho_r = 0$ (unresponsiveness), both retailer r and its manufacturers anticipate that r's retail pricing decisions will remain unaffected by unexpected bargaining outcomes. Therefore, (6) reduces to the first-order conditions of the Nash-in-Nash model with simultaneous timing (e.g., Draganska, Klapper and Villas-Boas, 2010; Ho and Lee, 2017). In most applied work, however, the timing of wholesale and retail price setting remains unobserved and "certainly neither timing assumption is completely accurate" (Crawford et al., 2018, page 911). Rather than taking a specific stance on the binary timing choice, (6) allow for more general form of conducts, i.e., neither the responsive nor the unresponsive state may arise with certainty ($0 \leq \rho_r \leq 1$ for $r = 1, \ldots, R$). The researcher can then identify and estimate the conduct (parameters ρ_r, $r = 1, ..., R$) from the data, mitigating misspecification risks. Finally, responsiveness parameters ρ_r have a different nature from the bargaining parameters λ_j. We illustrate their distinctive theoretical interpretations in the noncooperative game that microfounds our Nash-in-Nash bargaining (see Appendix A). In Section 3.2, we discuss their separable identification.

The role of retailers' responsiveness on counterfactual outcomes. In addition to the surplus division in the vertical chain, retailers' responsiveness in retail pricing decisions may also significantly affect counterfactual analysis. We now perform two Monte Carlo exercises to shed light on this point. In both exercises, we use a setting of bilateral oligopoly with $M = 3$, $R = 3$, and $J = 9$ (i.e., every manufacturer sells its brand to every retailer, interlocking relationships). In the first one, we analyze the model's predictions of an upstream merger and a downstream merger under three different parameter configurations: $\rho_r = \rho \in \{0, 0.5, 1\}$ for all $r = 1, 2, 3$. The model's other parameters, e.g., parameters of demand, bargaining, and costs, are the same. We refer to Section 4 for more details on the data generating process.

The results are depicted in Figure 1. The left panel shows the average percentage change in retail prices, and the right panel displays the average percentage change in wholesale prices for the merging firms. We find that the merger predictions differ remarkably across the three configurations of ρ, which aligns with the discrepancies between simultaneous and sequential timing in merger analysis found by some recent works (e.g., Moresi, 2020; Panhans, 2024; Bonnet, Bouamra-Mechemache and Molina, forthcoming). In the downstream merger case, the predicted average decrease in wholesale prices for the merging firms' products is much larger when $\rho = 1$ (-14.34%) than when $\rho = 0$ (-1.15%). This discrepancy implies that the relative increase in retail prices when $\rho = 1$ ($+1.71\%$) is half as large as when $\rho = 0$ ($+3.26\%$). The differences are even more striking in the upstream merger case. While the wholesale prices for the merging firms' products increase on average by 15.02% when $\rho = 0$, they decrease by -19.15% when $\rho = 1$. We obtain similar predictions for retail prices, which rise by 6.2% when $\rho = 0$ and fall by -5.27% when $\rho = 1$.

In the second exercise, we first generate the data under $\rho = 0$ and $\rho = 1$. Then, using the data generated under each case, we estimate the bargaining and

Figure 1: Merger predictions under different configurations of ρ

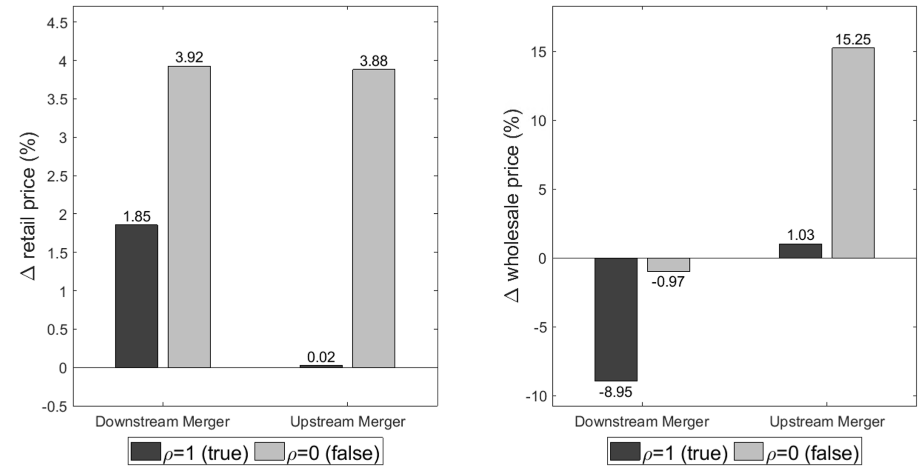


Notes: In the pre-merger market configuration, there are $M = 3$ manufacturers, $R = 3$ retailers, and $J = 9$ products (interlocking relationships). We simulate a mixed logit demand model with in which the indirect utility $U_{ij} = \beta_0 + \beta_i x_j - \alpha_i p_j + \xi_j + \epsilon_{ij}$, and the marginal cost function for product j is given by $\mu_j = \kappa_0 + \kappa_v v_j + \kappa_x x_j + \omega_j$. We draw (ξ_j, ω_j) from a mean-zero bivariate normal distribution with variances $\sigma_\xi^2 = \sigma_\omega^2 = 2$ and covariance $\sigma_{\xi\omega} = 0.1$, and x_j, v_j are i.i.d. according to $U(0,1)$. We set $\beta_0 = -3$, $\beta_i \sim \mathcal{N}(4,1)$, $\alpha_i \sim \ln \mathcal{N}(0.49, 0.64)$, and $\kappa_0 = \kappa_v = \kappa_x = 1$. We also set $\lambda = 0.5$ for each manufacturer-retailer pair. Given this configuration, we compute the retail prices by solving the equilibrium of our model (i.e., the bargaining equilibrium and the downstream price equilibrium described in (6) and (4), respectively). We consider mergers between two competing firms in the downstream market (bars corresponding to "Downstream Merger") and the upstream merger.(bars corresponding to "Upstream Merger"). The y-axis shows the average percentage change in retail (left figure) and wholesale prices (right figure) for the merging firms' products.

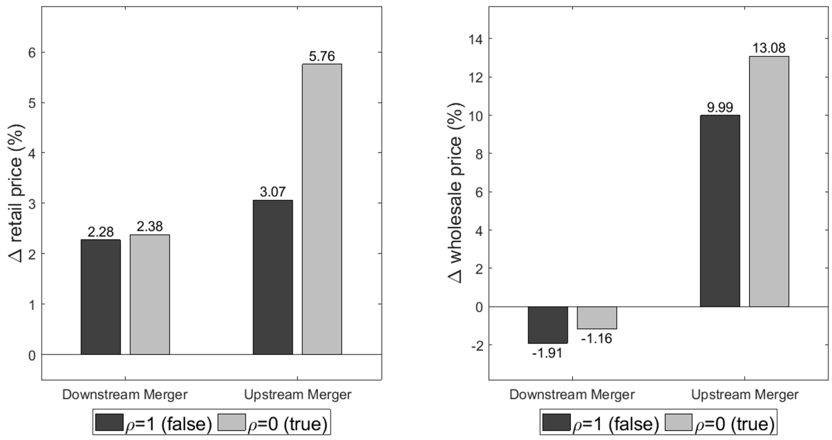
marginal cost parameters under both the true specification and the false one. Finally, we use the parameter estimates from both the true and false specifications to simulate upstream and downstream mergers as in our first exercise. Figure 2 presents the merger simulation results, with panel (a) showing the case where the true specification is $\rho = 1$, and panel (b) the case where the true specification is $\rho = 0$. We find important biases due to misspecifying the value of ρ. In the upstream merger case, the simulation under the misspecified model systematically overestimates (resp. underestimates) the wholesale and retail price increases when the true specification is $\rho = 1$ (resp. $\rho = 0$). For instance, panel (a) shows a larger wholesale price increase under the misspecified model (+15.25%) than under the true model (+1.03%). Similarly, the simulation results indicate a larger retail price increase under the misspecified model (+3.88%) compared to the true model where retail prices are nearly unaffected by the merger (+0.02%). These misspecification biaises also arise in the downstream merger case, particularly when the true model specification is $\rho = 1$ (panel (a)). In this scenario, the merger simulation suggests a lower wholesale price decrease under the misspecified model (-0.97%) than under the true model (-8.95%). This discrepancy leads to a much larger retail price increase under the misspecified model (+3.92%) compared to the true model

Figure 2: Counterfactual predictions, true and misspecified bargaining models

(a) True model specification: $\rho = 1$



(b) True model specification: $\rho = 0$



Notes: We generate $T = 500$ markets for each case, either $\rho = 1$ (sequential timing) or $\rho = 0$ (simultaneous timing), using a demand and supply specification similar to the one used in Figure 1. Given the true ρ (say, $\rho = 1$), we take the simulated data and estimate the bargaining, responsiveness, and marginal cost parameters under both the true specification ($\rho = 1$) and the false specification ($\rho = 0$) of the bargaining model. We then use parameter estimates to simulate both an upstream and a downstream merger under the true and the false model specification. The y-axis shows the average percentage change in retail and wholesale prices for the merging firms' products.

(+1.85%). To summarize, the second exercise highlights the potential risk of imposing a timing assumption (either $\rho = 1$ or 0) when one does not observe the true one. Misspecifying the value of ρ could even lead to qualitatively misleading counterfactual outcomes and flawed policy recommendations.

3 Identification and estimation

We now examine the identification and estimation of the Nash-in-Nash model described by (6). Suppose that demand functions $\mathbf{q}_t(\cdot)$ as well as the equilibrium retail prices $\mathbf{p}_t^*$ and wholesale prices $\mathbf{w}_t^*$ are known.[17] Denote by $\mathbf{q}_t^* = \mathbf{q}_t(\mathbf{p}_t^*)$ the observed vector of demand in market t. We consider that the number of markets grows asymptotically to infinity ($T \to \infty$), while the maximum number of products sold across all markets remains fixed. We aim to identify and estimate bargaining parameters $\boldsymbol{\lambda} = (\lambda_j)_{j=1}^J$, responsiveness parameters $\boldsymbol{\rho} = (\rho_r)_{r=1}^R$, and the parameters governing marginal costs $(\mu_{jt})_{j,t}$.

Our strategy relies on the set of first-order conditions from the Nash-in-Nash bargaining model (6) and proceeds in two steps. In the first step, we invert (6) to back out the J-dimensional vector of marginal costs $(\mu_{jt})_{j=1}^J$ for all $t = 1, \ldots, T$. Specifically, let $\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho})$ be a $J \times J$ matrix with the (j,k)th element given by:

$$d_{j,k} = \Bigg[\frac{1-\lambda_j}{\lambda_j}\left(q_{kt}^* - \rho_{r(j)} q_{kt}^{-j}(\mathbf{p}_{rt}^{-j}(\mathbf{w}_{rt,-j}^*; \mathbf{p}_{-rt}^*), \mathbf{p}_{-rt}^*) - (1-\rho_{r(j)}) q_{kt}^{-j}(\mathbf{p}_{-jt}^*)\right) q_{jt}^* \\ - \rho_{r(j)} \sum_{h \in \mathcal{J}_{r(j)}} \frac{\partial q_{kt}}{\partial p_{ht}} \frac{\partial p_{ht}}{\partial w_{jt}} \left(\pi_{r(j)t}(\mathbf{w}_{r(j)t}^*; \mathbf{p}_t^*) - \pi_{r(j)t}^{-j}(\rho_{r(j)})\right)\Bigg] \tag{7}$$

Using (7), we can rewrite (6) in vector-matrix notations as follows:

$$[\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M]\,(w_{jt}^* - \mu_{jt})_{j=1}^J = -\left[\left(\pi_{r(j)t}(\mathbf{w}_{r(j)t}^*; \mathbf{p}_t^*) - \pi_{r(j)t}^{-j}(\rho_{r(j)})\right) q_{jt}^*\right]_{j=1}^J,$$

where $\odot$ refers to the element-wise (Hadamard) product, and $\boldsymbol{\Omega}_M$ is the $J \times J$ ownership matrix of manufacturers where the (j,k)th element equals 1 if j and k are owned by the same manufacturer and 0 otherwise. Suppose that $\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M$ is invertible, the J-dimensional vector of marginal costs is given by:

$$\begin{aligned}(\mu_{jt})_{j=1}^J &= (\mu_{jt}(w_{jt}^*, \mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}))_{j=1}^J \\ &= (w_{jt}^*)_{j=1}^J + [\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M]^{-1}\left[\left(\pi_{r(j)t}(\mathbf{w}_{r(j)t}^*; \mathbf{p}_t^*) - \pi_{r(j)t}^{-j}(\rho_{r(j)})\right) q_{jt}^*\right]_{j=1}^J\end{aligned} \tag{8}$$

[17] Note that one can estimate demand and use the set of first-order conditions characterizing the retailers' pricing behavior in the downstream market to back out wholesale prices (e.g., Villas-Boas, 2007). According to our notations, these (inferred) wholesale prices incorporate marginal distribution costs.

where the second term on the right-hand side of (8) corresponds to the (negative) J-dimensional vector of manufacturers' markups $(-(w_{jt}^* - \mu_{jt})_{j=1}^J)$.

In the second step, we first introduce the following assumption on marginal costs.

Assumption 1.

(i). (Monotonicity) $\mu_{jt} = mc_j(v_{jt}; \omega_{jt})$, *where* v_{jt} *are observed cost shifters for product* j, ω_{jt} *is an unobserved cost shock, and* $mc_j(\cdot)$ *is an unknown function strictly increasing in* ω_{jt} *for any given* v_{jt}.

(ii). (Exogeneity) There exists random variables z_{jt} *such that* $\mathbb{E}[\omega_{jt} \mid z_{jt}] = 0$ *for* $j = 1, ..., J$.

Assumption 1(i) specifies the marginal cost μ_{jt} as a function of observed cost shifters v_{jt} and unobserved cost shocks ω_{jt}. It covers the linear and additive cost structure ($\mathrm{mc}_j(v; \omega) = \kappa v + \omega$) as well as nonlinear ones. However, the unobserved cost shocks are potentially correlated with the manufacturers' markup term in (8), introducing endogeneity issues that threaten the identification of $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$.[18] Assumption 1(ii) addresses this concern by requiring the existence of variables z_{jt} that explain the manufacturers' markups while remain orthogonal to the cost shocks. In other words, exogenous variation in manufacturers' markups is necessary for disentangling the contribution of the markup term from that of marginal costs in wholesale prices. As discussed below, such instruments may include exogenous cost and demand shifters.

Combining Assumption 1 and (8), we obtain conditional moment conditions on true parameters $(\boldsymbol{\lambda}_0, \boldsymbol{\rho}_0)$ and $(\mathrm{mc}_{j0}(\cdot; \cdot))_{j=1}^J$: for $j = 1, ..., J$,

$$\mathbb{E}\left[\mathrm{mc}_{j0}^{-1}\left(\mu_{jt}(w_{jt}^*, \mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}_0, \boldsymbol{\rho}_0); v_{jt}\right) \mid z_{jt}\right] = 0, \tag{9}$$

where $\mathrm{mc}_j^{-1}(\cdot; v)$ is the inverse function of $\mathrm{mc}_j(v; \cdot)$ with respect to ω given v.

To identify the parameters of interest from (9), we face two challenges. First, we must determine whether and under what conditions the system of equations in (6) is invertible. This requires establishing the conditions for the invertibility of the matrix $\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M$ (i.e., non-zero determinant). Second, what instrumental variables z_{jt} can provide identification power, especially for the separable identification of $\boldsymbol{\lambda}_0$ and $\boldsymbol{\rho}_0$. Given the invertibility of $\mathrm{mc}_{j0}(\cdot; v)$ for any v (due to the monotonicity in Assumption 1), one can establish some high-level

[18] The source of endogeneity arises from the dependence of manufacturers' markups on demand and its derivatives, which are correlated with unobserved cost factors.

requirements on such instruments. When mc_{j0} is parametrically specified (e.g., additively linear in v_{jt} and ω_{jt}), classic rank conditions around the true parameters (e.g., Rothenberg, 1971) can achieve at least local identification. When mc_{j0} is nonparametrically specified, one can rely on arguments such as completeness conditions on z.[19] This requires the variation of z_{jt} to be sufficiently rich so that for any $(\tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\rho}}, (\tilde{\mathrm{mc}}_j(\cdot;\cdot))_{j=1}^J) \neq (\boldsymbol{\lambda}_0, \boldsymbol{\rho}_0, (\mathrm{mc}_{j0}(\cdot;\cdot))_{j=1}^J)$, there exists $z_{jt} = z$ and some product j such that $\mathbb{E}\left[\tilde{\mathrm{mc}}_j^{-1}\left(\mu_{jt}(w_{jt}^*, \mathbf{q}_t^*, \mathbf{p}_t^*; \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\rho}}); v_{jt}\right) \mid z_{jt} = z\right] \neq 0$. We will show that exogenous demand and cost shifters can provide such identification power.

Our two-step strategy implies a Generalized Method of Moments (GMM) estimation procedure. For instance, suppose that $\mathrm{mc}_{j0}(v_{jt}; \omega_{jt}) = \tau_{j0} v_{jt} + \omega_{jt}$ and z_{jt} is a vector of K dimensions, one can then construct moment conditions based on (9): for $j = 1, ..., J$,

$$\mathbb{E}\left[(\mu_{jt}(w_{jt}^*, \mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}_0, \boldsymbol{\rho}_0) - \kappa_{j0} v_{jt}) z_{jt}\right] = 0,$$

adopt the usual parametric GMM estimator given by:

$$\begin{aligned}(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\rho}}, (\hat{\kappa}_j)_{j=1}^J) \equiv \underset{\boldsymbol{\lambda}, \boldsymbol{\rho}, (\kappa_j)_{j=1}^J}{\operatorname{argmin}} & \left(\left[\frac{1}{T}\sum_{t=1}^{T}(\mu_{jt}(w_{jt}^*, \mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) - \kappa_j v_{jt}) z_{jt}\right]_{j=1}^J\right)^\top \\ & \mathbf{W}\left(\left[\frac{1}{T}\sum_{t=1}^{T}(\mu_{jt}(w_{jt}^*, \mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) - \kappa_j v_{jt}) z_{jt}\right]_{j=1}^J\right)\end{aligned} \tag{10}$$

where $\mathbf{W}$ is a weighting matrix of dimension $JK \times JK$.[20]

In the rest of this section, we solve the two aforementioned challenges.

3.1 Invertibility of the Nash-in-Nash first-order conditions

Given $(\boldsymbol{\lambda}, \boldsymbol{\rho})$, the determinant of $\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M$, denoted as $\det(\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M)$, is a function of the equilibrium retail prices and the corresponding quantities $(\mathbf{p}_t^*, \mathbf{q}_t^*) \in \mathbb{R}^{2J}$. The set of $(\mathbf{p}_t^*, \mathbf{q}_t^*)$ that delivers $\det(\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M) = 0$ defines a lower-dimensional object in $\mathbb{R}^{2J}$. Intuitively, this set implies a constraint on the underlying demand and cost shocks and is often "thin". As long as demand and supply shocks exhibit sufficient variation across markets, one obtains almost surely $\det(\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M) \neq 0$.

[19] Such arguments have been used in the identification of nonparametric IV models (e.g., Newey and Powell, 2003) and the demand literature (e.g., Berry and Haile, 2014; Iaria and Wang, 2021).

[20] In practice, one can concentrate out cost parameters $(\kappa_j)_{j=1}^J$ of the objective function in (10) (see, e.g., Bonnet, Bouamra-Mechemache and Molina, forthcoming; Molina, 2024).

We propose an intertibility result to formalize this intuition in a class of demand systems routinely used in applied work. Recall that $\mathbf{p}_{-j}$ and $\boldsymbol{\xi}_{-jt}$ refer to the subvector $(p_k)_{k\neq j,1\leq k\leq J}$ and $(\xi_{kt})_{k\neq j,1\leq k\leq J}$, respectively. The proof can be found in Appendix E.

Theorem 1 (Invertibility)**.** *Suppose that Assumption 1(i) holds and demand functions $q_{jt}(\mathbf{p}) = q_j(\mathbf{p};\boldsymbol{\xi}_t)$ and $q_{kt}^{-j}(\mathbf{p}_{-j}) = q_k^{-j}(\mathbf{p}_{-j};\boldsymbol{\xi}_{-jt})$ are real analytic with respect to $\mathbf{p}$ and $\mathbf{p}_{-j}$ for $j = 1,...,J$ and $k \neq j$ respectively, given demand shocks $\boldsymbol{\xi}_t = (\xi_{jt})_{j=1}^J$. Then, under regularity conditions E.1, $det(\mathbf{D}_t(\mathbf{q}_t^*, \mathbf{p}_t^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \boldsymbol{\Omega}_M) \neq 0$ almost surely for $(\boldsymbol{\xi}_t, \boldsymbol{\omega}_t)$, where $\boldsymbol{\omega}_t = (\omega_{jt})_{j=1}^J$.*

The real-analytic condition in Theorem 1 requires that demand functions be sufficiently smooth with respect to prices. This condition is satisfied by common demand models such as the linear, multinomial logit, nested logit, mixed logit, and mixed probit models.[21] In addition, it accommodates flexible substitution patterns among products, including both substitutability and complementarity. Moreover, Theorem 1 imposes no restrictions on the upstream or downstream market structures (i.e., no constraints on the ownership matrices $\boldsymbol{\Omega}_M$ and $\boldsymbol{\Omega}_R$). Finally, it is possible to obtain the invertibility by using other conditions (e.g., on market structures or substitution patterns among products) rather than the real analyticity of the demand system. We provide some examples in Appendix F.

3.2 Sources of identification and instruments

To shed light on the sources of identification for the bargaining and responsiveness parameters, we focus on the bilateral negotiation between manufacturer m and retailer r over w_{jt}. The computational details can be found in Appendix D.

At the equilibrium wholesale and retail prices, the partial derivative of the joint profit for this pair with respect to w_{jt} is given by:

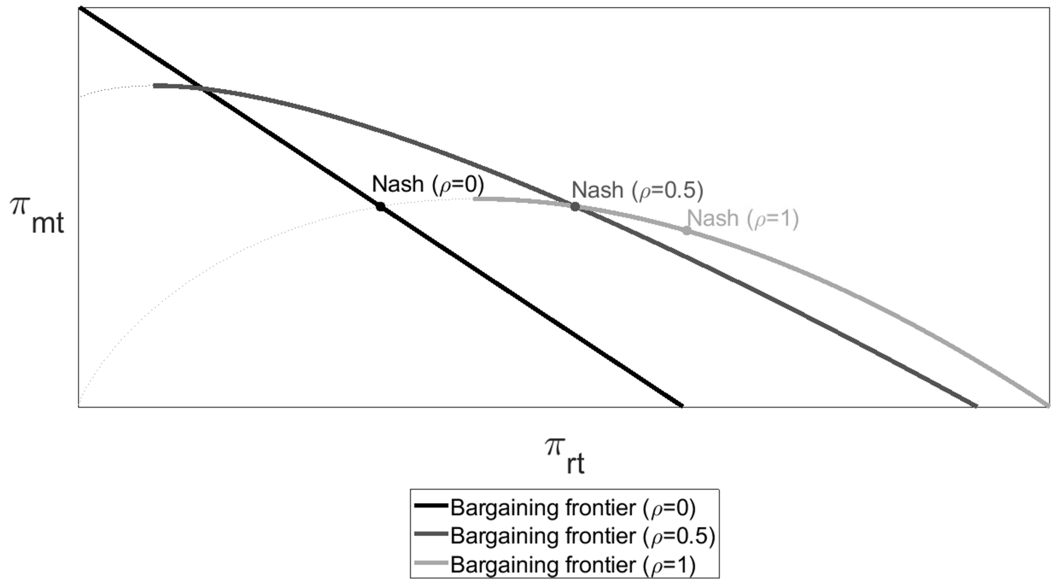
$$\frac{\partial\left(\pi_{mt}(\rho_r) + \pi_{rt}(\rho_r)\right)}{\partial w_{jt}} = \rho_r \sum_{k\in\mathcal{J}_m} (w_{kt}^* - \mu_{kt}) \sum_{h\in\mathcal{J}_r} \frac{\partial q_{kt}}{\partial p_{ht}} \frac{\partial p_{ht}}{\partial w_{jt}} \tag{11}$$

Hence, the responsiveness parameter ρ_r governs the extent to which the joint profit $\pi_{mt}(\rho_r) + \pi_{rt}(\rho_r)$ deteriorates when w_{jt}^* increases.[22] Interestingly, one can relate (11) to the slope of the bargaining frontier evaluated at equilibrium wholesale and

[21]See Iaria and Wang (2024) for a discussion on the real-analytic property of the mixed logit and mixed probit demand models with a linear index indirect utility structure.

[22]Keeping all other wholesale prices fixed, the decrease in joint profit resulting from an increase in w_{jt}^* stems from the so-called "double-marginalization" phenomenon (e.g., Spengler, 1950).

Figure 3: Responsiveness parameters and bargaining frontier



Notes: The x-axis represents the profit of retailer r and the y-axis represents the profit of manufacturer m. Each curve depicts the bargaining frontier (or Pareto frontier) for the bilateral negotiation between manufacturer m and retailer r over w_{jt}. The dot on each curve indicates the Nash bargaining solution for $\lambda = 0.5$.

retail prices:

$$\begin{aligned}\frac{\Delta\pi_{mt}(\pi_{rt}(w^*_{jt}, \mathbf{w}^*_{-jt}))}{\Delta\pi_{rt}(w^*_{jt}, \mathbf{w}^*_{-jt})} &= -1 - \rho_r \frac{\sum_{k\in\mathcal{J}_m}(w^*_{kt} - \mu_{kt})\sum_{l\in\mathcal{J}_r}\frac{\partial q_{kt}}{\partial p_{lt}}\frac{\partial p_{lt}}{\partial w_{jt}}}{q^*_{jt}} \\ &= -1 - \frac{1}{q^*_{jt}}\frac{\partial\left(\pi_{mt}(\rho_r) + \pi_{rt}(\rho_r)\right)}{\partial w_{jt}}.\end{aligned} \tag{12}$$

The term -1 reflects the mechanical inverse relationship between the changes in retailer r's and manufacturer m's profits due to a change in w_{jt} (i.e., a zero-sum relationship). The second term represents the rate of surplus transfer from retailer r to manufacturer m via w_{jt}. Hence, ρ_r determines the steepness of the bargaining frontier. We illustrate this interpretation in Figure 3. When $\rho_r = 0$ (unresponsiveness), the slope of the bargaining frontier equals -1, indicating that an increase in w_{jt} (around w^*_{jt}) does not affect the joint profit $\pi_{mt}(\rho_r) + \pi_{rt}(\rho_r)$. In contrast, when $\rho_r > 0$, the slope of the bargaining is greater than -1, implying any surplus transfer from retailer r to manufacturer m via w_{jt} deteriorates the joint profit. This negative effect on joint profit intensifies as ρ_r increases, with the case $\rho_r = 1$ (responsiveness) exhibiting the most significant negative effect. This interpretation of ρ_r contrasts with that of the bargaining parameter λ_j: while ρ_r is directly related to the slope (tangent) of the bargaining frontier at w^*_{jt}, λ_j governs the division of the joint profit between manufacturer m and retailer r, (i.e., the

Nash solution on the bargaining frontier corresponding to w_{jt}^*). Thus, intuitively, exogenous variation affecting the size of the joint profit should aid in identifying the responsiveness parameters, whereas exogenous variation affecting the split of the joint profit should help identify the bargaining parameters.

Based on these insights, we now proceed with a more formal analysis of the separable identification of the bargaining and responsiveness parameters.

Identification via approximative form. Let us rearrange the first-order conditions from the Nash-in-Nash bargaining model (6) as follows:

$$\sum_{k\in\mathcal{J}_{m(j)}}(w_{kt}^*-\mu_{kt})\Bigg[\frac{1-\lambda_j}{\lambda_j}\frac{q_{kt}^*-\rho_{r(j)}q_{kt}^{-j}(\mathbf{p}_{r(j)t}^{-j}(\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-r(j)t}^*)-(1-\rho_{r(j)})q_{kt}^{-j}(\mathbf{p}_{-jt}^*)}{\pi_{r(j)t}(\rho_{r(j)})-\pi_{r(j)t}^{-j}(\rho_{r(j)})}q_{jt}^* \\ -\rho_{r(j)}\sum_{h\in\mathcal{J}_{r(j)}}\frac{\partial q_{kt}}{\partial p_{ht}}\frac{\partial p_{ht}}{\partial w_{jt}}\Bigg]=q_{jt}^* \tag{13}$$

for $j=1,...,J$. We develop the Taylor expansion of (13) at $\left(\frac{1-\lambda_j}{\lambda_j}\right)_{j=1}^J=\mathbf{0}_J$ and $(\rho_r)_{r=1}^R=\mathbf{0}_R$ as follows: for $j=1,...,J$,

$$\sum_{k\in\mathcal{J}_{m(j)}}(w_{kt}^*-\mu_{kt})\Bigg[\frac{1-\lambda_j}{\lambda_j}\frac{q_{kt}(\mathbf{p}_t^*)-q_{kt}^{-j}(\mathbf{p}_{-jt}^*)}{\pi_{r(j)t}(\mathbf{w}_{r(j)t}^*,\mathbf{p}_t^*)-\pi_{r(j)t}^{-j}(\mathbf{w}_{r(j)t,-j}^*,\mathbf{p}_{-jt}^*)}q_{jt}(\mathbf{p}_t^*) \\ -\rho_r\sum_{h\in\mathcal{J}_{r(j)}}\frac{\partial q_{kt}}{\partial p_{ht}}\frac{\partial p_{ht}}{\partial w_{jt}}\Bigg]\simeq q_{jt}(\mathbf{p}_t^*). \tag{14}$$

By inverting (14) and using vector-matrix notations, we can approximate our bargaining model described in (8) as follows:[23]

$$(w_{kt}^*)_{k=1}^J\simeq(\mu_{kt})_{k=1}^J+\mathbf{A}_\lambda\Bigg[\left(\frac{\lambda_j}{1-\lambda_j}\right)_{j=1}^J \\ +\Big(\mathrm{diag}\Big(\left(\frac{\rho_{r(j)}\lambda_j}{1-\lambda_j}\right)_{j=1}^J\Big)\mathbf{B}_t\odot\mathbf{\Omega}_M\Big)\mathbf{A}_\lambda\left(\frac{\lambda_j}{1-\lambda_j}\right)_{j=1}^J\Bigg] \tag{15}$$

where $\mathrm{diag}\Big(\left(\frac{\rho_{r(j)}\lambda_j}{1-\lambda_j}\right)_{j=1}^J\Big)$ is a $J\times J$ diagonal matrix constructed from the vector $\left(\frac{\rho_{r(j)}\lambda_j}{1-\lambda_j}\right)_{j=1}^J$ using the diag$(\cdot)$ operator, and $\mathbf{A}_\lambda$ and $\mathbf{B}_t$ are two $J\times J$ matrices with the (j,k)th element respectively given by $a_{j,k}=\frac{\sum_{k\in\mathcal{J}_{m(j)}}q_{kt}(\mathbf{p}_t^*)-q_{kt}^{-j}(\mathbf{p}_{-jt}^*)}{\pi_{r(j)t}(\mathbf{w}_{r(j)t}^*,\mathbf{p}_t^*)-\pi_{r(j)t}^{-j}(\mathbf{w}_{r(j)t,-j}^*,\mathbf{p}_{-jt}^*)}$

[23]Concretely, we apply the following matrix operations: $\mathrm{Diag}(\mathbf{u})(\mathbf{E}\odot\mathbf{F})=(\mathrm{Diag}(\mathbf{u})\mathbf{E})\odot\mathbf{F}$, $(\mathbf{GE})^{-1}=\mathbf{E}^{-1}\mathbf{G}^{-1}$ and $(\mathbf{G}+\mathbf{E})^{-1}\simeq\mathbf{G}^{-1}-\mathbf{G}^{-1}\mathbf{E}\mathbf{G}^{-1}$, where $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ are square matrices, and $\mathbf{u}$ is a vector.

and $b_{j,k} = \frac{1}{q_{jt}(\mathbf{p}_t^*)} \sum_{h \in \mathcal{J}_{r(j)}} \frac{\partial q_{kt}}{\partial p_{ht}} \frac{\partial p_{ht}}{\partial w_{jt}}$.

The approximative form in (15) delivers useful insight for the identification of our bargaining, responsiveness, and cost parameters. First, the marginal cost shifters enter in $(\mu_k)_{k=1}^J$ and are excluded from $\mathbf{A}_\lambda$ and $\mathbf{B}_t$. We can thus identify the cost parameters in μ_{kt} via the variation in cost shifters. Second, when $\boldsymbol{\rho} = \mathbf{0}_R$ (unresponsiveness), (15) is linear in $\left(\frac{\lambda_j}{1-\lambda_j}\right)_{j=1}^J$. Therefore, the identification of bargaining parameters $\boldsymbol{\lambda}$ is achieved through the variation in $\mathbf{A}_\lambda$ that is unrelated to unobserved marginal cost factors in $(\mu_k)_{k=1}^J$. This includes, among others things, exogenous variation in demand shifters.[24] In the general case when $\boldsymbol{\rho} \neq \mathbf{0}_R$, the identification is analogous to that in a nonlinear least square model. The key insight is that $\left(\frac{\rho_{r(j)}\lambda_j}{1-\lambda_j}\right)_{j=1}^J$ interacts with $\mathbf{B}_t$, offering an additional source of variation beyond that provided by $\mathbf{A}_\lambda$. Thus, to identify $\boldsymbol{\lambda}$, one can vary $\mathbf{A}_\lambda$ while keeping $\left(\mathrm{diag}\left(\left(\frac{\rho_{r(j)}\lambda_j}{1-\lambda_j}\right)_{j=1}^J\right)\mathbf{B}_t \odot \boldsymbol{\Omega}_M\right)\mathbf{A}_\lambda$ unchanged. The variation in $\mathbf{B}_t \odot \boldsymbol{\Omega}_M$ is then used to further identify $\boldsymbol{\rho}$.

This approximative-form analysis demonstrates that commonly used instruments, such as demand and cost shifters, still provide identification power via equilibrium outcomes $(\mathbf{q}_t^*, \mathbf{p}_t^*, \mathbf{w}_t^*)$. Additionally, the derivatives $\sum_{h \in \mathcal{J}_{r(j)}} \frac{\partial q_{kt}}{\partial p_{ht}} \frac{\partial p_{ht}}{\partial w_{jt}}$ in $\mathbf{B}_t$ also provide a valuable source of variation for identification. This argument resembles that of the demand rotators in Bresnahan (1982), which change demand through slopes rather than intercepts. We illustrate these insights via the following examples.

Example 1 (Triangle vertical structure)**.** *Consider a vertical market with one manufacturer ($M = 1$) and two retailers ($R = 2$). The manufacturer sells its brand to both retailers, resulting in two products being offered to consumers. We suppose that the demand for each product is linear and given by:*

$$q_{1t}(p_{1t}, p_{2t}) = \delta_{1t} - p_{1t} + 0.5p_{2t},$$
$$q_{2t}(p_{1t}, p_{2t}) = \delta_{2t} - p_{2t} + 0.5p_{1t}.$$

Additionally, we make the simplifying assumption that $q_{1t}^{-2} = q_{2t}^{-1} = q_{2t}^* + q_{1t}^*$ at the equilibrium retail prices (p_{1t}^*, p_{2t}^*). The marginal cost structure is given by $\mu_{jt} = \kappa_j v_{jt} + \omega_{jt}$, where $\kappa_j \neq 0$ is an unknown cost parameter. From the sets of

[24] For instance, when the number of products and market structure ($\boldsymbol{\Omega}_M$ and $\boldsymbol{\Omega}_R$) vary across market in an exogenous way, the resulting variation in $\mathbf{A}_\lambda$ is also useful for the identification of $\boldsymbol{\lambda}$.

first-order conditions (4) *and* (6)*, we obtain (see Appendix G for details):*

$$\begin{pmatrix} w_{1t}^* \\ w_{2t}^* \end{pmatrix} = \frac{1}{15} \underbrace{\begin{bmatrix} \frac{1-\lambda_1}{\lambda_1} + \frac{8}{15}\rho_1 + \frac{7}{15} & -\left(\frac{1-\lambda_1}{\lambda_1} + \frac{4}{15}\rho_1 + \frac{2}{15}\right) \\ -\left(\frac{1-\lambda_2}{\lambda_2} + \frac{4}{15}\rho_2 + \frac{2}{15}\right) & \frac{1-\lambda_2}{\lambda_2} + \frac{8}{15}\rho_2 + \frac{7}{15} \end{bmatrix}^{-1}}_{\boldsymbol{\Theta}} \underbrace{\begin{pmatrix} 8\delta_{1t} + 2\delta_{2t} - 7\mu_{1t} + 2\mu_{2t} \\ 8\delta_{2t} + 2\delta_{1t} - 7\mu_{2t} + 2\mu_{1t} \end{pmatrix}}_{\tilde{\boldsymbol{\delta}}_t} + \begin{pmatrix} \mu_{1t} \\ \mu_{2t} \end{pmatrix} \quad (16)$$

First, one can shift $\tilde{\boldsymbol{\delta}}_t$ via $(\delta_{1t}, \delta_{2t})$ to identify the four elements of $\boldsymbol{\Theta}$. With these four identified elements, we have a system of equations from which we can identify the four unknown parameters λ_1, λ_2, ρ_1, and ρ_2. Finally, by varying the cost shifters v_{1t} and v_{2t} in μ_{1t} and μ_{2t}, one can identify the cost parameters κ_1 and κ_2, respectively.

In practice, the identification of our model using demand and cost shifters can be loosely summarized as comparing the number of unknowns parameters to the number of available moments. As in Example 1, suppose we have J product-specific cost parameters, J bargaining parameters, and $R \leq J$ responsiveness parameters. The total number of parameters is thus $2J+R$. Analogous to Example 1, each cost shifter provides a moment that just identifies the corresponding cost parameter. In addition, by varying J demand shifters, we can obtain J^2 moments from the Jacobian matrix $\frac{\partial(w_1^*,\ldots,w_J^*)}{\partial(\delta_1,\ldots,\delta_J)}$. As long as the number of moments, $J + J^2$, is at least as large as the number of parameters, $2J + R$, identification can in principle be achieved. The inequality $J + J^2 \geq 2J + R$ holds whenever $J \geq 2$. However, when $J = 1$ (implying that $R = 1$), we only have 2 moments to identify 3 parameters. In this case, as shown in the following example, one can rely on arguments such as demand rotators to generate additional moments.

Example 2 (Bilateral monopoly)**.** *Consider a bilateral monopoly ($M = 1$ and $R = 1$) where only one product is offered to consumers. We suppose that the demand for this product has an exponential form: $q_t(p_t) = \delta_t p_t^{-\varepsilon_t}$ with $\varepsilon_t > 1$. As in Example 1, the marginal cost structure is given by $\mu_t = \kappa v_t + \omega_t$, where $\kappa \neq 0$ is an unknown cost parameter. From the sets of first-order conditions* (4) *and* (6)*, we obtain (see Appendix G for details):*

$$w_t^* = \left(\frac{1-\lambda}{\lambda} - \frac{1}{\varepsilon_t - 1}\right)^{-1} \left(\frac{1-\lambda}{\lambda} + \rho\frac{\varepsilon_t}{\varepsilon_t - 1}\right)(\kappa v_t + \omega_t).$$

First, by varying v_t, one can identity the quantity

$$A(\varepsilon) := \kappa \left(\frac{1-\lambda}{\lambda} - \frac{1}{\varepsilon - 1}\right)^{-1} \left(\frac{1-\lambda}{\lambda} + \rho\frac{\varepsilon}{\varepsilon - 1}\right)$$

for each ε in the support of ε_t. Second, note that when ε tends to $\frac{1}{1-\lambda}$ from the *right, we have* $\frac{1-\lambda}{\lambda} - \frac{1}{\varepsilon-1}$ *decreases to zero and* $A(\varepsilon)$ *tends to infinity. Then, by observing that* $A(\varepsilon^*)$ *is equal to infinity at some observed* ε^**, we learn that* ε^* *is equal to* $\frac{1}{1-\lambda}$ *and thus identify* $\lambda = 1 - \frac{1}{\varepsilon^*}$*. In addition, we can identify* $A(\varepsilon_1)/A(\varepsilon_2)$ *for given* ε_1 *and* ε_2 *and this ratio only depends on* λ *and* ρ*, then we can identify* ρ *from this ratio and the identified* λ*. Finally,* κ *is identified by* $A(\varepsilon)\left(\frac{1-\lambda}{\lambda} - \frac{1}{\varepsilon-1}\right)\left(\frac{1-\lambda}{\lambda} + \rho\frac{\varepsilon}{\varepsilon-1}\right)^{-1}$.

In Example 2, the nonlinearity in the demand and the variation in ε_t allows us to rotate w_t^* around μ_t and achieve the identification of (λ, ρ). This stands in contrast to the linear demand case with $J = 1$, where it is impossible to rotate w^* around μ and separately identify λ and ρ (see Appendix G). Despite this singularity, empirical researchers frequently work with nonlinear demand models and multiple products ($J \geq 2$). As a result, identification is generally achieved in practice.

4 Monte Carlo study

In this section, we use Monte Carlo simulations to assess the finite sample properties of our GMM estimator (10) using the instrumental variables proposed in our identification arguments (e.g., demand and cost shifters, BLP-type instruments).

Based on Monte Carlo studies of supply models with oligopoly competition (e.g., Skrainka, 2012; Armstrong, 2016; Conlon and Gortmaker, 2020), we consider the following data generating process. We generate 100 datasets with $T \in \{500, 1000\}$ markets. In each market, we have $M = 3$ manufacturers and $R = 3$ retailers, where each manufacturer deals with each retailer (i.e., interlocking relationships). We define a product as a manufacturer-retailer combination, implying a total of $J = 9$ products offered to consumers. Hence, we have multi-product manufacturers and retailers, with $|\mathscr{J}_m| = |\mathscr{J}_r| = 3$ for all $r = 1, \ldots, R$ and $m = 1, \ldots, M$. We specify the demand in market t to be mixed-logit with the indirect utility given by:

$$U_{ijt} = \beta_i x_j - \alpha_i p_{jt} + \xi_{jt} + \epsilon_{ijt}$$

where $x_{jt} \overset{\text{iid}}{\sim} \mathrm{U}(0,1)$, $\epsilon_{ijt} \overset{\text{iid}}{\sim} \mathrm{GEV}(0,1,0)$, $\beta_i \sim \mathscr{N}(1,1)$, and $\alpha_i \sim \ln\mathscr{N}(0.49, 0.64)$, implying that the population mean of α_i is 2 and its variance is 2. We specify the marginal cost of product j in market t as follows:

$$\mu_{jt} = \kappa_0 + \kappa_v v_{jt} + \kappa_x x_{jt} + \omega_{jt}$$

where $v_{jt} \overset{\text{iid}}{\sim} \mathrm{U}(0,1)$. We set the marginal cost parameters to $\kappa_0 = \kappa_v = \kappa_x = 1$, and the bargaining and responsiveness parameters to $\lambda_j = \lambda = 0.5$ and $\rho_r = \rho = 0.5$, respectively. As in Conlon and Gortmaker (2020), we draw the structural error terms ξ_{jt} and ω_{jt} from the following bivariate normal distribution

$$\begin{pmatrix} \xi_{jt} \\ \omega_{jt} \end{pmatrix} \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}\right).$$

Finally, we generate retail prices by solving the bargaining and downstream pricing equilibrium characterized by (6) and (4), respectively.[25] We treat the demand parameters as known and focus on the finite sample estimates of the parameters in our bargaining model.

We compare the finite-sample performance of two estimators: nonlinear least squares (NLLS) and the generalized method of moments (GMM). The NLLS does not address the endogeneity of market shares that are embedded in the manufacturers' markup term. The GMM estimator outlined in (10) accounts for this endogeneity issue. In addition to a constant term and the demand/cost shifters (x_{jt}, v_{jt}), we use two sets of (excluded) instrumental variables:

$$\mathbf{z}_{jt}^{\text{BLP}} = \left[\sum_{k\in\mathcal{J}_{r(j)t}\setminus\{j\}} x_{kt} \quad \sum_{k\in\mathcal{J}_t\setminus\mathcal{J}_{r(j)t}} x_{kt} \quad \sum_{k\in\mathcal{J}_{m(j)t}\setminus\{j\}} x_{kt}\right]$$

$$\mathbf{z}_{jt}^{\text{BH}} = \left[\sum_{k\in\mathcal{J}_{r(j)t}\setminus\{j\}} \xi_{kt} \quad \sum_{k\in\mathcal{J}_t\setminus\mathcal{J}_{r(j)t}} \xi_{kt} \quad \sum_{k\in\mathcal{J}_{m(j)t}\setminus\{j\}} \xi_{kt} \right.$$
$$\left. \sum_{k\in\mathcal{J}_{r(j)t}\setminus\{j\}} v_{kt} \quad \sum_{k\in\mathcal{J}_t\setminus\mathcal{J}_{r(j)t}} v_{kt} \quad \sum_{k\in\mathcal{J}_{m(j)t}\setminus\{j\}} v_{kt}\right]$$

The first set of instruments $\mathbf{z}_{jt}^{\text{BLP}}$ corresponds to the traditional BLP instruments widely used in the literature. The second set of instruments $\mathbf{z}_{jt}^{\text{BH}}$ is based on insights from Berry and Haile (2014), which suggest exploiting variation in unobserved characteristics and cost shifters of competing products to discriminate between oligopoly models.[26]

We report the NLLS and GMM estimates results in Table 1. As expected, the NLLS estimator produces large bias for all parameters (e.g., the bias is 0.335 for λ and -0.437 for ρ); increasing the number of markets from $T = 500$ to $T = 1000$

[25] As emphasized in Conlon and Gortmaker (2020), Monte Carlo studies that do not generate prices and market shares from equilibrium play may raise concerns about the validity of BLP-type instruments, as markups are not "endogenous". Note that we discarded certain Monte Carlo draws due to convergence issues.

[26] Note that the unobserved characteristics of competing products ($\boldsymbol{\xi}_{-jt}$) are independent of ω_{jt} in our Monte Carlo design, ensuring the exogeneity of these variables.

Table 1: **Monte Carlo results**

	True	NLLS			GMM		
		Bias	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**
T = 500							
λ	0.5	0.335	0.003	0.335	0.047	0.151	0.158
ρ	0.5	-0.437	0.004	0.437	0.047	0.389	0.392
κ_0	1	-3.825	0.043	3.826	-0.049	0.215	0.221
κ_v	1	-0.697	0.012	0.697	0.003	0.047	0.047
κ_x	1	-0.720	0.011	0.720	-0.013	0.049	0.051
T = 1000							
λ	0.5	0.335	0.002	0.335	0.026	0.104	0.107
ρ	0.5	-0.436	0.003	0.436	0.040	0.304	0.307
κ_0	1	-3.829	0.033	3.829	-0.013	0.154	0.155
κ_v	1	-0.694	0.010	0.694	0.001	0.033	0.033
κ_x	1	-0.719	0.008	0.719	-0.007	0.049	0.051

Notes: "NLLS" stands for the nonlinear least squares estimator and "GMM" is the 2-step GMM estimator. The biases, standard errors (SE), and root mean square errors (RMSE) are obtained based on 100 replications.

does not reduce the bias. In contrast, the GMM estimator exhibits fairly low bias. When $T = 500$, the bias is 0.047 for both λ and ρ, which is less than 10%. The standard errors suggest that obtaining precise estimates for ρ is more difficult than other parameters, likely due to the greater nonlinearity of the model with respect to this parameter. Additionally, the root mean square errors are nearly identical to the standard errors, as the bias for each parameter is small relative to its standard error. As the number of markets increases from $T = 500$ to $T = 1000$, the bias, standard errors, and root mean square errors all decrease. For instance, the root mean square errors of λ and ρ drop by 32% and 22%, respectively. This result indicates an improvement in the performance of our GMM estimator as the number of markets increases, verifying our identification analysis and consistent with the asymptotic theory.

5 Conclusion

We develop an empirical framework of vertical relationships that incorporates a novel Nash-in-Nash bargaining model with uncertainty in retailers' pricing responses to wholesale prices. This allows us to generalize the timing assumptions routinely used in existing literature. We demonstrate that the proposed model can be microfounded by a noncooperative game along the lines of Rey and Vergé

(2020). We use a two-step strategy to identify and estimate the bargaining, responsiveness, and marginal cost parameters. We show that exogenous variations in demand shifters, rotators, and cost shifters provide identification power. Our Monte Carlo simulations support these instruments' validity. As a next step, we plan to apply our methodology to investigate the extent to which retailers are responsive in retail pricing decisions to the brewer-retailer bargaining in Chicago.

Our framework offers several valuable perspectives for future research of empirical Nash-in-Nash bargaining. A methodological question in our framework is how to improve the efficiency of the GMM estimation given the high nonlinearity. For instance, it remains to be explored whether and how one can construct optimal instruments analogously to those in demand estimation (Reynaert and Verboven, 2014; Gandhi and Houde, 2023) and testing firm conduct (Backus, Conlon and Sinkinson, 2021). Another promising direction is to endogenize the responsiveness parameters. In other words, the degree of uncertainty regarding a retailer's responsiveness in retail pricing decisions could be an outcome of some corporate decisions, such as the adoption of new information technologies (e.g., Holmes, 2001; Basker, 2012).

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Appendix

A Microfoundation

In this appendix, we propose a noncooperative microfoundation for our Nash-in-Nash bargaining model with uncertainty about retailers' responsiveness in retail pricing decisions.

To this purpose, we adapt the noncooperative game developed in Rey and Vergé (2020) to our framework of vertical relations. Consider an industry with a vertical structure as described in Section 2, where M (multi-product) upstream manufacturers deal with R (multi-product) downstream retailers. Assume that every manufacturer-retailer pair has positive gains from trade, resulting in an interlocking relationships distribution network (i.e., competing manufacturers deal with the same set of competing retailers). Taking this distribution network as given, we consider a game with manufacturer-retailer bargaining and retail price competition that follows the spirit of our model introduced in Section 2. Specifically, the timing of play can be outlined as follows. In the first stage, manufacturers and retailers engage in secret and bilateral negotiations over wholesale prices. In the second stage, retailers learn their state of nature (either responsive or unresponsive with probabilities ρ_r and $1-\rho_r$, respectively). In the last stage, retailers compete in retail prices for consumers. Although stages 2 and 3 remain as in Section 2, the first stage is modelled following the random-proposer bargaining game with delegated agents introduced in Rey and Vergé (2020). Specifically, we assume that both manufacturers and retailers rely on delegated agents, denoted by m_j and r_j with $j = 1\ldots,J$, to negotiate wholesale prices (we drop market subscript to simplify notation). Hence, each pair of delegated agents $m_j - r_j$ negotiates over w_j, with each agent acting on behalf of its firm by seeking to maximize its profit. Formally, the game is described as follows:

- **Stage 1: Manufacturer-retailer bargaining.**
  Each wholesale price w_j is determined through a bilateral negotiation between a $m_j - r_j$ pair according to the following protocol.
  - Stage 1.a: m_j proposes a wholesale price to r_j. If r_j accepts the offer, the bilateral negotiation concludes and the game proceeds to stage 2; otherwise, it moves to stage 1.b. All offers and acceptance decisions are simultaneous and secret.
  - Stage 1.b: If r_j rejects m_j's initial offer, nature selects one agent to make a new (and final) offer, with m_j chosen with probability ϕ and r_j with probability $1-\phi$. This selection process is independent across pairs of agents, and its outcome remains secret.
  - Stage 1.c: The selected agent (either r_j or m_j) makes a final offer to its counterpart, which either accepts or rejects it. If the offer is accepted, the bilateral negotiation concludes, and product j is sold to r_j's retailer at the agreed wholesale price. If the offer is rejected, a disagreement ensues, and product j is not offered on the market. All offers and acceptance decisions continue to occur simultaneously and secretly.
- **Stage 2: State of nature.**
  Each retailer learns its state of nature: with the probability ρ_r, retailer r is in a responsive state and, with probability $1-\rho_r$, it is in an unresponsive state.

- **Stage 3: Retail price competition.**
  Given contract secrecy and the realized state of the nature, retailers simultaneously set retail prices.

We look for the sequential equilibrium (Kreps and Wilson, 1982) of this three-stage game. The notion of sequential equilibrium requires firms' beliefs to be "consistent", meaning that any player observing a deviation believes all others will continue to follow the equilibrium strategies, assuming that this deviation stems from a "tremble" (Fudenberg and Tirole, 1991). Hence, no agent revises its beliefs regarding wholesale prices negotiated by other agents when an unexpected bargaining outcome occurs in stage 1. Likewise, when setting retail prices in stage 3, a retailer observing a deviant wholesale price still conjectures that its rivals have negotiated the equilibrium wholesale prices. Given any $\boldsymbol{\rho} \in [0,1]^R$ and $\boldsymbol{\lambda} \in [0,1]^J$, we show that there exist $\boldsymbol{\phi} \in [0,1]^J$ such that the Nash-in-Nash bargaining outcomes, as characterized by (6), and the retail prices set by retailers replicate the sequential equilibrium of the above noncooperative game, that is: $w_j^* = w_j^\star$ and $p_j^* = p_j^\star$ for all $j = 1, \dots, J$, where the superscript $\star$ denotes the equilibrium outcome of the noncooperative game and * is that of the model with Nash-in-Nash bargaining (Section 2).

As stages 2 and 3 are similar to those described in Section 2, we consider stage 1 by focusing on the $m_j - r_j$ pair that negotiates over w_j. For the sake of exposition, we omit $\mathbf{p}$ in π_r and π_m. Moreover, without loss of generality, we assume that $\pi_r(\cdot; \mathbf{w}_{-j}^\star)$ and $\pi_m(\cdot; \mathbf{w}_{-j}^\star)$ are at most single-peaked for all $m = 1 \dots, M$ and $r = 1, \dots, R$. We also focus on the case where $\frac{\partial \pi_r(w_j^\star, \mathbf{w}_{-j}^\star)}{\partial w_j} < 0$ and $\frac{\partial \pi_m(w_j^\star; \mathbf{w}_{-j}^\star)}{\partial w_j} > 0$, as the equilibrium of the noncooperative game must lie on the bargaining (or Pareto) frontier.[27]

Consider first that nature selects r_j to propose a final offer to m_j in stage 1.c. Taking all other wholesale prices as given ($\mathbf{w}_{-j} = \mathbf{w}_{-j}^\star$), r_j offers $\underline{w}_j$ to maximize retailer r's profit subject to each agent's participation constraint, that is:

$$\underline{w}_j \equiv \underset{w_j}{\operatorname{argmax}}\ \pi_r(w_j, \mathbf{w}_{-j}^\star, \rho_r) \quad \text{subject to} \quad \begin{cases} \pi_m(w_j, \mathbf{w}_{-j}^\star, \rho_r) \geq \pi_m^{-j}(\mathbf{w}_{-j}^\star, \rho_r) \\ \pi_r(w_j, \mathbf{w}_{-j}^\star, \rho_r) \geq \pi_r^{-j}(\mathbf{w}_{-j}^\star, \rho_r) \end{cases}$$

where $\pi_m(\cdot)$, $\pi_r(\cdot)$, $\pi_m^{-j}(\cdot)$, and $\pi_r^{-j}(\cdot)$ are defined as in (C.3a), (C.3b), (C.4a), and (C.4b), respectively. As $\pi_r(w_j; \mathbf{w}_{-j}^\star, \rho_r)$ has at most a single peak and is decreasing with respect to w_j at equilibrium wholesale prices, it follows that $w_j^* \geq \underline{w}_j$. Furthermore, as $[\underline{w}_j, w_j^*]$ lies on the bargaining (or Pareto) frontier, $\pi_r(w_j; \mathbf{w}_{-j}^\star; \rho_r)$ is decreasing and $\pi_m(w_j; \mathbf{w}_{-j}^\star; \rho_r)$ is increasing in $w_j \in [\underline{w}_j, w_j^*]$. As a result, we have $\pi_r(\underline{w}_j, \mathbf{w}_{-j}^\star, \rho_r) > \pi_r^{-j}(\mathbf{w}_{-j}^\star, \rho_r)$ and $\pi_m(\underline{w}_j, \mathbf{w}_{-j}^\star, \rho_r) = \pi_m^{-j}(\mathbf{w}_{-j}^\star, \rho_r)$. That is, r_j's final offer is such that m_j's participation constraint is binding.

Consider now that nature selects m_j to propose a final offer to r_j in stage 1.c. Taking all other wholesale prices as given ($\mathbf{w}_{-j} = \mathbf{w}_{-j}^\star$), m_j offers $\bar{w}_j$ to maximize manufacturer m's profit subject to each agent's participation constraint, that is:

$$\bar{w}_j \equiv \underset{w_j}{\operatorname{argmax}}\ \pi_m(w_j, \mathbf{w}_{-j}^\star, \rho_r) \quad \text{subject to} \quad \begin{cases} \pi_m(w_j, \mathbf{w}_{-j}^\star, \rho_r) \geq \pi_m^{-j}(\mathbf{w}_{-j}^\star, \rho_r) \\ \pi_r(w_j, \mathbf{w}_{-j}^\star, \rho_r) \geq \pi_r^{-j}(\mathbf{w}_{-j}^\star, \rho_r) \end{cases}$$

[27] Under contract secrecy, π_r is always decreasing in w_j. Due to the double-marginalization problem, π_m may also decrease in w_j. Intuitively, however, any scenario in which $\frac{\partial \pi_m(w_j; \mathbf{w}_{-j}^*)}{\partial w_j} < 0$ cannot constitute an equilibrium, as the $m_j - r_j$ pair would always be able to find an alternative wholesale price that is Pareto improving.

Following similar reasoning as above, we have $\bar{w}_j \geq w_j^*$, $\pi_m(\bar{w}_j, \mathbf{w}_{-j}^\star, \rho_r) > \pi_m^{-j}(\mathbf{w}_{-j}^\star, \rho_r)$, and $\pi_r(\bar{w}_j, \mathbf{w}_{-j}^\star, \rho_r) = \pi_r^{-j}(\mathbf{w}_{-j}^\star, \rho_r)$. That is, m_j's final offer is such that r_j's participation constraint is binding. In summary, we have $\bar{w}_j \geq w_j^* \geq \underline{w}_j$, with $\pi_r(w_j, \mathbf{w}_{-j}^\star, \rho_r)$ decreasing and $\pi_m(w_j, \mathbf{w}_{-j}^\star, \rho_r)$ increasing over $w_j \in [\underline{w}_j, \bar{w}_j]$.

Proceeding backwards, consider now stage 1.a. Given that nature selects r_j with probability $1-\phi_j$ to make a final (counter)offer, m_j's maximization problem is given by:

$$\max_{w_j} \pi_m(w_j; \mathbf{w}_{-j}^\star, \rho_r) \text{ subject to } \begin{cases} \pi_m(w_j; \mathbf{w}_{-j}^\star, \rho_r) \geq \phi_j \pi_m(\bar{w}_j; \mathbf{w}_{-j}^\star, \rho_r) + (1-\phi_j)\pi_m(\underline{w}_j; \mathbf{w}_{-j}^\star, \rho_r) \\ \pi_r(w_j; \mathbf{w}_{-j}^\star, \rho_r) \geq \phi_j \pi_r(\bar{w}_j; \mathbf{w}_{-j}^\star, \rho_r) + (1-\phi_j)\pi_r(\underline{w}_j; \mathbf{w}_{-j}^\star, \rho_r) \end{cases} \tag{A.1}$$

Hence, m_j seeks to maximize manufacturer m's profit taking into account that r_j may reject its offer and make an ultimate (counter)offer with probability $1-\phi_j$. Denote $w_j^\star$ as the solution to (A.1), which can be described as follows. First, given that $\phi_j \in [0,1]$, the participation constraints in (A.1) imply that $w_j^\star \in [\underline{w}_j, \bar{w}_j]$. Second, as $[\underline{w}_j, \bar{w}_j]$ is on the bargaining (or Pareto) frontier, $\pi_r(w_j; \mathbf{w}_{-j}^\star; \rho_r)$ is decreasing and $\pi_m(w_j; \mathbf{w}_{-j}^\star; \rho_r)$ is increasing over $w_j \in [\underline{w}_j, \bar{w}_j]$. Hence, $w_j^\star$ is such that r_j is indifferent between accepting or rejecting m_j's offer, that is $\pi_r(w_j^\star; \mathbf{w}_{-j}^\star, \rho_r) = \phi_j \pi_r(\bar{w}_j; \mathbf{w}_{-j}^\star, \rho_r) + (1-\phi_j)\pi_r(\underline{w}_j; \mathbf{w}_{-j}^\star, \rho_r)$. Note that any $w_j > w_j^\star$ would violate r_j's participation constraint, ruling it our as a solution to (A.1). Similarly, given that $\pi_m(w_j; \mathbf{w}_{-j}^\star, \rho_r)$ has at most a single peak and is increasing over $w_j \in [\underline{w}_j, \bar{w}_j]$, any $w_j < w_j^\star$ would reduce manufacturer m's profit. Therefore, $w_j = w_j^\star$ is the unique solution to (A.1).

In summary, we have $w_j^\star \in [\underline{w}_j, \bar{w}_j]$, with $w_j^\star = \underline{w}_j$ when $\phi_j = 0$ and $w_j^\star = \bar{w}_j$ when $\phi_j = 1$. As $w_j^* \in [\underline{w}_j, \bar{w}_j]$, we can replicate the Nash-in-Nash solution characterized in (6) by choosing an appropriate $\phi_j \in [0,1]$ such that $w_j^\star = w_j^*$ for all $j = 1, \ldots, J$. Hence, as stages 2 and 3 remain unchanged, it follows directly that the equilibrium outcomes of the model in Section 2 coincide with the sequential equilibrium of the noncooperative game developed in this appendix.

B Retailers' price-cost margins

At the equilibrium wholesale prices, the first-order conditions of (2) and (3) in stage 3 coincide:

$$q_{jt}(\mathbf{p}_t^*) + \sum_{k \in \mathscr{J}_{r(j)}} (p_{kt}^* - w_{kt}^* - c_{kt}) \frac{\partial q_{kt}}{\partial p_{jt}} = 0 \quad \forall j \in \mathscr{J}$$

We re-write these first-order conditions as follows:

$$\mathbf{q}_t + (\mathbf{\Omega}_R \odot \mathbf{Q}_\mathbf{p})(\mathbf{p}_t^* - \mathbf{w}_t^* - \mathbf{c}_t) = \mathbf{0}_J \tag{B.1}$$

where $\odot$ is the element-wise (Hadamard) product, $\mathbf{\Omega}_R$ denotes the $J \times J$ ownership matrix of retailers where the (j,k)th element equals 1 if products j and k are distributed by the same retailer in market t and 0 otherwise, and $\mathbf{Q}_\mathbf{p} = \left(\frac{\partial q_{kt}}{\partial p_{jt}}\right)_{j,k}$ is the $J \times J$ matrix of the partial derivatives of demand with respect to retail prices. We can then recover the J-dimensional vector of price-cost margins of retailers as follows:

$$\mathbf{p}_t^* - \mathbf{w}_t^* - \mathbf{c}_t = -(\mathbf{\Omega}_R \odot \mathbf{Q}_\mathbf{p})^{-1}\mathbf{q}_t \tag{B.2}$$

as long as $\mathbf{\Omega}_R \odot \mathbf{Q}_\mathbf{p}$ is invertible. The invertibility can be implied by diagonal dominance (Berry and Haile, 2014), a property that holds in commonly used demand models (e.g., nested or mixed logit).

C Manufacturers' price-cost margins

In stage 1, manufacturers and retailers engage in secret and bilateral negotiations to determine wholesale prices (and consequently, manufacturers' price cost margins), anticipating the potential impact of bargaining outcomes on retail price competition between retailers. Hence, before analyzing the wholesale price setting stage, we first review the retail price setting stage (stage 3). As wholesale contracts are kept secret, each retailer sets the retail prices of its products based on the belief that the wholesale prices paid by its rivals are at equilibrium. However, each retailer's pricing behavior depends on the realized state of nature in stage 2: "responsive " or "unresponsive".

Downstream retail pricing in the responsive state. When the "responsive" state arises, retailer r sets its retail prices based on the observed bargaining outcomes $(w_{jt})_{j\in\mathscr{J}_r}$, that is:

$$\mathbf{p}_{rt}(\mathbf{w}_{rt};\mathbf{p}^*_{-rt}) \equiv \underset{\{p_{jt}\}_{j\in\mathscr{J}_r}}{\operatorname{argmax}} \sum_{j\in\mathscr{J}_r}(p_{jt}-w_{jt}-c_{jt})q_{jt}(\mathbf{p}_{rt},\mathbf{p}^*_{-rt})$$

where $\mathbf{p}_{rt}$ denotes the vector of retail prices set be retailer r, $\mathbf{w}_{rt}$ is the vector of wholesale prices paid by retailer r, and $\mathbf{p}^*_{-rt}$ is the vector of equilibrium retail prices set by retailer r's rivals. Moreover, when a bargaining breakdown occurs over w_{jt} for $j\in\mathscr{J}_r$, retailer r sets the retail prices of its remaining products given the removal of product j from market t:

$$\mathbf{p}^{-j}_{rt}(\mathbf{w}_{rt,-j};\mathbf{p}^*_{-rt}) \equiv \underset{(p_{kt})_{k\in\mathscr{J}_r\setminus\{j\}}}{\operatorname{argmax}} \sum_{k\in\mathscr{J}_r\setminus\{j\}}(p_{kt}-w_{kt}-c_{kt})q^{-j}_{kt}(\mathbf{p}_{rt,-j},\mathbf{p}^*_{-rt}) \tag{C.1}$$

where $\mathbf{p}_{rt,-j}$ is the vector of out-of-equilibrium retail prices set by retailer r following a bargaining breakdown over w_{jt}, q^{-j}_{kt} is the demand for product k when product j is not offered in market t, and $\mathbf{w}_{rt,-j}$ is the vector of wholesale prices for all products distributed by retailer r except product $j\in\mathscr{J}_r$. In practice, we can use an algorithm similar to that described in Bonnet, Bouamra-Mechemache and Molina (forthcoming) to compute $\mathbf{p}^{-j}_{rt}$.

Downstream retail pricing in the unresponsive state. When the "unresponsive" state arises, retailer r sets its retail prices based on the belief that $(w^*_{jt})_{j\in\mathscr{J}_r}$, that is:

$$\mathbf{p}_{rt}(\mathbf{w}^*_{rt};\mathbf{p}^*_{-rt}) \equiv \underset{\{p_{jt}\}_{j\in\mathscr{J}_r}}{\operatorname{argmax}} \sum_{j\in\mathscr{J}_r}(p_{jt}-w^*_{jt}-c_{jt})q_{jt}(\mathbf{p}_{rt},\mathbf{p}^*_{-rt})$$

Hence, when a bargaining breakdown occurs over w_{jt} for $j\in\mathscr{J}_r$, retailer r never adjusts the retail prices of its remaining products accordingly.

Nash-in-Nash bargaining with uncertainty in retailers' responsiveness. Each manufacturer-retailer pair negotiate wholesale prices of products given the uncertainty about the retailer's state of responsiveness, where $\rho_r\in[0,1]$ denotes the probability that retailer r is in the "responsive" state. We focus on the bilateral negotiation between manufacturer m and retailer r over w_{jt} for $j \in \mathscr{J}_r \cap \mathscr{J}_m$. Taking other wholesale prices as given (i.e., $\mathbf{w}_{-jt} = \mathbf{w}^*_{-jt}$), the equilibrium wholesale price w^*_{jt} is defined as follows:

$$w^*_{jt} \equiv \underset{w_{jt}}{\operatorname{argmax}} \ \left(\pi_{mt}(\rho_r) - \pi^{-j}_{mt}(\rho_r)\right)^{\lambda_j} \left(\pi_{rt}(\rho_r) - \pi^{-j}_{rt}(\rho_r)\right)^{1-\lambda_j} \tag{C.2}$$

The terms $\pi_{mt}(\rho_r)$ and $\pi_{rt}(\rho_r)$ in (C.2) denote the profits of manufacturer m and retailer r in the case of an agreement, which are respectively given by:

$$\begin{aligned}\pi_{mt}(\rho_r) =& \rho_r \Bigg[(w_{jt} - \mu_{jt})\ q_{jt}(\mathbf{p}_{rt}(w_{jt}, \mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) \\ & \quad + \sum_{k \in \mathscr{J}_m \setminus \{j\}} (w^*_{kt} - \mu_{kt})\ q_{kt}(\mathbf{p}_{rt}(w_{jt}, \mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) \Bigg] \\ & + (1-\rho_r) \Bigg[(w_{jt} - \mu_{jt})\ q_{jt}(\mathbf{p}^*_t) + \sum_{k \in \mathscr{J}_m \setminus \{j\}} (w^*_{kt} - \mu_{kt})\ q_{kt}(\mathbf{p}^*_t) \Bigg]\end{aligned} \tag{C.3a}$$

$$\begin{aligned}\pi_{rt}(\rho_r) =& \rho_r \Bigg[\Big(p_{jt}(w_{jt}, \mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}) - w_{jt} - c_{jt}\Big)\ q_{jt}(\mathbf{p}_{rt}(w_{jt}, \mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) \\ & \quad + \sum_{k \in \mathscr{J}_r \setminus \{j\}} \Big(p_{kt}(w_{jt}, \mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}) - w^*_{kt} - c_{kt}\Big)\ q_{kt}(\mathbf{p}_{rt}(w_{jt}, \mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) \Bigg] \\ & + (1-\rho_r) \Bigg[(p^*_{jt} - w_{jt} - c_{jt}) q_{jt}(\mathbf{p}^*_t) + \sum_{k \in \mathscr{J}_r \setminus \{j\}} (p^*_{kt} - w^*_{kt} - c_{kt})\ q_{kt}(\mathbf{p}^*_t) \Bigg]\end{aligned} \tag{C.3b}$$

where $\mathbf{w}^*_{rt,-j}$ denotes the vector of equilibrium wholesale prices for all products distributed by retailer r except product $j \in \mathscr{J}_r$, and $\mathbf{p}^*_t$ is the vector of equilibrium retail prices. When $w_{jt} = w^*_{jt}$, these profits boil down to $\pi_{mt} = \sum_{j \in \mathscr{J}_m} (w^*_{jt} - \mu_{jt}) q_{jt}(\mathbf{p}^*_t)$ and $\pi_{rt} = \sum_{j \in \mathscr{J}_r} (p^*_{jt} - w^*_{jt} - c_{jt}) q_{jt}(\mathbf{p}^*_t)$ and the dependence on ρ_r disappears.

The terms $\pi^{-j}_{mt}(\rho_r)$ and $\pi^{-j}_{rt}(\rho_r)$ in (C.2) denote the status quo payoffs of manufacturer m and retailer r in the event of a disagreement, which are respectively given by:

$$\begin{aligned}\pi^{-j}_{mt}(\rho_r) =& \rho_r \sum_{k \in \mathscr{J}_m \setminus \{j\}} (w^*_{kt} - \mu_{kt})\ q^{-j}_{kt}(\mathbf{p}^{-j}_{rt}(\mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) \\ & + (1-\rho_r) \sum_{k \in \mathscr{J}_m \setminus \{j\}} (w^*_{kt} - \mu_{kt})\ q^{-j}_{kt}(\mathbf{p}^*_{-jt})\end{aligned} \tag{C.4a}$$

$$\begin{aligned}\pi^{-j}_{rt}(\rho_r) =& \rho_r \sum_{k \in \mathscr{J}_r \setminus \{j\}} \Big(p^{-j}_{kt}(\mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}) - w^*_{kt} - c_{kt}\Big)\ q^{-j}_{kt}(\mathbf{p}^{-j}_{rt}(\mathbf{w}^*_{rt,-j}; \mathbf{p}^*_{-rt}), \mathbf{p}^*_{-rt}) \\ & + (1-\rho_r) \sum_{k \in \mathscr{J}_r \setminus \{j\}} (p^*_{kt} - w^*_{kt} - c_{kt})\ q^{-j}_{kt}(\mathbf{p}^*_{-jt})\end{aligned} \tag{C.4b}$$

where $\mathbf{p}^*_{-jt}$ is the vector of equilibrium retail prices for all products except j.

Using (C.3a), (C.3b), (C.4a), and (C.4b), we can derive the first-order condition of (C.2) as follows:

$$
\begin{aligned}
&\frac{\frac{\partial \pi_{mt}(\rho_r)}{\partial w_{jt}}}{\pi_{mt}(\rho_r)-\pi_{mt}^{-j}(\rho_r)} = -\frac{1-\lambda_j}{\lambda_j}\frac{\frac{\partial \pi_{rt}(\rho_r)}{\partial w_{jt}}}{\pi_{rt}(\rho_r)-\pi_{rt}^{-j}(\rho_r)}\\
\Leftrightarrow &\frac{\rho_r q_{jt}(\mathbf{p}_{rt}(w_{jt}^*,\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-rt}^*)+(1-\rho_r)q_{jt}(\mathbf{p}_t^*)+\rho_r\sum_{k\in\mathcal{J}_m}(w_{kt}^*-\mu_{kt})\sum_{h\in\mathcal{J}_r}\frac{\partial q_{kt}}{\partial p_{ht}}\frac{\partial p_{ht}}{\partial w_{jt}}}{\sum_{k\in\mathcal{J}_m}(w_{kt}^*-\mu_{jt})\Big[\rho_r\big(q_{kt}(\mathbf{p}_{rt}(w_{jt}^*,\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-rt}^*)-q_{kt}^{-j}(\mathbf{p}_{rt}^{-j}(\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-rt}^*)\big)+(1-\rho_r)\big(q_{kt}(\mathbf{p}_t^*)-q_{kt}^{-j}(\mathbf{p}_{-jt}^*)\big)\Big]}\\
&= -\frac{\lambda_j}{1-\lambda_j}\frac{\frac{\partial \pi_{rt}(\rho_r)}{\partial w_{jt}}}{\pi_{rt}(\rho_r)-\pi_{rt}^{-j}(\rho_r)}
\end{aligned}
$$

where by convention $q_{jt}^{-j}(\mathbf{p}_{rt}^{-j}(\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-rt}^*) = q_{jt}^{-j}(\mathbf{p}_{-jt}^*) = 0$. At $w_{jt} = w_{jt}^*$, we have $q_{jt}(\mathbf{p}_{rt}(w_{jt}^*,\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-rt}^*) = q_{jt}(\mathbf{p}_t^*)$ and $\pi_{rt}(\rho_r) = \pi_{rt}$ (the dependence on ρ_r disappears). Moreover, as shown in Appendix D, we have $\frac{\partial \pi_{rt}}{\partial w_{jt}} = -q_{jt}(\mathbf{p}_t^*)$. Hence, the first-order condition of (C.2) boils down to:

$$
\begin{aligned}
&\frac{q_{jt}(\mathbf{p}_t^*)+\rho_r\sum_{k\in\mathcal{J}_m}(w_{kt}^*-\mu_{kt})\sum_{h\in\mathcal{J}_r}\frac{\partial q_{kt}}{\partial p_{ht}}\frac{\partial p_{ht}}{\partial w_{jt}}}{\sum_{k\in\mathcal{J}_m}(w_{kt}^*-\mu_{jt})\Big[q_{kt}(\mathbf{p}_t^*)-\rho_r q_{kt}^{-j}(\mathbf{p}_{rt}^{-j}(\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-rt}^*)-(1-\rho_r)q_{kt}^{-j}(\mathbf{p}_{-jt}^*)\Big]} = \frac{\lambda_j}{1-\lambda_j}\frac{q_{jt}(\mathbf{p}_t^*)}{\pi_{rt}-\pi_{rt}^{-j}(\rho_r)}\\
\Leftrightarrow &\left[\sum_{k\in\mathcal{J}_m}(w_{kt}^*-\mu_{kt})\rho_r\sum_{h\in\mathcal{J}_r}\frac{\partial q_{kt}}{\partial p_{ht}}\frac{\partial p_{ht}}{\partial w_{jt}}+q_{jt}(\mathbf{p}_t^*)\right]\left(\pi_{rt}-\pi_{rt}^{-j}(\rho_r)\right)\\
&=\sum_{k\in\mathcal{J}_m}(w_{kt}^*-\mu_{jt})\Big[q_{kt}(\mathbf{p}_t^*)-\rho_r q_{kt}^{-j}(\mathbf{p}_{rt}^{-j}(\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-rt}^*)-(1-\rho_r)q_{kt}^{-j}(\mathbf{p}_{-jt}^*)\Big]\frac{\lambda_j}{1-\lambda_j}q_{jt}(\mathbf{p}_t^*)\\
\Leftrightarrow &\sum_{k\in\mathcal{J}_m}(w_{kt}^*-\mu_{kt})\Bigg[\Big(q_{kt}(\mathbf{p}_t^*)-\rho_r q_{kt}^{-j}(\mathbf{p}_{rt}^{-j}(\mathbf{w}_{rt,-j}^*;\mathbf{p}_{-rt}^*),\mathbf{p}_{-rt}^*)-(1-\rho_r)q_{kt}^{-j}(\mathbf{p}_{-jt}^*)\Big)\frac{1-\lambda_j}{\lambda_j}q_{jt}(\mathbf{p}_t^*)\\
&-\rho_r\sum_{h\in\mathcal{J}_r}\frac{\partial q_{kt}}{\partial p_{ht}}\frac{\partial p_{ht}}{\partial w_{jt}}\left(\pi_{rt}-\pi_{rt}^{-j}(\rho_r)\right)\Bigg] = \left(\pi_{rt}-\pi_{rt}^{-j}(\rho_r)\right)q_{jt}(\mathbf{p}_t^*) \qquad \text{(C.5)}
\end{aligned}
$$

Manufacturers' price-cost margins. For each market t, we have a total of J price-cost margins of manufacturers, i.e., $w_{jt}^*-\mu_{jt}\ \forall j\in\mathcal{J}$. In a spirit similar to the price-cost margins of retailers, we rely on a system of J equations where (C.5) is the jth equation. More specifically, it can be shown that (C.5) is the jth equation of the following system of Nash-in-Nash first order conditions written in matrix form as follows:

$$
\begin{aligned}
&\Bigg[\Big(\mathbf{Q}_t\odot\mathbf{\Omega}_M\Big)\left(\text{diag}\left(\left(\frac{1-\lambda_j}{\lambda_j}\right)_{j=1}^J\right)\odot\left(\mathbf{q}_t\mathbf{1}_J^\top\right)\right)\\
&-\text{diag}\left(\left(\rho_{r(j)}\right)_{j=1}^J\right)\left(\left(\left(\boldsymbol{\pi}_{Rt}-\boldsymbol{\pi}_{Rt}^{out}\right)\mathbf{1}_J^\top\right)\odot\mathbf{I}\right)\Big(\left(\mathbf{P}_{\mathbf{w}_t}\mathbf{Q}_{\mathbf{p}_t}\right)\odot\mathbf{\Omega}_M\Big)\Bigg]\left(w_{jt}^*-\mu_{jt}\right)_{j=1}^J \qquad \text{(C.6)}\\
&= -\mathbf{q}_t\odot\left(\left(\boldsymbol{\pi}_{Rt}-\boldsymbol{\pi}_{Rt}^{out}\right)\mathbf{1}_J^\top\right)
\end{aligned}
$$

where $\top$ is the transpose operator, $\odot$ refers to the element-wise (Hadamard) product, $\mathbf{1}_J$ is an all-ones vector of dimension J (every element is equal to 1), $\mathbf{I}$ is the $J\times J$ identity matrix, and $\text{diag}\left(\left(\frac{1-\lambda_j}{\lambda_j}\right)_{j=1}^J\right)$ and $\text{diag}\left(\left(\rho_{r(j)}\right)_{j=1}^J\right)$ are two $J\times J$ diagonal matrices constructed from the J-dimensional vectors $\left(\frac{1-\lambda_j}{\lambda_j}\right)_{j=1}^J$ and $\left(\rho_{r(j)}\right)_{j=1}^J$, respectively. Additionally, $\mathbf{q}_t=(q_{1t},\ldots,q_{Jt})$ is the J-dimensional vector of demand at equilibrium retail prices, $\mathbf{\Omega}_M$ denotes the $J\times J$ ownership matrix of manufacturers where the (j,k)th element equals 1 if products j and k are owned by the same manufacturer in market t and 0 otherwise, and $\mathbf{Q}_{\mathbf{p}_t}$ is the $J \times J$ matrix of first partial derivatives of demand with respect to retail prices where the (j,k)th element equals $\frac{\partial q_{kt}}{\partial p_{jt}}$. We now describe the terms $\mathbf{P}_{\mathbf{w}_t}$, $\mathbf{Q}_t$, $\boldsymbol{\pi}_{Rt}$, and $\boldsymbol{\pi}_{Rt}^{out}$ in (C.6). The term $\mathbf{P}_{\mathbf{w}_t}$ refers to the $J \times J$ matrix of retail pass-through, where the (j,k)th element equals $\frac{\partial p_{kt}}{\partial w_{jt}}$ if $k \in \mathscr{J}_{r(j)}$ and 0 otherwise. This matrix solely depends on the downstream market structure and the Jacobian and Hessian matrices of demand with respect to retail prices (see, e.g., Bonnet, Bouamra-Mechemache and Molina, forthcoming, for computational details). The term $\mathbf{Q}_t$ is a $J \times J$ matrix given by:

$$
\begin{aligned}
\mathbf{Q}_t = &\left(\left(\rho_{r(j)}\right)_{j=1}^{J} \mathbf{1}_J^\top\right) \begin{pmatrix} q_{1t}(\mathbf{p}_t^*) & -q_{2t}^{-1}(\mathbf{p}_{r(1)t}^{-1}, \mathbf{p}_{-r(1)t}^*) & \cdots & -q_{Jt}^{-1}(\mathbf{p}_{r(1)t}^{-1}, \mathbf{p}_{-r(1)t}^*) \\ -q_{1t}^{-2}(\mathbf{p}_{r(2)t}^{-2}, \mathbf{p}_{-r(2)t}^*) & q_{2t}(\mathbf{p}_t^*) & \cdots & -q_{Jt}^{-2}(\mathbf{p}_{r(2)t}^{-2}, \mathbf{p}_{-r(2)t}^*) \\ \vdots & \vdots & \ddots & \vdots \\ -q_{1t}^{-J}(\mathbf{p}_{r(J)t}^{-J}, \mathbf{p}_{-r(J)t}^*) & -q_{2t}^{-J}(\mathbf{p}_{r(J)t}^{-J}, \mathbf{p}_{-r(J)t}^*) & \cdots & q_{Jt}(\mathbf{p}_t^*) \end{pmatrix} \\
&+ \left(\left(1-\rho_{r(j)}\right)_{j=1}^{J} \mathbf{1}_J^\top\right) \begin{pmatrix} q_{1t}(\mathbf{p}_t^*) & -q_{2t}^{-1}(\mathbf{p}_{-1t}^*) & \cdots & -q_{J_t t}^{-1}(\mathbf{p}_{-1t}^*) \\ -q_{1t}^{-2}(\mathbf{p}_{-2t}^*) & q_{2t}(\mathbf{p}_t^*) & \cdots & -q_{Jt}^{-2}(\mathbf{p}_{-2t}^*) \\ \vdots & \vdots & \ddots & \vdots \\ -q_{1t}^{-J}(\mathbf{p}_{-Jt}^*) & -q_{2t}^{-J}(\mathbf{p}_{-Jt}^*) & \cdots & q_{Jt}(\mathbf{p}_t^*) \end{pmatrix}
\end{aligned}
$$

Finally, the terms $\boldsymbol{\pi}_{Rt}$ and $\boldsymbol{\pi}_{Rt}^{out}$ are two J-dimensional vectors given by:

$$
\boldsymbol{\pi}_{Rt} = \left(\pi_{r(j)t}\right)_{j=1}^{J} = \left(\sum_{k \in \mathscr{J}_{r(j)}} (p_{kt}^* - w_{kt}^* - c_{kt}) q_{kt}(\mathbf{p}_t^*)\right)_{j=1}^{J} = \left(-\left(\boldsymbol{\Omega}_R \odot \mathbf{Q}_{\mathbf{p}_t}\right)^{-1} \mathbf{q}_t\right) \odot \mathbf{q}_t
$$

and

$$
\begin{aligned}
\boldsymbol{\pi}_{Rt}^{out} &= \left(\rho_{r(j)}\right)_{j=1}^{J} \odot \left(\pi_{r(j)t}^{-j}(\mathbf{p}_{r(j)t}^{-j}, \mathbf{p}_{-r(j)t}^*)\right)_{j=1}^{J} + \left(\left(1-\rho_{r(j)}\right)_{j=1}^{J}\right) \odot \left(\pi_{r(j)t}^{-j}(\mathbf{p}_{-jt}^*)\right)_{j=1}^{J} \\
\Leftrightarrow \boldsymbol{\pi}_{Rt}^{out} &= \left(\rho_{r(j)}\right)_{j=1}^{J} \odot \left(\sum_{k \in \mathscr{J}_{r(j)} \setminus \{j\}} (p_{kt}^{-j} - w_{kt}^* - c_{kt}) q_{kt}^{-j}(\mathbf{p}_{r(j)t}^{-j}, \mathbf{p}_{-r(j)t}^*)\right)_{j=1}^{J} \\
&\quad + \left(\left(1-\rho_{r(j)}\right)_{j=1}^{J}\right) \odot \left(\sum_{k \in \mathscr{J}_{r(j)} \setminus \{j\}} (p_{kt}^* - w_{kt}^* - c_{kt}) q_{kt}^{-j}(\mathbf{p}_{-jt}^*)\right)_{j=1}^{J} \\
\Leftrightarrow \boldsymbol{\pi}_{Rt}^{out} &= \left(\rho_{r(j)}\right)_{j=1}^{J} \\
&\quad \odot \left[\boldsymbol{\Omega}_R \odot \boldsymbol{\Delta}_{Rt}^{out} \odot \begin{pmatrix} 0 & q_{2t}^{-1}(\mathbf{p}_{r(1)t}^{-1}, \mathbf{p}_{-r(1)t}^*) & \cdots & q_{Jt}^{-1}(\mathbf{p}_{r(1)t}^{-1}, \mathbf{p}_{-r(1)t}^*) \\ q_{1t}^{-2}(\mathbf{p}_{r(2)t}^{-2}, \mathbf{p}_{-r(2)t}^*) & 0 & \cdots & q_{Jt}^{-2}(\mathbf{p}_{r(2)t}^{-2}, \mathbf{p}_{-r(2)t}^*) \\ \vdots & \vdots & \ddots & \vdots \\ q_{1t}^{-J}(\mathbf{p}_{r(J)t}^{-J}, \mathbf{p}_{-r(J)t}^*) & q_{2t}^{-J}(\mathbf{p}_{r(J)t}^{-J}, \mathbf{p}_{-r(J)t}^*) & \cdots & 0 \end{pmatrix} \mathbf{1}_J\right] \\
&\quad + \left(1-\rho_{r(j)}\right)_{j=1}^{J} \odot \left[\boldsymbol{\Omega}_R \odot \begin{pmatrix} 0 & q_{2t}^{-1}(\mathbf{p}_{-1t}^*) & \cdots & q_{Jt}^{-1}(\mathbf{p}_{-1t}^*) \\ q_{1t}^{-2}(\mathbf{p}_{-2t}^*) & 0 & \cdots & q_{Jt}^{-2}(\mathbf{p}_{-2t}^*) \\ \vdots & \vdots & \ddots & \vdots \\ q_{1t}^{-J}(\mathbf{p}_{-Jt}^*) & q_{2t}^{-J}(\mathbf{p}_{-Jt}^*) & \cdots & 0 \end{pmatrix} \left(-\left(\boldsymbol{\Omega}_R \odot \mathbf{Q}_{\mathbf{p}_t}\right)^{-1} \mathbf{q}_t\right)\right]
\end{aligned}
$$

where $\boldsymbol{\Delta}_{Rt}^{out}$ is a $J \times J$ matrix of out-of-equilibrium retail price-cost margins with the (j,k)th element being 0 if $j=k$, $p_{kt}^{-j} - w_{kt}^* - c_{kt}$ if $k \in \mathscr{J}_{r(j)} \setminus \{j\}$, and $p_{kt}^* - w_{kt}^* - c_{kt}$ otherwise. Similar to the vector of equilibrium retail price-cost margins $(p^*_{jt} - w^*_{jt} - c_{jt})^J_{j=1}$, every out-of-equilibrium retail price-cost margin in $\mathbf{\Delta}^{out}_{Rt}$ solely depends on demand functions and its derivatives as well as the downstream market structure. For example, $p^{-j}_{kt} - w^*_{kt} - c_{kt}$ is the kth element of the following $(|\mathscr{J}_{r(j)}|-1)$-dimensional vector $-\left(\mathbf{\Omega}_{r(j)} \odot \mathbf{Q}_{\mathbf{p}^{-j}_t}\right)^{-1} \mathbf{q}^{-j}_{r(j)t}(\mathbf{p}^{-j}_{r(j)t}, \mathbf{p}^*_{-r(j)t})$, where $\mathbf{\Omega}_{r(j)}$ and $\mathbf{Q}_{\mathbf{p}^{-j}_t}$ are two $(|\mathscr{J}_{r(j)}|-1) \times (|\mathscr{J}_{r(j)}|-1)$ matrices analogous to $\mathbf{\Omega}_R$ and $\mathbf{Q}_{\mathbf{p}_t}$, but restricted to retailer $r(j)$'s products.

Let $\mathbf{D}_t$ be the $J \times J$ matrix on the left-hand side of (C.6). Then, we can rewrite (C.6) as:

$$\mathbf{D}_t \left(w^*_{jt} - \mu_{jt}\right)^J_{j=1} = -\mathbf{q}_t \odot \left(\left(\boldsymbol{\pi}_{Rt} - \boldsymbol{\pi}^{out}_{Rt}\right) \mathbf{1}^\top_J \right) \tag{C.7}$$

When $\mathbf{D}_t$ is invertible (see Appendices E and F), the J-dimensional vector of price-cost margins of manufacturers can be derived from (C.7) as follows:

$$\left(w^*_{jt} - \mu_{jt}\right)^J_{j=1} = -\mathbf{D}^{-1}_t \mathbf{q}_t \odot \left(\left(\boldsymbol{\pi}_{Rt} - \boldsymbol{\pi}^{out}_{Rt}\right) \mathbf{1}^\top_J \right) \tag{C.8}$$

D Joint profit and slope of the bargaining frontier

Consider the bilateral negotiation between manufacturer m and retailer r over w_{jt}, taking all other wholesale prices as given. First, note that the partial derivatives of manufacturer m's and retailer r's profit with respect to w_{jt}, evaluated at the equilibrium wholesale and retail prices, are given by:

$$\begin{aligned}
\frac{\partial \pi_{rt}(\rho_r)}{\partial w_{jt}} &= \rho_r \left[-q^*_{jt} + \sum_{h \in \mathscr{J}_r} \frac{\partial p_{ht}}{\partial w_{jt}} q^*_{ht} + \sum_{k \in \mathscr{J}_r} (p^*_{kt} - w^*_{kt} - c_{kt}) \sum_{h \in \mathscr{J}_r} \frac{\partial q_{kt}}{\partial p_{ht}} \frac{\partial p_{ht}}{\partial w_{jt}} \right] - (1-\rho_r) q^*_{jt} \\
&= \rho_r \left[-q^*_{jt} + \sum_{h \in \mathscr{J}_r} \frac{\partial p_{ht}}{\partial w_{jt}} \underbrace{\left(q^*_{ht} + \sum_{k \in \mathscr{J}_r} (p^*_{kt} - w^*_{kt} - c_{kt}) \frac{\partial q_{kt}}{\partial p_{ht}} \right)}_{= \frac{\partial \pi_{rt}}{\partial p_{ht}} = 0} \right] - (1-\rho_r) q^*_{jt} \\
&= -q^*_{jt} \\
\frac{\partial \pi_{mt}(\rho_r)}{\partial w_{jt}} &= \rho_r \left[q^*_{jt} + \sum_{k \in \mathscr{J}_m} (w^*_{kt} - \mu_{kt}) \sum_{l \in \mathscr{J}_r} \frac{\partial q_{kt}}{\partial p_{lt}} \frac{\partial p_{lt}}{\partial w_{jt}} \right] + (1-\rho_r) q^*_{jt} \\
&= \rho_r \sum_{k \in \mathscr{J}_m} (w^*_{kt} - \mu_{kt}) \sum_{l \in \mathscr{J}_r} \frac{\partial q_{kt}}{\partial p_{lt}} \frac{\partial p_{lt}}{\partial w_{jt}} + q^*_{jt}
\end{aligned}$$

Hence, we have

$$\frac{\partial \left(\pi_{mt}(\rho_r) + \pi_{rt}(\rho_r)\right)}{\partial w_{jt}} = \rho_r \sum_{k \in \mathscr{J}_m} (w^*_{kt} - \mu_{kt}) \sum_{h \in \mathscr{J}_r} \frac{\partial q_{kt}}{\partial p_{ht}} \frac{\partial p_{ht}}{\partial w_{jt}}.$$

Second, note that at the equilibrium wholesale and retail prices, $\frac{\partial \pi_{rt}(\rho_r)}{\partial w_{jt}} < 0$. This strict monotonicity implies that one can write w_{jt} as a function of $\pi_{rt}(\rho_r)$ along the path of equilibrium outcomes. Then, the slope of the bargaining frontier at the equilibrium profit outcomes is:

$$\begin{aligned}
\frac{\Delta\pi_{mt}(\pi_{rt}(w_{jt}^*, \mathbf{w}_{-jt}^*))}{\Delta\pi_{rt}(w_{jt}, \mathbf{w}_{-jt}^*)} &= \frac{\Delta\pi_{mt}(\pi_{rt}(w_{jt}^*, \mathbf{w}_{-jt}^*))/\Delta w_{jt}}{\Delta\pi_{rt}(w_{jt}, \mathbf{w}_{-jt}^*)/\Delta w_{jt}} \\
&= \frac{q_{jt}^* + \rho_r \sum\limits_{k\in\mathscr{J}_m} (w_{kt}^* - \mu_{kt}) \sum\limits_{h\in\mathscr{J}_r} \frac{\partial q_{kt}}{\partial p_{ht}} \frac{\partial p_{ht}}{\partial w_{jt}}}{-q_{jt}^*} \\
&= -1 - \rho_r \frac{\sum_{k\in\mathscr{J}_m} (w_{kt}^* - \mu_{kt}) \sum_{l\in\mathscr{J}_r} \frac{\partial q_{kt}}{\partial p_{lt}} \frac{\partial p_{lt}}{\partial w_{jt}}}{q_{jt}^*} \\
&= -1 - \frac{1}{q_{jt}^*} \frac{\partial\left(\pi_{mt}(\rho_r) + \pi_{rt}(\rho_r)\right)}{\partial w_{jt}}.
\end{aligned}$$

The term -1 simply reflects the mechanical inverse relationship between the changes in retailer r's and manufacturer m's profits with respect to w_{jt} (i.e., a zero-sum relationship). The second term represents the rate of surplus transfer from retailer r to manufacturer m through w_{jt} (i.e., the steepness of the bargaining frontier), which is directly related to $\frac{\partial(\pi_{mt}(\rho_r)+\pi_{rt}(\rho_r))}{\partial w_{jt}}$.

E Proof of Theorem 1

Without loss of generality, we fix the observed cost variables v_j and abuse the notation $\mu_j = \text{mc}_j(\omega_j)$. Our result can be then understood as being conditioned on the observed cost variables. We remove notation t to simplify the exposition. We assume away the pathological case in which the determinant of $\mathbf{D}(\mathbf{q}^*, \mathbf{p}^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \Omega_M$ is a constant function.

At $\mathbf{p}^*$ and $\mathbf{w}^*$, we derive the FOCs for the downstream price competition: for $j \in \mathscr{J}_r$,

$$q_j(\mathbf{p}^*; \boldsymbol{\xi}) + \sum_{k\in\mathscr{J}_r} (p_k^* - w_k^*) \frac{\partial q_k(\mathbf{p}^*; \boldsymbol{\xi})}{\partial p_j} = 0. \tag{E.1}$$

Denote by $\mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w})$ the correspondence from $(\boldsymbol{\xi}, \mathbf{w})$ to $\mathbf{p}$ determined by (E.1). We then have $\mathbf{p}^* \in \mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w}^*)$. Denote by $\mathbf{w}^e(\boldsymbol{\xi}, \boldsymbol{\mu}; \boldsymbol{\lambda}, \boldsymbol{\rho})$ the correspondence from $(\boldsymbol{\xi}, \boldsymbol{\mu})$ to $\mathbf{w}$ determined by (6). We then have $\mathbf{w}^* \in \mathbf{w}^e(\boldsymbol{\xi}, \boldsymbol{\mu}; \boldsymbol{\lambda}, \boldsymbol{\rho})$.

Assumption E.1.

(i). $\frac{\partial \mathbf{q}}{\partial \mathbf{p}}$ *is negative definite.*

(ii). $(\boldsymbol{\xi}, \boldsymbol{\omega})$ *is absolutely continuous with respect to Lebesgue measure in* $\mathbb{R}^{2J}$*. Denote by* $f_{(\xi,\omega)}$ *their (conditional) density function. Moreover,* $\mu_j(\cdot)$ *is continuous in* ω_j*.*

(iii). $\mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w})$ *is a vector of continuous functions of* $\mathbf{w}$ *for any* $\boldsymbol{\xi}$ *in its domain* Ξ*.*

(iv). $\mathbf{w}^e(\boldsymbol{\xi}, \boldsymbol{\mu}; \boldsymbol{\lambda}, \boldsymbol{\rho})$ *is a vector of continuous functions of* $\boldsymbol{\mu}$ *for any* $\boldsymbol{\xi} \in \Xi$*.*

(v). $p_k^{-j}(\mathbf{w}_{\mathscr{F}_r/\{j\}}; \mathbf{p}_{-\mathscr{J}_r})$ *is a continuous function of* $\mathbf{w}_{\mathscr{F}_r/\{j\}}$ *for any* $k \neq j$*,* $k \in \mathscr{F}_{r(j)}$*.*

Assumption E.1(i) strengthens the negative semidefinite Slutsky matrix to negative definite. It is compatible with the connected substitutes conditions (Berry, Gandhi and Haile, 2013) as well as demand with complementarity (Wang, 2021). Assumption E.1(ii) is satisfied when the demand and supply shocks are continuous. Assumptions E.1(iii) and (iv) require the downstream pricing competition and the upstream Nash bargaining to have unique (and continuous) outcomes, both of which are often necessary for empirical research and counterfactual simulations. Similarly, Assumption E.1(v) requires retailer's re-optimization of the downstream prices in the case of bargaining disagreement to have unique (and continuous) outcomes. This is equivalent to that given the equilibrium prices of products owned by retailers other than r, the FOCs of the profit maximization when one of r's product is removed have a unique (and continuous) solution.

The proof proceeds in four steps.

Step 1. $\mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w})$ is almost everywhere (a.e.) differentiable with respect to $\mathbf{w}$ for given $\boldsymbol{\xi}$.
Using Assumption E.1(i), we can invert (E.1) and express the price-cost margins of retailers as:[28]

$$\mathbf{p} - \mathbf{w} = -\left(\mathbf{O}_R \circ \frac{\partial \mathbf{q}}{\partial \mathbf{p}}\right)^{-1} \mathbf{q}. \tag{E.2}$$

(E.2) defines a continuously differentiable mapping from $\mathbf{p} \in \mathbb{R}^J$ to $\mathbf{w} \in \mathbb{R}^J$. According to Sard's theorem, the set $\tilde{\mathscr{W}} = \left\{\mathbf{w} \in \mathbb{R}^J : \text{Det}\left(\frac{\partial \mathbf{w}}{\partial \mathbf{p}}(\mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w}))\right) = 0\right\}$ is of zero Lebesgue measure. Then, on $\mathbb{R}^J \backslash \tilde{\mathscr{W}}$, according to the Inverse Function Theorem, $\frac{\partial \mathbf{p}^e}{\partial \mathbf{w}}$ exists and equal to $\left[\frac{\partial \mathbf{w}}{\partial \mathbf{p}}(\mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w}))\right]^{-1}$.

Step 2. $\mathbf{p} = \mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w})$ is real analytic with respect to $\mathbf{w} \in \mathbb{R}^J \setminus \tilde{\mathscr{W}}$. Moreover, $\mathbb{R}^J \setminus \tilde{\mathscr{W}}$ is a union of up to countable connected open sets.
Because $q_j(\mathbf{p}; \boldsymbol{\xi})$ is real analytic with respect to $\mathbf{p} \in \mathbb{R}^J$, combining this with (E.2), we then obtain that $\mathbf{w}$ can be written as a real analytic function of $\mathbf{p}$. Then, according to the Inverse Function Theorem, $\mathbf{p} = \mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w})$ is real analytic with respect to $\mathbf{w} \in \mathbb{R}^J \setminus \tilde{\mathscr{W}}$.

Note that the set $\tilde{\mathscr{P}} = \left\{\mathbf{p} \in \mathbb{R}^J : \text{Det}\left(\frac{\partial \mathbf{w}}{\partial \mathbf{p}}(\mathbf{p})\right) = 0\right\}$ defines the zero set of the real analytic function $\text{Det}\left(\frac{\partial \mathbf{w}}{\partial \mathbf{p}}(\mathbf{p})\right)$. It can then be written as $\cup_{k=0}^{J-1} S_k$, where S_k is a k-dimensional submanifold in $\mathbb{R}^J$ (Theorem 6.3.3 on page 168 in Krantz and Parks (2002)). Consequently, $\mathbb{R}^J \setminus \tilde{\mathscr{P}}$ is an union of up to countable connected open sets. Because of the continuity of $\mathbf{p}^e(\boldsymbol{\xi}, \mathbf{w})$ with respect to $\mathbf{w}$ (Assumption E.1(iii)), $\mathbb{R}^J \setminus \tilde{\mathscr{W}} = (\mathbf{p}^e)^{-1}(\boldsymbol{\xi}; \mathbb{R}^J \setminus \tilde{\mathscr{P}})$ is also an union of up to countable connected open sets.

Analogously, we can prove that the best response function of retailer r for product $j \in \mathscr{F}_r$, $p_j(\mathbf{w}; \mathbf{p}_{-\mathscr{F}_r})$, is real analytic with respect to $\mathbf{w}$ for given $\mathbf{p}_{-\mathscr{F}_r}$. Then, $p_j(\mathbf{w}; \mathbf{p}^e_{-\mathscr{F}_r}(\boldsymbol{\xi}, \mathbf{w}))$ is real analytic with respect to $\mathbf{w}$. In addition, the price adjustment function for product $k \neq j$ and $k \in \mathscr{F}_{r(j)}$, $p_k^{-j}(\mathbf{w}_{\mathscr{F}_r/\{j\}}; \mathbf{p}_{-\mathscr{J}_r})$ is real analytic with respect to $\mathbf{w}_{\mathscr{F}_r/\{j\}}$ given $\mathbf{p}_{-\mathscr{J}_r}$. As a result, $p_k^{-j}(\mathbf{w}_{\mathscr{F}_r/\{j\}}; \mathbf{p}^e_{-\mathscr{F}_r}(\boldsymbol{\xi}, \mathbf{w}))$ is real analytic with respect to $\mathbf{w}$. To simplify the exposition, we abuse the notation $\tilde{W}$ to denote the finite unions of all the zero measure sets (which has still zero Lebesgue measure) of these real analytic functions. Then, $\mathbb{R}^J \setminus \tilde{W}$ that is still a union of up to countable connected open sets.

Step 3. The matrix $\mathbf{D}(\mathbf{q}^*, \mathbf{p}^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \Omega_M$ is a.e. (in the space of $\mathbf{w}$) non-singular.

Because of the result in Step 2, at the true $\boldsymbol{\lambda}_0$ and $\boldsymbol{\rho}_0$, each component in $\mathbf{D}(\mathbf{q}^*, \mathbf{p}^*; \boldsymbol{\lambda}_0, \boldsymbol{\rho}_0) \odot \Omega_M$ is a real analytic function of $\mathbf{w} \in \mathbb{R}^J \setminus \tilde{\mathscr{W}}$. Then, its determinant is also real analytic with respect to $\mathbf{w} \in \mathbb{R}^J \setminus \tilde{\mathscr{W}}$ and therefore, according to Mityagin (2020), the set of zeros of the

[28] The ownership matrix $\mathbf{O}_R$ is a block-diagonal matrix with each block being a square matrix of 1s. Because $\frac{\partial \mathbf{q}}{\partial \mathbf{p}}$ is negative definite, then $-\frac{\partial \mathbf{q}}{\partial \mathbf{p}}$ is positive definite and therefore a P-matrix (Gale and Nikaidô, 1965). Then, all the principal minors of $\frac{\partial \mathbf{q}}{\partial \mathbf{p}}$ are negative and $\mathbf{O}_R \circ \frac{\partial \mathbf{q}}{\partial \mathbf{p}}$ is invertible.

determinant in $\mathbb{R}^J \setminus \tilde{\mathscr{W}}$ is of zero Lebesgue measure. Moreover, $\tilde{\mathscr{W}}$ is also of zero Lebesgue measure. As a result, the determinant is a.e. non-zero in $\mathbb{R}^J$.

Step 4. Given $\boldsymbol{\xi}$, denote by $\mathscr{W}_{\text{nz}}$ the set of $\mathbf{w}$ such that the determinant of $\mathbf{D}(\mathbf{q}^*, \mathbf{p}^*; \boldsymbol{\lambda}_0, \boldsymbol{\rho}_0) \odot \Omega_M$ is non-zero. According to Step 3, $\mathscr{W}_{\text{nz}}$ has full Lebesgue measure in $\mathbb{R}^J$. Then, for any $\mathbf{w}' \in \mathscr{W}_{\text{nz}}$, by using (6), we can uniquely back out a $\boldsymbol{\mu}'$. Denote by Ω_μ the set of all such $\boldsymbol{\mu}'$. According to Assumption E.1(iv), $\mathbf{w} = \mathbf{w}^e(\boldsymbol{\xi}, \boldsymbol{\mu}; \boldsymbol{\lambda}, \boldsymbol{\rho})$ then defines an a.e. continuously differentiable bijection from Ω_μ to $\mathscr{W}_{\text{nz}}$. Using Assumption E.1(ii) and the monotonicity of $mc_j(\omega_j)$ in ω_j, we obtain:

$$\begin{aligned}
\Pr(\mathbf{w}^e(\boldsymbol{\xi}, \boldsymbol{\mu}; \boldsymbol{\lambda}, \boldsymbol{\rho}) \in \mathscr{W}_{\text{nz}}) &= \int d\boldsymbol{\xi} \int \mathbf{1}_{\mathbf{w}^e(\boldsymbol{\xi}, \mathbf{mc}(\boldsymbol{\omega}); \boldsymbol{\lambda}, \boldsymbol{\rho}) \in \mathscr{W}_{\text{nz}}} f_{(\xi,\omega)}(\boldsymbol{\xi}, \boldsymbol{\omega}) d\boldsymbol{\omega} \\
&= \int d\boldsymbol{\xi} \int \mathbf{1}_{\mathbf{w} \in \mathscr{W}_{\text{nz}}} f_{(\xi,w)}(\boldsymbol{\xi}, \mathbf{w}; \boldsymbol{\lambda}, \boldsymbol{\rho}) d\mathbf{w} \\
&= 1,
\end{aligned}$$

where $f_{(\xi,w)}(\cdot; \boldsymbol{\lambda}, \boldsymbol{\rho})$ is the density function of $(\boldsymbol{\xi}, \mathbf{w})$ given $(\boldsymbol{\lambda}, \boldsymbol{\rho})$. The proof is completed.

F Arguments for the Invertibility

In this appendix, we propose invertibility arguments that do not rely on the real analyticity in Theorem 1. To simplify the exposition, we remove the notation t. Consider the following restrictions on demand and supply.

Assumption F.1.

(i). For any $k \neq j$, $(\mathbf{w}, \mathbf{p})$ and $\rho \in \{0, 1\}$, $q_k^{-j}(\rho) \geq q_k(\rho)$ and $q_0^{-j}(\rho) > q_0(\rho)$. Moreover, $\frac{\partial q_j}{\partial p_j} < 0$ and $\frac{\partial q_j}{\partial p_k} \geq 0$ for $j, k = 1, ..., J$ and $j \neq k$.

(ii) (Single-product retailer) $\Omega_R = \mathbf{I}_{J \times J}$ and $\frac{\partial p_j}{\partial w_j} \geq 0$ for $j = 1, ..., J$.

Assumption F.1(i) restricts products to be weakly substitutes. It also captures the intuition that removing a product is at least a weakly positive demand shock for other products and a strictly positive shock for the outside option. Assumption F.1(ii) specifies downstream retailer to be single-product. Moreover, the dependence of each product's retail price on its wholesale price is non-negative.

Proposition 1. *Suppose that Assumption F.1 holds. Then, Det$(\mathbf{D}(\mathbf{q}^*, \mathbf{p}^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \Omega_M) \neq 0$.*

Proof. Note that at the equilibrium prices and the corresponding demand, $\frac{\partial \pi_r(\rho)}{\partial w_j} = -q_j^* < 0$ for any $\rho \in [0, 1]$ (see Appendix D). Besides, using Assumption F.1(i), we obtain that for any $\rho \in [0, 1]$ and $k \neq j$,

$$\begin{aligned}
q_k(\rho) - q_k^{-j}(\rho) &= \rho(q_k(1) - q_k^{-j}(1)) + (1 - \rho) q_k(0) - q_k^{-j}(0) \leq 0, \\
\sum_{k=1}^{J} \left(q_k(\rho) - q_k^{-j}(\rho)\right) &= \left[1 - \sum_{k=1}^{J} q_k(\rho)\right] - \left[1 - \sum_{k \neq j} q_k^{-j}(\rho)\right] \\
&= q_0^{-j}(\rho) - q_0^{-j}(\rho) \\
&> 0.
\end{aligned} \tag{F.1}$$

Then, matrix $\mathbf{D}_1 := \left[\frac{1-\lambda_j}{\lambda_j}\frac{\partial \pi_{r(j)}(\rho_{r(j)})}{\partial w_j}\left(q_k(\rho_{r(j)}) - q_k^{-j}(\rho_{r(j)})\right)\right]_{j,k=1,\dots,J}$ is strictly diagonally dominant with negative diagonal and non-negative off-diagonal elements.

When $\mathbf{O}_R$ is the identity matrix, (6) becomes

$$\sum_{k\in\mathcal{J}_m}(w_k^* - \mu_k)\Big[-\frac{1-\lambda_j}{\lambda_j}q_j^*\left(q_k(\rho_{r(j)}) - q_k^{-j}(\rho_{r(j)})\right) + \rho_{r(j)}\frac{\partial q_k}{\partial p_j}\frac{\partial p_j}{\partial w_j}(\pi_{r(j)}(\rho_{r(j)}) - \pi_{r(j)}^{-j}(\rho_{r(j)}))\Big] = -(\pi_{r(j)}(\rho_{r(j)}) - \pi_{r(j)}^{-j}(\rho_{r(j)}))q_j^*.$$

Because of Assumption F.1(i), then the matrix

$$\mathbf{D}_2 = \left(\rho_{r(j)}\frac{\partial q_k}{\partial p_j}\frac{\partial p_j}{\partial w_j}(\pi_{r(j)}(\rho_{r(j)}) - \pi_{r(j)}^{-j}(\rho_{r(j)}))\right)_{j,k=1,\dots,J}$$

has non-positive diagonals and non-negative off-diagonal elements. Then, $\mathbf{D}(\mathbf{q}^*, \mathbf{p}^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) = \mathbf{D}_1 + \mathbf{D}_2$ is strictly diagonally dominant and therefore a P-matrix (Gale and Nikaidô, 1965). Consequently, $\mathbf{D}(\mathbf{q}^*, \mathbf{p}^*; \boldsymbol{\lambda}, \boldsymbol{\rho}) \odot \Omega_M$, which is block-diagonal with the diagonal elements being the principal minors of $\mathbf{D}(\mathbf{q}^*, \mathbf{p}^*; \boldsymbol{\lambda}, \boldsymbol{\rho})$, has a positive determinant. The proof is completed. □

G Examples 1 and 2 : Details

Example 1. Under the linear demand condition, we have:

$$\begin{aligned} p_1(w_1;p_2) &= \frac{\delta_1 + w_1 + 0.5p_2}{2}, \quad p_1(w_1;p_2) - w_1 = \frac{\delta_1 + 0.5p_2 - w_1}{2}, \\ q_1(p_1(w_1;p_2), p_2) &= \frac{\delta_1 + 0.5p_2 - w_1}{2}, \\ p_2(w_2;p_1) &= \frac{\delta_2 + w_2 + 0.5p_1}{2}, \quad p_2(w_2;p_1) - w_2 = \frac{\delta_2 + 0.5p_1 - w_2}{2}, \\ q_2(p_1, p_2(w_2;p_1)) &= \frac{\delta_2 + 0.5p_1 - w_2}{2}. \end{aligned}$$

At $w = w*$, we have $p_1^* = p_1(w_1^*; p_2^*)$ and $p_2^* = p_2(w_2^*; p_1^*)$. Then,

$$\begin{aligned} p_1^* &= \frac{8}{15}(w_1^* + \delta_1) + \frac{2}{15}(w_2^* + \delta_2), \\ q_1^* &= p_1^* - w_1^* = \frac{8}{15}(\delta_1 + w_1^*) + \frac{2}{15}(\delta_2 + w_2^*) - w_1^*, \\ p_2^* &= \frac{2}{15}(w_1^* + \delta_1) + \frac{8}{15}(w_2^* + \delta_2), \\ q_2^* &= p_2^* - w_2^* = \frac{2}{15}(\delta_1 + w_1^*) + \frac{8}{15}(\delta_2 + w_2^*) - w_2^*. \end{aligned}$$

Plugging p_2^{-1}, p_1^{-2} and (q_1^*, q_2^*) in (6), we obtain (16).

Example 2. Under the exponential demand, we have

$$\begin{aligned} p(w) &= \frac{\varepsilon}{\varepsilon - 1}w, \quad p(w) - w = \frac{1}{\varepsilon - 1}w, \\ q(p(w)) &= \delta\left(\frac{\varepsilon}{\varepsilon - 1}\right)^{-\varepsilon} w^{-\varepsilon}. \end{aligned}$$

At $w = w^*$, we have $p^* = p(w^*)$, and $q^* = q(p*)$. Then, setting $w = w^*$ in (6) and plugging these quantities, we obtained the desired result.

Failure of identification: linear demand with $J = 1$. Consider a case in which there is a single manufacturer and a single retailer, producing and selling one product respectively. Moreover, the demand for this product is linear, $q(p) = \delta - \alpha p$. Then, using the FOC of the retailer's profit maximization in the frictionless state, we obtain

$$\begin{aligned} p(w) &= \frac{\beta + \alpha w}{2}, \quad p(w) - w = \frac{\beta - \alpha w}{2\alpha}, \\ q(p(w)) &= \frac{\beta - \alpha w}{2}. \end{aligned}$$

At $w = w^*$, we have $p^* = p(w^*)$, and $q^* = q(p^*)$. Then, setting $w = w^*$ in (6) and plugging these quantities, we obtain

$$w^* - \mu = \frac{1}{2}\left[\frac{1-\lambda}{\lambda} + \frac{\rho}{2} + \frac{1}{2}\right]^{-1}\left(\frac{\delta}{\alpha} - \mu\right).$$

The variation in demand shifter δ and/or rotator α can only identify $\frac{1-\lambda}{\lambda} + \frac{\rho}{2}$ but not separably λ and ρ. However, one can still identify τ in μ by varying the cost shifter in μ.