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Strategic commitment by an informed speculator*

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Abstract

We analyze speculation by an informed trader who can commit to her trading strategy in a Kyle-style dealership market. Market makers observe the exact parametric form of the speculator's trading strategy but not her private information and then price competitively given the net (informed plus noise trade) order flow. We derive necessary and sufficient conditions for the speculator not to profit from commitment. This imposes conditions on model primitives satisfied by Normally-distributed uncertainty that give rise to linear equilibria, but are generically not satisfied. With commitment the speculator may trade less aggressively after some signals, but more aggressively after others.

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1 Introduction

Consider a speculator who has designed a sophisticated trading algorithm. The algorithm may rely on information gleaned elsewhere, but otherwise has minimal real time human interaction—in essence, the trading algorithm is fixed over long durations although its information inputs arrive at a far higher frequency. This reflects that in the context of algorithmic high frequency trading, the trading algorithm is effectively "hardwired" into the code, and changing the code requires human participation and hence occurs at frequencies that are orders of magnitudes lower than the algorithmic trading itself (which typically occurs at millisecond or higher frequency). The speculator trades in a market with high frequency market makers who seek to reverse engineer the speculator's algorithm to uncover her trading strategy. One's first instinct might be that this reverse engineering should harm the speculator. But reflection suggests that this instinct is misplaced, as the market makers are learning the form[1] of the speculator's strategy, rather her information, itself. This means that, by unraveling the speculator's strategy, market makers may be providing the speculator a first-mover advantage, possibly yielding her higher profits.[2,3]

We analyze speculation by an informed trader who can commit to her trading strategy in a static Kyle-style (Kyle, 1985) competitive dealership market. In a dynamic setting, the informed trader anticipates the inferences market makers will draw regarding her intratemporal trading strategy and optimizes accordingly. Our static setting compresses this process to the limiting case where the speculator's strategy is common knowledge. Specifically, the speculator can commit to the functional form of her trading strategy; and market makers then price competitively given knowledge of this functional form (e.g., the linear parameter of a linear trading strategy), but not the trader's private information, setting the informationally-efficient pricing rule as a function of the net order flow from the speculator and noise traders. In the equilibrium of this Stackelberg setting, the speculator's trading strategy maximizes expected profits subject to market makers setting consistent informationally-efficient pricing rules. We show that in this Stackelberg setting,

[1] For example, if the strategy is linear in the speculator's signal, this form is characterized by the slope and intercept of this linear dependence.

[2] An illustration of an explicit open construction of strategies is RavenPack, which posts the algorithm used to read news and score sentiment on its website.

[3] Indeed, market makers have strong incentives to unravel a speculator's trading strategy, else they will misprice relative to other market makers who do unravel.

the speculator's optimal strategy maximizes the expected value of the error in the market makers' forecast of her trade. We establish existence of equilibrium and characterize the unique value of strategic commitment by the speculator.

We build on the analysis in Boulatov and Livdan (2024) establishing that a unique Nash equilibrium exists in the static trading model of Kyle (1984, 1985). The original Kyle (1985) model examines a Nash equilibrium in which a monopolistic informed trader chooses a possibly non-linear trading strategy to maximize profits and competitive market makers simultaneously choose a possibly non-linear pricing rule that generates zero expected market-maker profits conditional on any net order flow. Kyle (1985) shows that there is only one equilibrium in which the trading strategy and pricing rule are both linear functions. Using mild regularity conditions, Boulatov and Livdan prove the existence and uniqueness of a Nash equilibrium without imposing linearity assumptions and for a broad class of pdfs for the fundamental value v and random aggregate noise trader demand u. We extend this analysis to our Stackelberg setting, establishing that a payoff unique equilibrium exists.

We then characterize the value of strategic commitment. We ask: when can a speculator earn higher expected profits if she can commit to a particular trading strategy? In this setting, market makers break even in expectation, so the question becomes: when can committing to a trading strategy allow a speculator to extract greater profits from the noise traders? It is immediate that, with commitment, (i) the speculator's equilibrium payoffs are uniquely pinned down, and (ii) the speculator can do at least as well as in the standard Nash setting—the speculator can always commit to the Nash trading strategy to earn the same expected profit. The question becomes: under what circumstances can the speculator do no better with commitment?

We first establish constructively that the speculator can earn strictly higher expected profits in the Stackelberg equilibrium than in the Nash equilibrium. Specifically, we solve explicitly for equilibrium outcomes in the Cho and Karoui (2000) model, which maintains all assumptions of Kyle (1985) save that v has a Bernoulli distribution rather than a normal distribution, showing that the ability to commit to a trading strategy has value.

We then show that the speculator cannot always profit from an ability to commit to her trading strategy, but that these conditions are "rare". We derive closed-form analytic conditions on the pdfs of v and u for a risk-neutral speculator not to be able to profit from commitment, i.e., for the

Nash and Stackelberg equilibria to coincide. Importantly, these conditions are satisfied for linear equilibria, in particular when, as in Kyle (1985), the pdf f_v is equal to some linear rescaling of f_u. We then establish the knife-edge nature of this result: in the vicinity of linear equilibria, for almost all distributions f_v and f_u, commitment yields higher speculator profits. Finally, we show that even with normal uncertainty, relaxing risk neutrality is sufficient to ensure that a speculator can gain from committing to a less-aggressive trading strategy.

One's intuition may be that quite generally the source of the speculator's higher profits with commitment reflects that she commits to trading with reduced intensity on her private information thereby reducing the amount of information in net order flow. Indeed, this is what happens in the Cho and Karoui (2000) model. This intuition is misplaced. To prove this, we perturb the classic Kyle setting with normally distributed noise trade and fundamentals by introducing a small perturbation of the noise trade distribution that gives rise to a quadratic shift of the linear pricing function. We show that whether the speculator trades more aggressively or less depends on both the sign of the induced quadratic pricing shift and the size of the realization of the fundamental v. Specifically, the speculator wants to commit to a lower trading intensity than her BNE trading strategy *only* after signals for which the expected sensitivity of BNE pricing is *higher*. However, she wants to commit to a higher trading intensity than her BNE trading strategy following signals for which the expected sensitivity of BNE prices is *lower*.

The closest related research is Biais and Germain (2002), who analyze a setting with discrete (bad, zero, good) private information and equally likely liquidity trades of $-L$, 0, or L. The competitive market makers see a pair of orders, but do not know which one is from the informed trader. Biais and Germain characterize when commitment by the speculator to a mixed trading strategy (trading when she has private information with a probability $\alpha < 1$) has value. Commitment raises speculator profits whenever she is sufficiently likely to have information, as this less aggressive trading strategy causes price to move less when she trades L or $-L$.

2 Model

As a benchmark, recall the classic static Kyle (1985) model. In the model, a single risk-neutral speculator privately observes an asset's liquidation value v drawn from a distribution with mean zero and pdf $f_v(\cdot)$. Liquidity traders cumulatively trade a quantity u, drawn independently from a distribution with zero mean and pdf $f_u(\cdot)$. After observing v, the speculator chooses a quantity x to trade. The quantity x is a "market order," in the sense that it depends on v, but not the equilibrium price. The speculator does not see the level of noise trade u before trading. Market makers know the joint distribution of v and u but do not observe either realization. Instead, they only observe the net order flow $y = x + u$, and then set a competitive price that yields zero expected profits conditional on the net order flow observed.

The Bayesian Nash equilibrium (BNE) in Kyle (1985) is defined by two functions, a trading strategy $X^*(\cdot)$ and a pricing rule $P^*(\cdot)$ that satisfy, respectively, a profit-maximization condition and a market-efficiency condition. The profit-maximization condition states that the speculator's order $x = X^*(v)$ maximizes expected profits given (correct conjectures about) the pricing rule $P^*(\cdot)$:

$$X^*(v) = \arg\max_x E_u[(v - P^*(x + u))x|v], \tag{1}$$

where the notation $E_u[\mathcal{F}(v, x, u)|v]$ indicates that we calculate the expectation of the functional $\mathcal{F}(v, x, u)$ with respect to realizations of the noise trade u conditional on the observed fundamental realization v. In what follows, we simplify notation and remove conditioning on v for the speculator, as she observes v by assumption.

The market efficiency condition states that market makers expect zero profits given the observed net order flow $y = x + u$ and correct conjectures about the speculator's trading strategy:

$$P^*(y) = E[v|X^*(v) + u = y]. \tag{2}$$

With Normal pdfs $f_v(\cdot)$ and $f_u(\cdot)$, the proof that a unique BNE exists in which $X^*(\cdot)$ and $P^*(\cdot)$ are linear functions is simple (see Kyle (1985)). The equilibrium trading strategy and pricing rule take the forms

$$X^*(v) = \frac{\sigma_u}{\sigma_v} v \;\text{ and }\; P^*(y) = \frac{1}{2}\frac{\sigma_v}{\sigma_u} y. \tag{3}$$

Note that a linear trading strategy implies a linear pricing rule and vice versa. Thus, in any equilibrium, *either both* the trading strategy and pricing rule are linear *or neither* are.

To examine general non-linear trading strategies and pricing rules, we introduce notation to describe the reaction function of market makers to a possibly non-linear trading strategy of the speculator. We re-write pricing rule (2) to emphasize the functional dependence on the conjectured speculator's strategy $X_c(\cdot)$:

$$P(y, X_c) = E[v | X_c(v) + u = y]. \tag{4}$$

The notation $P(y, X_c)$ indicates that the price depends on both a scalar argument given by the aggregate order flow y and a function argument given by the demand function X_c that the market makers believe the speculator is using. Our analysis makes use of functionals, i.e. functions mapping both scalars and other functions into scalars. To keep notation clear, we place scalar arguments in front of functional arguments, as in (4). For clarity, we generally use lower-case letters to denote scalars and upper case letters to denote functions or functionals, except for the profit functionals π and $\overline{\pi}$ defined below, where we use lower case letters for consistency with standard notation (see, e.g., Kyle (1985)), and pdfs, where we use lower case letters to avoid confusion with cdfs.

Let $\overline{P}(x, X_c)$ denote the expected price obtained by the speculator when she trades x and market makers believe she is using trading strategy X_c. The functional $\overline{P}(x, X_c)$ is defined by

$$\overline{P}(x, X_c) = E_u\left[P(x + u, X_c)\right]. \tag{5}$$

When the trading strategy $X(\cdot)$ is linear, the functionals $P(y, X)$ and $\overline{P}(x, X)$ are identical linear functions because the zero-mean noise term u has no effect when P is linear in y. When, instead, the trading strategy $X(\cdot)$ is non-linear, the functions $P(y, X)$ and $\overline{P}(x, X)$ generally differ from each other and do not have simple closed-form expressions.

We analyze two strategic settings, the Nash equilibrium of Kyle (1985) and the Stackelberg equilibrium described informally by Kyle (1983).[4] To facilitate analysis, we redefine the equilibrium concepts as fixed point problems involving functionals.

Suppose the speculator observes the realization v and trades the quantity x, while market makers conjecture that the speculator follows strategy X_c and set the informationally-efficient pricing rule, $P(x+u, X_c)$. The speculator's expected payoff $\pi_I(v, x, P)$ is given by

$$\pi_I(v, x, P) = E_u[x(v - P(x+u, X_c))]. \tag{6}$$

When the speculator determines the *functional form* of her best response $X(\cdot)$ *before* seeing the specific realization of v,[5] her ex-ante expected payoff, integrating (6) over all realizations of v, is a functional of her actual strategy X and the market makers' conjecture X_c in (4). The speculator's *ex-ante expected* payoff, denoted $\bar{\pi}_I(X, X_c)$, takes the form

$$\overline{\pi}_I(X, X_c) \quad = \quad E_v[\pi_I(v, X(v), X_c)] \;=\; E_{v,u}[X(v)(v - P(X(v) + u, X_c))], \tag{7}$$

where $P(x+u, X_c)$ is the informationally-efficient price.

Definitions of BNE and BSE. The Nash equilibrium trading strategy, denoted X_N, is defined by the fixed-point condition

$$X_N = \arg\max_X \; \overline{\pi}_I(X, X_N). \tag{8}$$

That is, in a Nash equilibrium, when the speculator takes as given the (efficient) pricing by market makers based on their beliefs about the speculator's trading strategy, the speculator indeed chooses that same trading strategy. Although the reaction-function notation emphasizes the choice of the function $X(\cdot)$, condition (8) leads to a definition of Nash equilibrium that is logically equivalent to that in Kyle (1985), given by equations (1) and (2). The two definitions are equivalent because the speculator's optimization problem decomposes into separate state-by-state optimization problems for each realization of v.

Suppose now that the speculator can commit to her trading strategy. Then the speculator's

[4] We later discuss structural similarities and distinctions between this model and that of Rochet and Vila (1994) who consider an informed trader who also sees the level of noise trade u and can condition trade on u.

[5] After the realization of v is observed, substituting it into X yields the optimal traded quantity $x = X(v)$.

Stackelberg equilibrium trading strategy, denoted X_S, is given by the fixed-point

$$X_S = \arg\max_X \ \overline{\pi}_I(X, X). \tag{9}$$

Market makers still price according to the market-efficiency condition (4), but the speculator now accounts for the functional dependence of their pricing rule on her trading strategy.

3 First-Order Conditions

In the Stackelberg setting, we must account for the functional dependence of price when we derive the analogue of first-order conditions (FOC) for the speculator's payoff (7). This payoff depends on the pricing rule, which, in turn, depends on the functional form of her trading strategy. The speculator may be able to increase her payoff by adjusting the functional form of her trading strategy $X(\cdot)$.

Define the partial derivative of the price functional by

$$\overline{P}'(x, X) = \frac{\partial}{\partial x}\overline{P}(x, X). \tag{10}$$

To describe the price sensitivity to variation in the speculator's trading strategy, we use the notion of *functional differentiation*. The functional differential of the price at a strategy $X(\cdot)$ is defined by

$$\delta\overline{P}(x, X; \delta X) = \lim_{\varepsilon\to 0}\left\{\frac{\overline{P}(x, X + \varepsilon\delta X) - \overline{P}(x, X)}{\varepsilon}\right\}, \tag{11}$$

provided that the limit (11) exists for every *function* δX (from a Banach functional space), and defines a functional, linear and bounded in δX. This definition corresponds to the *weak*, or *Gateaux differential* (Kolmogorov and Fomin (1999), Saati (1981)). The differential (11) measures the price sensitivity to the *functional form* of the speculator's trading strategy $X(\cdot)$. In contrast, (10) describes the *quantity* variation component, which does not depend on the functional form of the strategy. The definition below of the full variation of the price functional can be viewed as an extension of the full differential of a function depending on several variables, when the full differential is obtained as a sum of partial differentials with respect to all variables. In our setting, one of the "variables" is the function representing the speculator's trading strategy. Thus, the full variation of the price functional is given by

$$D\overline{P}(x, X, \delta x; \delta X) = \overline{P}'(x, X)\delta x + \delta\overline{P}(x, X; \delta X). \tag{12}$$

For a given x and X, $D\overline{P}(x, X, \delta x; \delta X)$ defines a linear function of δx and a linear functional of δX.

When the speculator can commit to her trading strategy, her expected payoff (7) has two functional arguments, one corresponding to her actual strategy, and one corresponding to the strategy conjectured by market makers. Using the definition in (11), $\delta_1\overline{\pi}_I(X, Y; \delta X)$ and $\delta_2\overline{\pi}_I(X, Y; \delta Y)$ are the functional differentials of the speculator's expected payoff with respect to the first and second functional arguments, respectively. The full variation of the speculator's expected payoff is

$$\delta\overline{\pi}_I(X, Y; \delta X, \delta Y) = \delta_1\overline{\pi}_I(X, Y; \delta X) + \delta_2\overline{\pi}_I(X, Y; \delta Y).$$

Using these definitions, we obtain the following first-order conditions for the speculator's profit maximization problem in the Nash and Stackelberg settings. A necessary condition for a Nash equilibrium is that for all variations δX belonging to the functional space,

$$0 = \delta_1\overline{\pi}_I(X_N, X_N; \delta X) = E_v\left[\left\{v - \overline{P}(X_N(v), X_N) - X_N(v)\overline{P}'(X_N(v), X_N)\right\}\delta X(v)\right]. \tag{13}$$

A necessary condition for a Stackelberg equilibrium is that for all variations δX,

$$\begin{aligned} 0 &= \delta_1\overline{\pi}_I(X_S, X_S; \delta X) + \delta_2\overline{\pi}_I(X_S, X_S; \delta X) \\ &= E_v\left[\left\{v - \overline{P}(X_S(v), X_S) - X_S(v)\overline{P}'(X_S(v), X_S)\right\}\delta X(v)\right] - E_v\left[X_S(v)\delta_2\overline{P}(X_S(v), X_S)\right]. \end{aligned} \tag{14}$$

The first-order conditions (13) and (14) are analogous to, but distinct from, the first-order condition obtained in Rochet and Vila (1994), where an insider also observes uninformed (liquidity demand) that she can condition her strategy on. In our Kyle (1985) setting, insiders do not observe liquidity demand, rendering the analysis fundamentally different.

In the Stackelberg setting, the first-order condition (14) contains the additional structural derivative of the price functional $\delta_2\overline{P}(X_S(v), X_S)$, reflecting that when the speculator changes her trading strategy, she internalizes the effect on market maker pricing in her profit maximization problem. Define $Q(y, X) = E[X(v)|X(v) + u = y]$ to be the expected value of $X(v)$ from the perspective of market makers who observe the net order flow y; and define

$$J(v, x, X) = \frac{\partial}{\partial x} E_u [Q(x + u, X)\{v - P(x + u, X)\}] \tag{15}$$

to be the derivative of the speculator's profits with respect to x, accounting for how x affects market maker inferences about $X(\cdot)$. Using (13) and (14), we have

Proposition 1: *For any v, the first-order conditions for the speculator's strategy are given by*

$$0 = v - \overline{P}(X_N(v), X_N) - X_N(v)\overline{P}'(X_N(v), X_N), \tag{16}$$

for the Nash setting, and

$$0 = v - \overline{P}(X_S(v), X_S) - X_S(v)\overline{P}'(X_S(v), X_S) - J(v, X_S(v), X_S), \tag{17}$$

for the Stackelberg setting.

Proof: See the Appendix. □

The J term in (17) corresponds to the marginal expected profit of the speculator in the information set of market makers—the J-term equals $E_v\left[X_S(v)\delta_2\overline{P}(X_S(v), X_S)\right]$, reflecting the structural variation component $\delta_2\overline{P}(X(v), X(\cdot))$ present only in the Stackelberg setting. The J term (15) describes the expected profit consequences of how the speculator accounts for the market makers' reaction to her actions. To provide intuition, we rewrite the speculator's expected payoff in the Stackelberg setting as

$$\begin{aligned} \overline{\pi}_I(X(v), X) &= E_u[(X(v) - E[X(v)|X(v) + u = y])\{v - P(X(v) + u, X)\}] \\ &= E_u[(x - Q(x + u, X))(v - P(x + u, X))]. \end{aligned} \tag{18}$$

This rewriting makes the economic intuition for the speculator's expected payoff clear. With commitment, the speculator's optimization problem explicitly takes into account that market makers can partially "undo" the effect of speculation by anticipating her orders. The term

$$Q(x+u, X) = E\left[X(v)|X(v)+u=y\right]$$

is the market makers' best estimate of the speculator's order $X(v)$. That is, the term $x - Q(x+u, X)$ in (18) is the *unexpected* component of the speculator's order from the informational perspective of market makers. This means that the *speculator's expected payoff equals the expected value of the error in the market makers' forecast of her trade.*[6]

Result 1: *Maximizing* (18) *solves the speculator's optimization problem in the Stackelberg setting. Solving this problem does not require differentiation with respect to functional arguments.*

Proof: Substituting (15) for J into the speculator's first-order condition, (17), yields

$$\left.\frac{\partial}{\partial x}\right|_{x=X(v)} \left\{E_u\left[x\left(v - P(x+u, X)\right)\right] - E_u\left[Q(x+u, X)\left\{v - P(x+u, X)\right\}\right]\right\} = 0.$$

Factoring terms yields

$$\left.\frac{\partial}{\partial x}\right|_{x=X(v)} E_u\left[\left(x - Q(x+u, X)\right)\left(v - P(x+u, X)\right)\right] = 0, \tag{19}$$

which is the optimality condition for a speculator with objective (18) . □

The terms with functional differentiation vanish when we apply the first-order condition to (18) because

$$E_{u,v}\left[\delta_2 Q(y, X)\left(v - P(y, X)\right)\right] = E_y E_{v|y}\left[\delta_2 Q(y, X)\left(v - E_{v'|y}\left[v'\right]\right)\right] = 0,$$

and

$$E_{u,v}\left[\delta_2 P(y, X)\left(X(v) - Q(y, X)\right)\right] = E_y E_{v|y}\left[\delta_2 P(y, X)\left(X(v) - E_{v'|y}\left[X(v')\right]\right)\right] = 0.$$

Note that, as it follows from (18), we can also rewrite $\overline{\pi}_I(X, X)$ in the form

$$\overline{\pi}_I(X, X) = E_y Cov_{v|y}\left[X(v), v\right],$$

[6] For related forecast error results see Bernhardt and Miao (2004) or Bernhardt, Seiler and Taub (2010).

where the covariance $Cov_{v|y}$ conditional on the total order flow y is evaluated in the market makers' information set.

Returning to (16) and (17), we see that with a risk-neutral monopolist speculator, a necessary and sufficient for Nash and Stackelberg equilibria to coincide is that the J term must equal zero. We next illustrate the distinction between the two components of the full variation of the price functional (12) in the classical Normal uncertainty setting, where trading strategies are linear. We show that the J term vanishes, implying that Nash and Stackelberg payoffs coincide.

Example 1: Suppose $f_v \sim N(0, \sigma_v^2)$ and $f_u \sim N(0, \sigma_u^2)$ and strategies are linear, $X(v) = \beta v$.

Kyle (1985) shows that the conjectured linear speculator strategy $X_c(v) = \beta_c v$ leads to the linear informationally-efficient pricing rule

$$P(y, X_c) \;=\; \lambda\,(X_c)\,y, \qquad \text{where } \lambda(X_c) = \frac{\beta_c}{\beta_c^2 + \beta_0^2} \text{ and } \beta_0 = \sigma_u/\sigma_v. \tag{20}$$

The expected price functional is also linear

$$\overline{P}\,(x, X_c) = \lambda\,(X_c)\,x. \tag{21}$$

Combining (12), (20), and (21), the total derivative for the linear strategy $X(v) = \beta v$ is

$$\lambda dx + x\delta\lambda\,(X, \delta X)\,, \tag{22}$$

where $\delta\lambda\,(X;\delta X)$ only depends on the functional variations of the linear strategies $\delta X(v) = v\delta\beta$. That is, $\delta\lambda\,(X;\delta X)$ depends on the variation of β, but not v. Then (12) and (22) yield that for linear variations $\delta X(v) = v\delta\beta$, the price derivative and the structural variation components are given by

$$\overline{P}'(x, X) \;=\; \lambda(\beta) = \frac{\beta}{\beta^2 + \beta_0^2} \quad \text{and} \quad \delta\overline{P}\,(x, X;\delta X) \;=\; x\frac{\partial\lambda}{\partial\beta}\delta\beta. \tag{23}$$

Foreshadowing future results, we exploit the explicit linear solutions for the speculator's trading strategy and market maker pricing to show that commitment has no added value when both liquidation values and noise trade are normally distributed. Around the optimal linear strategy,

$\beta = \beta_0$, we have

$$\left[\frac{\partial \lambda}{\partial \beta}\right]_{\beta=\beta_0} = \frac{\partial}{\partial \beta}\left[\frac{\beta}{\beta^2 + \beta_0^2}\right]_{\beta=\beta_0} = 0. \tag{24}$$

Substituting this into (23) yields that the variation of the price functional with respect to linear variations of the informed trader's strategy, $\delta X(v) = \delta\beta v$, vanishes at $X(v) = \beta_0 v$:

$$\delta\overline{P}(x, X; \delta X) = 0. \tag{25}$$

Thus, $E_v\left[x\delta\overline{P}(x, X; \delta X)\right] = E_v\left[v^2\right]\beta^2\frac{\partial\lambda}{\partial\beta}\delta\beta = 0$, which implies that the Stackelberg term J vanishes in equilibrium. Directly evaluating, $Q(y, X) = E_{v|y}[X(v)] = \lambda\beta y$, and we confirm that $J = E_u\left[\lambda\beta v - 2\lambda^2\beta y\right] = \lambda\beta v(1 - 2\lambda\beta) = 0$, reflecting the efficient pricing.

In essence, the linearity of a risk-neutral speculator's trading strategy and hence the pricing rule mean that the market makers' best estimate of the speculator's market order $Q(x + u, X) = E[X(v)|X(v) + u = y]$ is also linear. This means that J inherits a linear structure, so the noise trade integrates out and, with efficient pricing, J reduces to zero. □

3.1 Equilibrium Existence

We next pose the informed trader's optimization in both Nash and Stackelberg settings as fixed-point problems. We impose regularity conditions on the distributions to ensure the existence of equilibria. Specifically, we impose assumptions on the densities $f_v(\cdot)$ and $f_u(\cdot)$ that rely on regularly-varying functions (RVF) and slowly-varying functions (SVF) in the Karamata sense (see, e.g., Seneta (2019), Takesi and Maric (2006), or Karamata (1962)).

DEFINITION 1A (Seneta, 2019): *A function $f(x)$ defined, positive, and measurable on $x \geq A > 0$ is regularly varying of index ρ if $\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)} = \lambda^\rho$ for some real $\rho \in R$ and any $\lambda > 0$.*

To establish the existence of Nash and Stackelberg equilibria we adopt the regularity conditions on the distributions of priors and noise trade imposed by Boulatov and Livdan (2024):

1. Probability density functions $f_v(\cdot)$ and $f_u(\cdot)$ are smooth and have infinite support.

2. $f_v(x) = C_v \exp(-\psi_v(|x|))$ and $f_u(x) = C_u \exp(-\psi_u(|x|))$ where (i) ψ_v is convex (the pdf f_v

is thus log-concave), (ii) both ψ_v and ψ_u are measurable with respect to f_v, and (iii) ψ_u is measurable with respect to f_u. In addition, both ψ_v and ψ_u are monotonically increasing RVF with indices $a_v > 1$ and a_u, respectively.

3. $f_u(u)$ is an analytic function for $u \in C$ (except for, possibly, the point $u = 0$) satisfying the condition $|f_u(x + iy)| \leq g_u(x)M(y)$, where the function $g_u(\cdot)$ is a pdf, ψ_u is measurable with respect to g_u, and $M(y)$ is finite, $|M(y)| < \infty$, for any finite $|y| < \infty$.

The conditions rule out fat-tailed distributions with infinite support and asymmetric distributions. Condition 3 says that the analytic extension of $f_u(\cdot)$ in a complex plane remains bounded and without fat tails along the real axis. In essence, equilibrium existence requires that expected trading profits be bounded—if the tails are too fat, speculators observe high values of v sufficiently frequently and with fat-tailed noise trade can submit aggressive orders without being easily detected by market makers, resulting in unbounded profits and hence an unraveling of equilibrium. Normal pdfs satisfy these regularity conditions, corresponding to $f_v(x) = C_v \exp(-\psi_v(|x|))$ and $f_u(x) = C_u \exp(-\psi_u(|x|))$ where $\psi_v(|x|) = \psi_u(|x|) = \frac{x^2}{2}$, implying that ψ_v and ψ_u are RVF with indices $a_v = a_u = 2$ and monotonically increasing in $|x|$.

We also adopt from Boulatov and Livdan (2024) a regularity condition on the speculator's trading strategy. Specifically, we restrict our analysis to *admissible* trading strategies, where.

Definition 1 *Trading strategy $X(\cdot)$ is admissible if it has a finite weighted L^1_ρ norm*

$$\|X\| = \int_{-\infty}^{+\infty} \frac{dv}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} |X(v)| < \infty.$$

Assumption 1 *The speculator's strategies are admissible.*

The normal pdf weighted norm is commonly used for strategies $X(v)$ with infinite support that may have infinite limits when the fundamental value v increases indefinitely. Then we have:

Theorem 1 *The Bayesian Nash and Stackelberg equilibria (X_N, P_N) and (X_S, P_S) exist given the regularity conditions 1–3 on the distributions. The speculator's optimal Stackelberg strategy X^* satisfies the first-order condition: $\nabla_{X^*(v)} \overline{\pi}_I (X^*, X^*) = 0$.*

Proof: See the Appendix. □

Boulatov and Livdan (2024) prove that a unique BNE exists given regularity conditions 1–3. To extend this result to establish existence of the Stackelberg equilibrium, we first observe that the functional $\overline{\pi}_I(X, X)$ is *weakly continuous*. Therefore, its values on each weakly converging sequence $\{X_n\}$ converge (Krasnoselskii, 1964). Because $\overline{\pi}_I$ is weakly continuous, it assumes its upper and lower bounds on any finite ball $\|X\| \le r$ (Krasnoselskii, 1964).

Second, exploiting the explicit form of the pricing rule, we observe that when the norm of the speculator's strategy $\|X\|$ increases unboundedly, i.e., when $\|X\| \to \infty$, then $\overline{\pi}_I(X, X) \to 0$. Economically, this means that strategies with infinite norm are not profitable, implying that the optimal strategy has a finite norm. This also means there exists an upper bound on the speculator's expected profit, $\overline{\pi}_I(X, X)$. Because $\overline{\pi}_I$ is weakly continuous, $\max\limits_{\|X\|\le h}\overline{\pi}_I(X, X) = \overline{\pi}_I^*$, where $\overline{\pi}_I^* < \infty$ is the maximal value of $\overline{\pi}_I(X, X)$ in the entire strategy space. Then, because the optimum $\overline{\pi}_I^*$ of $\overline{\pi}_I(X, X)$ exists and is finite, the corresponding optimal strategy X^* has a finite norm and $\overline{\pi}_I(X, X)$ takes its maximal value on this optimal strategy, $\overline{\pi}_I^* = \overline{\pi}_I(X^*, X^*)$. Finally, because $\overline{\pi}_I(X, X)$ is uniformly differentiable, its gradient at the optimal strategy X^* vanishes, $\nabla_{X^*(v)}\overline{\pi}_I(X^*, X^*) = 0$ (Krasnoselskii, 1964).

3.2 Market maker isoprofit curves

To provide intuition about the speculator's optimization problem in the Nash and Stackelberg settings, we now allow for pricing functions $P(y)$ that need not equal the informationally-efficient pricing function, $P(y, X_c) \equiv E[v|y = X_c(v) + u]$. We consider a speculator's profit functionals in this extended strategy space. The informationally-efficient pricing rules and corresponding conjectured strategies form an *isoprofit curve* in the functional space characterized by the expected profits of market makers being identically zero along the curve. Expected market maker profits equal

$$\begin{aligned}\overline{\pi}_M(X, X_c) &= E_{v,u}[y(P(y, X_c) - v)] = E_y E_{v|y}[y(P(y, X_c) - v)] \\ &= E_y[y(P(y, X_c) - P(y, X))],\end{aligned} \tag{26}$$

where $P(y, X) = E_{v|y}[v] = E[v|y = X(v) + u]$ is the expected value of v conditional on the net order flow y. Informational efficiency requires that the marginal profit be zero, i.e., the price is the conditional expectation of v given the correct conjecture, $X_c = X$. We prove in Proposition 2 that this condition holds locally, even for arbitrary variations of the pricing rule.

Consider the Stackelberg setting with commitment, $X_c = X$, and define the *isoprofit curve* $\Gamma_C(X, \hat{P})$ in the functional space of arbitrary (not necessarily informationally efficient) pricing rules $\hat{P}(y, X)$ as $\overline{\pi}_M\left(X, \hat{P}\right) \equiv const = C$, when $(X, \hat{P}) \in \Gamma_C\left(X, \hat{P}\right)$. In particular, the zero-profit isoprofit curve for market makers, $\overline{\pi}_M(X, P) \equiv 0$, contains all pairs of speculator trading strategies and associated informationally-efficient pricing rules, i.e., $(X, P) \in \Gamma_0(X, P)$. Suppose that a pair $(X, \hat{P})$ belongs to Γ_C, i.e., $(X, \hat{P}) \in \Gamma_0(X, \hat{P})$. Consider any small variation of the speculator's strategy δX. Then the resulting variation in the pricing rule $\delta\hat{P}$ should lead the point $\left(X + \delta X, \hat{P} + \delta\hat{P}\right)$ to still belong to the same isoprofit curve Γ_C. Evaluating the variation of (26) along Γ_C, and noting that $\delta y = \delta(X(v) + u) = \delta X(v)$, we obtain

$$\delta\overline{\pi}_M(X, \hat{P}) = E_{v,u}[\delta X(v)(\hat{P}(y, X) - v + y\hat{P}'(y, X)) + y\delta\hat{P}(y, X)] = 0.$$

We use the explicit form of the informationally-efficient pricing rule $P(y, X) = E[v|y]$ to obtain:

Proposition 2: *The informationally-efficient pricing rule $P(y, X)$ satisfies the isoprofit condition*

$$\delta\overline{\pi}_M(X, P) = E_{v,u}[\delta X(v)(P(y, X) - v + yP'_e(y, X)) + y\delta P(y, X)] = 0, \tag{27}$$

for expected market maker profits.

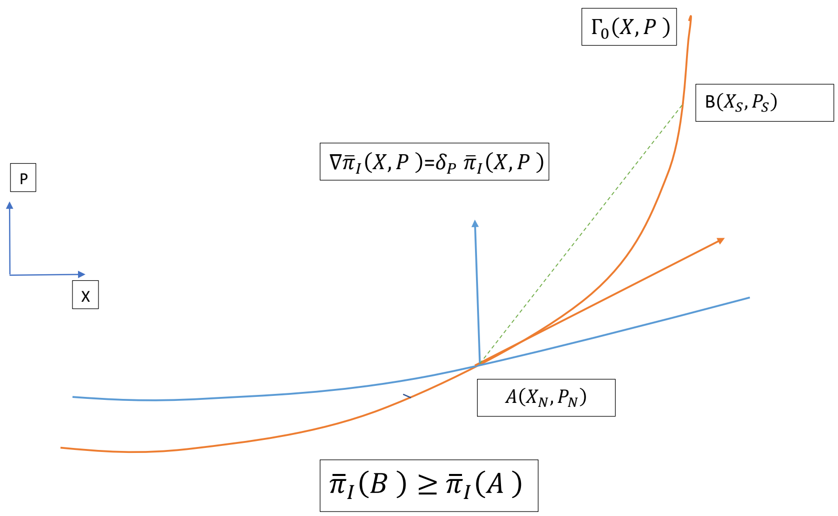
Proof: See the Appendix. □

Figure 1 illustrates the economics. Recall that the speculator's FOC in the Nash setting takes the form

$$\delta_1\overline{\pi}_I(X, X_c) = \delta_X E_{u,v}[X(v)(v - P(X(v) + u, X_c)] = 0. \tag{28}$$

This must hold also in equilibrium at the "fixed" point of (X, X_c) where $X = X_c$. In the Nash setting, the functional form of pricing rule is pinned down by the market makers' conjecture X_c; it is not a choice variable for the speculator who takes it as given in her optimization problem. The blue curve in Figure 1 is a "response function" of the speculator's trading strategy X to some arbitrary pricing rule $\hat{P}$ set by market makers. That is, the blue curve illustrates a set of pricing rules and

Figure 1: Optimization by speculator in Nash and Stackelberg settings



The Nash and Stackelberg equilibrium points are (X_N, P_N) and (X_S, P_S). The Nash equilibrium is given by the intersection of the blue curve representing the speculator's best response to an arbitrary pricing rule P and the red curve representing informationally-efficient pricing given an arbitrary informed trading strategy X. The Stackelberg equilibrium $(X_S, P_S) = \arg\max_{(X,P)\in\Gamma_0} \bar{\pi}_I(X, P)$, is obtained by maximizing the speculator's expected profit *along the zero profit curve* Γ_0 *for market makers.*

corresponding speculator trading strategies in the functional space associated with the FOC (28). The red curve in Figure 1 is $\Gamma_0(X, P)$, which represents a set (X, P) of arbitrary speculator trading strategies X and associated informationally-efficient pricing rules $P(y, X)$. It can be viewed as the response function P of market makers to an arbitrary speculator trading strategy X. The Nash equilibrium point (X_N, P_N) is given by the intersection of the two curves, i.e., by the intersection of the two reaction functions, and is characterized by both FOCs holding, i.e., by the speculator's optimality condition and the informational efficiency condition for the pricing rule set by market makers.

Now consider the Stackelberg equilibrium point (X_S, P_S) in Figure 1. The speculator is, in effect, constrained to choosing a point on the isoprofit curve Γ_0 for which market makers expect zero profit, i.e., the pricing rule is informationally efficient. That is, a speculator's optimization problem can be represented as maximizing her expected profit *along the zero profit curve* Γ_0 *for market makers*. Its solution, $(X_S, P_S) = \arg\max_{(X,P)\in\Gamma_0} \overline{\pi}_I(X, P)$, or, equivalently, $X_S = \arg\max_X \overline{\pi}_I(X, X)$ gives the Stackelberg equilibrium. Therefore, the speculator's optimal payoff is uniquely defined.

3.3 Speculator profit and commitment

The ability to commit to a trading strategy can never harm a speculator—she can always commit to her Nash strategy, in which case market maker pricing is unchanged, implying that the speculator's profits are unchanged state by state. We next constructively establish that a speculator can sometimes earn strictly higher expected profits in the Stackelberg equilibrium than in the Nash equilibrium. Further, since market makers expect zero profits in both types of equilibria, this implies that noise trader losses are strictly higher in the Stackelberg setting.

To do this, we compare speculator profits in the Nash and Stackelberg equilibria of the Cho and Karoui (2000) model. Cho and Karoui replace the assumption in Kyle (1985) that v has a normal distribution with the assumption that v has a symmetric Bernoulli distribution,

$$v = \begin{cases} a, & prob. = \frac{1}{2} \\ -a, & prob. = \frac{1}{2}, \end{cases} \tag{29}$$

maintaining all other assumptions of Kyle (1985). Boulatov and Livdan (2024) prove that this model has a unique Nash equilibrium, and that equilibrium trading strategies and pricing are non-linear.

The pricing rule in the Cho and Karoui (2000) model remains both a smooth analytic function of total order flow and a smooth analytic functional in the speculator's strategy (Boulatov and Livdan (2024)), even though the Bernoulli distribution does not satisfy our distributional assumptions. As a result, the model is (almost) analytically tractable.

With a symmetric Bernoulli prior (29), the speculator's strategy takes the form

$$X(v) = \begin{cases} \beta, & v = a \\ -\beta, & v = -a \end{cases} .$$

That is, the speculator's optimal strategy is an *odd* function of v. Following Boulatov and Livdan (2024), we obtain, as in Cho and Karoui (2000), the pricing rule

$$P(y) = a\frac{\frac{1}{2}\exp\left(\frac{(y-\beta)^2}{2\sigma_u^2}\right) - \frac{1}{2}\exp\left(\frac{(y+\beta)^2}{2\sigma_u^2}\right)}{\frac{1}{2}\exp\left(\frac{(y-\beta)^2}{2\sigma_u^2}\right) + \frac{1}{2}\exp\left(\frac{(y+\beta)^2}{2\sigma_u^2}\right)} = a\frac{\exp\left(\frac{y\beta}{\sigma_u^2}\right) - \exp\left(-\frac{y\beta}{\sigma_u^2}\right)}{\exp\left(\frac{y\beta}{\sigma_u^2}\right) + \exp\left(\frac{y\beta}{\sigma_u^2}\right)} = a\tanh\left(\frac{\beta y}{\sigma_u^2}\right). \tag{30}$$

To analyze the Nash equilibrium, we use the speculator's optimality condition

$$\overline{P}\left(X(v)\right) + X(v)\overline{P}'\left(X(v)\right) = v,$$

which involves the expected price functional $\overline{P}\left(X(v)\right) = \mathbb{E}_u\left[P\left(X(v) + u\right)\right]$. Using (30), we solve for the expected price functional

$$\overline{P}(x) = a\mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(x + u\right)\right)\right]. \tag{31}$$

Because $\tanh(\cdot)$ is an odd function and the normal $N(0, \sigma_u^2)$ distribution of the noise trade is symmetric, it follows from (31) that $\overline{P}(x)$ is an odd function, i.e., $\overline{P}\left(-x\right) = -\overline{P}(x)$. Rescaling the speculator's strategy and noise demand by σ_u, we obtain

$$\overline{P}(x) = a\mathbb{E}_u\left[\tanh\left(\beta\left(x + u\right)\right)\right]. \tag{32}$$

As in Cho and Karoui (2000), we obtain the speculator's FOCs in the form

$$\begin{aligned} 1 &= \mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(\beta + u\right)\right)\right] + \left(\frac{\partial}{\partial x}\right)_{x=\beta}\mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(x + u\right)\right)\right], \\ -1 &= \mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(-\beta + u\right)\right)\right] + \left(\frac{\partial}{\partial x}\right)_{x=-\beta}\mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(x + u\right)\right)\right], \end{aligned} \tag{33}$$

where the parameter a cancels out. Because the expected price (31) is an odd function of β, it follows that the two FOCs in (33) are equivalent (reflecting the symmetric prior and signal structure). After simplifying, evaluating the derivative, and rescaling u by σ_u, algebra yields the following fixed point condition

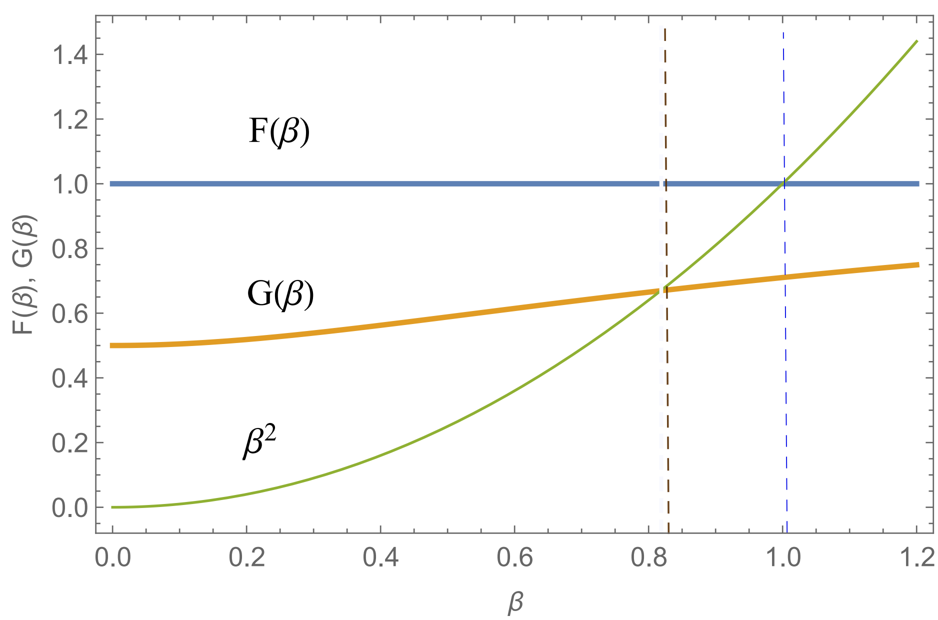
$$\beta^2 = \frac{1 - \mathbb{E}_u\left[\tanh\left(\beta^2 + \beta u\right)\right]}{1 - \mathbb{E}_u\left[\tanh^2\left(\beta^2 + \beta u\right)\right]}, \text{ where } u \sim N(0,1). \tag{34}$$

Integration yields that $\mathbb{E}_u\left[\tanh\left(\beta^2 + \beta u\right)\right] = \mathbb{E}_u\left[\tanh^2\left(\beta^2 + \beta u\right)\right]$ for all β and hence the right-hand side of (34) is given by

$$F(\beta) = \frac{1 - \mathbb{E}_u\left[\tanh\left(\beta^2 + \beta u\right)\right]}{1 - \mathbb{E}_u\left[\tanh^2\left(\beta^2 + \beta u\right)\right]} \equiv 1.$$

In Figure 1, the solution of (34) is given by the intersection of β^2 (thin green line) and $F(\beta)$ (blue line) at $\beta = 1$. This fully characterizes the Bayesian Nash equilibrium (BNE).

Figure 2: FOC for equilibrium with commitment and Nash



The functions β^2 and $F(\beta)$ are given by the thin green and blue lines, respectively, while $G(\beta)$ is given by the orange line. The solution of (34) is $\beta = 1$. The solution $\beta_S \approx 0.8$ of (36) is smaller than $\beta_N = 1$ solving (34), i.e., $\beta_S < \beta_N = 1$.

Now consider the Stackelberg setting. In this case, the speculator maximizes:

$$\overline{\pi}_I(\beta) = a\beta\left(1 - \mathbb{E}_u\left[\tanh\left(\beta\left(x + u\right)\right)\right]\right), \; u \sim N(0,1). \tag{35}$$

The associated FOCs are given by

$$\beta^2 = G(\beta), \tag{36}$$
$$G(\beta) = \frac{1 - \mathbb{E}_u\left[\tanh\left(\beta^2 + \beta u\right)\right]}{2\mathbb{E}_u\left[\left(1 - \tanh^2\left(\beta^2 + \beta u\right)\right)\left(1 - \tanh\left(\beta^2 + \beta u\right)\right)\right]}.$$

The function $G(\beta)$ is given by the orange line in Figure 2. The Stackleberg solution β_S of (36) is smaller than the Nash solution β_N solving (34), i.e., $\beta_S < \beta_N = 1$. Indeed, plugging $\beta = 1$ into the right-hand side of (36), yields $G(1) = 0.65 < 1$. Thus, as Figure 2 illustrates, the solution for β in (36) is less than 1, implying that the speculator's expected Stackelberg profits in equation (35) strictly exceed her Nash profits. Figure 3 plots those profits, revealing that they are maximized by $\beta_S \approx 0.8 < \beta_N = 1$, i.e., the speculator can increase profits by committing to a less aggressive trading strategy, loosely consistent with intuition from Biais and Germain (2002). Equation (32) also indicates that since $\beta_S < \beta_N$, price impacts are smaller in the Stackelberg equilibrium than in the Nash equilibrium, implying that the Stackelberg equilibrium features greater residual market uncertainty (lower information efficiency).

The speculator's equilibrium expected profits are *higher* when she can commit to her strategy than in the BNE, whenever the two equilibria differ. With commitment the speculator cannot increase her equilibrium expected profits by deviating from the Nash equilibrium allocation (X_N, P_N) to another allocation with zero expected market maker profit if and only if the two equilibria correspond, $(X_N, P_N) = (X_S, P_S)$. Conversely, for the Nash equilibrium (X_N, P_N) to correspond to the Stackelberg equilibrium, it means that

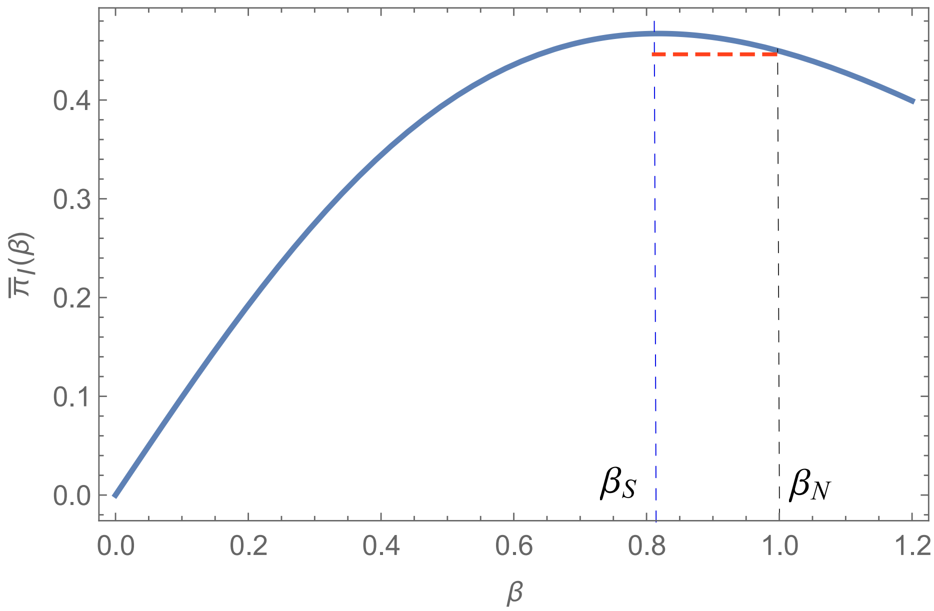
$$\delta_k \overline{\pi}_I (X_N, X_N; \delta X_k) = 0, \quad \text{for both } k = 1, 2. \tag{37}$$

From (27), the market price efficiency condition takes the form

$$\delta_1 \overline{\pi}_M (X_N, X_N; \delta X_1) + \delta_2 \overline{\pi}_M (X_N, X_N; \delta X_2) = 0.$$

We can now provide explicit conditions under which strategic commitment by the informed speculator does not have value:

Figure 3: Speculator profits in equilibrium with commitment



This figure shows the speculator's profits (35) as function of her trading intensity β. The speculator's expected profit has a maximum at $\beta_S \approx 0.8 < \beta_N = 1$.

Proposition 3: *The Stackelberg equilibrium outcome* (X_S, P_S) *is the same as the Nash equilibrium outcome* (X_N, P_N) *if and only if*

$$\left(\frac{\partial}{\partial x}\right)_{x=X_N(v)} E_u\left[Q\left(x+u, X_N\right)\left\{v - P_N\left(x+u, X_N\right)\right\}\right] \quad = \quad 0. \tag{38}$$

Proof: The equilibria correspond if and only if

$$\delta_k \overline{\pi}_I\left(X_N, X_N; \delta X_k\right) = 0, \quad k = 1, 2.$$

From $\delta_1 \overline{\pi}_I\left(X_N, X_N; \delta X_1\right) = 0$ for the Nash setting we have

$$\left(\frac{\partial}{\partial x}\right)_{x=X_N(v)} E_u\left[x\left\{v - P_N\left(x+u, X_N\right)\right\}\right] = 0,$$

which is equivalent to the result of Proposition 1 for the FOC in the Nash setting. The condition in

(38) reduces to $J(v, x, X) = 0$, which is required for the FOCs for the Nash and Stackelberg to be the same, and hence for equilibrium outcomes to match, which also follows from Proposition 1. □

The equivalence of the first-order conditions in the Nash and Stackelberg settings with a monopolist risk neutral speculator reduces to $J(v, x, X) = 0$, implying that their equilibrium outcomes are identical. Importantly, this correspondence of equilibrium outcomes holds whenever the Nash equilibrium takes a linear form, nesting the classical finance setting with normally-distributed uncertainty:

Proposition 4: *The Nash equilibrium is linear when the pdf $f_v(\cdot)$ is equal to a linear rescaling of $f_u(\cdot)$, i.e., when $\gamma f_u(\gamma u) = f_v(u)$ for some positive real* $\gamma > 0$ and any real u.[7]

The J term vanishes if the Nash equilibrium is linear, implying that Nash and Stackelberg equilibria yield the same equilibrium outcome.

Proof: We conjecture and verify that the equilibrium trading strategy is linear, $X(v) = \beta v$ with trading intensity parameter $\beta = \gamma$. The informationally-efficient pricing rule is then:

$$P_e(y, X) = \frac{\int v f_v(v) f_u(y - \beta v)\, dv}{\int f_v(v) f_u(y - \beta v)\, dv}. \tag{39}$$

Substituting $\beta = \gamma$, equation (39) yields

$$P_e(y, X) = \frac{\int v f_v(v) f_u\left(\gamma\left(\frac{y}{\gamma} - v\right)\right) dv}{\int f_v(v) f_u\left(\gamma\left(\frac{y}{\gamma} - v\right)\right) dv} = \frac{\int v f_v(v) f_v\left(\frac{y}{\gamma} - v\right) dv}{\int f_v(v) f_v\left(\frac{y}{\gamma} - v\right) dv}. \tag{40}$$

Using the new integration variable $v' = \frac{y}{\gamma} - v$, we have $v = \frac{y}{\gamma} - v'$, and (40) yields:

$$P_e(y, X) = \frac{\int v f_v(v) f_v\left(\frac{y}{\gamma} - v\right) dv}{\int f_v(v) f_v\left(\frac{y}{\gamma} - v\right) dv} = \frac{\int \left(\frac{y}{\gamma} - v'\right) f_v\left(\frac{y}{\gamma} - v'\right) f_v(v')\, dv'}{\int f_v\left(\frac{y}{\gamma} - v'\right) f_v(v')\, dv'} = \frac{y}{\gamma} - P_e(y, X).$$

Solving yields $P_e(y, X) = \lambda y$, with $\lambda = \frac{1}{2\gamma}$. Note that $\lambda = \frac{1}{2\gamma}$ is consistent with the speculator's FOC, $2\lambda\gamma = 1$. Thus, the linear equilibrium exists (and uniqueness follows from Boulatov and

[7] See Carre, Collin-Dufresne and Gabriel (2022).

Livdan (2024)).

We now prove that the J term vanishes if the Nash equilibrium is linear. From (38),

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)_{x=X_N(v)} E_u\left[x\left\{v-P\left(x+u,X_N\right)\right\}\right] &= 0,\\ \left(\frac{\partial}{\partial x}\right)_{x=X_N(v)} E_u\left[Q\left(x+u,X_N\right)\left\{v-P\left(x+u,X_N\right)\right\}\right] &= 0.\end{aligned}$$

The speculator's trading strategy is $X_N(v)=\gamma v$ and $P\left(y,X_N\right)=E_{v|y}\left[v\right]=\lambda y$. Therefore,

$$Q\left(y,X_N\right)=E_{v|y}\left[X_N(v)\right]=\lambda\gamma y.$$

Using the short-hand notation $Q(y)=Q(x+u,X_N)$ and $P(y)=P(x+u,X_N)$, we have

$$\left(\frac{\partial}{\partial x}\right)_{x=X_N(v)} E_u\left[Q\left(x+u,X_N\right)\left\{v-P\left(x+u,X_N\right)\right\}\right]=E_u\left[Q'(y)\left\{v-P(y)\right\}-Q(y)P'(y)\right],$$

where $y=X_N(v)+u$. Finally, we obtain

$$\begin{aligned}J = E_u\left[Q'(y)\left\{v-P(y)\right\}-Q(y)P'(y)\right] &= E_u\left[Q'(y)v-\frac{\partial}{\partial y}\left\{P(y)Q(y)\right\}\right]\\ &= E_u\left[\lambda\gamma v-2\lambda^2\gamma y\right]=\lambda\gamma v\left(1-2\lambda\gamma\right)=0. \qquad (41)\end{aligned}$$

The last equality in (41) follows from the speculator's FOC, $2\lambda\gamma=1$. □

Thus, in a common class of financial models—those with normally distributed uncertainty where the speculator's equilibrium trading strategy takes a linear form—a risk-neutral speculator is unable to profit from an ability to commit to the form of her trading strategy. To see the intuition, recall that one can pose the speculator's optimization problem with commitment as maximizing the difference between her actual profits and the market makers' forecast:

$$E_u\left[\left(x-Q\left(x+u,X\right)\right)\left(v-P\left(x+u,X\right)\right)\right]. \qquad (42)$$

Crucially, when $X(v)$ is linear in v, so is the market makers' best estimate of $X(v)$, i.e.,

$$Q(x+u, X) = E\left[\beta v | X(v) + u = y\right] = \beta P(x+u, X) = \beta\lambda(x+u).$$

The last equality follows because linear strategies lead to linear pricing rules. Consequently, maximizing (42) leads to the same trading strategy as with no commitment, where the Q term is absent.

We next show the knife-edge nature of the conditions for Nash and Stackelberg equilibria to correspond. We first show that if the Nash equilibrium strategy deviates just slightly from linearity then Nash and Stackelberg equilibria generically differ, implying that the ability to commit to a trading strategy has strictly positive value. The speculator's Nash equilibrium strategy can slightly deviate from linearity, $X(v) = \beta v + \delta X(v)$, when the pdfs slightly deviate from the rescaling condition that the pdfs f_v and f_u satisfy $\gamma f_u(\gamma u) = f_v(u)$ for some $\gamma > 0$ and any $u \in R$.

To show this, we consider a small variation of the noise distribution pdf,[8] $f_u^{(1)}(u) = f_u(u) + \delta f_u(u)$, such that the scaling condition $\gamma f_u(\gamma u) = f_v(u)$ with the same scaling parameter γ ceases to hold. From Proposition 3, Nash and Stackelberg equilibria correspond if and only if the J term vanishes at equilibrium, i.e. $J(v, x, X) = 0$. We have:

Proposition 5 *In the vicinity of linear equilibria, the set of variations δf_u that lead to a zero J term has measure zero. In contrast, the set of variations δf_u that lead to a non-zero J term has positive measure. Thus, in the vicinity of linear equilibria, speculator profit with commitment almost always exceeds the profit in Nash equilibria.*

Proof: See the Appendix. □

One may conjecture that the source of the speculator's higher profits with commitment reflects that she commits to trading with reduced intensity on her private information thereby reducing the amount of information in net order flow. Notably, we established earlier that this is what happens in the Cho and Karoui (2000) model that features normal liquidity trade and binary private information; and a similar qualitative result emerges in Biais and Germain (2002), where a speculator with binary private information commits to sometimes not trading on that information.

We now show that this reasoning is flawed. We prove that whether commitment leads the

[8]Equivalently, we could assume a small variation of f_v.

speculator to trade more aggressively or less depends on the precise details of the distributions of private information $f_v(\cdot)$ and noise trade $f_u(\cdot)$. From our analysis, it follows directly that whether a speculator trades more aggressively or less hinges on the sign of the J term—the speculator trades less aggressively when, as in the Cho and Karoui model, the J term is positive, and she trades more aggressively when that term is negative. We establish that even within the same environment, the speculator may commit to trade more aggressively than in the BNE after some signals, but less aggressively after others. To do this, we start with a classical Kyle model with $f_u(\cdot) \sim N(0,1)$ and $f_v(\cdot) \sim N(0,1)$, where the equilibrium features linear pricing and no value to commitment. We identify arbitrarily small variations of $\delta f_u(\cdot)$ that give rise to arbitrarily small quadratic deviations $\delta P(\cdot)$ to the BNE linear pricing rule. Given the perturbed BNE pricing rule, we explicitly solve for the J term that determines the exact difference in the BNE and Stackelberg trading intensities following any signal v. We find that the sign of J hinges on the sign of the perturbation and the size of the realization of v. This means that the conjecture that the ability to commit to the form of the trading strategy necessarily leads to reduced trading intensity is false, and we characterize exactly when the Stackelberg trading intensity on signal v is higher or lower than its BNE counterpart.

Proposition 6 *Consider a classic Kyle (1985) setting with $v \sim N(0,1)$ and $u \sim N(0,1)$ giving rise to BNE trading strategy $X_N(v) = v$ and pricing rule $P(y) = y/2$. Then a small perturbation of the noise trade density of $\delta \log(f_u) = Bu^3$ leads to a perturbed BNE pricing rule of the form $P(y) = y/2 + ay^2 + b$, where a has the same sign as B. The Stackelberg trading intensity $X_S(v)$ exceeds that in the BNE if $B > 0$ and $|v| > 1$ or $B < 0$ and $|v| < 1$, and are lower otherwise.*

Proof: See the Appendix. □

The construction and intuition is simple.[9] We show that a small perturbation of the noise trade distribution of $\delta log(f_u) = Bu^3$ leads to a small quadratic perturbation of the BNE pricing function of the form $\delta P(y) = ay^2 + b$, where a has the same sign as B, and b does not affect the relative sensitivity of pricing to the speculator's order.[10] Relative to linear pricing, the quadratic perturba-

[9]We normalize the distributions to have unit variance to ease presentation. This normalization does not affect the result.

[10]Within the adopted perturbative approach, the range of applicability of our results is limited by the requirement that the perturbative term is small compared to the initial quadratic term $u^2/2$, and therefore $B|u| << 1$. Consequently, this leads to $|\delta P(y)| = |ay^2 + b| << |P(y)| = |y|/2$. Roughly speaking, this implies that $|ay| << 1$, and hence $|y| << \frac{1}{|a|}$,

tion increases the expected sensitivity of pricing to the speculator's order in the BNE if and only if $a|v^2| > a|v|$. This happens if $a > 0$ and $|v| > 1$ or $a < 0$ and $|v| < 1$. Following signals for which the expected sensitivity of BNE pricing is higher (and hence the marginal trading profit is lower), the speculator commits to a lower trading intensity than her BNE trading strategy, and she does the opposite following signals for which the expected sensitivity of BNE prices is lower. The quadratic price impact parameter a has the same sign as the noise trade perturbation B, determining whether commitment leads to greater trading aggression after small signals ($a, B < 0$) or large ($a, B > 0$).

3.4 Risk Aversion

To conclude, we highlight another dimension of the knife-edged nature of the setting where both f_v and f_u are Normal. While a risk-neutral monopolist speculator does not benefit from commitment, we now show that in a CARA normal setting, a risk averse speculator always strictly gains from an ability to commit to her trading strategy. Even though the equilibrium remains linear, commitment now pays off for a risk-averse speculator, reflecting that a risk-averse speculator cares about more than just first moments. This leads the speculator to reduce her trading intensity in order to reduce the price impact and hence exposure to price risk.

In a CARA Normal setting, a speculator with risk aversion coefficient $\alpha > 0$ maximizes:

$$U_I\,(v, x, X_c) = -E_u[e^{-\alpha W}], \tag{43}$$

where $W = x(v - P\,(x + u, X_c))$. The pricing rule is still set by competitive market makers and hence is informationally efficient:

$$P(y, X_c) = E[v|X_c(v) + u = y].$$

Given a linear trading strategy for the speculator, $X_c(v) = \beta_c v$, the pricing rule is linear, $P(y, X_c) =$

which means that y can still be sufficiently large in the limit of small $a \to 0$. We also have the results for the perturbation with both cubic and quartic terms, $\delta log(f_u) = Bu^3 - Cu^4$, with $C > 0$, in which case the distribution formally remains normalizable for any $y \in R$ and the correction to pricing rule is $\delta P(y) = cy^3 + ay^2 + dy + b$. Still, the range of applicability is limited by $|\delta log(f_u)| << u^2/2$ and hence $|\delta P(y| << y/2$. With this richer set of distributions, the analysis becomes more complex, but the main results remain qualitative the same (see more details in Appendix).

λy, with $\lambda = \frac{\beta_c}{\beta_c^2+\beta_0^2}$. Substituting $X(v) = \beta v$ into (43) yields a certainty equivalent representation for the speculator's expected payoff,

$$\begin{aligned} W_{eff} &= E_u[x(v - \lambda(x+u))] - \frac{\alpha}{2} Var[x(v - \lambda(x+u))] \\ &= x(v - \lambda x) - \frac{\alpha}{2} Var[x(v - \lambda x) - \lambda x u] \\ &= x(v - \lambda x) - \frac{\alpha}{2} \sigma_u^2 x^2 \lambda^2. \end{aligned}$$

In a BNE, maximizing W_{eff} over x yields a linear optimal trading strategy $X(v) = \beta v$, where

$$\beta = \frac{1}{2\lambda + \alpha\sigma_u^2\lambda^2}.$$

Rearranging and substituting for $\lambda = \frac{\beta_c}{\beta_c^2+\beta_0^2}$ yields

$$\frac{1}{\beta} = 2\lambda + \alpha\sigma_u^2\lambda^2 = 2\frac{\beta_c}{\beta_c^2 + \beta_0^2} + \alpha\sigma_u^2 \left(\frac{\beta_c}{\beta_c^2 + \beta_0^2}\right)^2.$$

In a BNE, market makers' conjectures about the speculator's trading intensity are correct, $\beta_c = \beta = \beta_N$, yielding the fixed point equation

$$\frac{1}{\beta_N} = 2\frac{\beta_N}{\beta_N^2 + \beta_0^2} + \alpha\sigma_u^2 \left(\frac{\beta_N}{\beta_N^2 + \beta_0^2}\right)^2.$$

Getting a common denominator, this reduces to a quartic equation in $\beta_N \geq 0$:

$$\beta_0^4 - \beta_N^4 = \alpha\sigma_u^2\beta_N^3. \tag{44}$$

Because $\beta_N \geq 0$, it follows that $\beta_N \leq \beta_0$, i.e., ceteris paribus, a risk-averse monopolist trades less aggressively than a risk-neutral one.

In the Stackelberg setting, the speculator commits to the linear strategy $X(v) = \beta v$ that solves

$$\beta_S = \arg\max_{\beta} \left\{\beta(1 - \lambda(\beta)\beta) - \frac{\alpha}{2}\sigma_u^2\beta^2\lambda(\beta)^2\right\},$$

with $\lambda(\beta) = \frac{\beta}{\beta^2+\beta_0^2}$. Substituting for $\lambda(\beta)$, the speculator's optimal trading intensity solves

$$\beta_S = \arg\max_{\beta} \left\{ \beta_0^2 \frac{\beta}{\beta^2 + \beta_0^2} - \frac{\alpha}{2}\sigma_u^2 \left(\frac{\beta^2}{\beta^2 + \beta_0^2} \right)^2 \right\}.$$

The associated first-order condition yields

$$\beta_0^2 \frac{\beta_0^2 - \beta_S^2}{\left(\beta_S^2 + \beta_0^2\right)^2} - 2\alpha\sigma_u^2\beta_S \left(\frac{\beta_S^2}{\beta_S^2 + \beta_0^2} \right) \frac{\beta_0^2}{\left(\beta_S^2 + \beta_0^2\right)^2} = 0.$$

Multiplying by $\left(\beta_S^2 + \beta_0^2\right)^3$ yields:

$$\left(\beta_S^2 + \beta_0^2\right)\left(\beta_0^2 - \beta_S^2\right) - 2\alpha\sigma_u^2\beta_S^3 = 0,$$

which simplifies to

$$\beta_0^4 - \beta_S^4 = 2\alpha\sigma_u^2\beta_S^3. \tag{45}$$

To compare trading intensities in the BNE and Stackelberg settings, divide (45) by (44) to obtain

$$\frac{\beta_0^4 - \beta_S^4}{\beta_0^4 - \beta_N^4} = 2.$$

Thus, $\beta_S^4 < \beta_N^4$, and hence $\beta_S < \beta_N$, i.e., a risk-averse speculator who can commit adopts a less aggressive trading strategy than one who cannot. It follows that even though trading strategies are linear in this classical CARA Normal setting, commitment still has value for a risk-averse speculator. Inspection of (44) reveals why. Risk-aversion introduces an additional negative price-risk term $-\frac{\alpha}{2}\sigma_u^2 x^2 \lambda^2$ not present for a risk-neutral speculator who has $\alpha = 0$. The speculator commits to reducing her trading intensity below that in the BNE in order to reduce the equilibrium λ, and hence reduce her exposure to price risk.

4 Conclusion

We analyze strategic commitment by an informed speculator in a Kyle-style competitive dealership market. The speculator commits to a trading strategy and market makers price competitively given knowledge of the functional form of the trading strategy, but not the speculator's private information. We provide conditions under which a unique equilibrium obtains. We characterize the (non-negative) value of this strategic commitment, showing constructively that it is strictly positive in a setting where, rather than receiving a normal signal, the risk-neutral informed trader receives a binary signal about the asset's value and noise trade is normally distributed, so that the resulting informationally-efficient pricing function is non-linear. We then derive necessary and sufficient closed-form conditions for a risk neutral informed trader not to be able to profit from commitment. This imposes conditions on model primitives—the distributions of the fundamental value and noise trade—that are satisfied by linear equilibria, e.g., when both distributions are Normal as in the classical Kyle model, but not when the speculator is risk averse or if equilibria are only "almost linear".

5 Appendix A: Proofs

Proof of Proposition 1: The FOC in the Nash setting (16) follows directly from (13). We now derive the FOC (17) in the Stackelberg setting, which takes into account the price functional variation component. The FOC expressed in terms of the first functional variation of (14) yields

$$0 = E_v\left[\left\{v - \overline{P}(X_S(v), X_S) - X_S(v)\overline{P}' X_S(v), X_S\right\}\delta X(v)\right] - E_{v,u}\left[X_S(v)\delta P(y, X_S, \delta X)\right],$$

where $y = X_S(v) + u$ and we use $\delta P(y, X)$ as short-hand notation for the functional variation

$$\delta P(y, X) = \frac{\int (v - P(y, X)) f_v(v)\left(-\frac{\partial}{\partial y}\right) f_u(y - X(v))\,\delta X(v) dv}{\int f_v(v) f_u(y - X(v))\,dv},$$

and we omit the irrelevant argument δX. Defining $Q(y, X) = E_{v|y}[X(v)]$, we have

$$\begin{aligned}
E_{v,u}[X(v)\delta P(y, X)] &= \int\int f_v(v) f_u(y - X(v))\,X(v)\delta P(y, X)\,dydv \\
&= E_v\left[\int Q(y, X)(v - P(y, X))\left(-\frac{\partial}{\partial y}\right) f_u(y - X(v))\,\delta X(v) dy\right] \\
&= E_v\left[\delta X(v)\int\left(\frac{\partial}{\partial y} Q(y, X)(v - P(y, X))\right) f_u(y - X(v))\,dy\right] \\
&= E_v\left[\delta X(v) E_u\left[\frac{\partial}{\partial y} Q(y, X)(v - P(y, X))\right]\right] \\
&= E_{v,u}\left[\delta X(v)\frac{\partial}{\partial y} Q(y, X)(v - P(y, X))\right].
\end{aligned}$$

Thus,

$$E_{v,u}[X(v)\delta P(y, X)] = E_{v,u}\left[\delta X(v)\frac{\partial}{\partial y} Q(y, X)(v - P(y, X))\right]. \tag{46}$$

Substituting

$$J(v, x, X) = E_u\left[\frac{\partial}{\partial y} Q(y, X)(v - P(y, X))\right] = \frac{\partial}{\partial x} E_u\left[Q(x + u, X)(v - P(x + u, X))\right],$$

into (46) yields

$$E_{u,v}\left[X(v)\delta P(y, X)\right] = E_v\left[\delta X(v) J(v, x, X)\right], \tag{47}$$

for arbitrary variations $\delta X(v)$. Application of the main principle of variation calculus (Kolmogorov and Fomin, 1999) to (47) yields the result of Proposition 1. $\square$

Proof of Proposition 2: The informationally-efficient pricing rule and its functional derivative with respect to the informed trader's strategy are given by (Boulatov and Livdan, 2024):

$$P_e(y,X) = \frac{\int v f_v(v) f_u(y - X(v))\, dv}{\int f_v(v) f_u(y - X(v))\, dv},$$

and

$$\delta P_e(y,X) = \frac{\int (v - P_e(y,X))\, f_v(v) \left(-\frac{\partial}{\partial y}\right) f_u(y - X(v))\, \delta X(v) dv}{\int f_v(v) f_u(y - X(v))\, dv}.$$

Therefore,

$$\begin{aligned}
E_{v,u}[y\delta P_e(y,X)] &= \int\int f_v(v) f_u(y - X(v))\, y\delta P_e(y,X)\, dydv \\
&= \int\int y(v - P_e(y,X))\, f_v(v)\left(-\frac{\partial}{\partial y}\right) f_u(y - X(v))\, \delta X(v) dvdy \\
&= \int f_v(v)\delta X(v) \int \left(\frac{\partial}{\partial y} y(v - P_e(y,X))\right) f_u(y - X(v))\, dydv \\
&= E_v\left[\delta X(v) E_u\left[\frac{\partial}{\partial y} y(v - P_e(y,X))\right]\right] \\
&= E_{v,u}\left[\delta X(v) \frac{\partial}{\partial y} y(v - P_e(y,X))\right] \\
&= E_{v,u}\left[\delta X(v)\left(v - P_e(y,X) - yP'_e(y,X)\right)\right].
\end{aligned}$$

Thus,

$$E_{v,u}[y\delta P_e(y,X)] = -E_{v,u}\left[\delta X(v)\left(P_e(y,X) - v + yP'_e(y,X)\right)\right],$$

or

$$E_{v,u}[y\delta P_e(y,X) + \delta X(v)\left(P_e(y,X) - v + yP'_e(y,X)\right)] = 0. \qquad \square$$

Proof of Theorem 1. Boulatov and Livdan (2024) establish existence of the BNE given the regularity conditions 1–3.

For the Stackelberg equilibrium, first note that the functional $\overline{\pi}_I(X,X)$ is *weakly continuous*.

Hence, its values on each weakly converging sequence $\{X_n\}$ converge (Krasnoselskii, 1964). In fact, we know that $\overline{\pi}_I(X, X)$ has a uniformly continuous gradient and hence is uniformly differentiable.

Claim: The *gradient* of the functional $\overline{\pi}_I(X, X)$ is a completely continuous operator, and therefore the functional $\overline{\pi}_I$ is weakly continuous, and assumes its upper and lower bounds on any finite ball $\|X\| \leq r$ (Krasnoselskii, 1964).

Proof: By (14) and (17), the speculator's expected profit functional has gradient

$$\begin{aligned} \nabla_{X(v)}\overline{\pi}_I(X, X) &= v - A_S\left(v, X(v), X\right), \\ \text{where } A_S\left(v, X(v), X\right) &= \overline{P}\left(X\left(v\right), X\right) + X(v)\overline{P}'\left(X\left(v\right), X\right) + J\left(v, X(v), X\right), \end{aligned}$$

which is equivalent to saying that the full variation of $\overline{\pi}_I(X, X)$ takes the form

$$\delta\overline{\pi}_I\left(X, X; \delta X(v)\right) = E_v\left[\delta X(v)\left(v - \overline{P}\left(X(v), X\right) - X(v)\overline{P}'\left(X(v), X\right) - J\left(v, X(v), X\right)\right)\right],$$

obtained in Proposition 1. The operator $A_S(v, x, X) = \frac{\partial}{\partial x}\left(x\overline{P}(X, X)\right) + J\left(v, X(v), X\right)$ is completely continuous. Boulatov and Livdan (2024) show the first term $\frac{\partial}{\partial x}\left(x\overline{P}(X, X)\right)$ is completely continuous given regularity conditions 1–3.

The J-term given by (15) is also completely continuous. That is, (15) says that

$$J(v, x, X) = E_u\left[\frac{\partial}{\partial y}Q\left(y, X\right)\left(v - P\left(y, X\right)\right)\right] = \int_{-\infty}^{+\infty} f_u\left(y - x\right)\frac{\partial}{\partial y}Q\left(y, X\right)\left(v - P\left(y, X\right)\right)dy. \qquad (48)$$

Therefore, J is given by a superposition, $J = GR$, with $R = \frac{\partial}{\partial y}Q\left(y, X\right)\left(v - P\left(y, X\right)\right)$ and linear operator G acting on a vector $F \in L_1$ and defined by

$$G\left(x; F\right) = \int_{-\infty}^{+\infty} f_u\left(y - x\right)F(y)dy = \mathbb{E}_u\left[F\left(x + u\right)\right].$$

The linear operator G is characterized by a kernel $g\left(x, y\right) = f_u\left(y - x\right)$. As $g\left(x, y\right)$ is continuous, G is completely continuous (Krasnosel'skii, 1964, p. 19). The pricing rule $R\left(y, X\right)$ is continuous with respect to the functional argument X. It is even Frechet differentiable for any nontrivial strategy

$X \neq 0$, and hence is smooth in the functional sense. Thus, the operator J is completely continuous as a superposition of continuous and completely continuous operators (Krasnoselskii, 1964, p. 46). $\square$

Next, we use the explicit form of the pricing rule to establish:

Claim: If the norm of the speculator's trading strategy $\|X\|$ increases unboundedly, $\|X\| \to \infty$, then $\overline{\pi}_I(X, X) \to 0$.

Proof: We have

$$\overline{\pi}_I(X, X) = E_{u,v}[X(v)(v - P_e(X(v) + u, X))],$$

with

$$P_e(X(s) + u, X) = \frac{\int v f_v(v) f_u(X(s) - X(v) + u)\, dv}{\int f_v(v) f_u(X(s) - X(v) + u)\, dv}.$$

Suppose X belongs to a finite ball $\|X\| \leq r$, and consider a transformation $X \to tX$. In the limit $t \to \infty$, we have asymptotically $f_u(tz) \to \frac{1}{t}\delta(z)$. Taking into account that, for any differentiable function g, $\delta(g(s)) = \frac{\delta(s - s_0)}{g'(s_0)}$, with $g(s_0) = 0$ and $g'(s_0) \neq 0$, we obtain

$$P_e(tX(s) + u, tX) \;=\; \frac{\int v f_v(v) f_u(tX(s) - tX(v) + u)\, dv}{\int f_v(v) f_u(tX(s) - tX(v) + u)\, dv} = \frac{\int v f_v(v) f_u\left(t\left(X(s) - X(v) + \frac{u}{t}\right)\right) dv}{\int f_v(v) f_u\left(t\left(X(s) - X(v) + \frac{u}{t}\right)\right) dv} \to s,$$

and hence

$$\overline{\pi}_I(X, X) \to E_{u,v}[X(v)(v - v)] = 0.$$

As any vector with infinitely large norm, $\|X\| \to \infty$, can be obtained from the vector $\|X_0\| \leq r$ via a transformation $X = tX_0$, this result means that $\overline{\pi}_I(X, X) \to 0$ for any X with norm $\|X\| \to \infty$. $\square$

It follows that strategies with infinite norms are not profitable, and thus the optimal strategy must have a finite norm. Because the expected profit functional $\overline{\pi}_I$ is continuous and finite in any finite region $\|X\| < +\infty$, this also means that the speculator's expected profit is bounded from above, $\overline{\pi}_I(X, X) \leq a < +\infty$. If r is sufficiently large, then $\max_{\|X\| \leq r} \overline{\pi}_I(X, X) = \max_{\|X\| \leq h} \overline{\pi}_I(X, X)$, for some $h < r$ (Krasnoselskii, 1964). Because $\overline{\pi}_I$ is weakly continuous, $\max_{\|X\| \leq h} \overline{\pi}_I(X, X) = \overline{\pi}_I^*$, where $\overline{\pi}_I^* \leq a < +\infty$ is a maximal value of $\overline{\pi}_I(X, X)$ in the entire strategy space.

In sum, the optimum $\overline{\pi}_I^*$ of $\overline{\pi}_I(X, X)$ exists and is finite, and the associated optimal strategy X^*

has a finite norm, with $\overline{\pi}_I^* = \overline{\pi}_I(X^*, X^*)$. Last, because $\overline{\pi}_I(X, X)$ is uniformly differentiable, its gradient at the optimal strategy X^* vanishes, i.e., $\nabla_{X^*(v)}\overline{\pi}_I(X^*, X^*) = 0$ (Krasnoselskii, 1964). □

Proof of Proposition 5: We proceed with a series of lemmas. We first show that as a result of the variation, $\delta f_u(u) \neq 0$, the initial linear Nash equilibrium (X, P) defined by the speculator's strategy $X = \beta v$ and pricing rule $P = \lambda y$ transforms into the (typically nonlinear) equilibrium $\left(X^{(1)}, P^{(1)}\right)$ with $X^{(1)}(v) = \beta v + \delta X(v)$ and $P^{(1)}(y) = \lambda y + \delta P(y)$, where the variations δX and δP are uniquely defined, continuously and smoothly depending on the variation $\delta f_u(u)$. More precisely, the variations δX and δP are expressed through $\delta f_u(u)$ by means of continuous linear operators in a Banach space:

Lemma 1 *With a small variation of the noise distribution pdf, $f_u^{(1)}(u) = f_u(u) + \delta f_u(u)$, the initial linear Nash equilibrium (X, P) transforms into $\left(X^{(1)}, P^{(1)}\right)$ with $X^{(1)}(v) = \beta v + \delta X(v)$ and $P^{(1)}(y) = \lambda y + \delta P(y)$, as $\delta P = R_P \delta f_u$ and $\delta X = R_X \delta f_u$, where R_P and R_X are linear operators in Banach space defined by*

$$\delta P(y) = \int_{-\infty}^{+\infty} R_P(y, y')\, \delta f_u(y')\, dy', \text{ and } \delta X(v) = \int_{-\infty}^{+\infty} R_X(v, v')\, \delta f_u(v')\, dv',$$

with the linear operators R_P and R_X defined in the proof.

Proof: The pricing rule variation is given by

$$\begin{aligned} \delta P(y, X) &= \frac{\int (v - P(y, X))\, f_v(v) \left(\delta f_u(y - X(v)) - \frac{\partial}{\partial y} f_u(y - X(v))\, \delta X(v)\right) dv}{\int f_v(v) f_u(y - X(v))\, dv} \\ &= \frac{\int (v - \lambda y)\, f_v(v) \left(\delta f_u(y - \beta v) - f_u'(y - \beta v)\, \delta X(v)\right) dv}{\int f_v(v) f_u(y - \beta v)\, dv}. \end{aligned} \tag{49}$$

Because the FOC (16) always holds in the Nash equilibrium,

$$v = \left(\frac{\partial}{\partial x}\right)_{x = X(v)} x \overline{P}(x, X).$$

Using the inverse function $V(\cdot)$ property, $V(X(v)) = v$, we have

$$\delta X(v, X) = -\frac{1}{V'(X(v))}\left(\frac{\partial}{\partial x}\right)_{x=X(v)} x\delta\overline{P}(x, X) = -\frac{1}{2\lambda}\frac{\partial}{\partial v}v\delta\overline{P}(\beta v) = -\frac{1}{2\lambda}E_u\left[\frac{\partial}{\partial v}v\delta P(\beta v + u)\right], \quad (50)$$

where we use the fact that the variation is around the linear strategy $X(v) = \beta v$.

Combining (49) and (50) yields a closed-form linear nonuniform integral equation with respect to the pricing rule variation δP as

$$(1 - K)\delta P = L\delta f_u, \quad (51)$$

with the linear operators K and L.

We use the following standard short-hand notation. The action of linear operators K and L on variations δP and δf_u can be represented by linear kernels: $K\delta P(y) = \int K(y.y')\delta P(y')$, and $L\delta f_u(y) = \int L(y, y')\delta f_u(y')dy'$, respectively. [11] Specifically, :

$$\begin{aligned} K\delta P(y) &= \frac{\int (v - \lambda y) f_v(v) f_u'(y - \beta v) \frac{1}{2\lambda} E_u\left[\frac{\partial}{\partial v}v\delta P(\beta v + u)\right] dv}{\int f_v(v) f_u(y - \beta v) dv} \quad (52)\\ &= \frac{1}{2\lambda\beta}\frac{\int\int (v - \lambda y) f_v(v) f_u'(y - \beta v) f_u(y' - \beta v)\left(1 + \beta v\frac{\partial}{\partial y'}\right)\delta P(y')\, dv dy'}{\int f_v(v) f_u(y - \beta v) dv} \\ &= \frac{\int\int (v - \lambda y) f_v(v) f_u'(y - \beta v)\left(1 - \beta v\frac{\partial}{\partial y'}\right) f_u(y' - \beta v)\delta P(y')\, dv dy'}{\int f_v(v) f_u(y - \beta v) dv}, \end{aligned}$$

and

$$L\delta f_u(y) = \frac{\int (v - \lambda y) f_v(v)\delta f_u(y - \beta v) dv}{\int f_v(v) f_u(y - \beta v) dv} = \frac{1}{\beta}\frac{\int\left(\frac{y}{2} - \xi\right) f_u(y - \xi)\delta f_u(\xi) d\xi}{\int f_u(\xi) f_u(y - \xi) d\xi}, \quad (53)$$

where in the last equality we introduce a dummy integration variable $\xi = y - \beta v$ and use the scaling property $f_v\left(\frac{1}{\beta}u\right) = \beta f_u(u)$.

Now consider (51), which is a non-uniform Fredholm type integral equation[12] (Kolmogorov and Fomin, 1999, p.116-117). Kolmogorov and Fomin (1999, p.120) show that: **either** (1) equation

[11] Note that both results are particular cases of Hahn-Banach representation theorem (see, e.g., Balakrishnan, 1971).
[12] Also referred to as a Fredholm Alternative.

(51) has a unique solution δP for any right-hand side given by $L\delta f_u$, which means that the operator $I - K$ is invertible, **or** (2) the uniform integral equation $(1 - K)\,\delta P = 0$ has a non-trivial solution.

We can exclude the second possibility, as it would imply the existence of other Nash equilibrium infinitesimally close to the initial linear one, even though the distribution of noise trade did not change, contradicting the uniqueness of Nash equilibrium under conditions 1–3 established by Boulatov and Livdan (2024).

Thus, we have $\delta P = (1 - K)^{-1} L\delta f_u = R_P \delta f_u$ with the linear operator $R_P = (1 - K)^{-1} L$. Using (50), we also obtain $\delta X = M\delta P = MR_P\delta f_u = R_X\delta f_u$, with the linear operator $R_X = MR_P$ and $M\delta P = \frac{\beta}{2\lambda} \int f_u' (y - \beta v)\, \delta P(y)\, dy$. Based on the Riesz representation theorem (Balakrishnan, 1971), the linear operators R_P and R_X are represented as $R_P\delta f_u = \int\limits_{-\infty}^{+\infty} R_P(y, y')\, \delta f_u(y')\, dy'$ and $R_X\delta f_u = \int\limits_{-\infty}^{+\infty} R_X(v, v')\, \delta f_u(v')\, dv'$, respectively. □

We next observe that there are some "directions" in the functional space of variations δf_u along which the equilibrium remains linear so the J term remains zero, $J = 0$, and therefore commitment does not lead to higher expected speculator profits. We then show that these "directions" in the functional space are "rare" comprising a measure zero set, while variations δf_u for which the J term is nonzero comprise a positive measure set.

Recall that in our initial linear Nash equilibrium $X(v) = \beta v$, $P(y) = \frac{1}{2\beta} y$, the equilibrium trading intensity β equals the scaling parameter, $\beta = \gamma$, with the scaling condition $\gamma f_u(\gamma u) = f_v(u)$. Then we have the following result:

Lemma 2 *If the variation δf_u of the noise distribution satisfies the condition*

$$\delta f_u(u) = -\delta k \frac{\partial}{\partial u} u f_u(u)\,, \tag{54}$$

with a real parameter $\delta k = \frac{\delta\gamma}{\gamma}$, then the initial linear equilibrium $X(v) = \beta v$, $P(y) = \frac{1}{2\beta} y$ transfroms into a linear one with $\beta^{(1)} = \beta + \delta\gamma$, and hence the J term remains zero.

Proof: (54) implies that with the small deviation $\delta\gamma \to 0$, the scaling property of the distribution still holds, but with a "shifted" scaling parameter of $\gamma^{(1)} = \gamma + \delta\gamma$, as

$$(\gamma + \delta\gamma)\left(f_u\left((\gamma + \delta\gamma)u\right) + \delta f_u(\gamma u)\right) = f_v(u) + o(\delta\gamma), \tag{55}$$

where we use the notation $o(\delta\gamma)$ for the second-order terms associated with the small change, $\delta\gamma$. Indeed, (55) says that

$$(\gamma + \delta\gamma)\left(f_u(\gamma u) + f_u'(\gamma u)\, u\delta\gamma + \delta f_u(\gamma u)\right) = f_v(u) + o(\delta\gamma),$$

or

$$(\gamma + \delta\gamma)\left(f_u(\gamma u) + f_u'(\gamma u)\, u\delta\gamma\right) + \gamma\delta f_u(\gamma u) = f_v(u) + o(\delta\gamma).$$

Using $\gamma f_u(\gamma u) = f_v(u)$, this simplifies to

$$\delta\gamma\left(f_u(\gamma u) + \gamma u f_u'(\gamma u)\right) + \gamma\delta f_u(\gamma u) = 0,$$

equivalent to (54).

Now, we directly verify that under the transformation (54), the initial linear equilibrium transforms into a linear one with $\beta^{(1)} = \beta + \delta\gamma$. The variation of the pricing rule is

$$\begin{aligned}
\delta P(y, X) &= \frac{\int (v - \lambda y)\, f_v(v)\left(\delta f_u(y - \beta v) - f_u'(y - \beta v)\,\delta X(v)\right) dv}{\int f_v(v) f_u(y - \beta v)\, dv} \\
&= -\delta k \frac{\int (v - \lambda y)\, f_v(v) f_u(y - \beta v)\, dv}{\int f_v(v) f_u(y - \beta v)\, dv} + \frac{\int (v - \lambda y)\, f_v(v) f_u'(y - \beta v)\left(-\delta k y + \delta k \beta v - \delta X(v)\right) dv}{\int f_v(v) f_u(y - \beta v)\, dv},
\end{aligned}$$

which, with re-organization, becomes

$$\begin{aligned}
\delta P(y, X) &= -\delta k \frac{\int (v - \lambda y)\, f_v(v) f_u(y - \beta v)\, dv}{\int f_v(v)\, f_u(y - \beta v)\, dv} - \delta k y \frac{\int (v - \lambda y)\, f_v(v) f_u'(y - \beta v)\, dv}{\int f_v(v) f_u(y - \beta v)\, dv} \\
&\quad + \frac{\int (v - \lambda y)\, f_v(v) f_u'(y - \beta v)\left(\delta k \beta v - \delta X(v)\right) dv}{\int f_v(v) f_u(y - \beta v)\, dv}.
\end{aligned} \tag{56}$$

Consider the right-hand side of (56). The first term vanishes as it equals $-\delta k\left(E[v|y] - P(y)\right) = 0$. The second term equals $-\delta k y P'(y) = -\frac{\delta\gamma}{\gamma} P(y) = \frac{\delta\lambda}{\lambda} P(y)$, with $\frac{\delta\lambda}{\lambda} = -\frac{\delta\gamma}{\gamma} = -\frac{\delta\beta}{\beta}$, consistent with the linear equilibrium condition $\lambda = \frac{1}{2\beta}$. The third term also vanishes if the strategy variation satisfies $\delta k \beta v = \delta X(v)$, or $\frac{\delta X(v)}{X(v)} = \delta k = \frac{\delta\beta}{\beta}$. This holds if and only if the speculator's strategy remains linear, $X^{(1)}(v) = \beta v + \delta\beta v = (\beta + \delta\beta)\, v$. □

Finally we prove that the special "shifts" in the functional space like (54) along which the J term remains zero, are "rare":

Lemma 3 *For almost all variations δf_u, the J term is nonzero, implying that the equilibrium becomes nonlinear, and hence commitment almost always pays off.*

Proof: Define $I(v,x,X) = \frac{\partial}{\partial x} J(v,x,X)$. In equilibrium, $J(v,x,X) = J(X^*(v), X^*)$ and $I(v,x,X) = I(X^*(v), X^*)$, where $X^*(v)$ is an equilibrium trading strategy. To ease notation, we drop the star and write $X^*(v)$ as $X(v)$. With the notation $\overline{Q}(x,X) = E_u[Q(x+u,X)]$, the same transformations yield

$$\begin{aligned}
I(v,x,X) &= E_u\left[Q(x+u,X)\{v - P(x+u,X)\}\right] \\
&= E_u\left[\left(Q(x+u,X) - \overline{Q}(x,X) + \overline{Q}(x,X)\right)\{v - P(x+u,X)\}\right] \\
&= -E_u\left[\left(Q(x+u,X) - \overline{Q}(x,X)\right)\left(P(x+u,X) - \overline{P}(x,X)\right)\right] + \overline{Q}(x,X)\left\{v - \overline{P}(x,X)\right\} \\
&= -Cov_u\left[Q(x+u,X), P(x+u,X)\right] + \overline{Q}(x,X)\left\{v - \overline{P}(x,X)\right\}. \qquad (57)
\end{aligned}$$

When the pdf of noise trade distribution f_u shifts to $f_u + \delta f_u$, the initial linear Nash equilibrium shifts to a new one, with $Q(y,X) = \lambda\beta y + \delta Q(y,X)$, $P(y,X) = \lambda y + \delta P(y,X)$. Then the last equality of (57) yields, in the first order limit with respect to the variations δQ and δP,

$$\begin{aligned}
I(v,x,X) &= -Cov_u\left[(\lambda\beta(x+u) + \delta Q(x+u,X)), (\lambda(x+u) + \delta P(x+u,X))\right] + \overline{Q}(x,X)\left\{v - \overline{P}(x,X)\right\} \\
&= -\lambda E_u\left[u, (\delta Q(x+u,X) + \beta\delta P(x+u,X))\right] + \overline{Q}(x,X)\left\{v - \overline{P}(x,X)\right\} + o(\delta Q, \delta P).
\end{aligned}$$

We must evaluate the variation $\delta I = I - I_0$, where I_0 corresponds to the initial noise distribution f_u.

One can show that $\delta Q(y,X) = \beta\delta P(y,X) + \delta\varphi(y,X)$, where $\delta\varphi(y,X) = E_{v|y}[\delta X(v)]$. That is,

$$\begin{aligned}
\delta P(y,X) &= \frac{\int (v-\lambda y)\, f_v(v)\left(\delta f_u(y-\beta v) - f_u'(y-\beta v)\,\delta X(v)\right) dv}{\int f_v(v) f_u(y-\beta v)\, dv}, \\
\delta Q(y,X) &= \frac{\int \beta(v-\lambda y)\, f_v(v)\left(\delta f_u(y-\beta v) - f_u'(y-\beta v)\,\delta X(v)\right) dv}{\int f_v(v) f_u(y-\beta v)\, dv} + \frac{\int f_v(v) f_u(y-\beta v)\,\delta X(v) dv}{\int f_v(v) f_u(y-\beta v)\, dv},
\end{aligned}$$

and hence $\delta Q(y,X) = \beta\delta P(y,X) + \delta\varphi(y,X)$. In what follows, we omit the functional variable X and use the short-hand notation $\delta Q(y)$ and $\delta P(y)$ for variations of the expected trading strategy and pricing rule, respectively. With this notation, transforming the last term and taking into account that in equilibrium, $x = X(v)$, and the FOC is satisfied as $v - \overline{P}(x) = x\overline{P}'(x)$ with $2\lambda\beta = 1$, we obtain

$$\delta I = -E_u[u\delta P(x+u,X)] - \lambda E_u[u\delta\varphi(x+u,X)] + \delta\overline{\varphi}(x,X)\left(v - \overline{P}(x,X)\right) + o(\delta Q, \delta P),$$

where $\delta\overline{\varphi}(x,X) = E_u[\delta\varphi(x+u,X)]$. Differentiating with respect to x yields

$$\begin{aligned}
J &= -E_u[u\delta P'(x+u)] - \lambda E_u[u\delta\varphi'(x+u)] + \lambda x(v-\lambda x)\frac{\partial}{\partial x}\left(\frac{\delta\overline{\varphi}(x,X)}{x}\right) + o(\delta Q,\delta P) \quad (58)\\
&= -E_u[u\delta P'(x+u)] - \lambda E_u[u\delta\varphi'(x+u)] \\
&\quad -\lambda E_u[\beta\delta P(x+u) + \delta\varphi(x+u) - x(\beta\delta P'(x+u) + \delta\varphi'(x+u))] + o(\delta Q,\delta P).
\end{aligned}$$

Now we have auxilliary results

$$\begin{aligned}
E_u[u\delta P'(x+u)] = \int f_u(u)\, u\frac{\partial}{\partial x}\delta P(x+u)\, du &= \int f_u(u)\, u\frac{\partial}{\partial u}\delta P(x+u)\, du \\
&= -\int \delta P(y)\frac{\partial}{\partial y}((y-x) f_u(y-x))\, dy,
\end{aligned}$$

and $\delta\varphi(y) = -\frac{1}{2\lambda}E_{v|y}\left[\frac{\partial}{\partial v}\left(v\delta\overline{P}(\beta v)\right)\right]$, which yields

$$\delta\varphi(y) = -\frac{1}{2\lambda}\frac{\int f_v(v) f_u(y-\beta v)\frac{\partial}{\partial v}\left(v\delta\overline{P}(\beta v)\right) dv}{\int f_v(v) f_u(y-\beta v)\, dv} = \frac{1}{2\lambda}\frac{\int v\delta\overline{P}(\beta v)\frac{\partial}{\partial v}(f_v(v) f_u(y-\beta v))\, dv}{\int f_v(v) f_u(y-\beta v)\, dv},$$

where we integrate by parts to obtain the last equality. Introducing a new dummy integration variable $\xi = \beta v$ and using the scaling property of the noise distribution, we finally obtain

$$\begin{aligned}
\delta\varphi(y) &= -\beta \int K_\varphi(y, y')\,\delta P(y')\,dy', \\
K_\varphi(y, y') &= \frac{\int f_u(\xi)\, f_u(y-\xi)\, \frac{\partial}{\partial \xi}\left(\xi f_u(y'-\xi)\right) d\xi}{\int f_u(\xi)\, f_u(y-\xi)\, d\xi}.
\end{aligned}$$

Note that the kernel $K(y, y')$ defined by (52) and $K_\phi(y, y')$ are different, $K(y, y') \neq K_\phi(y, y')$.

Substituting the first-order variations δQ and δP into (58) yields:

$$\begin{aligned}
J &= -\frac{1}{2}E_u\left[2u\delta P'(y) - u\frac{\partial}{\partial y}\int dy' K_\varphi(y, y')\,\delta P(y')\right] - \frac{1}{2}E_u\left[\delta P(y) - \int dy' K_\varphi(y, y')\,\delta P(y')\right] \\
&\quad -\frac{1}{2}E_u\left[-x\delta P'(y) + x\frac{\partial}{\partial y}\int dy' K_\varphi(y, y')\,\delta P(y')\right], \\
&= -\int f_u(y-x)(y-x)\frac{\partial}{\partial y}\delta P(y)\,dy + \frac{1}{2}\int f_u(y-x)(y-x)\frac{\partial}{\partial y}\int dy' K_\varphi(y, y')\,\delta P(y')\,dy \\
&\quad -\frac{1}{2}\int f_u(y-x)\left(\delta P(y) - \int dy' K_\varphi(y, y')\,\delta P(y')\right) dy \\
&\quad +\frac{1}{2}x\int f_u(y-x)\left(\delta P'(y) - \frac{\partial}{\partial y}\int dy' K_\varphi(y, y')\,\delta P(y')\right) dy
\end{aligned}$$

Integrating the first and second terms by parts yields

$$\begin{aligned}
J &= \int \frac{\partial}{\partial y}\left((y-x)\, f_u(y-x)\right)\delta P(y)\,dy - \frac{1}{2}\int\int \frac{\partial}{\partial y}\left((y-x)\, f_u(y-x)\right) K_\varphi(y, y')\,\delta P(y')\,dy'\,dy \\
&\quad -\frac{1}{2}\int f_u(y-x)\left(\delta P(y) - \int dy' K_\varphi(y, y')\,\delta P(y')\right) dy \\
&\quad +\frac{1}{2}x\int f_u(y-x)\left(\delta P'(y) - \frac{\partial}{\partial y}\int dy' K_\varphi(y, y')\,\delta P(y')\right) dy,
\end{aligned}$$

which, defining

$$\begin{aligned}
\Gamma(y, x) &= 2\frac{\partial}{\partial y}\left((y-x)\, f_u(y-x)\right) + \int (y'-x)\, f_u(y'-x)\,\frac{\partial}{\partial y'}K_\varphi(y', y)\,dy' \qquad (59) \\
&\quad - f_u(y-x) + \int f_u(y'-x)\, K_\varphi(y', y)\,dy' - x\frac{\partial}{\partial y}f_u(y-x) - x\int f_u(y'-x)\,\frac{\partial}{\partial y'}K_\varphi(y', y)\,dy',
\end{aligned}$$

can be put in the form

$$J = \frac{1}{2}\int \Gamma(y,x)\,\delta P(y)\,dy.$$

Next, note that an arbitrary variation δf_u leads to a new equilibrium and causes a variation of the pricing rule $\delta P = R_P \delta f_u$, where R_P is a linear operator according to (49). Using the fact that a variation δf_u translates into a variation δP, we now show that for almost all variations δf_u the variation of δP is nonzero, and hence the J term is almost always nonzero.

This stability property follows from the Euler-Lagrange lemma (Young, 1969), which says that the integral (5) could be zero for arbitrary variation $\delta P(y)$ if and only if the function $\Gamma(y,x)$ is identically zero for any real y and x, i.e., $\Gamma(y,x) \equiv 0$, for $\forall y, x \in R$. From (59), this is equivalent to the following integro-differential equation with respect to the pdf f_u:

$$\begin{aligned} f_u(y-x) &= 2\frac{\partial}{\partial y}\left((y-x) f_u(y-x)\right) + \int (y'-x) f_u(y'-x)\frac{\partial}{\partial y'}K_\varphi(y',y)\,dy' \\ &\quad + \int f_u(y'-x) K_\varphi(y',y)\,dy' - x\frac{\partial}{\partial y} f_u(y-x) - x\int f_u(y'-x)\frac{\partial}{\partial y'}K_\varphi(y',y)\,dy'. \end{aligned} \tag{60}$$

Integrating both parts with respect to $x \in (-\infty; +\infty)$, we obtain

$$\begin{aligned} \int f_u(y-x)\,dx = -1 &= 2\frac{\partial}{\partial y}\int (y-x) f_u(y-x)\,dx \\ &\quad + \int\int (y'-x) f_u(y'-x)\,dx \frac{\partial}{\partial y'}\int K_\varphi(y',y)\,dy dy' \\ &\quad - \int K_\varphi(y',y)\,dy' - \int\int x f_u(y'-x)\frac{\partial}{\partial y'}K_\varphi(y',y)\,dx dy'. \end{aligned} \tag{61}$$

The first two terms on the right-hand side of (61) vanish since, by assumption, the noise distribution has zero first moment, $\int f_u(u)\,u du = 0$. Collecting the last two terms on the right-hand side yields

$$\begin{aligned} &\int K_\varphi(y',y)\,dy' + \int\int x f_u(y'-x)\frac{\partial}{\partial y'}K_\varphi(y',y)\,dx dy' \\ = &\int K_\varphi(y',y)\,dy' + \int\int (y'-(y'-x)) f_u(y'-x)\frac{\partial}{\partial y'}K_\varphi(y',y)\,dx dy' \\ = &\int K_\varphi(y',y)\,dy' + \int y'\frac{\partial}{\partial y'}K_\varphi(y',y)\,dy' = \int \frac{\partial}{\partial y'}\left(y' K_\varphi(y',y)\right) dy' = 0, \end{aligned}$$

where the last equality holds because $|y|\,K_\varphi(y',y) \to 0$, when $|y| \to \infty$.

Hence we conclude that $\int \Gamma(y,x)\,dx = -1$ and the equation (60) cannot hold for arbitrary variations δP. This means that (5) is not zero for price variations δP of nonzero measure, and it can be zero for only special variations of zero measure.

Further note that

$$\int K_{\varphi}(y',y)\,dy = \frac{\int f_u(\xi)\, f_u(y'-\xi)\, \frac{\partial}{\partial \xi}\left(\xi \int f_u(y-\xi)\,dy\right) d\xi}{\int f_u(\xi)\, f_u(y'-\xi)\,d\xi} = 1.$$

Using this, we show that, as above, $\int \Gamma(y,x)\,dy = 0$. In particular, integrating (60) with respect to y, note that all terms except for the third one on the right-hand side vanish, yielding

$$\int f_u(y-x)\,dy = 1 = \int f_u(y'-x) \int K_{\varphi}(y',y)\,dydy' = 1,$$

and hence $\int \Gamma(y,x)\,dy = 0$. One can also show that $\int \Gamma(y,x)\,ydy = 0$ reflecting that the J term vanishes in linear equilibria. Using (5), we investigate the condition $J = 0$ further. The condition

$$g(x) = \int \Gamma(y,x)\,\delta P(y)\,dy = 0, \tag{62}$$

can be viewed as a linear integral equation with respect to the variation δP. As we know, it has a non-trivial linear solution corresponding to the linear equilibrium, $\delta P(y) = \delta\lambda y$. However, the set of solutions of (62) forms a zero measure set in the space of all price variations δP. In particular, we now show that it can only have linear solutions of the form $\delta P(y) = \mu y + \nu$ with real parameters $\mu, \nu \in R$. Going back to (59), we observe that

$$\begin{aligned}
2J &= -2\int f_u(y-x)(y-x)\frac{\partial}{\partial y}\delta P(y)\,dy + x\int f_u(y-x)\frac{\partial}{\partial y}\delta P(y)\,dy \qquad (63)\\
&\quad + \int f_u(y-x)(y-2x)\frac{\partial}{\partial y}\int K_{\varphi}(y,y')\,\delta P(y')\,dy'dy\\
&\quad - \int f_u(y-x)\left(\delta P(y) - \int K_{\varphi}(y,y')\,\delta P(y')\,dy'\right)dy\\
&= -\int f_u(y-x)(y-x)\frac{\partial}{\partial y}\delta P(y)\,dy - \int f_u(y-x)(y-2x)\frac{\partial}{\partial y}\left(\delta P(y) - \int K_{\varphi}(y,y')\,\delta P(y')\,dy'\right)dy\\
&\quad - \int f_u(y-x)\left(\delta P(y) - \int K_{\varphi}(y,y')\,\delta P(y')\,dy'\right)dy,
\end{aligned}$$

Defining

$$\Psi(y,\delta P) = \delta P(y) - \int K_\varphi(y,y')\,\delta P(y')\,dy'. \tag{64}$$

and

$$\Phi(y,x,\delta P) = \Psi(y,\delta P) + (y-2x)\frac{\partial}{\partial y}\Psi(y,\delta P), \tag{65}$$

we write this as

$$2J = -\int f_u(y-x)(y-x)\frac{\partial}{\partial y}\delta P(y)\,dy - \int f_u(y-x)\left(\Psi(y,\delta P) + (y-2x)\frac{\partial}{\partial y}\Psi(y,\delta P)\right)dy. \tag{66}$$

Therefore, the condition $J = 0$ yields

$$\int f_u(y-x)(y-x)\frac{\partial}{\partial y}\delta P(y)\,dy + \int f_u(y-x)\,\Phi(y,x,\delta P)\,dy = 0. \tag{67}$$

One can show that the integral equation $\Phi(y,x,\delta P) = 0$ has only linear solutions, which also satisfy $\int f_u(y-x)(y-x)\frac{\partial}{\partial y}\delta P(y)\,dy = 0$. To summarize, the linear deviations $\delta P(y)$ satisfy (67), and there are no other solutions.

We conclude that (60) cannot hold. Therefore, (5) is not zero for price variations δP of nonzero measure, and is zero for only special variations of zero measure. □

Proof of Proposition 6: We first conjecture and verify that a small log-cubic perturbation of the noise pdf, $\delta\log(f_u) = \delta f_u/f_u = Bu^3$ leads to a BNE pricing rule of the form $P(y) = y/2 + ay^2 + b$, where a and b are real coefficients that depend on B that we solve for below.

Recall that we have (51): $(1-K)\,\delta P = L\delta f_u$. We first calculate $L\delta f_u$ and then $K\delta P$ and finally solve for δP in terms of B. Using our short-hand notation and substituting $\sigma_u = \sigma_v = \beta = 1$, and

$\lambda = 1/2$ into $L\delta f_u$ yields:

$$
\begin{aligned}
L\delta f_u(y) &= \frac{\int \left(v - \frac{1}{2}y\right) f_v(v) f_u(y-v) \frac{\delta f_u(y-v)}{f_u(y-v)} dv}{\int f_v(\eta) f_u(y-\eta) d\eta} \\
&= \frac{\int \left(\xi - \frac{y}{2}\right) f_u(y-\xi) f_u(\xi) B(y-\xi)^3 d\xi}{\int f_u(\eta) f_u(y-\eta) d\eta} \\
&= B\frac{\int \left(\frac{y}{2} - \xi\right) \xi^3 f_u(y-\xi) f_u(\xi) d\xi}{\int f_u(\eta) f_u(y-\eta) d\eta} = -B\left(\frac{3}{4} + \frac{3}{2}\left(\frac{y}{2}\right)^2\right).
\end{aligned} \tag{68}
$$

Here, the second equality follows from substitution for $f_u(\cdot) = f_v(\cdot)$ and the functional form of the perturbation. The third equality follows from expansion of $(y-\xi)^3$ and the fact that u and v are mean zero. To see the final equality expand the expression $(y/2-\xi)\xi^3$ being integrated:

$$
(y/2-\xi)\xi^3 = -(\xi - y/2)^4 - 3(y/2)^2(\xi - y/2)^2 - 3(y/2)(\xi - y/2)^3 - (y/2)^3(\xi - y/2).
$$

The integrals of this expansion vanish for all terms of odd powers in $\xi - y/2$ as:

$$
\frac{\int \left(\xi - \frac{y}{2}\right)^n f_u(y-\xi) f_u(\xi) d\xi}{\int f_u(\eta) f_u(y-\eta) d\eta} = \frac{\int \left(\frac{y}{2} - z\right)^n f_u(y-z) f_u(z) dz}{\int f_u(\eta) f_u(y-\eta) d\eta} = 0, n = 2k+1,
$$

where we use the integration dummy variable $z = y - \xi$. For $n = 2$ we have:

$$
-3(\frac{y}{2})^2 \frac{\int \left(\xi - \frac{y}{2}\right)^2 f_u(y-\xi) f_u(\xi) d\xi}{\int f_u(\eta) f_u(y-\eta) d\eta} = -3(\frac{y}{2})^2 \frac{\int \left(\xi - \frac{y}{2}\right)^2 e^{-\frac{1}{2}(y-\xi)^2} e^{-\frac{1}{2}\xi^2} d\xi}{\int e^{-\frac{1}{2}(y-\eta)^2} e^{-\frac{1}{2}\eta^2} d\eta} = -\frac{3}{2}\left(\frac{y}{2}\right)^2,
$$

and for $n = 4$, we have

$$
-\frac{\int \left(\xi - \frac{y}{2}\right)^4 f_u(y-\xi) f_u(\xi) d\xi}{\int f_u(\eta) f_u(y-\eta) d\eta} = -\frac{\int \left(\xi - \frac{y}{2}\right)^4 e^{-\frac{1}{2}(y-\xi)^2} e^{-\frac{1}{2}\xi^2} d\xi}{\int e^{-\frac{1}{2}(y-\eta)^2} e^{-\frac{1}{2}\eta^2} d\eta} = -\frac{3}{4}.
$$

Adding the terms for $n = 2$ and $n = 4$ (and multiplying by the term B outside) yields the final equality, i.e., that $L\delta f_u(y) = -B\left(\frac{3}{4} + \frac{3}{2}\left(\frac{y}{2}\right)^2\right)$.

We now calculate $K\delta P(y)$. From (52) we obtain:

$$
\begin{aligned}
K\delta P(y) &= \frac{\int \left(v - \frac{1}{2}y\right) f_v(v) f_u'(y-v) E_u\left[\frac{\partial}{\partial v} v\delta P(v+u)\right] dv}{\int f_v(v') f_u(y-v') dv'} \\
&= \frac{\int \left(v - \frac{1}{2}y\right) f_v(v) f_u'(y-v) E_u\left[\frac{\partial}{\partial v} v\left(a(v+u)^2 + b\right)\right] dv}{\int f_v(v') f_u(y-v') dv'} \\
&= b\frac{\int \left(v - \frac{1}{2}y\right)(v-y) f_v(v) f_u(y-v) dv}{\int f_v(v') f_u(y-v') dv'} + a\frac{\int \left(v - \frac{1}{2}y\right)(v-y) f_v(v) f_u(y-v) E_u\left[(v+u)^2 + 2(v+u)\right] dv}{\int f_v(v') f_u(y-v') dv'}.
\end{aligned}
$$

The first term on the right-hand side reduces to

$$
\begin{aligned}
b\frac{\int \left(v - \frac{1}{2}y\right)(v-y) f_v(v) f_u(y-v) dv}{\int f_v(v') f_u(y-v') dv'} &= b\frac{\int \left(v - \frac{y}{2}\right)(v-y) f_u(v) f_u(y-v) dv}{\int f_u(v') f_u(y-v') dv'} \\
&= b\frac{\int \left(v - \frac{y}{2}\right)^2 f_u(v) f_u(y-v) dv}{\int f_u(v') f_u(y-v') dv'} = \frac{b}{2},
\end{aligned} \tag{69}
$$

where the final equality follows from substituting the functional form of the normal densities and integrating. The second term reduces to:

$$
\begin{aligned}
& a\frac{\int \left(v - \frac{1}{2}y\right)(v-y) f_v(v) f_u(y-v) E_u\left[(v+u)^2 + 2v(v+u)\right] dv}{\int f_v(v') f_u(y-v') dv'} \\
&= a\frac{\int \left(v - \frac{1}{2}y\right)(v-y) f_v(v) f_u(y-v)\left(3v^2+1\right) dv}{\int f_v(v') f_u(y-v') dv'} \\
&= a\frac{\int \left(v - \frac{y}{2}\right)(v-y) f_u(v) f_u(y-v)\left(3v^2+1\right) dv}{\int f_u(v') f_u(y-v') dv'} \\
&= \frac{a}{2} + 3a\frac{\int \left(v - \frac{y}{2}\right)(v-y) v^2 f_u(v) f_u(y-v) dv}{\int f_u(v') f_u(y-v') dv'} = a\left(\frac{1}{2} + 3\left(\frac{3}{4} - \frac{y^2}{8}\right)\right) = a\left(\frac{11}{4} - \frac{3}{8}y^2\right).
\end{aligned} \tag{70}
$$

Here the first equality uses the unit variance of u and the final equality follows from substituting the functional form of the normal densities and integrating. Adding the two terms, (69) and (70), yields

$$
K\delta P(y) = \frac{b}{2} + \frac{11}{4}a - \frac{3a}{8}y^2. \tag{71}
$$

Substituting this solution for $K\delta P(y)$ and the solution, $L\delta f_u(y) = -B\left(\frac{3}{4} + \frac{3}{2}\left(\frac{y}{2}\right)^2\right)$, into equation (51), $\delta P - K\delta P = L\delta f_u$, yields a self-consistent equation that determines the coefficients a and b:

$$ay^2 + b - \left(\frac{1}{2}b + \frac{11}{4}a\right) + \frac{3a}{8}y^2 = -B\left(\frac{3}{4} + \frac{3}{2}\left(\frac{y}{2}\right)^2\right). \tag{72}$$

Matching the coefficients in (72), yields a linear system

$$\begin{aligned} \frac{11}{8}a &= -\frac{3}{8}B \\ \frac{1}{2}b - \frac{11}{4}a &= -\frac{3}{4}B, \end{aligned} \tag{73}$$

with unique solution: $a = -\frac{3}{11}B$ and $b = -3B$. Thus the pricing rule shift is $\delta P(y) = -3B\left(1 + \frac{1}{11}y^2\right)$.

Finally, we evaluate the J term in the perturbed BNE. Recall that:

$$2J = -\int f_u(y-x)(y-x)\frac{\partial}{\partial y}\delta P(y)\,dy - \int f_u(y-x)\left(\Psi(y,\delta P) + (y-2x)\frac{\partial}{\partial y}\Psi(y,\delta P)\right)dy, \tag{74}$$

where $\Psi(y)$ is given by (64).

Substituting for $\delta P(y) = ay^2 + b$ into (64) yields:

$$\begin{aligned} \Psi(y) &= ay^2 + b - \frac{\int f_u(\xi) f_u(y-\xi)\frac{\partial}{\partial \xi}\left(\xi \int f_u(y'-\xi)\left(a(y')^2 + b\right)dy'\right)d\xi}{\int f_u(\xi') f_u(y-\xi')\,d\xi'} \\ &= a\left(y^2 - \frac{\int f_u(\xi) f_u(y-\xi)\frac{\partial}{\partial \xi}\left(\xi \int f_u(\eta-\xi)\eta^2 d\eta\right)d\xi}{\int f_u(\xi') f_u(y-\xi')\,d\xi'}\right). \end{aligned} \tag{75}$$

Evaluating the last integral term on the right-hand side, i.e., evaluating $K_\phi \delta P$, yields:

$$\begin{aligned} K_\varphi \delta P &= a\frac{\int f_u(\xi) f_u(y-\xi)\frac{\partial}{\partial \xi}\left(\xi \int f_u(\eta-\xi)\left((\eta-\xi)^2 + 2\xi(\eta-\xi) + \xi^2\right)d\eta\right)d\xi}{\int f_u(\xi') f_u(y-\xi')\,d\xi'} \\ &= a\frac{\int f_u(\xi) f_u(y-\xi)\frac{\partial}{\partial \xi}\left(\xi\left(1+\xi^2\right)\right)d\xi}{\int f_u(\xi') f_u(y-\xi')\,d\xi'} = a\frac{\int f_u(\xi) f_u(y-\xi)\left(1+3\xi^2\right)d\xi}{\int f_u(\xi') f_u(y-\xi')\,d\xi'} \\ &= a\frac{\int f_u(\xi) f_u(y-\xi)\left(1 + 3\left(\xi - \frac{y}{2} + \frac{y}{2}\right)^2\right)d\xi}{\int f_u(\xi') f_u(y-\xi')\,d\xi'} = \frac{3a}{4}y^2 + \frac{5a}{2}, \end{aligned} \tag{76}$$

where the first line expands the argument being integrated and uses $\int f_u(\eta-\xi)(\eta-\xi)d\eta = 0$. Substitution back into (75) yields $\Psi(y) = a\left(\frac{y^2}{4} - \frac{5}{2}\right)$ and $\Psi'(y) = \frac{a}{2}y$. Substituting these solutions and the unit variance of u into (74) yields

$$-2J = aE_{u|y=x+u}\left[2yu + \frac{y^2}{4} - \frac{5}{2} + (y-2x)\frac{y}{2}\right] = -\frac{a}{4}\left(x^2-1\right), \tag{77}$$

and hence $J(v,X) = \frac{a}{8}\left(X(v)^2-1\right) = \frac{a}{8}\left(v^2-1\right)$. Thus, by Proposition 1, the speculator's equilibrium optimal trading strategy with commitment is shifted with respect to the BNE strategy by [13]

$$\delta X_S(v) - \delta X(v) \;=\; -J\left(v, X_S^*(v), X_S^*\right) = \frac{a}{8}\left(v^2-1\right). \quad \square \tag{78}$$

[13] We also have the results for the perturbation with both cubic and quartic terms, $\delta log(f_u) = Bu^3 - Cu^4$, with $C > 0$, in which case the distribution formally remains normalizable for any $y \in R$ and the correction to pricing rule is $\delta P(y) = cy^3 + ay^2 + dy + b$. Still, the range of applicability is limited by $|\delta log(f_u)| << u^2/2$ and hence $|\delta P(y| << y/2$. Specifically, we have $c = \frac{1}{5}C, a = -\frac{6}{19}B, d = \frac{17}{10}C, b = -\frac{66}{114}B$. Similar to the cubic perturbation, the J term depends on realization of the fundamental v. The analysis becomes more complex, but the results remain qualitative the same.

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